A UNIFORM FOURIER RESTRICTION THEOREM FOR SURFACES IN $\mathbb{R}^3$

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Abstract. We establish a uniform Fourier restriction estimate for certain hypersurfaces in $\mathbb{R}^3$.

The pupose of this note is to prove an analog of the Stein-Tomas Fourier restriction theorem for a class of surfaces in $\mathbb{R}^3$. Suppose γ is a nonnegative $C^{(3)}$ function on $[0, b)$ satisfying $\gamma(0) = \gamma'(0) = 0$, $\gamma^{(j)} > 0$ for $j = 1, 2, 3$. Define

$$\Gamma : \{x \in \mathbb{R}^2 : |x| < b\} \to \mathbb{R}^3$$

by

$$\Gamma(x) = \big(x, \gamma(|x|)\big).$$

Let the surface S in $\mathbb{R}^3$ be the range of Γ and let σ be the image under Γ of the measure

$$\Big(\frac{\gamma'(|x|)}{|x|}\Big)^{1/2} dx.$$

Theorem. *There is an absolute constant C such that the inequality*

$$\Big(\int_S |\widehat{f}(\xi)|^2 d\sigma(\xi)\Big)^{1/2} \leq C\ \|f\|_{4/3}$$

holds for measurable f on $\mathbb{R}^3$.

Carbery and Ziesler ([CZ]) raise the possibility of a universal restriction theorem for convex hypersurfaces in $\mathbb{R}^n$. They point out that, in the radial case we consider and when $n = 3$, the natural choice for a weight is

$$\Big(\frac{\gamma''(|x|)\gamma'(|x|)}{|x|}\Big)^{1/4},$$

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the 1/4 power of the affine curvature on S. Our choice

$$\Big(\frac{\gamma'(|x|)}{|x|}\Big)^{1/2}$$

is (unfortunately) smaller since $\gamma'(t) \leq t\gamma''(t)$.

Proof of Theorem: By duality it is enough to prove the inequality

$$\|\widehat{gd\sigma}\|_4 \leq C\ \|g\|_{L^2(d\sigma)}$$

for functions g on S. By Plancherel's theorem, this is equivalent to

$$\|(gd\sigma) * (gd\sigma)\|_2 \leq C\|g\|^2_{L^2(d\sigma)}$$

and so to

$$\int\int g(x)g(y)h\big(x+y, \gamma(|x|)+\gamma(|y|)\big)\Big(\frac{\gamma'(|x|)\gamma'(|y|)}{|x||y|}\Big)^{1/2} dx\ dy \leq C\ \|g\|^2_{L^2(d\sigma)}\|h\|_2,$$

where the integrals are over $\{x \in \mathbb{R}^2 : |x| < b\}$. Applying the Schwartz inequality shows that it is enough to establish the inequality

$$\int\int h^2\big(x+y, \gamma(|x|)+\gamma(|y|)\big)\Big(\frac{\gamma'(|x|)\gamma'(|y|)}{|x||y|}\Big)^{1/2} dxdy \leq C\ \|h\|_2^2. \tag{1}$$

Writing $x = re^{i\theta}$ and $y = se^{i\phi}$, the left hand side of (1) is

$$\int\cdots\int h^2\big(re^{i\theta}+se^{i\phi}, \gamma(r)+\gamma(s)\big)\Big(r\gamma'(r)s\gamma'(s)\Big)^{1/2} d\theta\ d\phi\ dr\ ds. \tag{2}$$

A Jacobian computation shows that

$$\int\int h^2\big(re^{i\theta}+se^{i\phi}, \gamma(r)+\gamma(s)\big)d\theta\ d\phi = 4\int_{\{|r-s|<|x|<r+s\}} \frac{h^2\big(x, \gamma(r)+\gamma(s)\big)}{\Big(\big[(r+s)^2-|x|^2\big]\big[|x|^2-(r-s)^2\big]\Big)^{1/2}}dx.$$

Define

$$\omega(r,s;x) = \frac{4\chi_{\{|r-s|<|x|<r+s\}}(x)}{\Big(\big[(r+s)^2-|x|^2\big]\big[|x|^2-(r-s)^2\big]\Big)^{1/2}}$$

so that (2) is

$$\int\int\int h^2\big(x, \gamma(r)+\gamma(s)\big)\omega(r,s;x)dx\ \Big(r\gamma'(r)s\gamma'(s)\Big)^{1/2} dr\ ds. \tag{3}$$

For fixed $y \in (0,\infty)$ the equation $\gamma(r)+\gamma(s) = y$ defines a curve in $\{(r,s) : r,s > 0\}$. With x fixed as well, write $C_{x,y}$ for the intersection of that curve with $\{|r-s| < |x| < r+s\}$. If t denotes arclength on $C_{x,y}$, then the coarea formula ([F], Theorem 3.2.12) shows that (3) is equal to

$$\int\int h^2(x,y)\int_{C_{x,y}} \omega(r,s;x)\frac{\big(r\gamma'(r)s\gamma'(s)\big)^{1/2}}{\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/2}}dt\ dx\ dy.$$

Thus, in order to bound (3) by $C\,\|h\|_2^2$ and therefore to establish (1), it is sufficient (and necessary) to prove that

$$\int_{C_{x,y}} \omega(r,s;x)\frac{\big(r\gamma'(r)s\gamma'(s)\big)^{1/2}}{\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/2}}dt \le C \tag{4}$$

for some absolute constant C. And by symmetry it is enough to replace $C_{x,y}$ by $C_{x,y}\cap\{r>s\}$. So fix $x\in\mathbb{R}^2$, $y\in(0,\infty)$. Recall that t denotes arclength on $C_{x,y}$ and let $\big(r(t),s(t)\big)$ be a parametrization of $C_{x,y}$ normalized so that $r(0)=s(0)$ and $r'(t)\ge 0$. We will eventually need some facts about the derivatives of $r(t)$ and $s(t)$: Since $\gamma\big(r(t)\big)+\gamma\big(s(t)\big)\equiv y$ it follows that $\gamma'\big(r(t)\big)r'(t)+\gamma'\big(s(t)\big)s'(t)\equiv 0$. Since $\gamma'>0$ and $r'(t)>0$ we conclude that $s'(t)<0$. And since $r(t)>s(t)$ and $\gamma''>0$ it also follows that $|s'(t)|>r'(t)$. Thus

$$r'(t)+s'(t)<0. \tag{5}$$

Differentiating $r'(t)^2+s'(t)^2\equiv 1$ shows that $\big[r''(t),s''(t)\big]$ is parallel to $\big[\gamma'\big(r(t)\big),\gamma'\big(s(t)\big)\big]$, and it follows that

$$r''(t)+s''(t)<0. \tag{6}$$

Now define

$$C_1 = C_{x,y}\cap\{r^2+s^2\ge|x|^2\},$$
$$C_2 = C_{x,y}\cap\{r^2+s^2<|x|^2\}.$$

We will establish the analogs of (4) for C_1 and C_2. If $r^2+s^2\ge|x|^2$ then

$$(r+s)^2-|x|^2\ge 2rs$$

and so, since $2\gamma'(r)\gamma'(s)\le\gamma'(r)^2+\gamma'(s)^2$, the integral

$$\int_{C_1}\omega(r,s;x)\frac{\big(r\gamma'(r)s\gamma'(s)\big)^{1/2}}{\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/2}}dt$$

is controlled by

$$C\int_{C_1}\frac{1}{\big(|x|^2-(r-s)^2\big)^{1/2}}dt,$$

where C denotes an absolute constant which may, however, increase from line to line. Since $r'(t)^2+s'(t)^2\equiv 1$ and $s'(t)\le 0\le r'(t)$, it follows that

$$1\le r'(t)-s'(t)\le 2^{1/2}. \tag{7}$$

Thus the change of variable $u=r(t)-s(t)$ bounds the last integral by

$$\int_0^{|x|}\frac{1}{(|x|^2-u^2)^{1/2}}du=\frac{\pi}{2}.$$

To treat the integral over C_2 we begin similarly by noting that if $r^2+s^2<|x|^2$ then $|x|^2-(r-s)^2\geq 2rs$. It follows that

$$\int_{C_2}\omega(r,s;x)\frac{\big(r\gamma'(r)s\gamma'(s)\big)^{1/2}}{\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/2}}dt$$

is bounded by

(8)

$$\int_0^{t_0}\frac{\big(r(t)s(t)\big)^{1/2}}{\big(r(t)+s(t)\big)^{1/2}|(r(t)+s(t)-r(t_0)-s(t_0)|^{1/2}}\Big(\frac{\gamma'\big(r(t)\big)\gamma'\big(s(t)\big)}{r(t)s(t)}\Big)^{1/2}\frac{dt}{\big(\gamma'(r(t))^2+\gamma'(s(t))^2\big)^{1/2}}$$

for some t_0. It follows from (5) and (6) that $|r'(t)+s'(t)|$ is an increasing function of t so that, if $0<t<t_0$, we have

$$|r(t)+s(t)-r(t_0)-s(t_0)|\geq|t_0-t|\;|r'(t)+s'(t)|. \tag{9.}$$

The equations $\gamma'\big(r(t)\big)r'(t)+\gamma'\big(s(t)\big)s'(t)\equiv 0$ and $r'(t)^2+s'(t)^2\equiv 1$ show that

$$|r'(t)+s'(t)|=\frac{\gamma'\big(r(t)\big)-\gamma'\big(s(t)\big)}{\big[\gamma'\big(r(t)\big)^2+\gamma'\big(s(t)\big)^2\big]^{1/2}}. \tag{10}$$

The normalization $r(0)=s(0)$ and (7) together imply that $|r(t)-s(t)|\geq t$. With (9) and (10), this gives

$$|r(t)+s(t)-r(t_0)-s(t_0)|\geq\frac{\gamma'\big(r(t)\big)-\gamma'\big(s(t)\big)}{|r(t)-s(t)|}\frac{|t_0-t|t}{\big[\gamma'\big(r(t)\big)^2+\gamma'\big(s(t)\big)^2\big]^{1/2}}.$$

Writing r and s for $r(t)$ and $s(t)$, the integrand in (8) is thus bounded by

$$\frac{(rs)^{1/2}}{(r+s)^{1/2}}\Big(\frac{(r-s)\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/2}}{t(t_0-t)\big(\gamma'(r)-\gamma'(s)\big)}\Big)^{1/2}\Big(\frac{\gamma'(r)\gamma'(s)}{rs}\Big)^{1/2}\frac{1}{\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/2}}=$$

$$\frac{(rs)^{1/2}}{(r+s)^{1/2}}\Big(\frac{\gamma'(r)\gamma'(s)}{rs}\Big)^{1/4}\frac{1}{\big(\gamma'(r)^2+\gamma'(s)^2\big)^{1/4}}\frac{1}{\big(t(t-t_0)\big)^{1/2}}\Big(\frac{(r-s)}{(\gamma'(r)-\gamma'(s)}\Big)^{1/2}\Big(\frac{\gamma'(r)\gamma'(s)}{rs}\Big)^{1/4}.$$

Now, since γ'' is increasing,

$$\Big(\frac{1}{r}\int_0^r\gamma''(u)du\cdot\frac{1}{s}\int_0^s\gamma''(u)du\Big)^{1/2}\leq\frac{1}{r-s}\int_s^r\gamma''(u)du.$$

It follows that

$$(r-s)\big(\gamma'(r)\gamma'(s)\big)^{1/2}\leq(rs)^{1/2}\big(\gamma'(r)-\gamma'(s)\big).$$

Therefore (8) is bounded by C times

$$\int_0^{t_0}\frac{1}{\big(t(t_0-t)\big)^{1/2}}dt=\pi.$$

References

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[F] H. Federer, Geometric Measure Theory, Springer, 1969.