A CONVOLUTION PROPERTY OF THE CANTOR-LEBESGUE MEASURE, II

Daniel M. Oberlin

Department of Mathematics, Florida State University

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Let $\mathbb{T}$ be the circle group $\mathbb{R}/\mathbb{Z}$ and, for $1 \leq p \leq \infty$, write L^p for the Lebesgue space formed using normalized Lebesque measure on $\mathbb{T}$. Let λ be the usual Cantor-Lebesgue measure on $\mathbb{T}$. We are interested in determining the $L^p - L^q$ mapping properties of the convolution operator defined by λ: we would like to know the indices $p, q \in [1, \infty]$ for which there is an inequality

$$\|\lambda * f\|_{L^q} \leq C(p,q)\ \|f\|_{L^p} \tag{1}$$

for $f \in L^p$. Since (1) is trivial if $q \leq p$, our interest is in the case $p < q$. The following results are in [O].

Lemma 1. *Suppose $1 \leq p < q \leq \infty$. If the inequality*

$$\Big(\frac{1}{3}\Big[\big(\frac{a+b}{2}\big)^q + \big(\frac{b+c}{2}\big)^q + \big(\frac{a+c}{2}\big)^q\Big]\Big)^{\frac{1}{q}} \leq \Big(\frac{a^p + b^p + c^p}{3}\Big)^{\frac{1}{p}} \tag{2}$$

holds for all $a, b, c \geq 0$, then (1) holds with $C(p,q) = 1$.

Lemma 2. *Inequality (2) holds for $q = 2$ when $p \geq 2/(1 + 3^{-1/2}) \approx 1.2679$.*

It follows from duality and interpolation that if $1 < p < \infty$ then there is q satisfying $p < q < \infty$ and such that (1) holds with $C(p,q) = 1$. Similar results for more general measures are in [BJJ] and [R], while [C] establishes the "L^p-improving" property for a larger class of singular measures using a quite different method.

The known cases of (1) are all applications of Lemma 1 and so satisfy $C(p,q) = 1$. The main result of this note is that if convolution with λ maps L^p into L^q, then it does so as a contraction:

Theorem 3. *If (1) holds for $p, q \in [1, \infty]$ then (1) holds with $C(p,q) = 1$.*

A more difficult and interesting problem is to determine exactly the indices for which (1) holds. Here we focus on the case $q = 2$. In addition to the information above, there are the following results.

Proposition 4 ([B],[BJJ]). *Inequality (2) holds for $q = 2$ exactly when $p \geq \frac{\log 4}{\log 3} \approx 1.2619$.*

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Proposition 5. *If (1) holds then*

$$\frac{1}{p} + (1 - \frac{\log 2}{\log 3})(1 - \frac{1}{q}) \leq 1.$$

Proposition 5 is checked by testing (1) on the indicator functions of small intervals. It shows in particular that if (1) holds with $q = 2$ then $p \geq 2(1 + \log 2/\log 3)^{-1} \approx 1.2263$, providing a necessary condition to pair with the sufficient condition provided by Lemma 1 and Proposition 4. The second result of this note narrows the gap between these two conditions.

Proposition 6. *Suppose (1) holds with $q = 2$. Then the following inequality holds whenever $0 < a < b < 1$ and $2b < 1 + a$*

$$\Big(\frac{2^a}{6 \cdot a^a (b-a)^{2(b-a)}(1+a-2b)^{(1+a-2b)}}\Big)^{\frac{1}{2}} \leq \Big(\frac{2^b}{3 \cdot b^b (1-b)^{(1-b)}}\Big)^{\frac{1}{p}}. \tag{3}$$

Numerical calculations indicate that (3) fails when $p = 1.244$, $b = .0770$, and $a = .0105$. This rules out the possibility that the condition provided by Proposition 5 is sufficient as well as necessary (but leaves open the interesting possibility that the sufficient condition supplied by Beckner's Proposition 4 is necessary). In the remainder of this note we give the proofs of Theorem 3 and Proposition 6.

Proof of Theorem 3: We will show that if (1) holds for $C(p,q) \in [1, \infty)$ then (1) also holds when $C(p,q)$ is replaced by $\sqrt{C(p,q)}$. It is convenient to replace λ with its translate by $\frac{1}{2}$. We will need the facts that then the Fourier transform of λ is given by

$$\widehat{\lambda}(n) = \prod_{j=0}^{\infty} \cos(2\pi 3^{-j} n)$$

and that when f is 1-periodic and continuous on $\mathbb{R}$ we have, for integral M,

$$\lim_{M \to \infty} \int_0^1 f(\theta) f(M\theta)\, d\theta = \Big(\int_0^1 f(\theta)\, d\theta\Big)^2. \tag{4}$$

Fix a trigonometric polynomial

$$t(\theta) = \sum_{n=-L}^{L} \widehat{t}(n) e^{2\pi i n \theta}.$$

Then, for positive integers N,

$$\lambda * t(\theta)\ \lambda * t(3^N \theta) = \sum_{n_1, n_2} \prod_{j=0}^{\infty} \Big(\cos(2\pi 3^{-j} n_1) \cos(2\pi 3^{-j} n_2)\Big) \widehat{t}(n_1) \widehat{t}(n_2) e^{2\pi i (n_1 + 3^N n_2)\theta}.$$

Also, $t(\theta) t(3^N \theta) = \sum_{n_1, n_2} \widehat{t}(n_1) \widehat{t}(n_2) e^{2\pi i (n_1 + 3^N n_2)\theta}$ and so

$$\lambda * \big(t(\cdot) t(3^N \cdot)\big)(\theta) = \sum_{n_1, n_2} \prod_{j=0}^{\infty} \cos\big(2\pi 3^{-j}(n_1 + 3^N n_2)\big) \widehat{t}(n_1) \widehat{t}(n_2) e^{2\pi i (n_1 + 3^N n_2)\theta}.$$

Now

$$\prod_{j=0}^{\infty}\cos\big(2\pi 3^{-j}(n_1+3^N n_2)\big)=\prod_{j=0}^{N}\cos\big(2\pi 3^{-j}n_1\big)\prod_{j=N+1}^{\infty}\cos\big(2\pi[3^{-j}n_1+3^{N-j}n_2]\big)=$$

$$\prod_{j=0}^{N}\cos\big(2\pi 3^{-j}n_1\big)\prod_{j=N+1}^{\infty}\big[\cos(2\pi 3^{-j}n_1)\cos(2\pi 3^{N-j}n_2)-\sin(2\pi 3^{-j}n_1)\sin(2\pi 3^{N-j}n_2)\big].$$

For $M\geq N+1$,

$$\prod_{j=N+1}^{M}\big[\cos(2\pi 3^{-j}n_1)\cos(2\pi 3^{N-j}n_2)-\sin(2\pi 3^{-j}n_1)\sin(2\pi 3^{N-j}n_2)\big]=\prod_{j=N+1}^{M}\cos(2\pi 3^{-j}n_1)\cos(2\pi 3^{N-j}n_2)+e$$

where the error term $e=e(n_1,n_2,N,M)$ satisfies

$$|e|\leq\prod_{j=N+1}^{M}\big[1+|\sin(2\pi 3^{-j}n_1)|\big]-1=O(3^{-N}L)$$

since $|n_1|\leq L$. Then

$$\Big|\prod_{j=0}^{\infty}\cos\big(2\pi 3^{-j}(n_1+3^N n_2)\big)-\prod_{j=0}^{\infty}\Big(\cos(2\pi 3^{-j}n_1)\cos(2\pi 3^{-j}n_2)\Big)\Big|=O(3^{-N}L)$$

and it follows that

$$|\lambda*t(\theta)\ \lambda*t(3^N\theta)-\lambda*\big(t(\cdot)t(3^N\cdot)\big)(\theta)|\leq C(t)\cdot 3^{-N},$$

where $C(t)$ is a positive constant depending on the trigonometric polynomial t. Thus

$$\Big|\|\lambda*t(\theta)\ \lambda*t(3^N\theta)\|_{L^q}-\|\lambda*\big(t(\cdot)t(3^N\cdot)\big)(\theta)\|_{L^q}\Big|\to 0$$

as $N\to\infty$. Since

$$\|\lambda*t(\theta)\ \lambda*t(3^N\theta)\|_{L^q}\to\|\lambda*t\|_{L^q}^2$$

by (4) and also

$$\|\lambda*\big(t(\cdot)t(3^N\cdot)\big)(\theta)\|_{L^q}\leq C(p,q)\ \|t(\theta)t(3^N\theta)\|_{L^p}\to C(p,q)\ \|t\|_{L^p}^2,$$

it follows that

$$\|\lambda*t\|_{L^q}\leq\sqrt{C(p,q)}\ \|t\|_{L^p}$$

as desired. Thus the proof of Theorem 3 is complete.

Proof of Proposition 6: If (1) holds it is easy to see that convolution with λ yields a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. If $q=2$ it follows that the inequality

$$\langle\lambda*\widetilde{\lambda},\chi_E*\chi_{-E}\rangle\leq C(p)\ |E|^{\frac{2}{p}}\tag{6}$$

holds for Borel $E \subseteq \mathbb{R}$. To discretize (6) define

$$C_N = \{\sum_{j=0}^{N-1} \epsilon_j 3^j : j \in \{0,2\}\ \}.$$

With "$*$" now representing the usual convolution on the group of integers and "$|\cdot|$" now standing for cardinality, (6) implies that

$$\frac{1}{12^N}\langle \chi_{C_N} * \chi_{-C_N}, \chi_F * \chi_{-F}\rangle \leq C(p)\ \left(\frac{|F|}{3^N}\right)^{\frac{2}{p}} \tag{7}$$

holds whenever $F \subseteq \mathbb{Z}$. We will establish (3) by applying (7) to certain sets $F_{N,k}$. Fix a positive integer N. For $J \subseteq \{0,1,\dots,N-1\}$ put

$$F_J = \{\sum_{j \in J} \epsilon_j 3^j : j \in \{-2,2\}\ \}$$

so that

$$\chi_{F_J} = \mathop{*}_{j\in J}(\delta_{-2\cdot 3^j} + \delta_{2\cdot 3^j}).$$

For $1 \leq k \leq N-1$ define

$$F_{N,k} = \cup\{F_J : J \subseteq \{0,1,\dots,N-1\},\ |J| = k\}.$$

Note that if J_1 and J_2 are disjoint then

$$\chi_{F_{J_1}} * \chi_{F_{J_2}} = \chi_{F_{J_1 \cup J_2}}$$

and that, in general,

$$\chi_{F_{J_1}} * \chi_{F_{J_2}} = \mathop{*}_{j \in J_1 \cap J_2}(\delta_{-4\cdot 3^j} + 2\delta_0 + \delta_{4\cdot 3^j}) * \chi_{F_{(J_1 \cup J_2)\setminus(J_1\cap J_2)}}.$$

It follows that

$$\chi_{F_{J_1}} * \chi_{F_{J_2}} \geq 2^{|J_1 \cap J_2|}\chi_{F_{(J_1 \cup J_2)\setminus(J_1\cap J_2)}}.$$

Thus, since F_{J_1} and F_{J_2} are disjoint if $J_1 \neq J_2$,

$$\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{N,k}} * \chi_{F_{N,k}}\rangle = \sum_{|J_1|=|J_2|=k} \langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{J_1}} * \chi_{F_{J_2}}\rangle \geq$$

$$\sum_{|J_1|=|J_2|=k} 2^{|J_1\cap J_2|}\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{(J_1 \cup J_2)\setminus(J_1\cap J_2)}}\rangle.$$

Now

$$\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_J}\rangle = \sum_{f \in F_J} |f + C_N \cap C_N| = 2^{|J|}2^{N-|J|} = 2^N$$

so

$$\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{N,k}} * \chi_{F_{N,k}}\rangle \geq 2^N \sum_{|J_1|=|J_2|=k} 2^{|J_1\cap J_2|} = 2^N \binom{N}{k}\sum_{l=0}^{k} 2^l \binom{k}{l}\binom{N-k}{k-l}.$$

Thus (7) implies that for $l = 0, \ldots, k$ there is the inequality

$$\frac{2^l}{6^N}\binom{N}{k}\binom{k}{l}\binom{N-k}{k-l} \leq C(p)\Big(\frac{\binom{N}{k}2^k}{3^N}\Big)^{\frac{2}{p}}. \tag{8}$$

By continuity, it is enough to establish (3) in when a and b are rational. With such a and b fixed, N will now stand for a positive integer such that both aN and bN are integers. Take $k = bN$ and $l = aN$ in (8), estimate the binomial coefficients using Stirling's formula, take Nth roots of both sides of the resulting inequality, and then let $N \to \infty$. This gives

$$\Big(\frac{2^a}{6 \cdot a^a (b-a)^{2(b-a)}(1+a-2b)^{(1+a-2b)}}\Big) \leq \Big(\frac{2^b}{3 \cdot b^b (1-b)^{(1-b)}}\Big)^{\frac{2}{p}},$$

the conclusion of Proposition 6.

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