Elliptic Diffusions, Exit Times and the Quasihyperbolic Metric

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Abstract

We show that the expected lifetime of a conditioned elliptic diffusion in a bounded domain in $\mathbb{R}^n$ is bounded above by an absolute constant times the sum of the volume of the domain and the p^{th} power of the L^p-norm of the quasihyperbolic metric all raised to the power $2/n$, with $p > 2/n$. It follows in an L^p-averaging domain that this lifetime is bounded above by a constant times the volume to the power $2/n$

1 Introduction

Throughout $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 2$ and X_t is the diffusion associated with a given divergence form operator, L, in Ω. Similar to Doob's h-conditioned Brownian motion the diffusion is conditioned by an L-harmonic function h. We write $P_t^h(x, y)$ for the conditioned transition density of the diffusion X_t killed when it leaves Ω, P_x^h for the measure on path space induced by P_t^h for X_t starting at $x \in \Omega$ and E_x^h for the corresponding expectation. Also τ_Ω^X denotes the first exit time of X_t from Ω and τ_Ω that exit time when X_t is Brownian motion. (See Section 2.) Also throughout, $|\Omega|$ is the n-dimensional Lebesgue measure of Ω.

In 1983, Cranston and McConnell published the following result [4].

Theorem A *If $n = 2$, then $E_x^h(\tau_\Omega) \leq C$ area(Ω) where C is a universal constant.*

As such the expected exit time scales directly with $|\Omega|$.

In 1985 Cranston showed that if $n \geq 3$ and Ω is a bounded Lipschitz domain, then

$$E_x^h(\tau_\Omega) \leq C(\Omega)$$

where $C(\Omega)$ depends only on n and Ω, see [3].

Then in 1987 Bañuelos extended these results to elliptic diffusions and, for $n \geq 3$, for uniform domains [1].

A domain $\Omega \subset \mathbb{R}^n$ is a uniform domain if there exists constants a and b such that each pair of points $x_1, x_2 \in \Omega$ can be joined by a rectifiable arc $\gamma \in \Omega$ for which

$$s(\gamma) \leq a|x_1 - x_2|$$

and

$$\min_{j=1,2} s(\gamma(x_j, x)) \leq bd(x, \partial\Omega)$$

for all $x \in \gamma$. Here $s(\gamma)$ denotes the euclidean length of γ, $\gamma(x_j, x)$ the part of γ between x_j and x, and $d(x, \partial\Omega)$ denotes the euclidean distance between x and the boundary of Ω, $\partial\Omega$. See [6].

Theorem B *If $n = 2$, then*

$$E_x^h(\tau_\Omega^X) \leq \frac{4}{\lambda\pi}\ \mathrm{area}(\Omega)$$

where λ is the ellipticity constant.

If $n \geq 3$ and if Ω is a uniform domain, then

$$E_x^h(\tau_\Omega^X) \leq C|\Omega|^{2/n}.$$

Here C is a constant that depends on n, the uniform domain constants, the ellipticity constants and the radius and diameter of Ω.

Theorem C, our main result, will show that for $n \geq 3$, the bounds on the expected lifetime will scale as $|\Omega|^{2/n}$ in a wide class of domains, which includes uniform domains as a subclass.

We emphasize that the constant in (1.1) depends only on the dimension n, the integrability exponent p, of the quasihyperbolic metric in Ω and the ellipticity constants of the diffusion.

We first define the quasihyperbolic metric in Ω:

$$k(x_0, x) = \inf_\gamma \int\limits_\gamma \frac{1}{d(\xi, \partial\Omega)}\, ds(\xi).$$

Here ds is arclength measure and the infimum is over all rectifiable curves γ in Ω that join $x_0 \in \Omega$ to $x \in \Omega$.

Also for $F : \Omega \to \mathbb{R}$ we write

$$\|F\|_{p,\Omega} = \|F(x)\|_{p,\Omega} = \left(\int_\Omega |F(x)|^p \, dx \right)^{1/p} .$$

Theorem C *If $n \geq 3$ and $p > n/2$, then*

$$E_y^h(\tau_\Omega^X) \leq C(n,p)(|\Omega| + \|k(x_0,x)\|_{p,\Omega}^p)^{2/n} \tag{1.1}$$

for all $y, x_0 \in \Omega$. Here $C(n,p)$ is a constant depending only on n, p and the ellipticity constants λ and Λ.

In view of a result of Sue Staples' [10] we make the following definition.

Definition 1.1 *We call Ω an L^p-averaging domain for $p > 1$, if there exists a constant C such that*

$$\|k(x_0,x)\|_{p,\Omega}^p \leq C|\Omega| \tag{1.2}$$

for some $x_0 \in \Omega$.

Corollary D *If Ω is an L^p-averaging domain for some $p > n/2$, then for all $y \in \Omega$*

$$E_y^h(\tau_\Omega^X) \leq C|\Omega|^{2/n}$$

where C is a constant that depends only on n, p, the constant in inequality (1.2) and the ellipticity constants λ and Λ.

Many important types of domains satisfy (1.2). To put Theorem C in perspective, we briefly discuss this. We call Ω a John domain if there is a point $x_0 \in \Omega$ and a constant $\delta > 0$ that satisfy the condition:

Every point $x \in \Omega$ can be joined to x_0 by a rectifiable curve $\gamma \subset \Omega$ where

$$d(\xi, \partial\Omega) \geq \delta|\xi - x|$$

for all $\xi \in \gamma$.

In a John domain the quasihyperbolic metric is in fact exponentially integrable with a bound that depends only on n and the John constant δ. This is true more generally for Hölder domains.

We call Ω a Hölder domain if there are constants c_1 and c_2 such that

$$k(x_0, x) \leq c_1 \log \frac{d(x_0, \partial\Omega)}{d(x, \partial\Omega)} + c_2$$

for all $x \in \Omega$ and some fixed $x_0 \in \Omega$. See [9].

In contrast to uniform domains, John domains may have internal cusps while Hölder domains and L^p-averaging may also have certain external cusps. The following diagram shows the relationships between these domains.

$$\begin{array}{c}
\textit{\textbf{Bounded Uniform Domain}} \\
\\
\Downarrow \quad \not\Uparrow \\
\\
\textit{\textbf{John Domain}} \\
\\
\Downarrow \quad \not\Uparrow \\
\\
\textit{\textbf{Hölder Domain}} \\
\\
\Downarrow \quad \not\Uparrow \\
\\
\mathbf{L^p}\textit{\textbf{-averaging for }}\mathbf{p > 1}
\end{array}$$

Corollary E *If Ω is a John domain, then*

$$E_y^h(\tau_\Omega^X) \leq C|\Omega|^{2/n} \tag{1.3}$$

for $y \in \Omega$ where C is a constant that depends only on n, the John constant δ and the ellipticity constants λ and Λ.

If Ω is a Hölder domain, then (1.3) holds where C depends only on n, the Hölder constants c_1 and c_2 and the ellipticity constants λ and Λ.

2 Elliptic Diffusions

We consider operators L in divergence form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right).$$

Here a_{ij} are bounded measurable functions, $a_{ij} = a_{ji}$ and there are positive constants λ, Λ satisfying the ellipticity condition

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost all $x \in \mathbb{R}^n$.

A function h in the Sobolev space $W^1_{2,\mathrm{loc}}$ is a weak solution to

$$Lh = 0$$

if

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial}{\partial x_i} \varphi \frac{\partial}{\partial x_j} h = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. These solutions h are called L-harmonic.

A uniformly elliptic diffusion X_t is associated with L. The diffusion can be constructed using the theory of Dirichlet forms, [5]. If $P_t(x,y)$ is the transition density killed upon leaving Ω, then the h-conditioned transition density is given by

$$P_t^h(x,y) = \frac{1}{h(x)} P_t(x,y) h(y)$$

for a positive L-harmonic function h. Moreover, τ_Ω^X denotes the first exit time of X_t from Ω.

3 Proof of Theorem C

Fix an L-harmonic function $h(x)$ and a point $x_0 \in \Omega$. By the form of the transition function we may normalize so that $h(x_0) = 1$. Let W be a Whitney decomposition of Ω and suppose that $x_0 \in Q_0 \in W$. Here W is a collection of closed cubes $\{Q\}$ with disjoint interiors such that

$$\frac{1}{4} \text{ diam } Q \leq d(Q, \partial\Omega) \leq 4 \text{ diam } Q \tag{i}$$

and

$$\Omega = \bigcup Q. \tag{ii}$$

Here diam Q is the euclidean diameter of Q and $d(Q, \partial\Omega)$ is the euclidean distance between Q and $\partial\Omega$. See [11]. We write $d(Q_0, Q_j)$ for the number of Whitney cubes in the shortest Whitney path joining Q_0 to Q_j, including Q_0 and Q_j. It follows from Moser's Harnack inequality [8] that there exists a constant $C_0 > 2$, depending only on n and the ellipticity constants λ and Λ, such that

$$C_0^{-m+1} < h(x) < C_0^{m+1} \tag{3.1}$$

if $x \in Q_j \in W$ with $d(Q_0, Q_j) = m$. Define the layers

$$\Omega_m = \left\{ x \in \Omega | C_0^{m-1} < h(x) < C_0^{m+1} \right\}.$$

Using an argument of Chung, Bañuelos obtains

$$E_x^h(\tau_\Omega^X) \leq C \sum_{m=-\infty}^{\infty} |\Omega_m|^{2/n} \tag{3.2}$$

where C is a constant that depends only on n, λ and Λ. See [1] and [2].

Using the notation

$$A_S = \{x \in \Omega | x \in Q_j \in W \text{ with } d(Q_0, Q_j) > S\},$$

$S = 0, 1, 2, \ldots$, it follows from (3.1) that

$$\Omega_{m+2} \subset A_{|m|} \text{ for } m \in \mathbb{Z}.$$

With this (3.2) becomes

$$E_x^h(\tau_\Omega^X) \leq CN|\Omega|^{2/n} + C \sum_{m=N}^{\infty} |A_m|^{2/n} \tag{3.3}$$

for any $N \geq 1$. See [1].

Lemma 3.1 *Let $x_0 \in Q_0 \in W$. If $x \in Q_j$, then*

$$d(Q_0, Q_j) \leq c(n)k(x, x_0) + 1$$

where $c(n)$ is a constant that depends only on n.

This result appears in [7]. With $B_t = \{x \in \Omega | k(x, x_0) \geq (t-1)/c(n)\}$, for $t \geq 1$, Lemma 3.1 implies that $|A_m| \leq |B_m|$.

As such, with $N = c(n) + 1$,

$$\begin{aligned} \sum_{m=N}^{\infty} |A_m|^{2/n} &\leq \sum_{m=N}^{\infty} |B_m|^{2/n} \\ &\leq \sum_{m=N}^{\infty} \int_{m-1}^{m} |B_t|^{2/n} \, dt \\ &\leq \int_{N-1}^{\infty} |B_t|^{2/n} \, dt \\ &= c(n) \int_{1}^{\infty} |\{x \in \Omega | k(x, x_0) \geq \lambda\}|^{2/n} \, d\lambda. \end{aligned} \tag{3.4}$$

Recall that $n > 2$ and $p > \frac{n}{2}$. Define $\varepsilon > 0$ by $p = \frac{n}{2} + \varepsilon \left(\frac{n-2}{2}\right)$. By Hölder's inequality

$$\begin{aligned} \int_{1}^{\infty} |\{x \in \Omega | k(x, x_0) \geq \lambda\}|^{2/n} \, d\lambda &\leq \left(\int_{1}^{\infty} \lambda^{-(1+\varepsilon)} \, d\lambda \right)^{(n-2)/n} \\ &\quad \cdot \left(\int_{1}^{\infty} \lambda^{p-1} \, |\{x \in \Omega | k(x, x_0) \geq \lambda\}| \, d\lambda \right)^{2/n} \\ &\leq \varepsilon^{(2-n)/n} \|k(x, x_0)\|_p^{2p/n}. \end{aligned} \tag{3.5}$$

Theorem C follows by combining (3.3), (3.4) and (3.5).

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