Calculation of Uncertainty Propagation Using Adjoint Equations

A.K. Alekseev
Dept. of Aerodynamics and Heat Transfer
RSC ``ENERGIA"
Korolev, Moscow Region
141070 RUSSIA

I. Michael Navon
Department of Mathematics and CSIT.
Florida State University
Tallahassee, FL 32306-4130
USA

Abstract

The uncertainty of flow parameters depending on the error of input data (initial, boundary conditions, coefficients) may be efficiently calculated using adjoint equations. This approach is extremely effective for uncertainty estimation at certain checkpoint because it needs to solve only single (adjoint) system of equations above the system describing the flow-field. The fields of "adjoint "temperature", adjoint "density" etc. are used to calculate the transfer of uncertainty from all input data.

Introduction

The uncertainty of flow-field parameters depending on the error of initial conditions, boundary conditions, and coefficients may be calculated by a number of approaches such as Monte-Carlo methods or sensitivity equations. Unfortunately, the algorithms that estimate both the flow parameters and their uncertainty are very rare because they usually need a great computation time. For example, the dispersion of a result ε may be calculated from the input data f_i dispersion using the sensitivity coefficients $\partial\varepsilon/\partial f_i$ [1].

$$\sigma_\varepsilon^2 = \sum_{i=1}^{N} \left(\frac{\partial \varepsilon}{\partial f_i} \sigma_{f_i} \right)^2 \qquad (1)$$

The time for $\partial\varepsilon/\partial f_i$ calculation either using sensitivity equations or the direct numerical differentiation of ε is proportional to time of ε calculation multiplied by the number N of input parameters. The total time of $\partial\varepsilon/\partial f_i$ calculation is practically unacceptable if the time of ε calculation is high and N is great.

The present paper addresses to the estimation of uncertainty using coefficients $\partial\varepsilon/\partial f_i$ calculated via adjoint equations. If the uncertainty is estimated in single checkpoint, this approach needs minimal computational resources because only one adjoint system should be solved independently on number N. The time for $\partial\varepsilon/\partial f_i$ calculation is approximately equal to double time of ε computation in this event.

The fields of adjoint "temperature", adjoint "density" etc. depend on flow-field, estimated parameter, checkpoint location and do not depend on the set of input data, which have an uncertainty. So, they are universal and permit the calculation of uncertainty caused by any parameter of the system of equations.

The adjoint approach is demonstrated herein for parabolized Navier-Stokes.

Flow parameter uncertainty estimation

We consider the uncertainty estimation in supersonic viscous flow, Fig. 1. The flow parameters are calculated by the finite-difference approximation of parabolized Navier-Stokes

[2,3]. The march along X coordinate was used.

$$\frac{\partial(\rho U)}{\partial X} + \frac{\partial(\rho V)}{\partial Y} = 0 \qquad (2)$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + \frac{1}{\rho}\frac{\partial P}{\partial X} = \frac{1}{\mathrm{Re}\,\rho}\frac{\partial^2 U}{\partial Y^2} \qquad (3)$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} + \frac{1}{\rho}\frac{\partial P}{\partial Y} = \frac{4}{3\rho\,\mathrm{Re}}\frac{\partial^2 V}{\partial Y^2} \qquad (4)$$

$$U\frac{\partial e}{\partial X} + V\frac{\partial e}{\partial Y} + (\kappa-1)e\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) = \frac{1}{\rho}\left(\frac{\kappa}{\mathrm{Re\,Pr}}\frac{\partial^2 e}{\partial Y^2} + \frac{4}{3\,\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)^2\right) \qquad (5)$$

$$P=\rho RT;\ e=C_v T;\ (X,Y)\in\Omega=(0<X<X_{max};\ 0<Y<1);$$

The entrance boundary (A ($X=0$), Fig. 1) conditions follow:
$e(0,Y)=e_\infty(Y);\ \rho(0,Y)=\rho_\infty(Y));\ U(0,Y)=U_\infty(Y);\ V(0,Y)=V_\infty(Y);$ (6)
the outflow conditions $\partial f/\partial Y=0$ are used on B, D ($Y=0$, $Y=1$).

Let the inflow parameters to contain the uncertainty. We assume the discrete analogues of these parameters to contain a normally distributed error $(\sigma_\rho,\sigma_U,\sigma_V,\sigma_e)$.

Let we seek for total flow-field and the accuracy of certain parameter (let it be temperature) in some check point $T(t_{est},x_{est})$, more precisely: a dependence of the temperature standard deviation from the input data deviations $\sigma_T = f(\sigma_\rho,\sigma_U,\sigma_V...)$.

Note $\boldsymbol{T}(X^{est},Y^{est})$ as $\varepsilon(f_\infty(Y),\mathrm{Re})$. If the estimated parameter is located on the outflow boundary we may express it as

$$\varepsilon(\boldsymbol{f}_\infty(Y)) = \int \boldsymbol{T}(X_{\max},Y)\,\delta(Y-Y^{est})\,dy \qquad (7)$$

If $\boldsymbol{T}(X^{est},Y^{est})$ is located within the field we write

$$\varepsilon(\boldsymbol{f}_\infty(Y)) = \int_\Omega \boldsymbol{T}(X,Y)\,\delta(Y-Y^{est})\,\delta(X-X^{est})\,dxdy \qquad (8)$$

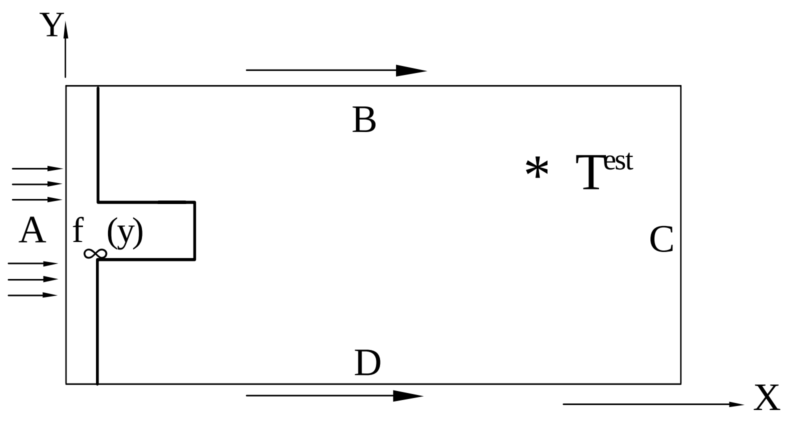

Fig. 1

The input data dispersion is transformed to the result dispersion by gradients [1], in our case:

$$\sigma_\varepsilon^2 = \sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial T_\infty^i}\sigma_{T_i}\right)^2 + \sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial\rho_\infty^i}\sigma_{\rho_i}\right)^2 + \sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial U_\infty^i}\sigma_{U_i}\right)^2 + \sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial V_\infty^i}\sigma_{V_i}\right)^2 + \left(\frac{\partial\varepsilon}{\partial(1/\mathrm{Re})}\sigma_{(1/\mathrm{Re})}\right)^2 \qquad (9)$$

The most efficient way for gradient calculation is based on adjoint equations. For these equations inference we introduce the Lagrangian $L(f_\infty(Y),\Psi)$, composed of the estimated value and weak statement of problem (2-5).

$$L(f_\infty(Y),\mathrm{Re}) = \varepsilon(f_\infty(Y)) = \int_\Omega T(X,Y)\delta(X-X^{est})\delta(Y-Y^{est})dXdY +$$

$$+\int_X\int_Y\left(\frac{\partial(\rho U)}{\partial X}+\frac{\partial(\rho V)}{\partial Y}\right)\Psi_\rho(X,Y)dXdY +$$

$$+\int_X\int_Y\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+\frac{1}{\rho}\frac{\partial P}{\partial X}-\frac{1}{\mathrm{Re}\,\rho}\frac{\partial^2 U}{\partial Y^2}\right)\Psi_U(X,Y)dXdY +$$

$$+\int_X\int_Y\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+\frac{1}{\rho}\frac{\partial P}{\partial Y}-\frac{4}{3\,\mathrm{Re}\,\rho}\frac{\partial^2 V}{\partial Y^2}\right)\Psi_V(X,Y)dXdY +$$

$$+\int_X\int_Y\left(U\frac{\partial e}{\partial X}+V\frac{\partial e}{\partial Y}+(\kappa-1)e\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right)\Psi_e(X,Y)dXdY +$$

$$+\int_X\int_Y\left(-\frac{\kappa}{\rho\mathrm{RePr}}\frac{\partial^2 e}{\partial Y^2}-\frac{4}{3\mathrm{Re}\,\rho}\left(\frac{\partial U}{\partial Y}\right)^2\right)\Psi_e(X,Y)dXdY\,. \qquad (10)$$

Consider the influence of inflow data variation $\Delta f_\infty(X)$ and coefficient variation $\Delta(1/\mathrm{Re})$. By subtracting the undisturbed solution we get the linear tangent model:

$$U\frac{\partial(\Delta\rho)}{\partial X}+\rho\frac{\partial(\Delta U)}{\partial X}+\rho\frac{\partial(\Delta V)}{\partial Y}+V\frac{\partial(\Delta\rho)}{\partial Y}+$$
$$\Delta\rho\frac{\partial U}{\partial X}+\Delta U\frac{\partial\rho}{\partial X}+\Delta\rho\frac{\partial V}{\partial Y}+\Delta V\frac{\partial\rho}{\partial Y}=0\,, \qquad (11)$$

$$U\frac{\partial\Delta U}{\partial X}+\Delta U\frac{\partial U}{\partial X}+\Delta V\frac{\partial U}{\partial Y}+V\frac{\partial\Delta U}{\partial Y}-\frac{1}{\rho\mathrm{Re}}\frac{\partial^2\Delta U}{\partial Y^2}+\frac{\Delta\rho}{\rho^2\,\mathrm{Re}}\frac{\partial^2 U}{\partial Y^2}-$$
$$-\frac{\Delta\rho}{\rho^2}\frac{\partial P}{\partial X}+\frac{(\kappa-1)}{\rho}\left(\Delta\rho\frac{\partial e}{\partial X}+e\frac{\partial\Delta\rho}{\partial X}+\rho\frac{\partial\Delta e}{\partial X}+\Delta e\frac{\partial\rho}{\partial X}\right)-\frac{1}{\rho}\frac{\partial^2 U}{\partial Y^2}\Delta(1/\mathrm{Re})=0 \qquad (12)$$

$$U\frac{\partial\Delta V}{\partial X}+\Delta U\frac{\partial V}{\partial X}+\Delta V\frac{\partial V}{\partial Y}+V\frac{\partial\Delta V}{\partial Y}-\frac{4}{3\rho\,Re}\frac{\partial^2\Delta V}{\partial Y^2}+\frac{4\Delta\rho}{3\rho^2\,Re}\frac{\partial^2 V}{\partial Y^2}-$$
$$-\frac{\Delta\rho}{\rho^2}\frac{\partial P}{\partial Y}+\frac{(\kappa-1)}{\rho}\left(\Delta\rho\frac{\partial e}{\partial Y}+e\frac{\partial\Delta\rho}{\partial Y}+\rho\frac{\partial\Delta e}{\partial Y}+\Delta e\frac{\partial\rho}{\partial Y}\right)-\frac{4}{3\rho}\frac{\partial^2 V}{\partial Y^2}\Delta(1/\mathrm{Re})=0 \qquad (13)$$

$$U\frac{\partial\Delta e}{\partial X}+\Delta U\frac{\partial e}{\partial X}+\Delta V\frac{\partial e}{\partial Y}+V\frac{\partial\Delta e}{\partial Y}+(\kappa-1)\Delta e\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)+(\kappa-1)e\left(\frac{\partial\Delta U}{\partial X}+\frac{\partial\Delta V}{\partial Y}\right)-$$

$$-\frac{1}{\rho}\left(\frac{\kappa}{\mathrm{RePr}}\frac{\partial^2\Delta e}{\partial Y^2}-\frac{\kappa}{\mathrm{RePr}}\frac{\Delta\rho}{\rho}\frac{\partial^2 e}{\partial Y^2}\right)-\frac{1}{\rho}\left(\frac{8}{3\mathrm{Re}}\left(\frac{\partial\Delta U}{\partial Y}\right)\left(\frac{\partial U}{\partial Y}\right)-\frac{\Delta\rho}{\rho}\frac{4}{3\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)^2\right)-.$$

$$-\frac{1}{\rho}\left(\frac{\kappa}{\mathrm{Pr}}\frac{\partial^2 e}{\partial Y^2}+\frac{4}{3}\left(\frac{\partial U}{\partial Y}\right)^2\right)\Delta(1/\mathrm{Re})=0 \tag{14}$$

On the boundaries the variations $\Delta\rho,\Delta U,\Delta V,\Delta e$ should satisfy to:

Inflow boundary conditions: $\Delta e(0,Y)=\Delta e_\infty(Y)$, $\Delta\rho(0,Y)=\Delta\rho_\infty(Y)$, $\Delta\rho(0,Y)=\Delta\rho_\infty(Y)$, $\Delta V(0,Y)=\Delta V_\infty(Y)$ (15)

Lateral boundary conditions: B $(Y=1)$: $\Delta e(X,1)=0$, $\Delta\rho(X,1)=0$, $\Delta U(X,1)=0$, $\Delta V(X,1)=0$.

D $(Y=0)$: $\Delta e(X,0)=0$, $\Delta\rho(X,0)=0$, $\Delta U(X,0)=0$, $\Delta V(X,0)=0$.

We use equations (11-15) for Lagrangian (10) variation statement.

$$\Delta\varepsilon(Q_w(t)) = \int_\Omega \Delta T\,\delta(t-t_{est})\,\delta(x-x_{est})\,dtdx + \dots \tag{16}$$

Integrating the Lagrangian variation (16) by parts taking into account of (11-15) allows estimating the variation of target parameter in dependence on the disturbed parameters (17).

$$\Delta L(\boldsymbol{f}_\infty(Y),\boldsymbol{f},\Psi)=\Delta\varepsilon(f_\infty(Y))=\int_Y\left(\left(\Psi_e U+(\kappa-1)\Psi_U\right)\Delta e_\infty(Y)\right)\Big|_{X=0}dY+$$

$$+\int_Y\left(\left(\Psi_\rho U+(\kappa-1)\Psi_U e/\rho\right)\Delta\rho_\infty(Y)\right)\Big|_{X=0}dY+$$

$$+\int_Y\left(\left(\Psi_U U+\rho\Psi_\rho+(\kappa-1)\Psi_e e\right)\Delta U_\infty(Y)\right)\Big|_{X=0}dY+\int_Y\left(\Psi_V U\Delta V_\infty(Y)\right)\Big|_{X=0}dY-$$

$$-\Delta(1/\mathrm{Re})\int_\Omega\left(\frac{1}{\rho}\frac{\partial^2 U}{\partial Y^2}\Psi_U+\frac{4}{3\rho}\frac{\partial^2 V}{\partial Y^2}\Psi_V+\frac{\kappa}{\rho\,\mathrm{Pr}}\frac{\partial^2 e}{\partial Y^2}\Psi_e+\frac{4}{3\rho}\left(\frac{\partial U}{\partial Y}\right)^2\Psi_e\right)dXdY \tag{17}$$

Eq. (17) is valid if the remaining terms of $\Delta L(\boldsymbol{f}_\infty(Y),\boldsymbol{f},\Psi)$ equal zero, i.e. on the solution of the adjoint problem (18-22).

Adjoint problem

$$U\frac{\partial\Psi_\rho}{\partial X}+V\frac{\partial\Psi_\rho}{\partial Y}+(\kappa-1)\frac{\partial(\Psi_V e/\rho)}{\partial Y}+(\kappa-1)\frac{\partial(\Psi_U e/\rho)}{\partial X}-$$

$$-\frac{\kappa-1}{\rho}\left(\frac{\partial e}{\partial Y}\Psi_V+\frac{\partial e}{\partial X}\Psi_U\right)+\left(\frac{1}{\rho^2}\frac{\partial P}{\partial X}-\frac{1}{\rho^2\,\mathrm{Re}}\frac{\partial^2 U}{\partial Y^2}\right)\Psi_U+\frac{1}{\rho^2}\left(\frac{\partial P}{\partial Y}-\frac{4}{3\mathrm{Re}}\frac{\partial^2 V}{\partial Y^2}\right)\Psi_V-$$

$$-\frac{1}{\rho^2}\left(\frac{\kappa}{\mathrm{RePr}}\frac{\partial^2 e}{\partial Y^2}+\frac{4}{3\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)^2\right)\Psi_e=0 \tag{18}$$

$$U\,\frac{\partial\Psi_U}{\partial X}+\frac{\partial(\Psi_U V)}{\partial Y}+\rho\,\frac{\partial\Psi_\rho}{\partial X}-\left(\frac{\partial V}{\partial X}\Psi_V+\frac{\partial e}{\partial X}\Psi_e\right)+\frac{\partial}{\partial X}\left(\frac{P}{\rho}\Psi_e\right)+$$

$$+\frac{\partial^2}{\partial Y^2}\left(\frac{1}{\rho\;\mathrm{Re}}\Psi_U\right)-\frac{\partial}{\partial Y}\left(\frac{8}{3\;\mathrm{Re}}\frac{\partial U}{\partial Y}\Psi_e\right)=0 \qquad (19)$$

$$\frac{\partial(U\,\Psi_V)}{\partial X}+V\,\frac{\partial\Psi_V}{\partial Y}-\left(\frac{\partial U}{\partial Y}\Psi_U+\frac{\partial e}{\partial Y}\Psi_e\right)+$$

$$+\rho\,\frac{\partial\Psi_\rho}{\partial Y}+\frac{\partial}{\partial Y}\left(\frac{P}{\rho}\Psi_e\right)+\frac{4}{3\;\mathrm{Re}}\frac{\partial^2}{\partial Y^2}\left(\frac{\Psi_V}{\rho}\right)=0 \qquad (20)$$

$$\frac{\partial(U\Psi_e)}{\partial X}+\frac{\partial(V\Psi_e)}{\partial Y}-\frac{\kappa-1}{\rho}\left(\frac{\partial\rho}{\partial Y}\Psi_V+\frac{\partial\rho}{\partial X}\Psi_U\right)-(\kappa-1)\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\Psi_e+$$

$$+(\kappa-1)\frac{\partial\Psi_V}{\partial Y}+(\kappa-1)\frac{\partial\Psi_U}{\partial X}+\frac{\kappa}{\mathrm{Re}\;\mathrm{Pr}}\frac{\partial^2}{\partial Y^2}\left(\frac{\Psi_e}{\rho}\right)-\delta(X-X^{est})\delta(Y-Y^{est})=0 \qquad (21)$$

The source term in equation describing Ψ_e (21) corresponds the checkpoint location within the flow-field.

Initial conditions *C* (*X*=*X*$_{max}$): $\Psi_{\rho,U,V}\Big|^{X=X_{\max}}=0;\;\; U\Psi_e+(\kappa-1)\Psi_U+\delta(Y-Y^{est})=0;$ (22)

Expression for Ψ_e in (22) corresponds the checkpoint location on the inflow boundary X_{max}.

Boundary conditions *B,D* (*Y*=0*; Y=1*): $\frac{\partial\Psi_f}{\partial Y}=0;$ (23)

The statement (18-23) differs from the adjoint equations used in Inverse CFD problems by the form of the target functional and, respectively, by the source term form in (21,22). The adjoint problem is solved in the reverse direction along *X*. Its statement is determined by the forward problem, check point position, and the choice of the estimated parameter. The adjoint problem does not depend on the choice of parameters containing the uncertainty. So, the same field of adjoint parameters may be used for the calculation of uncertainty propagation from any parameters (initial, boundary conditions, coefficients, sources). The gradients used for the uncertainty propagation (see Eq. (9)) have the following form:

$$\partial\varepsilon/\partial e_\infty(Y)=\Psi_e U+(\kappa-1)\Psi_U\,,\;\;\partial\varepsilon/\partial\rho_\infty(Y)=\Psi_\rho U+(\kappa-1)\Psi_U e/\rho$$

$$\partial\varepsilon/\partial U_\infty(Y)=\Psi_U U+\rho\Psi_\rho+(\kappa-1)\Psi_e e\,,\;\;\partial\varepsilon/\partial V_\infty(Y)=\Psi_V U$$

$$\nabla\varepsilon_{1/\mathrm{Re}}=-\int_\Omega\left(\frac{1}{\rho}\frac{\partial^2 U}{\partial Y^2}\Psi_U+\frac{4}{3\rho}\frac{\partial^2 V}{\partial Y^2}\Psi_V+\frac{\kappa}{\rho\,\mathrm{Pr}}\frac{\partial^2 e}{\partial Y^2}\Psi_e+\frac{4}{3\rho}\left(\frac{\partial U}{\partial Y}\right)^2\Psi_e\right)dXdY \qquad (24)$$

The calculation of the gradient implies the consequent solution of the direct and adjoint problems. So, the time for uncertainty calculation of the single parameter in the single checkpoint equals approximately two times of the flow-field calculation. The uncertainty estimation of every additional parameter needs the solution of additional adjoint equation.

Test results

The singular source in (21) is integrated over the cell and thus transformed to the finite source term $\delta_{ij} /(\Delta X \Delta Y)$, if check point is located within flow-field, and $\delta_{ij} / \Delta Y$ if the check point is on the boundary (22), where δ_{ij} is the unit matrix.

The uncertainty propagation is calculated by the adjoint equations and Monte-Carlo method (averaged over 100 trials) for the comparison. The inflow parameters contain the normally distributed error with the standard deviation in the range of 0.01-0.1. The standard deviations of temperature in the middle point (*N=50*) on the outflow boundary calculated by both methods are presented in the Table 1 for the case of uniform flow.

Uniform flow-field *(N_{est}=50)*

Table 1

σ_{f_∞}	σ_T , adjoint approach	σ_T , averaged over 100 runs
0.01	0.0024	0.002613
0.05	0.012	0.0144
0.1	0.024	0.03

Adjoint “density” field is presented in Figure 2, adjoint “temperature” is presented in Figure 3.

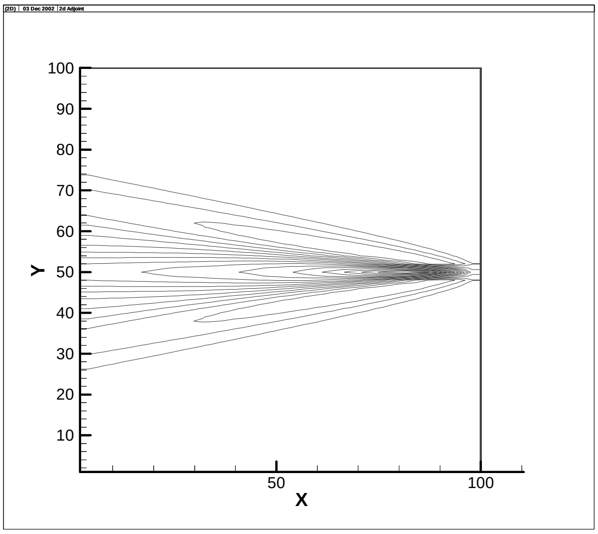

Fig. 2

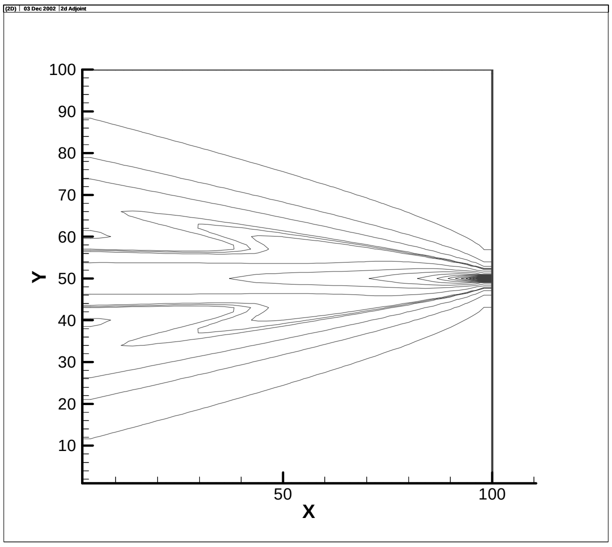

Fig. 3

The flow-field parameters and their gradients form sources and coefficients of adjoint equations. So, the following tests are conducted for non-uniform flow corresponding to underexpanded jet with the temperature ratio *Tj/T=3* (density isolines are provided in Fig. 4). The coincidence of adjoint and Monte-Carlo approaches is of the same quality. The temperature uncertainty estimations are presented in Tables 2 and 3 for different checkpoint locations *(N_{est}=50 and N_{est}=20)*. By comparing Fig. 5 and Fig. 6 we may see the difference of regions from which the main part of uncertainty is propagated.

Underexpanded jet (N_{est}=50)

Table 2

σ_{f_∞}	σ_T , adjoint approach	σ_T , averaged over 100 runs
0.01	0.0029	0.0029
0.05	0.0145	0.0152
0.1	0.029	0.03

Underexpanded jet, (N_{est}=20)

Table 3

σ_{f_∞}	σ_T , adjoint approach	σ_T averaged over 100 runs
0.01	0.00284	0.00304
0.05	0.0142	0.0148
0.1	0.0284	0.034

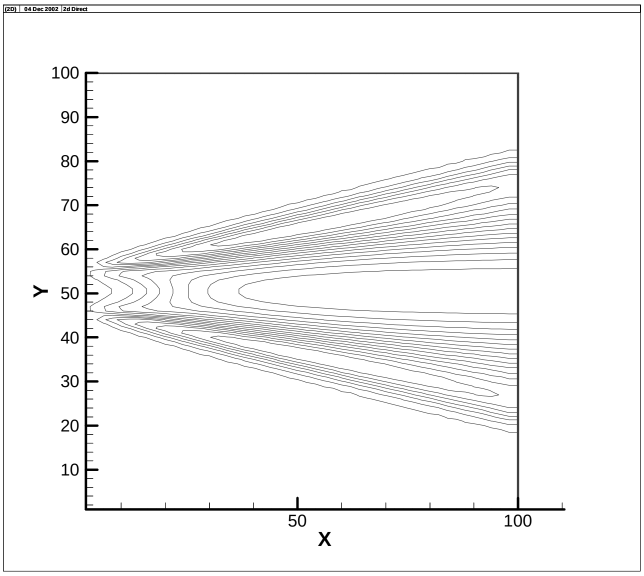

Fig. 4

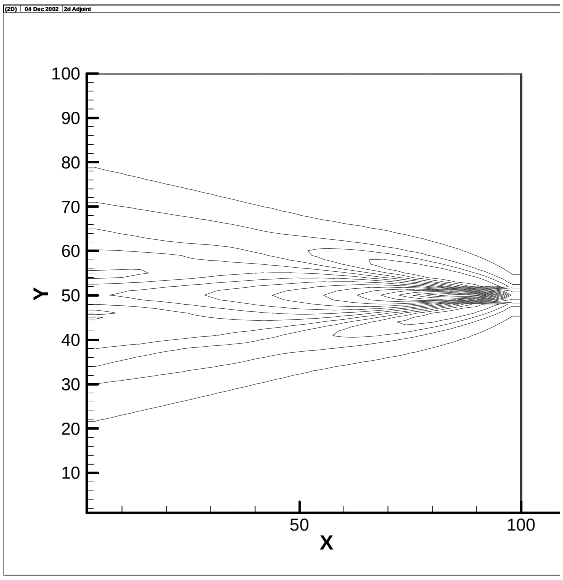

Fig. 5

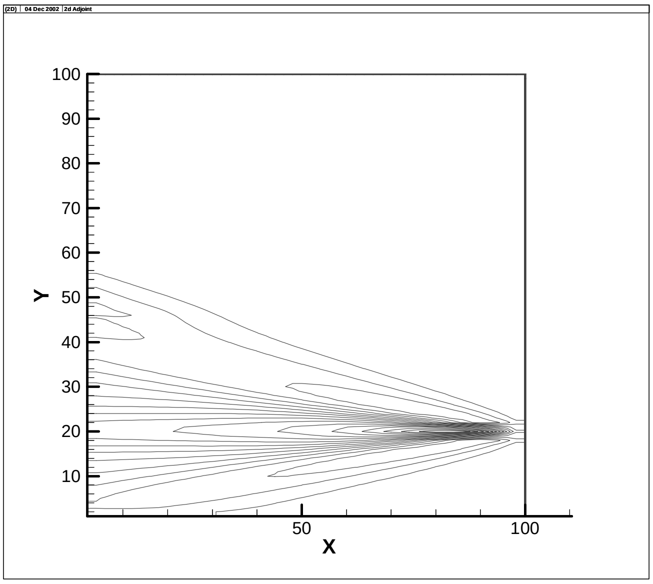

Fig. 6

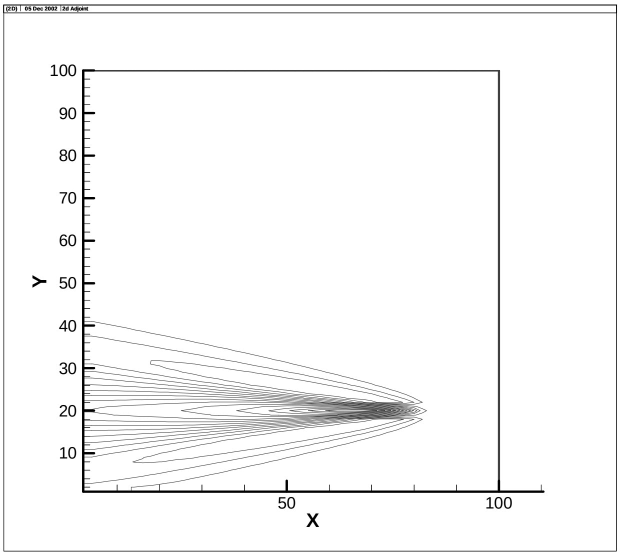

Fig. 7

Figures 2,3,5,6 correspond the checkpoint located on the boundary, (Eq. (22)), Fig. 7 corresponds the check point within the flow-field (Eq. (21)). Generally, the results of both approaches (Monte-Carlo and adjoint) correlate, while the consumed computer time differs by two orders of magnitude for these tests.

The computation of the single parameter *(T)* at the single checkpoint needs calculation of the flow-field and the adjoint field. The estimation of another parameter (or the same in another check point) needs calculation of the new adjoint field with the same flow-field.

Discussion

Let's compare different approaches for the uncertainty calculation from the viewpoint of necessary computer resources.

The temperature dispersion for total flow-field may be calculated by the sensitivity equations written for values $S_k(X,Y,X_k) = \frac{\partial T(X,Y)}{\partial f^i_\infty(X_k)}$. This approach implies the solution of the system of equations of higher dimension (by the dimension of the space of control parameters) in comparison with the system (2-5). For problem under consideration it may mean $4N_y$ calculations of (2-5). The adjoint equations need N_xN_y+1 calculations of (2-5) for the temperature dispersion in the total flow-field. Thus, the sensitivity equations are less expensive if the total field of dispersion is calculated. Nevertheless, the adjoint equations are far more efficient from computation time viewpoint for relatively small set of estimated parameters.

Another approach to the uncertainty estimation based on the adjoint equations of the second order is described in [4]. The target functional for this approach has a form:

$$\varepsilon(\delta f_\infty(Y)) = \int_X \left(T^{exact}_{Y_{est}}(X) - T^{error}_{Y_{est}}(X)\right)^2 dX$$

Second order adjoint approach uses calculation of Hessian (or part of its spectrum) and needs computation time proportional to the number of parameters containing error, which is for considered problem of the order of N_y. In general, the second order approach seems to be most suitable for the calculation of the uncertainty in the inverse CFD problem.

Monte-Carlo Methods are also expensive from the computational time standpoint, although they may be implemented much more simply because do not need the solution of any auxiliary problem.

In general, the considered adjoint method is most efficient from computational time viewpoint if we calculate the uncertainty of parameters in the small set of checkpoints.

The adjoint approach implies the simultaneous solution of the set of adjoint equations having slightest differences. So, the computation of the uncertainty may be easily paralleled.

Conclusion

The uncertainty of the flow parameter in the checkpoint from input data error may be calculated using adjoint equations under the total computer time consumption corresponding the double calculation of the flow-field.

The calculation of the uncertainty of *n* parameters needs the calculation of *n+1* fields (flow-field + *n* adjoint fields).

References

1. *Putko M.M., Newmann P.A., Taylor III A.C., and Green L.L.*, Approach for Uncertainty Propagation and Robust Design in CFD Using Sensitivity Derivatives, AIAA paper, 2001-2528, 2001, pp. 1-14
2. *A.K. Alekseev*, On Estimation of entrance boundary parameters from downstream measurements using adjoint approach, Int. J. for Numerical Methods in Fluids, v. 36, pp. 971-982, 2001
3. *A.K. Alekseev, I.M. Navon,* The analysis of an ill-posed problem using multiscale resolution and second order adjoint techniques, Computer Methods in Appl. Mech. and Eng., v.190, issue 15-17, pp. 1937-1953, 2001
4. *A.K. Alekseev, I.M. Navon,* On Estimation of Temperature Uncertainty Using the

Second Order Adjoint Problem, Int. J. of Comput. Fluid Dynamics, 2002 v. 16 (2), pp. 113–117