# Weighted Poincaré Inequalities for Solutions to A-Harmonic Equations

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**Abstract.** We first prove a local $A_r$-weighted Poincaré inequalities for solutions to $A$-harmonic equations of the form: $d^\star A(x, d\omega) = B(x, d\omega)$. Then, as an application of the local result, we prove a global $A_r$-weighted Poincaré inequality for functions that are solutions to the such equations in John domains.

## 1. Introduction

Poincaré inequalities are now ubiquitous in analysis. We mention only [9], [2] and especially [3] for geometric applications of these inequalities.

In contrast, we show here that, for certain $A$-harmonic tensors, a weak local Poincaré inequality holds in $\mathbb{R}^n$ for all positive exponents. This borrows results from [4], [5], [7] and [8].

Using this result we obtain a global weighted Poincaré inequality of $A$-harmonic functions in John domains for all positive exponents.

We assume $\Omega$ is a connected open subset of $\mathbb{R}^n$ throughout this paper. Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $l = 0, 1, \cdots, n$, the linear space of $l$-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_l}$, corresponding to all ordered $l$-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathbb{R}^n)$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all $l$-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the Hodge star operator $\star : \wedge \to \wedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all $\alpha, \beta \in \wedge$.

Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \langle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbb{R}$.

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The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star(-1)^{l(n-l)} : \wedge^l \to \wedge^l$. Let $0 < p < \infty$, we denote the weighted $L^p$-norm of a measurable function $f$ over $E$ by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x) dx\right)^{1/p}.$$

A differential $l$-form $\omega$ on $\Omega$ is a Schwartz distribution on $\Omega$ with values in $\wedge^l(\mathbb{R}^n)$. We denote the space of differential $l$-forms by $D'(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the $l$-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered $l$-tuples $I$. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,E} = \left(\int_E |\omega(x)|^p dx\right)^{1/p} = \left(\int_E (\sum_I |\omega_I(x)|^2)^{p/2} dx\right)^{1/p}.$$

Similarly, $W_p^1(\Omega, \wedge^l)$ are those differential $l$-forms on $\Omega$ whose coefficients are in $W_p^1(\Omega, \mathbb{R})$. The notations $W_{p,loc}^1(\Omega, \mathbb{R})$ and $W_{p,loc}^1(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$.

We consider here solutions to the equation

$$d^\star A(x, d\omega) = B(x, d\omega), \tag{1.1}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the following conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1}, \quad \langle A(x,\xi), \xi\rangle \ge |\xi|^p \text{ and } |B(x,\xi)| \le b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). From now on, the exponent $p$ will represent this exponent throughout this paper. A solution to (1.1) is an element of the Sobolev space $W_{p,loc}^1(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle + \langle B(x, d\omega), \varphi\rangle = 0$$

for all $\varphi \in W_{p,\mathrm{loc}}^1(\Omega, \wedge^{l-1})$ with compact support.

**Definition 1.3.** We call $u$ an $A$-harmonic tensor in $\Omega$ if $u$ satisfies the $A$-harmonic equation (1.1) in $\Omega$.

**Example 1.4.** We call $u$ a $p$-harmonic function if $u$ satisfies the $p$-harmonic equation

$$\mathrm{div}(\nabla u |\nabla u|^{p-2}) = 0$$

with $p > 1$.

## 2. The local weighted Poincaré inequality

For a measurable set $E \subset \mathbb{R}^n$ we write $|E|$ for the $n$ dimensional Lebesgue measure of $E$. Throughout $Q \subset \mathbb{R}^n$ is a cube and $\sigma Q$, $\sigma > 0$, denotes the cube with the same center as $Q$ and $|\sigma Q| = \sigma^n |Q|$.

**Definition 2.1.** We say the weight $w(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies the $A_r$ condition, $r > 1$, write $w \in A_r$, if $w(x) > 0$ a.e., and

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w dx \right) \left( \frac{1}{|Q|} \int_Q \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)} < \infty$$

for all $Q \subset \mathbb{R}^n$.

See [1] and [2] for the basic properties of $A_r$-weights. We also need the following lemma which is a reverse Hölder inequality [1].

**Lemma 2.2.** *If $w \in A_r$, then there exist constants $\beta > 1$ and $C$, independent of $w$, such that*

$$\| w \|_{\beta, Q} \leq C |Q|^{(1-\beta)/\beta} \| w \|_{1,Q}$$

*for all $Q \subset \mathbb{R}^n$.*

The following Lemma 2.3 appears in [8].

**Lemma 2.3.** *Let $u$ be an $A$-harmonic tensor in $\Omega$, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant $C$, independent of $u$, such that*

$$\|u\|_{s,Q} \leq C |Q|^{(t-s)/st} \|u\|_{t, \sigma Q}$$

*for all $Q$ with $\sigma Q \subset \Omega$.*

Lemma 2.4 contains the classical Poincaré inequality as well as a generalization to differential forms given in [4]. When $\omega$ is a function we denote its average value over $Q$ by

$$\omega_Q = |Q|^{-1} \int_Q \omega(y) \, dy.$$

Otherwise $\omega_Q$ is the exterior derivative of a suitable transform of $\omega$ and plays the role of average value in the Poincaré inequality. See [4].

**Lemma 2.4.** *Let $u \in D'(Q, \wedge^l)$ and $du \in L^q(Q, \wedge^{l+1})$. Then $u - u_Q$ is in $W^1_q(Q, \wedge^l)$ with $1 < q < \infty$ and*

$$\|u - u_Q\|_{q,Q} \le C(n,q)|Q|^{1/n}\|du\|_{q,Q}$$

*for $Q$ in $\mathbb{R}^n$, $l = 0, 1, \cdots, n$.*

We next state a Caccioppoli-type inequality.

**Lemma 2.5.** *Let $u$ be a solution to (1.1) in $\Omega$ and $\sigma > 1$. There exists a constant $C$, depending only on $a$, $b$, $p$ and $n$, such that*

$$\|du\|_{p,Q} \le C|Q|^{-1/n}\|u\|_{p,\sigma Q} \tag{2.6}$$

*for all $Q$ with $\sigma Q \subset \Omega$.*

For this result see [8] and [7]. We also need a result from [5].

**Lemma 2.7.** *Suppose that $|v| \in L^s_{\mathrm{loc}}(\Omega)$, $\sigma > 1$ and $0 < t < s$. If there exists a constant $A$ such that*

$$\|v\|_{s,Q} \le A|Q|^{(t-s)/st}\|v\|_{t,2Q} \tag{2.8}$$

*for all cubes $Q$ with $2Q \subset \Omega$, then for all $r > 0$, there exists a constant $B$, depending only on $\sigma$, $n$, $s$, $t$, $r$ and $A$, such that*

$$\|v\|_{s,Q} \le B|Q|^{(r-s)/sr}\|v\|_{r,\sigma Q}$$

*for all $Q$ with $\sigma Q \subset \Omega$.*

**Lemma 2.9.** *Suppose that $u$ is a solution to (1.1), $\sigma > 1$ and $q > 0$. There exists a constant $C$, depending only on $\sigma$, $n$, $p$, $a$, $b$ and $q$, such that*

$$\|du\|_{p,Q} \le C|Q|^{(q-p)/pq}\|du\|_{q,\sigma Q} \tag{2.10}$$

*for all $Q$ with $\sigma Q \subset \Omega$.*

**Proof.** By Lemmata 2.5, 2.3 and 2.4 with $p' = (p+1)/2$

$$\begin{aligned}\|du\|_{p,Q} &\le C_1|Q|^{1/n}\|u - u_{\sigma Q}\|_{p,\sqrt{\sigma}Q} \\ &\le C_2|Q|^{(p'-p)/pp'}\|u - u_{\sigma Q}\|_{p',\sigma Q} \\ &\le C_3|Q|^{(p'-p)/pp'}\|du\|_{p',\sigma Q}.\end{aligned}$$

As such $du$ satisfies the reverse Hölder inequality (2.8) and (2.10) follows from Lemma 2.7.

We also require a result from [7].

**Lemma 2.11.** *There exists a constant $C$, depending only on $n$ and $q$, such that*

$$\|v - v_Q\|_{q,Q} \le C\|v - c\|_{q,Q}$$

*for all $v \in L^q(Q,\Lambda)$ and all $c \in \mathcal{D}'(Q,\Lambda)$ with $dc = 0$. Here $1 < q < \infty$ and $v_Q$ is defined in Definition 1.7.*

Now we have the following local weighted Poincaré inequality for $A$-harmonic tensors.

**Theorem 2.12.** *Let $u \in D'(\Omega, \wedge^l)$ be an $A$-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $du \in L^s(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$. Assume that $\sigma > 1$, $0 < s < \infty$ and $w \in A_r$ for some $r > 1$. Then*

$$\|u - u_Q\|_{s,Q,w} \le C|Q|^{1/n}\|du\|_{s,\sigma Q,w} \tag{2.13}$$

*for all cubes $Q$ with $\sigma Q \subset \Omega$. Here $C$ is a constant independent of $u$.*

**Proof.** Choose $t = s\beta/(\beta - 1)$ where $\beta$ is the exponent in Lemma 2.2. Then $1 < s < t$ and $\beta = t/(t - s)$. Using Lemma 2.2 and Hölder's inequality,

$$\begin{aligned}\|u - u_Q\|_{s,Q,w} &= \left(\int_Q \left(|u - u_Q|w^{1/s}\right)^s\right)^{1/s} \\ &\le \|u - u_Q\|_{t,Q}\|w\|_{\beta,Q}^{1/s} \\ &\le C|Q|^{(1-\beta)/\beta s}\|w\|_{1,Q}^{1/s}\|u - u_Q\|_{t,Q}.\end{aligned} \tag{2.14}$$

Next choose $\alpha = s/r$ so that $\alpha < s < t$. If $\alpha > 1$ and $t > 1$, then using Lemmata 2.11, 2.3 and 2.4,

$$\begin{aligned}\|u - u_Q\|_{t,Q} &\le C\|u - u_{\sigma Q}\|_{t,Q} \\ &\le C|Q|^{(\alpha - t)/\alpha t}\|u - u_{\sigma Q}\|_{\alpha,\sigma Q} \\ &\le C|Q|^{(\alpha t + n\alpha - nt)/n\alpha t}\|du\|_{\alpha,\sigma Q}.\end{aligned} \tag{2.15}$$

If $t \le 1$, then first

$$\begin{aligned}\|u - u_Q\|_{t,Q} &\le C|Q|^{(2-t)/2t}\|u - u_Q\|_{2,Q} \\ &\le C|Q|^{(2-t)/2t}\|u - u_{\sigma Q}\|_{2,\sqrt{\sigma}Q} \\ &\le C|Q|^{(\alpha - t)/\alpha t}\|u - u_Q\|_{\alpha,\sigma Q}\end{aligned}$$

and again (2.15) follows.

If $\alpha \le 1$, then using Lemmata 2.3, 2.11 and 2.4,

$$
\begin{aligned}
\|u-u_Q\|_{t,Q} &\le C|Q|^{(p-t)/pt}\|u-u_Q\|_{p,\sqrt{\sigma}Q} \\
&\le C|Q|^{(p-t)/pt}\|u-u_{\sqrt{\sigma}Q}\|_{p,\sqrt{\sigma}Q} \\
&\le C|Q|^{(p-t)/pt}|Q|^{1/n}\|du\|_{p,\sqrt{\sigma}Q}.
\end{aligned}
\tag{2.16}
$$

Applying (2.10), (2.16) becomes

$$
\|u-u_Q\|_{t,Q} \le C|Q|^{(\alpha t+n\alpha-nt)/n\alpha t}\|du\|_{\alpha,\sigma Q}. \tag{2.17}
$$

Next

$$
\|du\|_{\alpha,\sigma Q} \le \|du\|_{s,\sigma Q,w}\|1/w\|^{1/s}_{\alpha/(s-\alpha),\sigma Q}. \tag{2.18}
$$

Combining (2.14), (2.15), (2.17) and (2.18),

$$
\|u-u_Q\|_{s,Q,w} \le C|Q|^{(\alpha-n)/n\alpha}\left(\|w\|_{1,Q}\|1/w\|_{\alpha/(s-\alpha),\sigma Q}\right)^{1/s}\|du\|_{s,\sigma Q,w}. \tag{2.19}
$$

Finally Definition 2.1 gives the result

$$
\|u-u_Q\|_{s,Q,w} \le C|Q|^{1/n}\|du\|_{s,\sigma Q,w}.
$$

## 3. A global result in John domains

We here consider solutions $u$ to $\operatorname{div} A(x,\nabla u) = B(x,\nabla u)$ in $\Omega \subset \mathbb{R}^n$, which we call $A$-harmonic functions. We write, for $d\mu = w\,dx$, and denote the $\mu$-average of the function $u$ over the cube $Q$:

$$
u_{Q,\mu} = \frac{1}{\mu(Q)}\int_Q u\,d\mu.
$$

We assume that $0 < \mu(Q) < \infty$ for all $Q$.

**Definition 3.1.** A $\delta$-John domain is a bounded domain $\Omega \subset \mathbb{R}^n$ with John center $x_0$ if:

every point $x \in \Omega$ can be joined to
$x_0$ by a continuous curve $\gamma \subset \Omega$ for
which $d(\xi,\partial\Omega) \ge \delta|\xi - x|$ for all $\xi \in \gamma$.

Also we define the sharp norm of a real-valued function $f$ over $E$ by

$$
\|f\|^{\sharp}_{p,E,w} = \inf_{a\in\mathbf{R}}\left(\int_E |f-a|^p\,d\mu\right)^{1/p}.
$$

To obtain a global result we apply a result from [6]:

**Theorem 3.2** *Suppose that $f$ and $g$ are measurable in a $\delta$-John domain $\Omega$ with distinguished cube $Q_0 \subset \Omega$ and $0 < q < \infty$. If, for some constant $A$,*

$$\|f\|^{\sharp}_{q,Q,w} \leq A\|g\|_{q,\sigma Q,w}$$

*for all cubes $Q$ with $\sigma Q \subset \Omega$, then there exists a constant $B$, depending only on $n$, $q$, $\sigma$ and $\delta$, such that*

$$\|f\|^{\sharp}_{q,\Omega,w} \leq AB\|g\|_{q,\Omega,w}.$$

*(Also see [5].)*

Together with the local result, this gives us a Poincaré inequality over John domains.

**Theorem 3.3.** *Suppose that $u$ is an $A$-harmonic function in a $\delta$-John domain $\Omega$, $0 < q < \infty$ and $w \in A_r(\Omega)$. There exists a constant $C$, depending only $q$, $\delta$, $n$, $p$ and $r$, such that*

$$\|u\|^{\sharp}_{q,\Omega,w} \leq C|\Omega|^{1/n}\|\nabla u\|_{q,\Omega,w}.$$

We remark that in the case $q \geq 1$ and $0 < \mu(E) < \infty$, we have

$$\begin{aligned}\|f\|^{\sharp}_{q,E,\mu} &\leq \|f - f_{E,\mu}\|^{\sharp}_{q,E,\mu} \\ &\leq 2\|f\|^{\sharp}_{q,E,\mu}.\end{aligned}$$

See [6]. Using this we have Corollary 3.4.

**Corollary 3.4.** *Along with the hypotheses of Theorem 3.3, we assume that $q \geq 1$ and $u_{\Omega,\mu} < \infty$. We have*

$$\|u - u_{\Omega,\mu}\|_{q,\Omega,w} \leq 2C|\Omega|^{1/n}\|\nabla u\|_{q,\Omega,w}.$$

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