Systoles of Two Extremal Riemann Surfaces

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1. Introduction

The purpose of this note is to study the π_1 and H_1 systoles of two Riemann surfaces, the Klein genus 3 surface, $y^7 = x^2(1-x)$, which we will denote S_K, and Bolza's surface of genus 2, $y^2 = x(1-x^4)$, which we will denote S_B. The π_1-systoles for these surfaces and many others were identified by Schmutz [Schm]. They are closed geodesics corresponding to conjugacy classes in the fundamental group. The H_1 systoles are more properly called the L_2H^1 systoles [G], but we will call them H_1 systoles for short. The H_1 systoles correspond to cycles on the surface. Using period matrices, we identify the H_1 systoles and show for these examples that the π_1 and H_1 systoles are in one to one correspondence (Theorem 1). It is unclear to what extent this behaviour is typical of extremal surfaces. We hope that the computations will be useful in studying the relationship between the length and the Jacobi spectra for other triangle surfaces and for Riemann surfaces in general.

Our main interest stems from the fact that in their respective Teichmüller spaces, S_B is known to be extremal and S_K locally extremal for these systoles. Both have the maximum number of automorphisms, 48 and 168 respectively, for their genus. When this work was begun, it was thought that S_K would be extremal for the H_1 systole. A subsequent computation has shown, however, that the H_1 systole of the Picard curve $y^3 = x^4 - 1$ is larger.

Both of these surfaces can be represented as $\Gamma \setminus H$, where H is the upper half plane and Γ is a normal subgroup of an appropriate oriented triangle group T^*. It follows that the automorphism group is T^*/Γ. For both surfaces the Jacobi lattices can be identified by a cyclotomic construction due to M. Craig [C]. In this construction the lattice is a ring of cyclotomic integers with a trace (sum of conjugates) norm.

We compute the H_1-systoles by computing a period matrix. Period matrices for these surfaces have been computed by many authors. For S_K see [Ba], [RL], [W], [EN], [R], [Ma] for S_B see [Schi]. Two different techniques are used. One is to choose cycles and one forms and do the integrations explicitly using the beta function [B][W]. One uses the action of the automorphism group on both the one forms and one cycles. The action on one forms is seen in the transformation formulas for the beta functions. The other technique is to choose a canonical basis cycles and find a period matrix from the action on the cycles by the automorphism group. We choose the former because it more directly gives the cyclotomic construction by which we recognize these lattices. Both techniques amount to determining the symplectic Jacobian lattice from the subgroup of its automorphism group determined by the automorphism group of the surface. The method can be used to compute period matrices for the Fermat curves [L]. For examples such as the Bring curve in genus 4, the automorphism group of the surface is not enough to determine the lattice [RR].

The goal of our construction is to identify the cycles on minimum norm in terms of a fundamental domain in the unit disk and correlate them with the π_1-systoles. Previous computations of the period matrices have not been concerned with the metric structure on the Jacobian. As an example, the lattice (form) $A_6^{(2)}$ can be computed from the period matrix of Klein's surface. This matrix was computed as early as 1907 by Baker [Bak]. However, the discovery of the extremal form $A_6^{(2)}$ is credited to Barnes in 1955 [Bar].

2. Length and Jacobi spectra

Let $\Gamma \subset PSL_2(\mathbb{R})$ be a group of hyperbolic isometries of the upper half-plane H such that $S = \Gamma \setminus H$ is a compact Riemann surface of genus g. We know that Γ may be identified with the fundamental group $\pi_1(S)$. Consider the map $\ell : \Gamma \to \mathbb{R}$ defined by $\ell = 2\,\mathrm{arccosh}(|\mathsf{tr}|/2)$, where tr is the trace of a matrix representative in $SL_2(\mathbb{R})$, then $\ell(A)$ is the length of the shortest closed geodesic which is freely homotopic to A [El] [Mc]. The function $\ell(A)$ depends only on the conjugacy class of A in Γ. The image of ℓ on conjugacy classes, with corresponding multiplicity, is referred to as the ***length spectrum*** S.

We will define the Jacobi spectrum in terms of the Jacobian of S. We recall the definition of the Jacobian. Let $\Omega(S)$ he the holomorphic 1-forms on S, or differentials of the first kind. It is a complex vector space of dimension g. Let $\Omega^*(S)$ be its complex dual. Since cycles in $H_1(S,\mathbb{Z})$ act linearly on $\Omega(S)$ by integration, we may consider $H_1(S,\mathbb{Z})$ as a subset of $\Omega^*(S)$. A Hermitian inner product is defined on $\Omega(S)$ by

$$\langle \omega, \sigma \rangle = \frac{i}{2} \int_S \omega \wedge \overline{\sigma}\,.$$

By duality this defines a Hermitian inner product $\langle\ \ ,\ \ \rangle$ on $\Omega^*(S)$. Since $H_1(S,\mathbb{Z})$ is a $2g$ dimensional sublattice of $\Omega^*(S)$, $\mathsf{Re}\langle\ \ ,\ \ \rangle$ defines a quadratic form. By the Riemann bilinear relations $\mathsf{Re}\langle\ \ ,\ \ \rangle$ has determinant 1, and $\mathsf{Im}\langle\ \ ,\ \ \rangle$ is the intersection form on $H_1(S,\mathbb{Z})$. Thus, it is skew symmetric and integer valued. A quadratic form of determinant 1 which comes as the real part of a Hermitian form whose imaginary part is skew symmetric, non-singular, and integer valued, is called ***symplectic***. Likewise the corresponding lattice is called symplectic, hence $H_1(S,\mathbb{Z})$ has the structure of a symplectic lattice. We call $H_1(S,\mathbb{Z})$ with its inner product the ***Jacobian lattice***. The torus $\Omega^*(S)/H_1(S,\mathbb{Z})$ is called the ***Jacobian***. We consider the function $N_J : H_1(S,\mathbb{Z}) \to \mathbb{R}$ defined by $N_J(\gamma) = \|\gamma\|^2 = \langle \gamma, \gamma \rangle =$ the norm of γ in the Jacobi Lattice. We refer to the image of N_J with multiplicity as the ***Jacobi spectrum***.

There is a natural map $p : \Gamma \to H_1(S,\mathbb{Z})$ given by sending each homotopy class to the corresponding homology class. Clearly $p(A)$ depends only on the conjugacy class of A in Γ. This map p gives a natural correspondence between the Jacobi spectrum and the length spectrum.

3. Extremal surfaces

We will consider Riemann surfaces which are extremal in terms of the length and Jacobi spectra. For a surface S define

$$N(S) = \min_{\gamma \neq 0} N_J(\gamma)$$

$$\ell(S) = \min_{A \neq I} \ell(A).$$

Also define

$$N_g = \max_S N(S)$$

$$\ell_g = \max_S \ell(S)$$

where the maximum is taken over all compact surfaces S of genus g. We consider extremal surfaces S for which $N(S)$ or $\ell(S)$ is a local or global maximum in the Teichmüller space for compact surfaces of genus g. If $A \in \Gamma$ is such that $\ell(A) = \ell(S)$ we say that the conjugacy class pair $\{A, A^{-1}\}$ is a ***π_1-systole of S***. Thus a π_1-systole can be thought of as a directionless closed geodesic of minimal length. Similarly if $\gamma \in H_1(S, \mathbb{Z})$ is such that $N_J(\gamma) = N(S)$ we say that the pair $\{\gamma, -\gamma\}$ is a ***H_1-systole of S*** [G].

In the paper by Buser and Sarnak [BS], the upper bound

$$N_g \leq \frac{3}{\pi} \log(4g+3)$$

is given. Mention is made of the surfaces of Bolza nd Klein, S_B and S_K, for which

$$N_2 = N(S_B) = \sqrt{2}$$

and

$$N_3 \geq N(S_K) = \frac{4}{\sqrt{7}}.$$

The Jacobian lattice for S_B is $2^{-1/4} D_4$ and for S_K is $7^{-1/6} A_6^{(2)}$, as we show below.

We note that D_4 is the extreme form in dimension 4 [CS2]. This shows that $N_3 = N(S_B)$.

The form $A_6^{(2)}$ is one of six extremal forms in dimension 6 [Bar] [CS2]. This shows that $N(S_K)$ is a local maximum for genus 3. It can be shown that none of the other extreme forms are symplectic. However, since this paper was first written, we have show that for the Picard curve, S_P, we have $N(S_P) = 1 + \sqrt{3}/3 > 4/\sqrt{7}$. Extremals for $\ell(S)$ have been studied by P. Schmutz [Schm]. He shows, for example, that

$$\ell_2 = \ell(S_B) = 2 \operatorname{arccosh}(1 + \sqrt{2})$$
$$\ell_3 > \ell(S_K) = 8 \operatorname{arccosh}\left(\frac{3}{2} - 2\sin^2\frac{\pi}{7}\right)$$

and he shows that $\ell(S_K)$ is a local maximum. He has also shown that $\ell(S_P)$ is a local maximum and $\ell(S_P) > \ell(S_K)$.

It is known that S_B has 12 π_1-systoles and 12 H_1-systoles and that S_K has 21 π_1-systoles and 21 H_1-systoles. The main object of this note is to prove the following:

Theorem 1. *For the surface S_B, $p : \pi_1(S_B) \to H_1(S_B, \mathbb{Z})$ maps the 12 π_1-systoles one-to-one onto the 12 H_1-systoles. Similarly for the 21 systoles of the surface S_K.*

We now describe the cyclotomic constructions for the lattices D_4 and $A_6^{(2)}$.

4. Cyclotomic Construction of D_4

The simplest description of D_4 is

$$D_4 = \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \big| x_0 + x_1 + x_2 + x_3 \text{ is even}\} \tag{1}$$

where the standard inner product on $\mathbb{R}^4$ is used. The minimum norm is 2 and the vectors of minimum norm for D_4 are seen to be the ones obtained from $(1, 1, 0, 0)$ and $(1, -1, 0, 0)$ by permutations and changes of sign. There are 24 of these. The symplectic structure, which is useful in identifying the vectors of minimum norm in the Jacobian of S_B is more easily seen from the following cyclotomic construction.

Let $\zeta = \exp(\pi i/4)$ and consider the ring of cyclotomic integers $\mathbb{Z}[\zeta]$, that is, the set of polynomials of the form

$$f(\zeta) = \sum_{k=0}^{3} x_k \zeta^k, \qquad x_k \in \mathbb{Z},$$

with multiplication using the identity $\zeta^4 + 1 = 0$. Define the trace by

$$\mathsf{Tr}\ f(\zeta) = f(\zeta) + f(\zeta^3).$$

(This is the sum of two of the four conjugates of $f(\zeta)$.) We note that

$$\begin{aligned} \mathsf{Tr}(\zeta) = \mathsf{Tr}(\zeta^3) &= \sqrt{2}\, i \\ \mathsf{Tr}(\zeta^2) &= 0 \\ \mathsf{Tr}(1) &= 2. \end{aligned} \tag{2}$$

We define a Hermitian inner product on $\mathbb{Z}[\zeta]$ using the trace:

$$\langle f, g\rangle = \frac{1}{2}\,\mathsf{Tr}(f(\zeta)\overline{g(\zeta)}),$$

where the bar denotes the complex conjugate. We refer to $\langle f, f\rangle$ as the ***trace norm*** of f. The trace norm is simply the length of the vector $(1/\sqrt{2})(f(\zeta), f(\zeta^3))$ in $\mathbb{C}^2$, and the corresponding Hermition inner product is the same as the one defined by the trace.

Let $\mathcal{B}$ be the basis for $\mathbb{Z}[\zeta]$ given by ζ^k, $k = 0, \ldots, 3$. Using (2) above we see that the matrix of inner products, or Gram matrix, of $\mathbb{Z}[\zeta]$ with respect to this basis is

$$\mathrm{Gram}(\mathbb{Z}[\zeta], \mathcal{B}) = \frac{1}{2}\bigtriangledown(2, -\sqrt{2}\,i, 0, -\sqrt{2}\,i).$$

Here $\bigtriangledown(\alpha_0, \ldots, \alpha_{n-1})$ denotes the $n \times n$ Hermitian matrix given by

$$\bigtriangledown(\alpha_0, \ldots, \alpha_{n-1})_{ij} = \begin{cases} \alpha_{j-i}, & j \geq i \\ \overline{\alpha_{i-j}} & j < i. \end{cases}$$

We see from this that the real part of the Gram matrix is the identity, so

$$\|f\|^2 = \sum_{k=0}^{3} x_k^2.$$

Thus the map $\sum_{k=0}^{3} x_k\zeta^k \to (x_0, x_1, x_2, x_4)$ shows that as a real lattice $\mathbb{Z}[\zeta]$ with the trace norm is equal to $\mathbb{Z}^4$.

Now it is easy to see that $f(\zeta)$ can be written in the form $(1-\zeta)g(\zeta)$ if and only if $f(1) = 0 \pmod 2$. Thus by (1) the ideal $((1-\zeta))$ in $\mathbb{Z}[\zeta]$ corresponds to D_4. Using (2) again gives that if $\mathcal{B}' = \{\zeta^k(1-\zeta) | k = 0, \ldots, 3\}$ is a basis for $D_4 = ((1-\zeta))$ and we have the Gram matrix

$$\mathrm{Gram}\big(((1-\zeta)), \mathcal{B}'\,\big) = \bigtriangledown(2, -1+\sqrt{2}\,i, -\sqrt{2}\,i, 1+\sqrt{2}\,i).$$

The real part, $\bigtriangledown(2, -1, 0, 1)$, has determinant 4. The corresponding Gram matrix for $2^{-1/4}D_4$ is $2^{-(1/2)}\bigtriangledown(2, -1, 0, 1) + \bigtriangledown(0, i, -i, i)$. The determinant of the real part is 1 and the imaginary part is non-singular with integer entries. It follows that $2^{-1/4}D_4$ is symplectic.

The vectors of minimum norm correspond in $\mathbb{Z}[\zeta]$ to $\zeta^k(1+\zeta)$, $\zeta^k(1-\zeta)$, and $\zeta^k(1+\zeta^2)$, $k = 0, \ldots, 7$.

5. Cyclotomic construction of $A_6^{(2)}$

This construction is useful in identifying the vectors of minimum norm in the Jacobian lattice of S_K, The simplest description of the 6 dimensional lattice $A_6^{(2)}$ is

$$A_6^{(2)} = \left\{ (x_0, \ldots, x_6) \in \mathbb{Z}^7 \,\middle|\, \sum_{k=0}^{6} x_k = 0, \sum_{k=0}^{6} k x_k = 0 \pmod 7 \right\} \tag{3}$$

using the standard inner product on $\mathbb{R}^7$. From this description it can be seen that the minimum norm is 4 and the vectors of minimum norm are rotations of $(1, -1, -1, 1, 0, 0, 0)$, $(1, -1, 0, -1, 1, 0, 0)$, and $(1, 1, 0, -1, 0, -1, 0)$ together with their negatives. There are 42 of these.

For the cyclotomic construction of $A_6^{(2)}$, let $\zeta = \exp(\pi i/7)$ and let $\mathbb{Z}[\zeta]$ be the ring of cyclotomic integers, the set of all $f(\zeta) = \sum_{k=0}^{6} x_k \zeta^k$, $x_k \in \mathbb{Z}$ with multiplication using the identity $1 + \zeta + \zeta^2 + \ldots + \zeta^6 = 0$. Define

$$\begin{aligned} \mathsf{Tr}\; f(\zeta) &= f(\zeta) + f(\zeta^2) + f(\zeta^4) \\ &= \sum_{k=0}^{2} f(\zeta^{2^k}) \end{aligned}$$

and note that

$$\begin{aligned} \mathsf{Tr}\; 1 &= 3 \\ \mathsf{Tr}\; \zeta = \mathsf{Tr}\; \zeta^2 = \mathsf{Tr}\; \zeta^4 &= -\frac{1}{2} + \frac{\sqrt{7}}{2} i \\ \mathsf{Tr}\; \zeta^3 = \mathsf{Tr}\; \zeta^5 = \mathsf{Tr}\; \zeta^6 &= -\frac{1}{2} - \frac{\sqrt{7}}{2} i. \end{aligned} \tag{4}$$

Now define the inner product by

$$\langle f, g \rangle = \frac{2}{7}\, \mathsf{Tr}(f(\zeta)\overline{g(\zeta)}).$$

With respect to the generating set $\mathcal{B} = \{\zeta^k | k = 0, \ldots, 6\}$, we see by (4) that the matrix of inner products is

$$\mathrm{Gram}(\mathbb{Z}[\zeta], \mathcal{B}) = \frac{1}{7} \bigtriangledown(6, -1, -1, -1, -1, -1, -1) + \frac{1}{\sqrt{7}} \bigtriangledown(0, -i, -i, i, -i, i, i)$$

and the resulting norm is

$$\|f\|^2 = \sum_{k=0}^{6} x_k^2 - \frac{1}{7}\left(\sum_{k=0}^{6} x_k\right)^2. \tag{5}$$

If $f \in ((1-\zeta))$ then f can be represented uniquely as $f(\zeta) = \sum\limits_{k=0}^{6} x_k \zeta^k$, with $\sum\limits_{k=0}^{6} x_k = 0$, and the norm by (5) becomes $\sum\limits_{k=0}^{6} x_k^2$. For $f \in ((1-\zeta))$, $f \in ((1-\zeta)^2)$ if and only if $f'(1) = 0$ (mod 7), that is, $\sum\limits_{k=0}^{6} k x_k = 0$ (mod 7). Thus by (3), $((1-\zeta)^2)$ with the trace norm can be identified with $A_6^{(2)}$. The Gram matrix with respect to the basis $\mathcal{B} = \{\zeta^k(1-\zeta)^2 | k = 0, \ldots, 5\}$ is

$$\mathrm{Gram}\big(((1-\zeta)^2), \mathcal{B}\big) = \triangledown(6, -4, 1, 0, 0, 1) + \sqrt{7}\triangledown(0, 0, 1, -2i, -2i, -1).$$

The determinant of the real part is 7^3. Thus $7^{-1/4} A_6^{(2)}$ symplectic lattice.

In $((1-\zeta)^2) \subseteq \mathbb{Z}[\zeta]$ the vectors of minimum norm correspond to

$$\zeta^k(1-\zeta)^2(1+\zeta), \quad \zeta^k(1-\zeta)^2(1+\zeta+\zeta^2), \quad \zeta^k(1-\zeta)^2(1+\zeta)^2(1+\zeta^2), \quad k = 0, \ldots, 6$$

and their negatives.

6. The surface S_B

The equation for S_B is $y^2 = x(1-x^4)$ and we wish to describe a fundamental domain for S_B in the hyperbolic disc. Let F_B be the interior of the equilateral octagon with interior angles $\pi/4$ centered at the origin as in figure 1. The sides are numbered $1, \ldots, 8$. It can be divided into 96 triangles with angles $\pi/2$, $\pi/3$ and $\pi/8$ as indicated. Let $F_{B,0}$ be the sector of F_B with arg in the interval $(-\pi/8, 0)$, then $F_{B,0}$ can be mapped conformally onto $\mathcal{S} = \{x | 0 \le \arg x \le \pi/4\}$ such that $0 \to \infty$ and the part of side 1 in the lower half plane goes to $[0, 1]$. Think of $\mathcal{S}$ as being the sheet of the surface $y^2 = x(1-x^4)$ over $\mathcal{S}$, with the image of side 1 corresponding to $\sigma = \{(x, x^{1/2}(1-x^4)^{1/2}) | 0 \le x \le 1\}$ where positive square roots are taken. Extending the map by reflection sends F_B with opposite edges identified onto the Riemann surface of $y^2 = x(1-x^4)$.

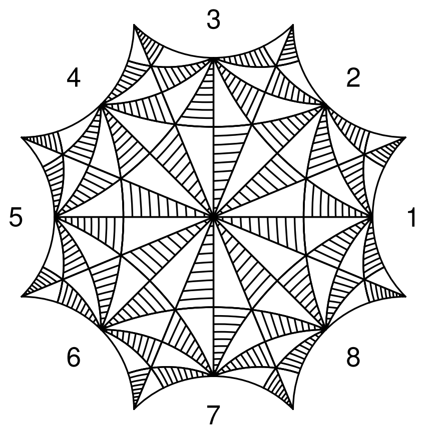

Figure 1

The side pairing maps can be expressed in terms of the triangle group $T(2,3,8)$. Let T be rotation on angle $\pi/4$ about the origin, $T(z) = \zeta z, \quad \zeta = \exp\left(\frac{\pi i}{4}\right)$. Let U be the rotation an angle π about the first right angle vertex on the positive real axis. Using hyperbolic trigonometry [Be], we compute that U is given as a fractional linear transformation by the matrix

$$U = M^{-1} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} M \tag{6}$$

where

$$M = \frac{1}{1-a^2} \begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix} \qquad\qquad a = \sqrt{(1-s)/(1+s)}$$

and $s = 2\sin(\pi/8)$.

The side pairing maps can now be described in terms of T and U. Let $A_k = T^k(UT^4)^2T^{8-k}$, $k = 0,\ldots,7$, then A_k maps side $k+1$ to $k+5$ and $A_{k+4} = A_k^{-1}$ (where subscripts are mod 8). The group Γ generated by the side pairing maps $\{A_0, A_1, A_2, A_3\}$ is free on these generators with the one relation $A_0{A_1}^{-1}A_2{A_3}^{-1}{A_0}^{-1}A_1{A_2}^{-1}A_3 = I$. The relation can be written as $\prod_{k=0}^{7} A_{5k} = I$ if we use the convention that products are written from left to right starting with the bottom index.

Note that A_k is represented in $\pi_1(S_B)$ by the closed geodesic segment from the center of side $k+1$ through the origin, to the center of side $k+5$. This projects to a closed geodesic on the surface. As shown in Schmutz [Schm], these give 4 of the 12 π_1-systoles for S_B, and the rest are images of these under automorphisms of S_B (figure 2). One can verify by computation in the triangle group that the conjugacy classes of elements $\{A_k, A_kA_{k+5}A_{k+10}, A_kA_{k+3} \big| k = 0, \ldots, 3\}$ are invariant under the automorphism group $T^*(2,3,8)/\Gamma$ of S_B (you need check only conjugation by T and U), and these give the 12 π_1-systoles. They correspond to the curves in figure 2.

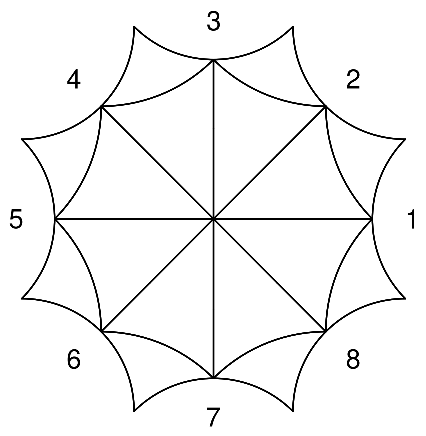

Figure 2

To verify the above statement that the conjugacy classes are invariant one uses the identities

$$TA_kT^{-1} = A_{k+1} \quad k = 0, \ldots, 7$$

$$UA_0U = A_4, \quad UA_1U = A_4A_7$$

$$UA_2U = A_5A_2A_7, \quad UA_3U = A_5A_0.$$

We remark that computations can be done in the triangle group either using matrix representatives for T and U, or using the relations $T^8 = U^2 = (TU)^3 = I$. We can also compute in Γ using the action on the vertices of F_B. Any element of Γ leaving a vertex fixed must be the identity.

Let $B_k = A_{k+5}A_{k+2}A_{k+7}$, so that B_k corresponds to the boundary edge $k-1$ oriented counterclockwise. The π_1-systoles are then given by

$$B_k, B_{k+2}B_{k+1}B_k, B_{k+1}B_k, \quad k = 0, \ldots, 7. \tag{7}$$

We next compute a period matrix for S_B. We identify S_B, as above, with $\{(x,y)|y^2 = x(1-x^4)\}$ together with a point at infinity. For $k = 0, \ldots, 7$ let β_k be the boundary cycle corresponding to B_k. Consider the holomorphic differentials on the curve defined by $\omega_0 = dx/y$ and $\omega_1 = \zeta^3 x\,dx/y$. We compute the period matrix with respect to the cycles β_k, $k = 0, \ldots, 3$ and forms ω_j, $j = 0, 1$.

First note that the automorphism T is given on the curve by $T(x,y) = (-ix, \zeta^3 y)$, so that

$$\begin{aligned} T^*\omega_0 &= \zeta^3\omega_0 \\ T^*\omega_1 &= \zeta\omega_1. \end{aligned} \tag{8}$$

Since T^* is an isometry on $\Omega(S_B)$, we deduce from (8) that $\langle\omega_0, \omega_1\rangle = 0$. By a change of variables $x \to 1/x$,

$$\begin{aligned} \langle\omega_0, \omega_0\rangle &= \frac{i}{2}\int\limits_{x\in\mathbb{C}} \frac{dx \wedge d\overline{x}}{|x|\,|1-x^4|} \\ &= \frac{i}{2}\int\limits_{x\in\mathbb{C}} \frac{dx \wedge d\overline{x}}{|1-x^4|} \\ &= \langle\omega_1, \omega_1\rangle \end{aligned}$$

where the integration above is over the complex plane. Thus $\{\omega_0, \omega_1\}$ is a multiple of an orthonormal basis on $\Omega(S_B)$.

Let $\sigma = \{x, x^{1/2}(1-x^4)^{1/2} | 0 \le x \le 1\}$ be the part of side 1 in the lower half plane as above. Then $\beta_k = T^k(\sigma - T^4\sigma)$ and hence by (8),

$$\begin{aligned} \int\limits_{\beta_k} \omega_0 &= 2\zeta^{3k}\int\limits_{\sigma}\omega_0 \\ \int\limits_{\beta_k} \omega_1 &= 2\zeta^{k}\int\limits_{\sigma}\omega_1. \end{aligned}$$

Now we compute that

$$\begin{aligned}
\int_\sigma \omega_0 &= \int_0^1 x^{-(1/2)}(1-x^4)^{-(1/2)}\,dx \\
&= \frac{1}{4}\int_0^1 t^{-(7/8)}(1-t)^{-(1/2)}\,dt \\
&= \frac{1}{4}B\left(\frac{1}{8},\frac{1}{2}\right) \\
&= \frac{1}{4\pi}\Gamma_1\Gamma_3\Gamma_4 \sin\frac{3\pi}{8} \\
&= \frac{\exp(\pi i/8)\Gamma_1\Gamma_3\Gamma_4}{8\pi i}(\zeta+\zeta^2),
\end{aligned}$$

where $\Gamma_k = \Gamma(k/8)$ and $B(x,y)$ is the beta function given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Similarly,

$$\int_\sigma \omega_1 = \frac{\exp(\pi i/8)\Gamma_1\Gamma_3\Gamma_4}{8\pi i}(\zeta^3-\zeta^2).$$

So for $C = \Gamma_1\Gamma_3\Gamma_4\exp(\pi i/8)/4\pi i$ we have

$$\begin{aligned}
\int_{\beta_k} \omega_0 &= C\zeta^{3k}(\zeta+\zeta^2) \\
\int_{\beta_k} \omega_1 &= C\zeta^{k}(\zeta^3-\zeta^2).
\end{aligned}$$

Note that the integral of ω_1 over β_k is obtained from the integral of ω_0 over β_k by substitution of ζ^3 for ζ, i.e., by conjugation in $\mathbb{Q}(\zeta)$. Since $\{\omega_1,\omega_2\}$ is a multiple of an orthonormal basis, so is the dual basis. Since the coordinates are conjugates, the norm of a cycle in the Jacobian is a constant multiple of the trace norm.

Consider the map

(9) $$\beta_k \to \zeta^k(\zeta^3-\zeta^2) = -\zeta^{k+2}(1-\zeta).$$

This identifies $H_1(S_B, \mathbb{Z})$ with the ideal $((1-\zeta)) \subset \mathbb{Z}[\zeta]$. The following correspondence can be computed from (9):

$$\beta_k + \beta_{k+1} \mapsto -\zeta^{k-2}(1+\zeta^2)$$
$$\beta_k + \beta_{k+1} + \beta_{k+2} \mapsto -\zeta^{k+1}(1+\zeta).$$

These correspond to the vectors of minimum norm in D_4, and by (7), they are the image of the conjugacy classes belonging to the π_1-systoles. This verifies the first part of Theorem 1.

7. The surface S_K

Klein's surface S_K has equation $y^7 = x^2(1-x)$. Let F_K be the interior of the equilateral 14-gon with interior angles $2\pi/7$, centered at the origin as in figure 3. It can be divided into 336 triangles with angles $\pi/2$, $\pi/3$, $\pi/7$ as indicated. If $F_{K,0}$ is the part of F_K in the sector $0 \leq \arg z \leq \pi/7$, then $F_{K,0}$ can be mapped conformally onto the upper half-plane with 0 going to ∞ and the side 1 going to $[0,1]$. Think of $F_{K,0}$ as being mapped to the sheet of the Riemann surface of $y^7 = x^2(1-x)$ over the upper half-plane, with side 1 corresponding to $\{(x, x^{2/7}(1-x)^{1/7}) | 0 \leq x \leq 1\}$, where positive 7th roots are taken. Extending by reflection, F_K with identification of sides $k \to k+5$ for k odd is mapped conformally onto the Riemann surface of the curve $y^7 = x^2(1-x)$.

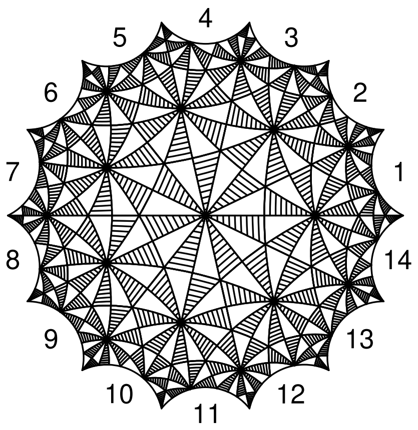

Figure 3

The side pairing maps can be expressed in terms of the triangle group $T(2,3,7)$. Let T be rotation an angle $\pi/7$ about the origin, given by $Tz = \zeta z$, with $\zeta = \exp(\pi i/7)$, and let U be rotation an angle π about the first right angle vertex on the positive real axis; i.e., given as in (6), with $s = 2\sin(\pi/7)$. The side pairing maps are seen to be $A_k = T^k(T^3U)^4T^{7-k}$, $k = 0,\dots,6$, with A_k mapping side $2k+1$ to $2k+6$ (indices taken mod 7). The group Γ generated by $\{A_k\}_{k=0}^6$ is free on these generators with the two relations, $\prod_{k=0}^{6} A_{4k} = I$ and $\prod_{k=0}^{6} A_{5k} = I$.

Note that A_k in $\pi_1(S_K)$ is given by the geodesic segment from side $2k+1$ to $2k+6$ which is perpendicular to both sides. This projects to a closed geodesic on the surface. These give 7 of the 21 π_1-systoles [Schm], the others being obtained from these by automorphisms of S_K . Figure 4 shows the images in the fundamental region of curves representing the 21 systoles. One group of 7 is seen as one geodesic segment. The other two groups are broken into 2 and 3 segments respectively. The boundary segments are shown, but they are not on any of the systoles.

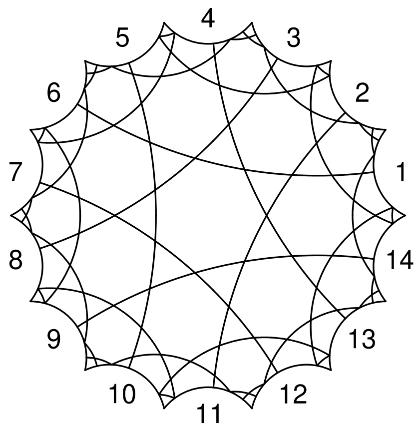

Figure 4

One verifies that the conjugacy classes of the elements $\{A_k, A_k^{-1}A_{k+3}^{-1}, A_k^{-1}A_{k+2}^{-1}A_{k+3}^{-1}\}$, $k = 0, \ldots, 6$ are closed under the automorphism group, $T^*(2,3,7)/\Gamma$, of S_K. These give the 21 π_1-systoles. Some useful identities in verifying the above are:

$$UA_0U = A_4, \quad UA_1U = A_0^{-1}A_3^{-1}, \quad UA_2U = A_0^{-1}A_2^{-1}A_4^{-1},$$
$$UA_3U = A_1^{-1}A_4^{-1}, \quad UA_4U = A_0, \quad UA_5U = A_5^{-1}, \quad UA_6U = A_6^{-1}$$

These computations can be done in terms of the elements B_k, $k = 0, \ldots, 6$, of Γ corresponding to the boundary curves $1+2, 3+4, \ldots, 13+14$. Let $B_k = A_{k+1}^{-1}A_{k+4}^{-1}$, then B_k corresponds to the curve $(2k+1)+(2k+2)$, $k = 0, \ldots, 6$. We compute that $B_{k+2}B_{k+1}B_k = A_k$, $\quad k = 0, \ldots, 6$. Thus in terms of the B_k the π_1-systoles are given by

$$\begin{gathered} B_k = A_{k+1}^{-1}A_{k+4}^{-1}, \quad B_{k+2}B_{k+1}B_k = A_k \\ B_k^{-1}B_{k+1}^{-1}B_{k+2}^{-2}B_{k+3}^{-1}B_{k+4}^{-2}B_{k+5}^{-1}B_{k+6}^{-1} = A_k^{-1}A_{k+2}^{-1}A_{k+4}^{-1} \end{gathered} \tag{10}$$

To compute a period matrix for S_K, identify it as above with the Riemann surface $\{(x,y)|y^7 = x^2(x-1)\}$ together with a point at ∞. For $k = 0, \ldots, 6$, let β_k be the cycles in $H_1(S_B, \mathbb{Z})$ corresponding to the elements $B_k \in \Gamma$. Consider the holomorphic differentials on S_K defined by $\omega_0 = y\,dx/x(x-1)$, $\omega_1 = y^2\,dx/x(x-1)$, $\omega_2 = y^4\,dx/x^2(x-1)$. We compute the period matrix with respect to the cycles β_j, $j = 0, \ldots, 6$ and forms ω_k, $k = 0, 1, 2$.

Note that T is given on the Riemann surface by $T(x,y)=(x,\zeta^2 y)$, $\zeta=e^{2\pi i/7}$. Thus

$$T^*\omega_0=\zeta^4\omega_0,\quad T^*\omega_1=\zeta\omega_1,\quad T^*\omega_2=\zeta^2\omega_2 \tag{11}$$

The above three formulas may be summarized by

$$T^*\omega_k=\zeta^{2^{k+2}}\omega_k,\quad k=1,2.$$

Using the beta function as before we compute that

$$\int_{\beta_0}\omega_k=C\mu^{2j+2}(1+\mu)^2(1+\mu^2)(1-\mu)^2$$
$$\mu=\zeta^{2^k}$$

for some constant C.

As in the case of S_B, we can see from (11) that the forms ω_k are mutually orthogonal. Further, the change of variables $x\to x/(1-x)$ in the integrals shows that the norms are equal. Now $(1+\zeta)^2(1+\zeta^2)$ is a unit in $\mathbb{Z}[\zeta]$ so the map

$$\beta_k\to\zeta^{2k+2}(1+\zeta)^2(1+\zeta^2)(1-\zeta)^2 \tag{12}$$

identifies $H_1(S_K,\mathbb{Z})$ with the ideal $((1-\zeta)^2)\subset\mathbb{Z}[\zeta]$. The above shows that the norm in the Jacobian is a multiple of the trace norm in $\mathbb{Z}[\zeta]$ defined in section 5. From (11) we compute that

$$\beta_k+\beta_{k+1}+\beta_{k+2}\mapsto\zeta^{2k+5}(1+\zeta+\zeta^2)(1-\zeta)^2$$
$$\beta_k+\beta_{k+2}\mapsto\zeta^{2k+2}(1+\zeta)(1-\zeta)^2.$$

These correspond to the vectors of minimum norm in $A_6^{(2)}$ and by (10) they are images of the conjugacy classes belonging to the π_1-systoles. This verifies the second part of Theorem 1.

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