A Test for Admissibility for Two-Dimensional, Continuous Fields on Planar Surfaces

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Abstract: Radar scatterometers on oceanic satellites provide wind stress data over the oceans except for a 180° ambiguity in direction. Data sets consist of sticks, 2-dimensional vectors of known amplitude and direction but undetermined orientation. In [2] it was shown that, for admissible stick fields, the assignment of an orientation to a single stick determines a unique realization of a vector field from the given continuous stick field. In this note a criterion for determining admissibility of a given continuous stick field is given. This criterion depends only on the boundary curves of the region on which the stick field is given and it can be effectively checked.

1. Introduction

Typically, wind data over the oceans provided by scatterometers consist of stick fields, where the length of a stick at a point measures wind speed and the direction of the stick determines the direction of the wind up to a 180° ambiguity. Since the precise direction of the wind can be measured at various points in the ocean one would like to take these sample measurements to remove the ambiguity in the stick field. For this, regions of weak winds are removed from the map, since these regions tend to be quite disorderly and discontinuous and do not significantly affect the stick field in regions with stronger winds. This results in a region with holes on the 2-sphere; i.e., a planar surface F. We assume that the given stick field is just the visible part of an underlying continuous field. In [2] we called a stick field *admissible* if for each closed curve C on F the total change of the angles of the sticks along C is a multiple of 2π.

It was then shown in [2] that for a given admissible field of sticks, the assignment of an orientation to a single stick determines a unique realization as a continuous vector field. The given data sets will in general provide stick fields that are not admissible, but by cutting out regions of low wind amplitudes one usually obtains an admissible field. Since in practice it is impossible to check the admissibility of the field along *every* simple closed curve, the question arises whether it suffices to check this for a finite number of curves. In this note we show that in fact it suffices to check admissibility along the (finitely many) boundary curves of the planar surface F.

2. The Criterion for Admissibility

First, we give a more precise definition of an admissible stick field. Considering the planar surface F as a compact region in the complex plane $\mathbb{C}$ bounded by finitely many curves, we can describe a continuous stick field V by pairs (z, α), where $z = x + iy$ is a point of F and α is the angle that the stick at P forms with the x-axis (measured in anticlockwise direction from the x-axis) for $0 \leq \alpha < \pi$. We consider closed curves C on F that have only at most finitely many (transverse) self-intersections. For such a curve we choose a parametrization $\phi : S^1 \to F$ (where S^1 is the unit circle in $\mathbb{C}$) so that the points $\phi(e^{it})$ traverse C exactly once as t increases from 0 to 2π. Consider now the angle-measure map $f_c : S^1 \to S^1$ defined by $f_c(z) = e^{2i\alpha}$, where α is the angle that the stick at $\phi(z)$ forms with the x-axis. The map f is continuous since ϕ and V are continuous. (This map is similar to the Gauss mapping of plane curves, see e.g. [1], except that the sticks of V are used instead of the normal vectors of C.) There is a unique integer $n = \deg f_c$ (the degree of f_c) so that f_c is homotopic to the map $g : S^1 \to S^1$ given by $g(z) = z^n$ (see e.g. [3]). Let us call the curve C *admissible* if $\deg f_c$ is even. Then the continuous stick field is *admissible* if every (simple closed) curve c in F is admissible.

Criterion: *A continuous field of sticks on a planar surface F can be realized by a continuous vector field on F if and only if every boundary curve of F is admissible.*

By Corollary 1 of [2] it suffices to show that if every boundary curve of F is admissible then every (simple closed) curve of F is admissible.

First, note that if curves C_1 and C_2 are homotopic, then C_1 is admissible if and only if C_2 is. For let C_1 and C_2 be parametrized by ϕ_1 and ϕ_2 as above, let f_{c_1} and f_{c_2} be the corresponding maps that measure the angles of the sticks along C_1 and C_2, and let $G : S^1 \times I \to F$ be a homotopy with $G(z, 0) = \phi_1(z)$, $G(z, 1) = \phi_2(z)$. Then $H : S^1 \times I \to S^1$, defined by $H(z, t) = e^{2i\alpha}$, where α is the angle that the stick at $G(z, t)$ forms with the x-axis, is a homotopy between f_{c_1} and f_{c_2} and therefore $\deg f_{c_1} = \deg f_{c_2}$.

Now every (simple closed) curve C of F is homotopic to a product $\gamma_1 \cdots \gamma_m$ of curves

γ_i, where each γ_i is homotopic to a boundary curve of F (see Fig. 7 of [2]). Thus we need to show that $\gamma_1 \cdots \gamma_m$ is admissible (assuming every γ_i is admissible). Let $\phi_1, \ldots, \phi_m$ be parametrizations of $\gamma_1, \ldots, \gamma_m$ as above with the corresponding angle-measure maps $f_1, \ldots, f_m$, and let $\phi : S^1 \to F$ be the map given by $\phi(e^{it}) = \phi_k(e^{mit-(k-1)2\pi i})$ for $\frac{(k-1)2\pi}{m} \leq t \leq \frac{k2\pi}{m}$, $k = 1, \ldots, m$. Then ϕ is a parametrization for $\gamma_1 \cdots \gamma_m$ and for the angle-measure map f of $\gamma_1 \cdots \gamma_m$ we have $\deg f = \deg f_{\gamma_1} + \ldots + \deg f_{\gamma_m}$. Since each $\deg f_{\gamma_k}$ is even it follows that $\deg f$ is even, and hence that $\gamma_1 \cdots \gamma_m$ is admissible.

3. Conclusion

In practice the data set is discrete and it is assumed that it is just the visible part of an underlying dense field. The existence of wind shearing is typical of calm regions. Removing the low-wind areas typically leads to a continuous field on a planar surface F. The criterion is easily checked by looking at the stick field on the boundary curves of F. To decide whether such a boundary curve C of F is admissable, one chooses an arbitrary initial orientation of a stick S at an arbitrary point of C and transports this orientation once around C with maximum smoothness (i.e., less than 90° of vector rotation between consecutive points on C). This process yields two orientations of S: the initial orientation and the transported orientation. The curve C is admissible if and only if the initial orientation coincides with the transported orientation.

insert figure here

The figure shows part of a stick field for which the curve C_1 is not admissible whereas the curve C_2 is admissible. In general, if the criterion fails to hold for a boundary curve C then one may have wind-shearing near the curve C and one may try to remove a larger region near C to obtain a new planar surface F on which the given field is admissible.

References

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