CONVOLUTION AND RESTRICTION ESTIMATES FOR A 3-SURFACE IN $\mathbb{R}^5$

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Define $\phi : \mathbb{R}^3 \to \mathbb{R}^5$ by

$$\phi(x,y,z) = (x,y,z,x^2+y^2,y^2+z^2)$$

and let S be the 3-surface in $\mathbb{R}^5$ which is the range of ϕ. Let σ be the measure induced on S by 3-dimensional Lebesgue measure. Let T be the convolution operator defined, for functions f on $\mathbb{R}^5$, by

$$Tf = \sigma * f,$$

and let R^* be the adjoint Fourier restriction operator defined for functions g on S by

$$R^* g(\xi) = \widehat{g d\sigma}(\xi), \quad \xi \in \mathbb{R}^5.$$

Let T_0 and R_0^* be the localized operators obtained by replacing σ with its restriction σ_0 to the image under ϕ of the unit ball B in $\mathbb{R}^3$. We are interested in the problem of obtaining $L^p \to L^q$ bounds for the operators T, R^*, T_0 and R_0^* (though, unfortunately, we have positive results only for the local operators). Our approach is based on the idea, introduced in [O], that convolution inequalites of the form

$$\|\sigma_0 * \chi_E\|_{L^s(d\sigma_0)} \le C(s,t)\ |E|^{1/t}, \tag{1}$$

where $|E|$ is the Lebesgue measure of $E \subseteq \mathbb{R}^5$, can lead to estimates for the operators T_0 and R_0^*. An interesting aspect of the present setting is that the inequalities (1) which we shall obtain are consequences of certain mixed norm estimates for the x-ray transform [C].

We consider first the convolution operators. Let P be the point $(\frac{5}{7},\frac{2}{7})$. Let Δ be the triangle in $\mathbb{R}^2$ which is the convex hull of $(0,0)$, $(1,1)$, and P. If there is an estimate

$$\|T_0 f\|_{L^q(\mathbb{R}^5)} \le C(p,q)\|f\|_{L^p(\mathbb{R}^5)}, \tag{2}$$

then testing on the characteristic function of a small ball shows that $(\frac{1}{p},\frac{1}{q}) \in \Delta$. If there is a similar estimate for T, then a homogeneity argument shows that $\frac{1}{p}-\frac{1}{q}=\frac{3}{7}$ and thus that $(\frac{1}{p},\frac{1}{q}) = P$. Our result here is

Theorem 1. *If $(\frac{1}{p},\frac{1}{q})$ is a point in Δ different from P, then (2) holds.*

Turning to the restriction operators, if there is an estimate

$$\|R^*g\|_{L^q(\mathbb{R}^5)} \leq C(p,q)\|g\|_{L^p(\sigma)}, \tag{3}$$

then another homogeneity argument shows that $\frac{3}{p}+\frac{7}{q}=3$. The natural conjecture seems to be (see (4) in [M]) that this condition and $q>\frac{10}{3}$ are necessary and sufficient for (3). Our result is

Theorem 2. *If $p>\frac{12}{5}$ and $q>4$ then there is an estimate*

$$\|R_0^*g\|_{L^q(\mathbb{R}^5)} \leq C(p,q)\|g\|_{L^p(\sigma)}.$$

To place this result in perspective, observe that the point $(\frac{1}{p},\frac{1}{q})=(\frac{5}{12},\frac{1}{4})$ lies on the line $\frac{3}{p}+\frac{7}{q}=3$. Of course 4 is larger than the conjectured critical value $\frac{10}{3}$ for q, but the proof of Theorem 2 relies directly (and in the standard way) on the Hausdorff-Young Theorem, and $q=4$ is the best that can be expected from such proofs. The result of Theorem 2 can be interpolated with the trivial L^1-L^1 estimate for R_0^* to obtain bounds for other values of p and q.

As mentioned, the proofs of Theorems 1 and 2 depend on an estimate (1):

Theorem 3. *The estimate (1) holds whenever $s>\frac{5}{2}$ and $\frac{3}{s}=\frac{2}{t}$.*

To see how Theorem 1 follows from Theorem 3, let $E\subseteq\mathbb{R}^5$ be a Borel set and argue (as in [O?])

$$\|T_0\chi_E\|_{L^{s+1}(\mathbb{R}^5)}^{s+1} = \int_{\mathbb{R}^5}\int \chi_E(a-x)d\sigma_0(x)\Big(\int \chi_E(a-x_1)d\sigma_0(x_1)\Big)^s da =$$

$$\int_{\mathbb{R}^5}\chi_E(a)\int\Big(\int \chi_E(a+x-x_1)d\sigma_0(x_1)\Big)^s d\sigma_0(x)da.$$

With s and t as in Theorem 3, the last integral is dominated by $C(s,t)|E|^{(s+t)/t}$. With $\frac{1}{p}=\frac{5}{2(s+1)}$ and $\frac{1}{q}=\frac{2}{5}\frac{1}{p}$, this gives the estimate

$$\|T_0\chi_E\|_{L^q(\mathbb{R}^5)} \leq C(p)|E|^{1/p}$$

whenever $(\frac{1}{p},\frac{1}{q})$ is on the open segment with endpoints $(0,0)$ and P. The trivial L^1-L^1 estimate and standard duality and interpolation arguments then complete the proof of Theorem 1.

Recall that B is the unit ball in $\mathbb{R}^3$. As mentioned above, the proof of Theorem 2 relies on the Hausdorff-Young Theorem. Thus, aiming towards an estimate of

$$\|(\chi_{\phi(A)}d\sigma_0)*(\widetilde{\chi_{\phi(A)}d\sigma_0})\|_{L^{2,\infty}(\mathbb{R}^5)}$$

whenever A is a Borel subset of B, we begin by studying inequalities of the form

$$\int_A\int_A \chi_E\big(\phi(x_1,y_1,z_1)-\phi(x,y,z)\big)dx_1y_1z_1xyz \leq C(\alpha,\beta)|A|^\alpha|E|^\beta \tag{4}$$

where $A \subseteq B$, $E \subseteq \mathbb{R}^5$ are Borel sets and $|\cdot|$ indicates Lebesgue measure in the appropriate dimension. Such an inequality clearly holds for the pair $(\alpha, \beta) = (2, 0)$. Theorem 3 and Hölder's inequality imply the case

$$(\alpha, \beta) = \Big(\frac{1}{s'}, \frac{3}{2}\big(1 - \frac{1}{s'}\big)\Big)$$

of (4) whenever $1 \leq s' < \frac{5}{3}$. From these two estimates it follows that (4) holds for

$$(\alpha, \beta) = \Big(\frac{4s' - 5}{3s' - 3}, \frac{1}{2}\Big).$$

whenever $\frac{3}{2} < s' < \frac{5}{3}$. That is equivalent to the estimate

$$\|(\chi_{\phi(A)} d\sigma_0) * (\widetilde{\chi_{\phi(A)} d\sigma_0})\|_{L^{2,\infty}(\mathbb{R}^5)} \leq C(s')\ |A|^{\frac{4s'-5}{3s'-3}}.$$

It follows, because of compactness of support, that the inequality

$$\|(\chi_{\phi(A)} d\sigma_0) * (\widetilde{\chi_{\phi(A)} d\sigma_0})\|_{L^{r'}(\mathbb{R}^5)} \leq C(r', s')\ |A|^{\frac{4s'-5}{3s'-3}}$$

holds whenever $1 \leq r' < 2$, $\frac{3}{2} < s' < \frac{5}{3}$. Thus the Hausdorff-Young inequality yields

$$\|\widehat{\chi_{\phi(A)} d\sigma_0}\|_{2r}^2 \leq C(r', s')\ |A|^{\frac{4s'-5}{3s'-3}}$$

for $r > 2$ and $\frac{3}{2} < s' < \frac{5}{3}$. It follows that

$$\|\widehat{\chi_{\phi(A)} d\sigma_0}\|_q \leq C(p, q)\ |A|^{1/p}$$

whenever $q > 4$, $p > \frac{12}{5}$. An interpolation now gives Theorem 2.

We turn to the proof of Theorem 3. We adopt the convention that C denotes a positive constant which may change from line to line but depends only on the relevant indices. Theorem 3 follows from the estimate

$$\int_B \int_B f(x,y,z)\chi_E(x-x_1, y-y_1, z-z_1, x^2+y^2-x_1^2-y_1^2, y^2+z^2-y_1^2-z_1^2)dx_1 y_1 z_1 dxyz$$

$$\leq C\ \|f\|_{L^{s'}(\mathbb{R}^3)} |E|^{1/t} \tag{5}$$

for nonnegative Borel functions f on $\mathbb{R}^3$ whenever $s > \frac{5}{2}$, $\frac{3}{s} = \frac{2}{t}$ (and where B is the unit ball in $\mathbb{R}^3$). Changing notation, for the rest of the proof, B will be a ball centered at the origin in $\mathbb{R}^3$ and whose radius may increase from line to line. The proof of (5) begins with a series of changes of variable. Thus the LHS of (5) is equal to

$$\int_B \int_B f(x,y,z)\chi_E(-x_1, -y_1, -z_1, -2xx_1 - x_1^2 - 2yy_1 - y_1^2, -2yy_1 - y_1^2 - 2zz_1 - z_1^2)dx_1 y_1 z_1 dxyz.$$

Replacing E by a set with $\frac{1}{4}$ the measure of E, we can rewrite this as

$$\int_B \int_B f(x,y,z)\chi_E(x_1, y_1, z_1, xx_1 + yy_1, yy_1 + zz_1)dx_1 y_1 z_1 dxyz. \tag{6}$$

Letting $u = xx_1 + yy_1$, $du = |x_1|dx$, $v = yy_1 + zz_1$, $dv = |z_1|dz$ and then $\tau = \frac{y}{x_1 z_1}$ and $d\tau = \frac{dy}{|x_1 z_1|}$, it follows that (6) is bounded by

$$\int \int_B \int_{-\infty}^{\infty} f\big(\frac{u}{x_1} - y_1 z_1 \tau, x_1 z_1 \tau, \frac{v}{z_1} - x_1 y_1 \tau\big) d\tau \ \chi_E(x_1, y_1, z_1, u, v)\ dx_1 y_1 z_1\ duv.$$

The τ integral is clearly related to the x-ray transform of f. To exploit this, let $a = y_1 z_1$, $b = x_1 z_1$, $c = x_1 y_1$. Then

$$\frac{\partial(a,b,c)}{\partial(x_1,y_1,z_1)} = 2|x_1 y_1 z_1| = 2\sqrt{abc}$$

and it is enough to control integrals of the form

$$\int \int_B \int_{-\infty}^{\infty} f\Big(\frac{\pm ua}{\sqrt{abc}} - a\tau, b\tau, \frac{\pm vc}{\sqrt{abc}} - c\tau\Big) d\tau \ \chi_E\Big(\frac{\sqrt{abc}}{\pm a}, \frac{\sqrt{abc}}{\pm b}, \frac{\sqrt{abc}}{\pm c}, u, v\Big) \frac{1}{\sqrt{abc}}\ dabc\, duv$$

where the choice of $\pm$ is dictated by the signs of x_1, y_1, and z_1. The choice "all $+$'s" is typical. Then a change of variables leads to

$$\int \int_B \int_{-\infty}^{\infty} f(u - a\tau, b\tau, v - c\tau) d\tau \ \chi_E\Big(\frac{\sqrt{abc}}{a}, \frac{\sqrt{abc}}{b}, \frac{\sqrt{abc}}{c}, \frac{\sqrt{abc}}{a}u, \frac{\sqrt{abc}}{c}v\Big) \frac{|b|}{\sqrt{abc}}\ dabc\, duv.$$

Introducing polar coordinates $(a,b,c) = \rho(\eta_1, \eta_2, \eta_3) = \rho\eta$ in $\mathbb{R}^3$ leads to

$$\int \int_B \int_{-\infty}^{\infty} f(u - \rho\eta_1\tau, \rho\eta_2\tau, v - \rho\eta_3\tau) d\tau \cdot$$

$$\chi_E\Big(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_2}, \rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_3}, \rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}u, \rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_3}v\Big) \rho^{3/2} d\rho \frac{|\eta_2|}{\sqrt{|\eta_1\eta_2\eta_3|}} d\eta\, duv =$$

(7)

$$\int \int_{\Sigma_2} \int_{-\infty}^{\infty} f(u - \eta_1\tau, \eta_2\tau, v - \eta_3\tau) d\tau \cdot \int_0^{\infty} \chi_E\ \Big(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \dots\Big) \rho^{1/2} d\rho \frac{|\eta_2|}{\sqrt{|\eta_1\eta_2\eta_3|}} d\eta\, duv$$

where Σ_2 is the unit sphere in $\mathbb{R}^3$. The function $\frac{1}{\rho}$ is in $L^{5/2,\infty}(\rho^{3/2}d\rho)$. Thus

$$\int \chi_E\Big(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \dots\Big)\ \rho^{1/2} d\rho$$

is bounded by

$$C\Big(\int \chi_E(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \dots)\ \rho^{3/2} d\rho\Big)^{3/5}$$

and so, since ρ varies only over some finite interval $[0, M(B)]$, by

$$C\Big(\int \chi_E(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \dots)\ \rho^{3/2} d\rho\Big)^{1/t}$$

if $t > \frac{5}{3}$ (recall that $\frac{3}{s} = \frac{2}{t}$ and $s > \frac{5}{2}$). For such t, (7) is bounded by

$$C \int \int_{\Sigma_2} \int_{-\infty}^{\infty} f(u-\eta_1\tau, \eta_2\tau, v-\eta_3\tau)d\tau \Big(\int_0^{\infty} \chi_E\Big(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \dots\Big)\rho^{3/2}d\rho\Big)^{1/t} \frac{|\eta_2|}{\sqrt{|\eta_1\eta_2\eta_3|}}d\eta\, duv.$$

Hölder's inequality controls the integral above by the product of

$$\Big(\int \int_{\Sigma_2} \big(\int_{-\infty}^{\infty} f(u-\eta_1\tau, \eta_2\tau, v-\eta_3\tau)d\tau\big)^{t'} \frac{|\eta_2|}{\sqrt{|\eta_1\eta_2\eta_3|}}d\eta\, duv\Big)^{1/t'} \tag{8}$$

and

$$\Big(\int \int_{\Sigma_2} \int_0^{\infty} \chi_E\big(\rho^{1/2}\frac{\sqrt{|\eta_1\eta_2\eta_3|}}{\eta_1}, \dots\big)\ \rho^{3/2}d\rho \frac{|\eta_2|}{\sqrt{|\eta_1\eta_2\eta_3|}}d\eta\, duv\Big)^{1/t}. \tag{9}$$

Reversing the last two changes of variable shows that the integral in (9) is

$$\int \chi_E(x_1, y_1, z_1, x_1u, z_1v)|x_1|du|z_1|dvdx_1y_1z_1$$

and thus that (9) is bounded by $C\ |E|^{1/t}$. To estimate (8) we will apply a result of Christ on the x-ray transform: For a function f on $\mathbb{R}^3$ and for $\eta = (\eta_1, \eta_2, \eta_3) \in \Sigma_2$ and $x \in \{\eta\}^{\perp} \subseteq \mathbb{R}^3$ define

$$Xf(\eta, x) = \int_{-\infty}^{\infty} f(x-\tau\eta)d\tau.$$

For $\eta \in \Sigma_2$, let $\|Xf(\eta, x)\|_{L_x^{t'}}$ denote the $L^{t'}$ norm with respect to 2-dimensional Lebesgue measure on $\{\eta\}^{\perp}$. It follows from Theorem A in [C] that

$$\big(\int_{\Sigma_2} \|Xf(\eta, x)\|_{L_x^{t'}}^q d\eta\big)^{1/q} \leq C\ \|f\|_{L^{s'}(\mathbb{R}^3)} \tag{10}$$

if

$$\frac{3}{s} = \frac{2}{t},\ 1 \leq s' \leq 2,\ q = 2s. \tag{11}$$

Projecting the plane $\{\eta\}^{\perp}$ onto $\{(u, 0, v) : (u, v) \in \mathbb{R}^2\}$, the integral in (8) can be rewritten

$$\int_{\Sigma_2} \int_{\{\eta\}^{\perp}} \big(Xf(\eta, x)\big)^{t'} dx\ \frac{d\eta}{\sqrt{|\eta_1\eta_2\eta_3|}} = \int_{\Sigma_2} \|Xf(\eta, x)\|_{L_x^{t'}}^{t'} \frac{d\eta}{\sqrt{|\eta_1\eta_2\eta_3|}}.$$

If $q > t'$, (10) implies that (8) is bounded by

$$\Big(\int_{\Sigma_2} \|Xf(\eta, x)\|_{L_x^{t'}}^q d\eta\Big)^{\frac{1}{q}} \Big(\int_{\Sigma_2} \frac{d\eta}{|\eta_1\eta_2\eta_3|^{(q/t')'/2}}\Big)^{\frac{1}{t'(q/t')'}} \leq C\ \|f\|_{L^{s'}(\mathbb{R}^3)}$$

as long as (11) and $(\frac{q}{t'})' < 2$ hold. Thus, modulo these conditions, our bounds for (8) and (9) yield the desired bound

$$C\ \|f\|_{L^{s'}(\mathbb{R}^3)}|E|^{1/t}$$

for the LHS of (5). The hypotheses of Theorem 3 are $\frac{3}{s} = \frac{2}{t}$ and $s > \frac{5}{2}$. With $q = 2s$ it follows that $\frac{1}{q} = \frac{1}{3}\frac{1}{t}$ and so $\frac{t'}{q} = \frac{1}{3}\frac{t'}{t} = \frac{1}{3}(t'-1)$. Finally, since $t' < \frac{5}{2}$, we have $(\frac{q}{t'})' < 2$.

References

[C] M. Christ, Estimates for the k-plane transform, *Indiana Univ. Math. J.* 33 (1984), 891-910.

[O] D. Oberlin, Some convolution inequalities and their applications, *Trans. Amer. Math. Soc.* 354 (2002), 2541-2548.