# A RESTRICTION THEOREM FOR A $k$–SURFACE IN $\mathbb{R}^n$

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Fix a positive integer $k$ and let $n = k(k+3)/2$. If $x \in \mathbb{R}^k$, write $x = (x_1, \cdots, x_k)$, and define $\phi : \mathbb{R}^k \to \mathbb{R}^n$ by

$$\phi(x) = (x_1, \cdots, x_k, x_1^2, \cdots, x_k^2, x_1x_2, \cdots, x_1x_k, x_2x_3, \cdots, x_2x_k, \cdots, x_{k-1}x_k).$$

Write $S$ for the $k$-surface in $\mathbb{R}^n$ which is the range of $\phi$ and let $\sigma$ be the measure induced on $S$ by Lebesgue measure on $\mathbb{R}^k$. We are interested in the operator $R^*$ taking functions $f \in C_c^\infty(S)$ to functions on $\mathbb{R}^n$ which is given by

$$R^*(f)(\xi) = \widehat{fd\sigma}(\xi).$$

The operator $R^*$ is the adjoint of the Fourier restriction operator associated with the surface $S$ and the measure $\sigma$. The natural problem is to determine the indices $p, q \in [1, \infty]$ such that there is an a priori estimate

$$\|R^*f\|_{L^q(\mathbb{R}^n)} \leq C(p,q)\|f\|_{L^p(\sigma)}. \tag{1}$$

There is also the analogous problem for the localized operator $R_0^*$ given by $R_0^*(f)(\xi) = \widehat{f\psi d\sigma}(\xi)$ where $\psi$ is fixed in $C_c^\infty(S)$. For $k = 1$ these operators are associated with a parabola in $\mathbb{R}^2$. Their mapping properties are well understood and are analogous to those of the corresponding operator associated with the circle. For $k \geq 2$ the first result is due to Christ [C2], who obtained estimates for $R_0^*$ when $p = 2$. Mockenhaupt ([M1],[M2]) extended Christ's results to the cases $k \geq 3$. De Carli and Iosevich [CI] obtained a sharp $L^2$ result. Bak and Lee [BL] adapted Mockenhaupt's method to obtain the following nearly sharp result. (Their paper also contains a more detailed history of these problems.)

**Theorem 1.([BL])** *If $k = 2$ or $3$ then $R^*$ is bounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if and only if $\frac{1}{p} + \frac{k+2}{q} = 1$, $q > 2(k+1)$. If $k \geq 4$, then $R_0^*$ is bounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if $\frac{1}{p} + \frac{k+2}{q} < 1$, $q > 2(k+1)$, and $R_0^*$ is unbounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if $q \leq 2(k+1)$ or $\frac{1}{p} + \frac{k+2}{q} > 1$.*

The purpose of this note is to present a slight improvement on Theorem 1:

**Theorem 2.** *For $k \geq 2$, the operator $R^*$ is bounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if and only if $\frac{1}{p} + \frac{k+2}{q} = 1$, $q > 2(k+1)$.*

Quoting Christ ([C2], p.224): "The strategy of our proof is not new: following Prestini [P], we utilize an argument originating in Fefferman [F] and Carleson and Sjölin [CS], based on a change of variables and the Hausdorff-Young inequality, to reduce (1) to an easier problem concerning estimates for positive integral operators". The same strategy is utilized in [BL]. The proof of Theorem 2 is a bit simpler than that of Theorem 1, depending on a change of variables different from that in [BL].

The author is grateful to Jong-Guk Bak for interesting and helpful correspondence. In particular, the author was in possession of a proof of Theorem 2 for the case $k = 2$ when he received the preprint [BL], and he appreciates Bak's encouraging him to extend that argument to $k \geq 3$.

*Proof of Theorem 2:* We adopt the convention that $C$ denotes a positive constant which may depend only on the relevant dimensions and/or indices. Writing $\|\cdot\|_r$ for $\|\cdot\|_{L^r(\mathbb{R}^n)}$, the Hausdorff-Young inequality shows that it is enough to prove the inequality

$$\|(f d\sigma) * \cdots * (f d\sigma)\|_{(\frac{q}{k+1})'} \leq C\, \|f\|^{k+1}_{L^p(S)},$$

where the convolution is $(k+1)$-fold. This is equivalent to the inequality

$$\int_{(\mathbb{R}^k)^{k+1}} \prod_{l=1}^{k+1} f(x^l) h\big(\phi(x^1) + \cdots + \phi(x^{k+1})\big)\, dx^1 \cdots x^{k+1}$$

$$\leq C\, \|f\|^{k+1}_{L^p(\mathbb{R}^k)} \|h\|_{\frac{q}{k+1}} \tag{2}$$

for functions $f$ on $\mathbb{R}^k$. For $j = 1, \ldots, k$, write $v_j = (x_j^1, \ldots, x_j^{(k+1)})$ and let $d$ be the $(k+1)$-vector $(1, \ldots, 1)$. For fixed $v_2, \ldots, v_k$, define

$$\Phi(x_1^1, \ldots, x_1^{(k+1)}) = (v_1 \cdot d, |v_1|^2, v_1 \cdot v_2, \ldots, v_1 \cdot v_k).$$

Then the Jacobian $J$ of $\Phi$ is the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2x_1^1 & 2x_1^2 & \cdots & 2x_1^{(k+1)} \\ x_2^1 & x_2^2 & \cdots & x_2^{(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^1 & x_k^2 & \cdots & x_k^{(k+1)} \end{bmatrix}.$$

For $1 \leq s < 2$ we will estimate (2) by the product of

$$\Big(\int_{(\mathbb{R}^k)^{k+1}} h^{s'}\big(\phi(x^1) + \cdots + \phi(x^{k+1})\big) \cdot J\; dx^1 \cdots x^{k+1}\Big)^{1/s'} \tag{3}$$

and

$$\Big(\int_{(\mathbb{R}^k)^{k+1}} \prod_{l=1}^{k+1} f^s(x^l) \cdot J^{-s/s'} dx^1 \cdots x^{k+1}\Big)^{1/s}. \tag{4}$$

Beginning with (3), write

$$h\Big(\phi(x^1)+\cdots+\phi(x^{k+1})\Big) =$$

$$h\Big(\Phi(x_1^1,\ldots,x_1^{(k+1)}),v_2\cdot d,\ldots,v_k\cdot d,|v_2|^2,\ldots,|v_k|^2,v_2\cdot v_3,v_2\cdot v_4,\ldots,v_{k-1}\cdot v_k\Big)\doteq$$

$$h\big(\Phi(x_1^1,\ldots,x_1^{(k+1)}),\Psi(v)\big)$$

where $v=(v_2,\ldots,v_k)$. Thus $(3)^{s'}$ is

$$\int_{(\mathbb{R}^{k+1})^{k-1}}\int_{\mathbb{R}^{k+1}} h^{s'}\left(\Phi(x_1^1,\ldots,x_1^{(k+1)}),\Psi(v)\right)\cdot J\ dx_1^1\cdots x_1^{(k+1)}dv.$$

The map $\Phi$ has multiplicity at most 2 for almost all $v$. For such $v$

$$\int_{\mathbb{R}^{k+1}} h^{s'}\left(\Phi(x_1^1,\ldots,x_1^{(k+1)}),\Psi(v)\right)\cdot J\ dx_1^1\cdots x_1^{(k+1)}\le 2\int_{\mathbb{R}^{k+1}} h^{s'}\big(y,\Psi(v)\big)\ dy$$

and so it follows that $(3)^{s'}$ is bounded by

$$2\int_{(\mathbb{R}^{k+1})^{k-1}}\int_{\mathbb{R}^{k+1}} h^{s'}\big(y,\Psi(v)\big)\ dy\ dv. \tag{5}$$

To bound (5) (by $C\ \|h\|_{s'}^{s'}$), recall that

$$\Psi(v_2,\ldots,v_k)=(v_2\cdot d,\ldots,v_k\cdot d,|v_2|^2,\ldots,|v_k|^2,v_2\cdot v_3,v_2\cdot v_4,\ldots,v_{k-1}\cdot v_k)$$

where $d=(1,1,\ldots,1)$. We write $d'=d/\sqrt{k+1}$ and $v_j=d_jd'+c_j$ with $c_j\perp d'$. Then if $g$ is a function on $\mathbb{R}^{(k-1)(k+2)/2}$ we have

$$\int_{(\mathbb{R}^{k+1})^{k-1}} g\big(\Psi(v_2,\ldots,v_k)\big)dv_2\cdots dv_k$$

$$=\int_{\mathbb{R}^{k-1}}\int_{(\mathbb{R}^{k})^{k-1}} g\big(\sqrt{k+1}d_2,\ldots,\sqrt{k+1}d_k, \tag{6}$$

$$(d_2)^2+|c_2|^2,\ldots,(d_k)^2+|c_k|^2,d_2d_3+c_2\cdot c_3,d_2d_4+c_2\cdot c_4,\ldots,d_{k-1}d_k+c_{k-1}\cdot c_k\big)\ d\,c_2\cdots c_k\ d\,d_2\cdots d_k.$$

We require a lemma (whose proof is postponed until after the main argument).

**Lemma 1.** *The inequality*

$$\int_{\mathbb{R}^{k(k-1)}}\alpha\big(|c_2|^2,\ldots,|c_k|^2,c_2\cdot c_3,c_2\cdot c_4,\ldots,c_{k-1}\cdot c_k\big)dc_2\cdots c_k\le C\int_{\mathbb{R}^{k(k-1)/2}}\alpha(x)dx$$

*holds for nonnegative Borel functions $\alpha$ on $\mathbb{R}^{k(k-1)/2}$.*

An application of this lemma to (6) now shows that (if $g$ is a function on $\mathbb{R}^{(k-1)(k+2)/2}$)

$$\int_{(\mathbb{R}^{k+1})^{k-1}} g\big(\Psi(v_2,\ldots,v_k)\big)dv_2\cdots dv_k\le C\int_{\mathbb{R}^{(k-1)(k+2)/2}} g(z)dz.$$

Applying this to (5) shows that

$$(7) \qquad \Big(\int_{(\mathbb{R}^k)^{k+1}} h^{s'}\big(\phi(x^1)+\cdots+\phi(x^{k+1})\big)\cdot J\ dx^1\cdots x^{k+1}\Big)^{1/s'} \le C\ \|h\|_{s'},$$

completing the process of bounding (3).

To bound (4) we will need another lemma. Recall that $x^l = (x_1^l,\dots,x_k^l)$ and write $D$ for the absolute value of the determinant of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1^1 & x_1^2 & \cdots & x_1^{(k+1)} \\ x_2^1 & x_2^2 & \cdots & x_2^{(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^1 & x_k^2 & \cdots & x_k^{(k+1)} \end{bmatrix}$$

**Lemma 2 ([C1], Theorem B).** *Suppose $0<\gamma<1$, $1\le r<2$, and $\frac{1}{r}=1-\frac{\gamma}{k+1}$. Then the inequality*

$$\int_{(\mathbb{R}^k)^{k+1}} \prod_1^{k+1} f_l(x^l)D^{-\gamma}dx^1\cdots x^{k+1} \le C\prod_1^{k+1}\|f_l\|_{L^r(\mathbb{R}^k)}$$

*holds for nonnegative Borel functions $f_l$.*

An application of Lemma 2 to the integral in (4) bounds (4) by $C\ \|f\|^{k+1}_{L^{rs}(\mathbb{R}^k)}$ so long as the $r$ defined by $\frac{1}{r}=1-\frac{s}{s'}\frac{1}{k+1}$ satisfies $1\le r<2$ (which follows from $1\le s<2$). That is, (4) is bounded by $C\ \|f\|^{k+1}_{L^p(\mathbb{R}^k)}$ where $p=rs$ and so $\frac{1}{p}=(\frac{k+2}{s}-1)\frac{1}{k+1}$ (since $\frac{1}{r}=\frac{k+2-s}{k+1}$). With $q=s'(k+1)$, it follows that $\frac{1}{p}+\frac{k+2}{q}=1$. Thus the bound (7) for (3) now yields (2) whenever $\frac{1}{p}+\frac{k+2}{q}=1$ and $q>2(k+1)$.

*Proof of Lemma 1:* The proof depends on a particular parametrization of $(\mathbb{R}^k)^{k-1}$. We begin by introducing notation.

Write $e_j$ for the $j$th standard unit vector in $\mathbb{R}^k$. Fix a Borel mapping

$$\omega_{k-1}\mapsto O(\omega_{k-1})$$

of the unit sphere $\Sigma_{k-1}$ in $\mathbb{R}^k$ into the orthogonal group on $\mathbb{R}^k$ in such a way that $O(\omega_{k-1})e_1=\omega_{k-1}$. For $\omega_{k-1}\in\Sigma_{k-1}$, set

$$\omega'_{k-1}=O(\omega_{k-1})e_2\in\{\omega_{k-1}\}^{\perp}\cap\Sigma_{k-1},$$

realize $\Sigma_{k-2}$ as $\{\omega_{k-1}\}^{\perp}\cup\Sigma_{k-1}$, and, as above, for $\omega_{k-2}\in\Sigma_{k-2}$, let $O(\omega_{k-1},\omega_{k-2})$ be an orthogonal map on $\mathbb{R}^k$ which fixes $\omega_{k-1}$ and takes $\omega'_{k-1}$ to $\omega_{k-2}$. Then $O(\omega_{k-1},\omega_{k-2})O(\omega_{k-1})$ takes $e_1$ to $\omega_{k-1}$ and $e_2$ to $\omega_{k-2}$. For such $\omega_{k-1},\omega_{k-2}$ set

$$\omega'_{k-2}=O(\omega_{k-1},\omega_{k-2})O(\omega_{k-1})e_3\in\{\omega_{k-1},\omega_{k-2}\}^{\perp}\cap\Sigma_{k-1},$$

realize $\Sigma_{k-3}$ as $\{\omega_{k-1},\omega_{k-2}\}^{\perp}\cap\Sigma_{k-1}$, and, for $\omega_{k-3}\in\Sigma_{k-3}$, let $O(\omega_{k-1},\omega_{k-2},\omega_{k-3})$ be an orthogonal map on $\mathbb{R}^k$ which fixes $\omega_{k-1},\omega_{k-2}$ and takes $\omega'_{k-2}$ to $\omega_{k-3}$. Continue this way until $O(\omega_{k-1},\dots,\omega_1)$ is defined. Write $\omega=(\omega_1,\dots,\omega_{k-1})$ and

$$O(\omega)=O(\omega_{k-1},\dots,\omega_1)O(\omega_{k-1},\dots,\omega_2)\cdots O(\omega_{k-1}).$$

The notation $d\omega_1 \cdots \omega_{k-1}$ will represent integration with respect to the product of the surface area measures on (the realizations) of the spheres $\Sigma_1, \ldots, \Sigma_{k-1}$.

For $\theta_i \in [0, \pi]$ define

$$\sigma_1(\theta_1) = (\cos\theta_1, \sin\theta_1) \in \Sigma_1$$

and

$$\sigma_j(\theta_1, \ldots, \theta_j) = \big(\cos\theta_1; \sin\theta_1 \sigma_{j-1}(\theta_2, \ldots, \theta_j)\big) \in \Sigma_j.$$

For $j = 1, \ldots, k-2$, the notation $\big(\sigma_j(\theta_1, \ldots, \theta_j), 0\big)$ stands for the $k$-vector obtained by following $\sigma_j(\theta_1, \ldots, \theta_j)$ with $(k-j-1)$ 0's.

The parametrization of $(\mathbb{R}^k)^{k-1}$ is now

$$\big(c_0; c_1; \ldots; c_{k-2}\big) =$$

$$\Big(\rho_0 O(\omega) e_1; \rho_1 O(\omega)\big(\sigma_1(\theta_1^1), 0\big); \ldots; \rho_{k-2} O(\omega)\big(\sigma_{k-2}(\theta_1^{k-2}, \ldots, \theta_{k-2}^{k-2}), 0\big)\Big),$$

where the $\rho_j$'s are positive. The volume element which corresponds to Lebesgue measure $dc_0 \cdots c_{k-2}$ on $(\mathbb{R}^k)^{k-1}$ is

$$\prod_0^{k-2} \rho_j^{k-1} d\rho_0 \cdots \rho_{k-2}\, d\omega_1 \cdots \omega_{k-1} \prod_1^{k-2} (\sin\theta_1^j)^{k-2} \cdot \prod_2^{k-2} (\sin\theta_2^j)^{k-3} \cdots \prod_{k-2}^{k-2} (\sin\theta_{k-2}^j) d\theta_1^1 \theta_1^2 \theta_2^2 \cdots \theta_1^{k-2} \cdots \theta_{k-2}^{k-2}.$$

The ranges of integration for the $\rho$ and $\theta$ variables are $(0, \infty)$ and $[0, \pi]$, respectively. (The ranges for the $\omega$'s were described above.) For example, in the case $k = 4$ we have

$$c_0 = \rho_0 \omega_3, \quad c_1 = \rho_1(\cos\theta_1^1 \omega_3 + \sin\theta_1^1 \omega_2),$$
$$c_2 = \rho_2(\cos\theta_1^2 \omega_3 + \sin\theta_1^2 \cos\theta_2^2 \omega_2 + \sin\theta_1^2 \sin\theta_2^2 \omega_1).$$

The volume element can be written

$$\rho_2^3 (\sin\theta_1^2)^2 \sin\theta_2^2 d\rho_2 d\theta_1^2 \theta_2^2 d\omega_1 \cdot \rho_1^3 (\sin\theta_1^1)^2 d\rho_1 d\theta_1^1 d\omega_2 \cdot \rho_0^2 d\rho_0 d\omega_3.$$

For fixed $\omega_3, \rho_0, \omega_2, \theta_1^1$, and $\rho_1$, $dc_2$ is $\rho_2^3 (\sin\theta_1^2)^2 \sin\theta_2^2 d\rho_2 d\theta_1^2 \theta_2^2 d\omega_1$ since $d\omega_1$ gives "surface area" on $\{\omega_3, \omega_2\}^\perp \cap \Sigma_3$. And for fixed $\omega_3$ and $\rho_0$, $dc_1$ is $\rho_1^3 (\sin\theta_1^1)^2 d\rho_1 d\theta_1^1 d\omega_2$ since $d\omega_2$ gives surface area on $\{\omega_3\}^\perp \cap \Sigma_3$. Finally $dc_0$ is $\rho_2^3 d\rho_2 d\omega_3$.

Lemma 1 is the statement that

$$\int_{\mathbb{R}^{k(k-1)}} \alpha\big(|c_0|^2, \ldots, |c_{k-2}|^2, c_0 \cdot c_1, c_0 \cdot c_2, \ldots, c_{k-3} \cdot c_{k-2}\big) dc_0 \cdots c_{k-2} \le C \int_{\mathbb{R}^{k(k-1)/2}} \alpha(x) dx.$$

Since the orthogonal mappings $O(\omega)$ have no effect on the inner products $c_i \cdot c_j$, we define

$$\big(c'_0; c'_1; \ldots; c'_{k-2}\big) =$$

$$\Big(\rho_0 e_1; \rho_1\big(\sigma_1(\theta_1^1), 0\big); \ldots; \rho_{k-2}\big(\sigma_{k-2}(\theta_1^{k-2}, \ldots, \theta_{k-2}^{k-2}), 0\big)\Big).$$

Lemma 1 then follows by observing that

$$\int \alpha\big(|c'_0|^2, \ldots, |c'_{k-2}|^2, c'_0 \cdot c'_1, c'_0 \cdot c'_2, \ldots, c'_{k-3} \cdot c'_{k-2}\big) \prod_0^{k-2} \rho_j^{k-1} d\rho_0 \cdots \rho_{k-2}.$$

$$\prod_1^{k-2}(\sin\theta_1^j)^{k-2}\cdot\prod_2^{k-2}(\sin\theta_2^j)^{k-3}\cdots\prod_{k-2}^{k-2}(\sin\theta_{k-2}^j)d\theta_1^1\theta_1^2\theta_2^2\cdots\theta_1^{k-2}\cdots\theta_{k-2}^{k-2}$$

$$\leq C\int_{\mathbb{R}^{k(k-1)/2}}\alpha(x)dx. \tag{8}$$

We will explain this in the case $k=4$, the general case being completely analogous. If $k=4$ then

$$\big(|c'_0|^2,|c'_1|^2,|c'_2|^2,c'_0\cdot c'_1,c'_0\cdot c'_2,c'_1\cdot c'_2\big)=$$

$$\big(\rho_0^2,\rho_1^2,\rho_2^2,\rho_0\rho_1\cos\theta_1^1,\rho_0\rho_2\cos\theta_1^2,\rho_1\rho_2(\cos\theta_1^1\cos\theta_1^2+\sin\theta_1^1\sin\theta_1^2\cos\theta_2^2)\big),$$

while the volume element can be written

$$\rho_1\rho_2\sin\theta_1^1\sin\theta_1^2\sin\theta_2^2d\theta_2^2\cdot\rho_0\rho_2\sin\theta_1^2d\theta_1^2\cdot\rho_0\rho_1\sin\theta_1^1d\theta_1^1\cdot\rho_2d\rho_2\cdot\rho_1d\rho_1\cdot\rho_0d\rho_0.$$

Thus (8) is evident.

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