LIMITS OF TRANSLATES OF PLANE CURVES — ON A PAPER OF ALDO GHIZZETTI

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Abstract. We study the limits of PGL(3) translates of an arbitrary plane curve, giving a description of all possible limits of a given curve and computing the multiplicities of corresponding components in the normal cone to the base scheme of a related linear system. This information is a key step in the computation of the degree of the closure of the *linear orbit* of an arbitrary plane curve.

Our analysis recovers and extends results obtained by Aldo Ghizzetti in the 1930's.

1. Introduction

Let $\mathcal{C}$ be an arbitrary complex plane curve of degree d. Consider $\mathcal{C}$ together with all its translates: the orbit of $\mathcal{C}$ for the natural action of PGL(3) on the projective space $\mathbb{P}^n$ of all plane curves of degree d. Which plane curves appear in the orbit closure of $\mathcal{C}$? Or in other words, what are the limits of translates of $\mathcal{C}$? In this article we answer a refined form of this question.

Harris and Morrison ([HM98], p.138) define the flat completion problem for embedded families of curves as the determination of all curves in $\mathbb{P}^n$ that can arise as flat limits of a family of embedded stable curves over the punctured disc. The problem mentioned in the first paragraph contains the isotrivial case of the flat completion problem for plane curves, and a solution to it can in fact be found in the marvelous article [Ghi36b] by the Italian mathematician Aldo Ghizzetti (a summary of the results is contained in [Ghi36a]). However, as we will explain below, our main application requires a more refined type of information; thus our aim is somewhat different than Ghizzetti's, and we cannot simply lift his results. Consequently, our work in this paper is independent of [Ghi36b]. In any case, Ghizzetti's approach has substantially influenced ours; see §3.26 for a description of his work and a comparison with ours.

The enumerative geometry of families of plane curves with prescribed singularities presents a notoriously difficult problem. Spectacular progress was made in the last decade in several special cases; we mention the work of Kontsevich [Kon95] and of Caporaso-Harris [CH98]. Consider the special case where the family consists of a completely arbitrary plane curve and all its translates. In our paper [AF00a] we explained how the degree of this family, in other words, the number of curves in the family passing through the appropriate number of general points in the plane, can be computed. For example, for a nonsingular curve $\mathcal{C}$ this family is the set of all possible embeddings of $\mathcal{C}$ in $\mathbb{P}^2$; our motivation in [AF93], [AF00b], [AF00c], [AF00a], and the present article is the study of this set, and of its natural generalization for arbitrary plane curves.

The starting point of our method is to view the action map on $\mathcal{C}$ as a rational map c from the $\mathbb{P}^8$ of 3×3 matrices to the $\mathbb{P}^n$ of plane curves of degree d. We require a precise description of the closure of the graph of c, specifically of the scheme-theoretic inverse image of the locus of indeterminacy. This is the exceptional divisor of the blow-up of $\mathbb{P}^8$ along the base scheme of c, the projective normal cone (PNC). A set-theoretic description of the components of the PNC amounts to a solution of the (isotrivial) flat completion problem, together with careful bookkeeping of the different arcs in $\mathbb{P}^8$ used to obtain each limit. The PNC can be viewed as an arc space associated to the rational map c; it is probably possible to recast our analysis in §3 in the light of recent work on arc spaces (cf. for example [DL01]), and it would be interesting to do so.

In fact a set-theoretic description of the PNC does not suffice for our enumerative application in [AF00a]. This requires the full knowledge of the PNC *as a cycle,* that is, the determination of the multiplicity of its different components. Thus, we determine not only the limits of one-dimensional families of translates of $\mathcal{C}$, but we also classify such families up to a natural notion of equivalence and we keep track of the behavior of a family near the limit. The determination of the limits and the classification are contained in §3. As may be expected, the determination of the multiplicities is quite delicate; this is worked out in §4. Preliminaries, and a more detailed introduction, can be found in §2.

The final result of our analysis is stated in §2 of [AF00a], in the form of five 'Facts'. The proofs of these facts are spread over the present text; we recommend comparing loc.cit. with §3.2 and §4.2 to establish a connection between the more detailed statements proved here and the summary in [AF00a].

Caporaso and Sernesi use our determination of the limits in [CS03] (Theorem 5.2.1). Hassett [Has99] and Hacking [Hac03] study the limits of the family of nonsingular plane curves of a given degree, by methods different from ours: they allow the plane to degenerate together with the curve. It would be interesting to compare their results to ours.

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2. Preliminaries

2.1. We work over $\mathbb{C}$. Roughly speaking, the question we (and Ghizzetti) address is the following: given a plane curve $\mathcal{C}$, what plane curves can be obtained as limits of translates of $\mathcal{C}$? By a translate we mean the action on $\mathcal{C}$ of an invertible linear transformation of $\mathbb{P}^2$, that is, an element of PGL(3). We view PGL(3) as an open set in the space $\mathbb{P}^8$ parametrizing 3×3 matrices α; if $F(x,y,z)$ is a generator of the homogeneous ideal of $\mathcal{C}$, α acts on $\mathcal{C}$ by composition: we denote by $\mathcal{C}\circ\alpha$ the curve with ideal $(F\circ\alpha) = (F(\alpha(x,y,z)))$, and we call the set of all translates $\mathcal{C}\circ\alpha$ the *linear orbit* of $\mathcal{C}$. Our guiding question here concerns the limits of families $\mathcal{C}\circ g(a)$, for $g:\mathcal{A}\to\mathbb{P}^8$ any map from a smooth curve $\mathcal{A}$ to $\mathbb{P}^8$, centered at a point mapping to a singular transformation.

Since the flat limit is determined by the completion of the local ring of $\mathcal{A}$ at the center, we may replace $\mathcal{A}$ with Spec $\mathbb{C}[[t]]$. Thus, a 'curve germ in $\mathbb{P}^8$' (*germ* for short) in this article will simply be a $\mathbb{C}[[t]]$-valued point $\alpha(t)$ of $\mathbb{P}^8$. Our 'germs' are often called 'arcs' in the literature.

The limit of $\mathcal{C}\circ\alpha(t)$ as $t\to 0$:

$$\lim_{t\to 0}\mathcal{C}\circ\alpha(t)$$

is the flat limit over the punctured t-disk; concretely, this is obtained by clearing common powers of t in the expanded expression $F(\alpha(t))$ and then setting $t=0$.

It will always be assumed that the center $\alpha=\alpha(0)$ of a germ $\alpha(t)$ is a singular transformation. Further, we may and will assume that $\alpha(t)$ is invertible for some $t\neq 0$: indeed, this condition may be achieved by perturbing every $\alpha(t)$ achieving a limit, using terms of high enough power in t so as not to affect the limit.

We note that, by the same token, every limit attained by one of our 'germs' can conversely be realized as the flat limit of a family parametrized by a curve $\mathcal{A}$ mapping to $\mathbb{P}^8$, as above; and in fact we can even assume $\mathcal{A}=\mathbb{A}^1$. Indeed, truncating $\alpha(t)$ at a high enough power does not affect $\lim_{t\to 0}\mathcal{C}\circ\alpha(t)$, and a polynomial $\alpha(t)$ describes a map $\mathbb{A}^1\to\mathbb{P}^8$.

2.2. Here is one example showing that rather interesting limits may occur: let $\mathcal{C}$ be the 7-ic curve with equation

$$x^3z^4 - 2x^2y^3z^2 + xy^6 - 4xy^5z - y^7 = 0$$

and the family

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^8 & t^9 & 0 \\ t^{12} & \frac{3}{2}t^{13} & t^{14} \end{pmatrix} \quad ;$$

then

$$F\circ\alpha(t) = -\frac{1}{16}t^{52}(x^3(8x^2+3y^2-8xz)(8x^2-3y^2+8xz)) + \text{ higher order terms.}$$

That is,

$$\lim_{t\to 0} F\circ\alpha(t) = -\frac{1}{16}x^3(8x^2+3y^2-8xz)(8x^2-3y^2+8xz) \quad ,$$

a pair of quadritangent conics (see [AF00b], §4.1), union the distinguished tangent line taken with multiplicity 3. Note that the connected component of the identity in the stabilizer of this curve is the additive group.

2.3. Our primary objective is essentially to describe the possible limits of a plane curve $\mathcal{C}$, starting from a description of certain features of $\mathcal{C}$. We now state this goal more precisely.

In the process of computing the degree of the closure of the linear orbit of an arbitrary curve, [AF00a], we are led to studying the closure of the graph of the rational map

$$\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$$

mapping an invertible α to $\mathcal{C} \circ \alpha$, viewed as a point of the space $\mathbb{P}^N$ parametrizing plane curves of degree $\deg \mathcal{C}$. This graph may be identified with the blow-up of $\mathbb{P}^8$ along the base scheme S of this rational map. Much of the enumerative information we seek is then encoded in the *exceptional divisor* E of this blow-up, that is, in the projective normal cone of S in $\mathbb{P}^8$. In fact, the information can be obtained from a description of the components of E (viewed as 7-dimensional subsets of $\mathbb{P}^8 \times \mathbb{P}^N$) and from the multiplicities with which these components appear in E. A more thorough discussion of the relation between this information and the enumerative results, as well as of the general context underlying our study of linear orbits of plane curves, may be found in §1 of [AF00a].

Our goal in this paper is the description of the components of the projective normal cone of S, and the computation of the multiplicities with which the components appear in the projective normal cone.

2.4. This goal relates to the one stated more informally in §2.1 in the sense that the components of the projective normal cone dominate subsets of the boundary of the linear orbit of $\mathcal{C}$. Our technique will consist of studying an arbitrary $\alpha(t)$, aiming to determine whether the limit $(\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$ is a general point of the support of a component of the normal cone; we will thus obtain a description of all components of the normal cone. We should warn the reader that we will often abuse the language and refer to the point $(\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$ (in $\mathbb{P}^8 \times \mathbb{P}^N$) by the typographically more convenient $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ (which is a point of $\mathbb{P}^N$).

We should also point out that our analysis will not exhaust the boundary of a linear orbit: one component of this boundary may arise as the closure of the set of translates $\mathcal{C} \circ \alpha$ with α a rank-2 transformation, and a general such α does not belong to S. Indeed, the rational map mentioned above *is* defined at the general α of rank 2. To be more precise, if α is a rank-2 matrix whose image is not contained in $\mathcal{C}$ (for example, if $\mathcal{C}$ has no linear components), then $\mathcal{C} \circ \alpha$ may be described as a 'star' of lines through $\ker \alpha$, reproducing projectively the tuple of points cut out by $\mathcal{C}$ on the image of α. It would be interesting to provide a precise description of the set of stars arising in this manner. As this set does not contribute components to E, this study is not within the scope of this paper.

2.5. One of our main tools in the set-theoretic determination of the components of E will rely precisely on the fact that limits along rank-2 transformations do not contribute components to E. We will argue that if a limit obtained by a germ $\alpha(t)$ can also be obtained as a limit by a germ $\beta(t)$ contained in the rank-2 locus, then we can 'discard' $\alpha(t)$. Indeed, such limits will have to lie in the exceptional divisor of the blow-up of the rank-2 locus; as the rank-2 locus has dimension 7, such limits will span loci of dimension at most 6. We will call such limits 'rank-2 limits' for short. The form in which this observation will be applied is given in Lemma 3.1.

Incidentally, germs $\alpha(t)$ centered at a rank-2 transformation whose image is contained in $\mathcal{C}$ *do* contribute a component to E (cf. §3.6); by the argument given in the previous paragraph, however, contributing $\alpha(t)$ will necessarily be invertible for $t \neq 0$.

2.6. In §3 we will determine the components of E set-theoretically, as subsets of $\mathbb{P}^8 \times \mathbb{P}^N$. As a preliminary observation (cf. [AF00a], p. 8) we can describe the whole of E set-theoretically in terms of limits, as follows. Recall that S denotes the base locus of the rational map $c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N$ defined by $\alpha \mapsto \mathcal{C} \circ \alpha$.

Lemma 2.1. *As a subset of $\mathbb{P}^8 \times \mathbb{P}^N$, the support of E is*

$$\begin{aligned} |E| = \{(\alpha, X) \in \mathbb{P}^8 \times \mathbb{P}^N : X \textit{ is a limit of } c(\alpha(t)) \\ \textit{for some curve germ } \alpha(t) \subset \mathbb{P}^8 \textit{ centered at } \alpha \in S \textit{ and not contained in } S\} \quad . \end{aligned}$$

Proof. Let $\widetilde{\mathbb{P}}^8$ be the closure of the graph of the rational map c defined above. This is an 8-dimensional irreducible variety, mapping to $\mathbb{P}^8$ by the restriction π of the projection on the first factor of $\mathbb{P}^8 \times \mathbb{P}^N$, and identified with the blow-up of $\mathbb{P}^8$ along the base scheme S of c. The set E is the inverse image $\pi^{-1}(S)$ in $\widetilde{\mathbb{P}}^8$.

Any curve germ $\alpha(t)$ in $\mathbb{P}^8$ centered at $\alpha \in S$ and not contained in S lifts to a germ in $\mathbb{P}^8$ centered at a point of E; this yields the $\supset$ inclusion.

For the other inclusion, let $\tilde{\alpha}(t)$ be a germ centered at a point $\tilde{\alpha}$ of E, and such that $\tilde{\alpha}(t_0) \notin E$ for t_0 near 0; such a germ may be obtained (for example) by successively intersecting $\widetilde{\mathbb{P}}^8$ with general divisors of type $(1, 1)$ through $\tilde{\alpha}$. As π is 1-to-1 in the complement of E, $\tilde{\alpha}(t)$ is the lift of a (unique) curve germ $\alpha(t)$ in $\mathbb{P}^8$, giving the other inclusion. □

2.7. As mentioned in §1, our application in [AF00a] requires the knowledge of E as a cycle, that is, the computation of the multiplicities of the components of E. This information is obtained in §4. Our method will essentially consist of a local study of the families determined in §3: the multiplicity of a component will be computed by analyzing certain numerical information carried by germs $\alpha(t)$ 'marking' that component (cf. Definition 4.4).

We will in fact study the inverse image $\overline{E}$ of E in the normalization $\overline{\mathbb{P}}$ of $\widetilde{\mathbb{P}}^8$: roughly speaking, the multiplicity of the components of $\overline{E}$ is determined by the order of vanishing of $\mathcal{C} \circ \alpha(t)$ for corresponding germs $\alpha(t)$. The number of components of $\overline{E}$ dominating a given component D of E is computed by distinguishing the contribution of different marking germs.

Finally, the last key numerical information consists of the degree of the components of $\overline{E}$ over the corresponding components of E; this will be obtained by studying the PGL(3) action on $\widetilde{\mathbb{P}}^8$, and in particular the stabilizer of a general point of each component D of E. Given a component $\overline{D}$ of $\overline{E}$ dominating D, we identify a subgroup of this stabilizer (the *inessential* subgroup, cf. §4.6), determined by the interaction between different parametrizations of a corresponding marking germ; the degree of $\overline{D}$ over D is the index of this subgroup in the stabilizer (Proposition 4.12).

3. Set-theoretic description of the normal cone

3.1. In this section we determine the different components of the projective normal cone (PNC for short) E described in Lemma 2.1, for a given, arbitrary plane curve $\mathcal{C}$ with homogeneous ideal (F), where $F \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree d.

The PNC can be embedded in $\mathbb{P}^8 \times \mathbb{P}^N$, where $\mathbb{P}^N$ parametrizes all plane curves of degree d. A typical point of a component of the PNC is in the form

$$(\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$$

where $\alpha(t)$ is a curve germ centered at a point $\alpha(0)$ such that $\operatorname{im}\alpha(0)$ is contained in $\mathcal{C}$. We will determine the components of the PNC by determining a list of germs $\alpha(t)$ which exhaust the possibilities for pairs $(\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$ for a given curve. Roughly speaking, we will say that two germs are equivalent if they determine the same data $(\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$ (see Definition 3.2 for the precise notion). For a given curve $\mathcal{C}$ and a given germ $\alpha(t)$, we will construct an equivalent germ in a standardized form; we will determine which germs $\alpha(t)$ in these standard forms 'contribute' components of the PNC, in the sense that $(\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$ belongs to exactly one component (and that a sufficiently general such $\alpha(t)$ yields a general point of that component), and describe that component.

3.2. The end-result of the analysis can be stated without reference to specific germs $\alpha(t)$. We will do so in this subsection, by listing general points $(\alpha, \mathcal{X})$ on the components of the PNC, for a given $\mathcal{C}$.

We will find five types of components: the first two will depend on global features of $\mathcal{C}$, while the latter three will depend on features of special points of $\mathcal{C}$ (inflection points and singularities of the support of $\mathcal{C}$). The terminology employed here matches the one in §2 of [AF00a]. In four of the five components α is a rank-1 matrix, and the line $\ker\alpha$ plays an important rôle; we will call this 'the kernel line'.

- *Type I.*
  - α: a rank-2 matrix whose image is a linear component ℓ of $\mathcal{C}$;
  - $\mathcal{X}$: a fan consisting of a star of lines through the kernel of α and cutting out on the residual line ℓ' a tuple of points projectively equivalent to the tuple cut out on ℓ by the residual to ℓ in $\mathcal{C}$. The multiplicity of ℓ' in the fan is the same as the multiplicity of ℓ in $\mathcal{C}$.

  Fans and stars are studied in [AF00c]; they are items (3) and (5) in the classification of curves with small linear orbits, in §1 of loc. cit.
- *Type II.*
  - α: a rank-1 matrix whose image is a nonsingular point of the support $\mathcal{C}'$ of a nonlinear component of $\mathcal{C}$;

  - $\mathcal{X}$: a nonsingular conic tangent to the kernel line, union (possibly) a multiple of the kernel line. The multiplicity of the conic component in $\mathcal{X}$ equals the multiplicity of $\mathcal{C}'$ in $\mathcal{C}$.

  Such curves are items (6) and (7) in the classification of curves with small orbit. The extra kernel line is present precisely when $\mathcal{C}$ is not itself a multiple nonsingular conic.
- *Type III.*
  - α: a rank-1 matrix whose image is a point p at which the tangent cone to $\mathcal{C}$ is supported on at least three lines;
  - $\mathcal{X}$: a fan with star reproducing projectively the tangent cone to $\mathcal{C}$ at p, and a multiple residual kernel line.

  These limit curves are also fans, as in type I components; but note that type I and type III components are different, since for the typical $(\alpha, \mathcal{X})$ in type I components α has rank 2, while it has rank 1 for type III components.
- *Type IV.*
  - α: a rank-1 matrix whose image is a singular or inflection point p of the support of $\mathcal{C}$.
  - $\mathcal{X}$: a curve determined by the choice of a line in the tangent cone to $\mathcal{C}$ at p, and by the choice of a side of a corresponding Newton polygon. This procedure is explained more in detail below.

  The curves $\mathcal{X}$ arising in this way are items (7) through (11) in the classification in [AF00c], and are studied enumeratively in [AF00b].
- *Type V.*
  - α: a rank-1 matrix whose image is a singular point p of the support of $\mathcal{C}$.
  - $\mathcal{X}$: a curve determined by the choice of a line ℓ in the tangent cone to $\mathcal{C}$ at p, the choice of a formal branch for $\mathcal{C}$ at p tangent to ℓ, and the choice of a certain 'characteristic' rational number. This procedure is explained more in detail below.

  The curves $\mathcal{X}$ arising in this way are item (12) in the classification in [AF00c], and are studied enumeratively in [AF00b], §4.1.

Here are the details of the determination of the limit curves $\mathcal{X}$ for components of type IV and V.

Type IV: Let $p = \operatorname{im}\alpha$ be a singular or inflection point of the support of $\mathcal{C}$; choose a line in the tangent cone to $\mathcal{C}$ at p, and choose coordinates $(x : y : z)$ so that $x = 0$ is the line $\ker\alpha$, $p = (1 : 0 : 0)$, and that the selected line in the tangent cone has equation $z = 0$. The *Newton polygon* for $\mathcal{C}$ in the chosen coordinates is the boundary of the convex hull of the union of the positive quadrants with origin at the points (j, k) for which the coefficient of $x^i y^j z^k$ in the generator F for the ideal of $\mathcal{C}$ in the chosen coordinates is nonzero (see [BK86], p.380). The part of the Newton polygon consisting of line segments with slope strictly between -1 and 0 does not depend on the choice of coordinates fixing the flag $z = 0$, $p = (1 : 0 : 0)$.

The possible limit curves $\mathcal{X}$ determining components of type IV are then obtained by choosing a side of the polygon with slope strictly between -1 and 0, and setting to 0 the coefficients of the monomials in F *not* on that side. These curves are studied in [AF00b]; typically, they consist of a union of cuspidal curves. The kernel line is part of the distinguished triangle of such a curve, and in fact it must be one of the distinguished tangents.

This procedure determines a component of the PNC, unless the limit curve $\mathscr{X}$ is supported on a conic union (possibly) the kernel line.

Type V: Let $p = \operatorname{im}\alpha$ be a singular point of the support of $\mathcal{C}$, and let m be the multiplicity of $\mathcal{C}$ at p. Again choose a line in the tangent cone to $\mathcal{C}$ at p, and choose coordinates $(x:y:z)$ so that $x=0$ is the kernel line, $p=(1:0:0)$, and $z=0$ is the selected line in the tangent cone.

We may describe $\mathcal{C}$ near p as the union of m 'formal branches'; those that are tangent to $z=0$ may be written

$$z = f(y) = \sum_{i\geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_i} \neq 0$.

The choices made above determine a finite set of rational numbers, which we call the 'characteristics' for $\mathcal{C}$ (w.r.t. the line $z=0$): these are the numbers C such that at least two of the branches tangent to $z=0$ agree modulo y^C, differ at y^C, and have $\lambda_0 < C$.

For a characteristic C, the initial exponents λ_0 and the coefficients γ_{λ_0}, $\gamma_{\frac{C+\lambda_0}{2}}$ for the corresponding branches must agree. Let $\gamma_C^{(1)},\dots,\gamma_C^{(S)}$ be the coefficients of y^C in these S branches (so that at least two of these numbers are distinct, by the choice of C). Then $\mathscr{X}$ is defined by

$$x^{d-2S}\prod_{i=1}^{S}\left(zx - \frac{\lambda_0(\lambda_0-1)}{2}\gamma_{\lambda_0}y^2 - \frac{\lambda_0+C}{2}\gamma_{\frac{\lambda_0+C}{2}}yx - \gamma_C^{(i)}x^2\right) \quad .$$

This is a union of 'quadritangent' conics—that is, nonsingular conics meeting at exactly one point—with (possibly) a multiple of the distinguished tangent, which must be supported on the kernel line.

3.3. The following simple example illustrates the components described in §3.2: all five types are present for the curve

$$y((y^2+xz)^2 - 4xyz^2) = 0 \quad .$$

We will list five germs $\alpha(t)$, and the corresponding limits. (This is not an exhaustive list of all the components of the PNC for this curve.) The specific germs used here were obtained by applying the procedures explained in the rest of the section.

- *Type I.* A germ $\alpha(t)$ centered at a rank-2 matrix with image the linear component of the curve:

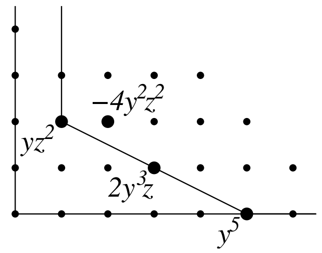
$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  This yields the limit

$$y\,x^2z^2 \quad ,$$

  a fan consisting of the line $y=0$ and a star through the point $(0:1:0)$, that is, the kernel of $\alpha(0)$.

- *Type II.* We 'aim' a one-parameter subgroup with weights $(1,2)$ at a nonsingular point of the curve and its tangent line:
$$\alpha(t)=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&t&0\\0&0&t^2\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&-1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\1-t&t&0\\1-t&t-t^2&t^2\end{pmatrix}\quad.$$
(The curve is nonsingular at $(1:1:1)=\operatorname{im}\alpha(0)$, and tangent to the line $y=z$.) This germ yields a limit
$$x^3((x+y)^2-4xz)\quad,$$
that is, a nonsingular conic union a (multiple) tangent line supported on the kernel line $\ker\alpha(0)$.
- *Type III.* We aim a one-parameter subgroup with weights $(1,1)$ at $(0:0:1)$:
$$\alpha(t)=\begin{pmatrix}t&0&0\\0&t&0\\0&0&1\end{pmatrix}$$
obtaining a limit of
$$xy(x-4y)z^2\quad:$$
a fan consisting of the tangent cone to $\mathcal{C}$ at p, union a multiple kernel line.
- *Type IV.* Considering now $\mathcal{C}$ at $p=(1:0:0)$, here is the Newton polygon w.r.t. the line $z=0$:



It has one side with slope between -1 and 0. The corresponding germ will be a one-parameter subgroup with weights $(1,2)$:
$$\alpha(t)=\begin{pmatrix}1&0&0\\0&t&0\\0&0&t^2\end{pmatrix}$$
yielding as limit the monomials of F situated on the selected side:
$$x^2yz^2+2xy^3z+y^5=y(y^2+xz)^2\quad.$$
This is a double nonsingular conic, union a transversal line.
- *Type V.* Finally, write formal branches for $\mathcal{C}$ at p:
$$\begin{cases}y=\quad 0\\ z=-y^2-2y^{5/2}-\dots\\ z=-y^2+2y^{5/2}-\dots\end{cases}$$

We find one characteristic $C = \frac{5}{2}$, corresponding to the second and third branch. These branches truncate to $-y^2$; as will be explained in §3.24, this information determines the germ

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^4 & t^5 & 0 \\ -t^8 & -2t^9 & t^{10} \end{pmatrix}$$

yielding the limit

$$x(zx + y^2 + 2x^2)(zx + y^2 - 2x^2)$$

prescribed by the formula given in §3.2. This is a pair of quadritangent conics union a distinguished tangent supported on the kernel line.

3.4. The rest of the section consists of the detailed analysis yielding the list given in §3.2. Our approach will be in the spirit of Ghizzetti's paper, and indeed was inspired by reading it. The general strategy consists of an elimination process: starting from an arbitrary germ $\alpha(t)$, we determine the possible components that arise unless $\alpha(t)$ is of some special kind, and keep restricting the possibilities for $\alpha(t)$ until none is left.

Here is a guided tour of the successive reductions. First of all, we will determine germs leading to type I components (§3.7); this will account for all germs centered at a rank-2 matrix, so we will then be able to assume that $\alpha(0)$ has rank 1. Next, we will show (Proposition 3.7, §3.8–§3.11) that every α centered at a rank-1 matrix is equivalent, in suitable coordinates, to one in the form

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $1 \le b \le c$ and q, r, and s polynomials satisfying certain conditions. Considering the case $q \equiv r \equiv s \equiv 0$ (that is, when α is a 'one-parameter subgroup') leads to components of type II, III, and IV; this is done in §3.13–§3.15. The subtlest case, leading to type V components, takes the remaining §3.16–§3.24. The key step here consists of showing that germs leading to new components are equivalent to germs in the form

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

where $C = \frac{c}{a}$ is one of the characteristics considered in §3.2, f is a corresponding branch, b is determined by the other data, and $\underline{\dots}$ stands for the truncation modulo t^c. This key reduction is accomplished in Proposition 3.15, after substantial preparatory work. A refinement of the reduction, given in Proposition 3.20, leads to the definition of 'characteristics' and to the description of components of type V given above (cf. Proposition 3.26).

As the process accounts for all possible germs, this will show that the list of components given in §3.2 is exhaustive, concluding our description of the PNC.

3.5. Before starting on the path traced above, we discuss three results that will be applied at several places in the discussion. The first two are discussed in this subsection, and the third one in the subsection which follows.

The first concerns a recurrent tool in establishing that a germ $\alpha(t)$ does not contribute a component to the PNC. As we discussed in §2.5, this is the case for 'rank-2 limits', that is, limits that can also be obtained by germs entirely contained within the locus of rank-2 transformations in $\mathbb{P}^8$. Since we are looking for germs determining components of E, we may ignore such 'rank-2 limits' and the germs that lead to them.

Lemma 3.1. *Assume that* $\alpha(0)$ *has rank* 1*. If* $\lim_{t\to 0} \mathcal{C}\circ\alpha(t)$ *is a star with center on* $\ker\alpha(0)$*, then it is a rank-2 limit.*

Proof. Assume $\mathcal{X} = \lim_{t\to 0} \mathcal{C}\circ\alpha(t)$ is a star with center on $\ker\alpha(0)$. We may choose coordinates so that $x=0$ is the kernel line and the generator for the ideal of $\mathcal{X}$ is a polynomial in x, y only. If

$$\alpha(t) = \begin{pmatrix} a_{00}(t) & a_{01}(t) & a_{02}(t) \\ a_{10}(t) & a_{11}(t) & a_{12}(t) \\ a_{20}(t) & a_{21}(t) & a_{22}(t) \end{pmatrix} \quad ,$$

then $\mathcal{X} = \lim_{t\to 0} \mathcal{C}\circ\beta(t)$ for

$$\beta(t) = \begin{pmatrix} a_{00}(t) & a_{01}(t) & 0 \\ a_{10}(t) & a_{11}(t) & 0 \\ a_{20}(t) & a_{21}(t) & 0 \end{pmatrix} \quad .$$

Since $\alpha(0)$ has rank 1 and kernel line $x=0$,

$$\alpha(0) = \begin{pmatrix} a_{00}(0) & 0 & 0 \\ a_{10}(0) & 0 & 0 \\ a_{20}(0) & 0 & 0 \end{pmatrix} = \beta(0) \quad .$$

Now $\beta(t)$ is contained in the rank-2 locus, verifying the assertion. □

A limit $\lim_{t\to 0} \mathcal{C}\circ\alpha(t)$ as in the lemma will be called a 'kernel star'.

A second tool will be at the root of our reduction process: we will replace a given germ $\alpha(t)$ with a different one in a more manageable form, but leading to the same component of the PNC in a very strong sense.

Definition 3.2. Two germs are 'equivalent' with respect to $\mathcal{C}$ if they are (possibly up to an invertible change of parameter) fibers of a family of germs with constant center and limit. More precisely, two germs $\alpha_0(t)$, $\alpha_1(t)$ are equivalent if there exists a connected curve $\mathcal{H}$, a regular map $A : \mathcal{H}\times \operatorname{Spec}\mathbb{C}[[t]] \to \mathbb{P}^8$, two points h_0, h_1 of $\mathcal{H}$, and a unit $\nu(t)\in\mathbb{C}[[t]]$ such that:

- $A(h_0,t) = \alpha_0(t)$;
- $A(h_1,t) = \alpha_1(t\nu(t))$;
- $A(\_,0) : \mathcal{H}\to\mathbb{P}^8$ is constant;
- If $\mathcal{A} : \mathcal{H}\times\operatorname{Spec}\mathbb{C}[[t]] \to \mathbb{C}^9$ is a lift of A, then $F\circ\mathcal{A}(h,t) \equiv \rho(h)Gt^w \mod t^{w+1}$ for some w, with $\rho : \mathcal{H}\to\mathbb{C}^*$ a nowhere vanishing function and G a nonzero polynomial in x, y, z independent of h.

It is clear that Definition 3.2 gives an equivalence relation on the set of germs: reflexivity and symmetry are immediate, and transitivity is obtained by joining two families along a common fiber (note that the limit G and the weight w are determined by any of the fibers of a family, hence they are the same for any two families extending a given germ).

Note the possible parameter change in the second condition: this guarantees that a germ $\alpha(t)$ is equivalent to any of its reparametrizations $\alpha(t\nu(t))$. This flexibility will play an important rôle in part of our discussion, especially in §4.6 and ff.

By Lemma 2.1, the third and fourth conditions amount to the statement that all germs $\alpha_h(t) = A(h,t)$ lift to germs in $\widetilde{\mathbb{P}^8}$, the closure of the graph of c, centered at the same point. If α_0, α_1 are centered at a point of S, then α_0 and α_1 (and in fact all the intermediate α_h) determine the same point in the projective normal cone. Thus, equivalent germs can be 'continuously deformed' one into the other while holding the center of their lift in $\mathbb{P}^8 \times \mathbb{P}^N$ fixed.

Typically, the curve $\mathcal{H}$ will simply be a chain of affine lines, minus some points.

Given an arbitrary germ $\alpha(t)$, we will want to produce an equivalent and 'simpler' germ. Our basic tool to produce an equivalent germ will be the following.

Lemma 3.3. *Assume $\alpha(t) \equiv \beta(t) \circ m(t)$, and that $M = m(0)$ is invertible. Then α is equivalent to $\beta \circ M$ (w.r.t. any curve $\mathcal{C}$).*

Proof. Write $m(t) = M + tm_1(t)$, and take $\mathcal{H} = \mathbb{A}^1$, $\nu(t) = 1$. Let

$$\alpha_h(t) = A(h,t) := \beta(t) \circ (M + htm_1(t)) \quad .$$

Then

- $\alpha_0 = \beta \circ M$,
- $\alpha_1 = \alpha$, and
- $\alpha_h(0) = \beta(0) \circ M$ does not depend on h.

For any given F,

$$F \circ \beta(t) = t^w G + t^{w+1} G_1(t)$$

with $G = \lim_{t\to 0} F \circ \beta(t)$. Then

$$F\circ\alpha_h(t) = F\circ\beta(t)\circ(M+htm_1(t)) = t^w(G+tG_1(t))\circ(M+htm_1(t)) = t^w G\circ M + \text{h.o.t.}$$

The term $G \circ M$ is not 0, since M is invertible, and does not depend on h.

Thus $\alpha = \alpha_1$ is equivalent to $\beta \circ M = \alpha_0$ according to Definition 3.2, as needed. □

3.6. The third preliminary item concerns formal branches of $\mathcal{C}$ at a point p, cf. [BK86] and [Fis01], Chapter 6 and 7. Choose affine coordinates $(y,z) = (1 : y : z)$ so that $p = (0,0)$, and let $\Phi(y,z) = F(1 : y : z)$ be the generator for the ideal of $\mathcal{C}$ in these coordinates. Decompose $\Phi(y,z)$ in $\mathbb{C}[[y,z]]$:

$$\Phi(y,z) = \Phi_1(y,z) \cdot \dots \cdot \Phi_r(y,z)$$

with $\Phi_i(y,z)$ irreducible power series. These define the *irreducible branches* of $\mathcal{C}$ at p. Each Φ_i has a unique tangent line at p; if this tangent line is *not* $y = 0$, by the Weierstrass preparation theorem we may write (up to a unit in $\mathbb{C}[[y,z]]$) Φ_i as a monic polynomial in z with coefficients in $\mathbb{C}[[y]]$, of degree equal to the multiplicity m_i of the branch at p (cf. for example [Fis01], §6.7). If Φ_i *is* tangent to $y = 0$, we may likewise write it as a polynomial in y with coefficients in $\mathbb{C}[[z]]$; *mutatis mutandis,* the discussion which follows applies to this case as well.

Concentrating on the first case, let

$$\Phi_i(y,z) \in \mathbb{C}[[y]][z]$$

be a monic polynomial of degree m_i, defining an irreducible branch of $\mathcal{C}$ at p, not tangent to $y=0$. Then Φ_i splits (uniquely) as a product of linear factors over the ring $\mathbb{C}[[y^*]]$ of power series with *rational nonnegative* exponents:

$$\Phi_i(y,z) = \prod_{j=1}^{m_i} (z - f_{ij}(y)) \quad ,$$

with each $f_{ij}(y)$ in the form

$$f(y) = \sum_{k\geq 0} \gamma_{\lambda_k} y^{\lambda_k}$$

with $\lambda_k \in \mathbb{Q}$, $1 \leq \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_k} \neq 0$. We call each such $z = f(y)$ a *formal branch* of $\mathcal{C}$ at p. The branch is *tangent to* $z=0$ if the dominating exponent λ_0 is >1. The terms $z - f_{ij}(y)$ in this decomposition are the Puiseux series for $\mathcal{C}$ at p.

Summarizing: if $\mathcal{C}$ has multiplicity m at p then $\mathcal{C}$ splits into m formal branches at p. In §3.16 and ff. we will want to determine $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ as a union of 'limits' of the individual formal branches at p. The difficulty here resides in the fact that we cannot perform an arbitrary 'change of variable' in a power series with fractional exponents. In the case in which we will need to do this, however, $\alpha(t)$ will have the following special form:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $a < b \leq c$ positive integers and $r(t)$, $s(t)$ polynomials (satisfying certain restrictions, which are immaterial here). We will circumvent the difficulty we mentioned by the following *ad hoc* definition.

Definition 3.4. The *limit* of a formal branch $z = f(y)$, along a germ $\alpha(t)$ as above, is defined by the dominant term in

$$(r(t) + s(t)t^b y + t^c z) - f(t^a) - f'(t^a)t^b y - f''(t^a)t^{2b}\frac{y^2}{2} - \cdots$$

where $f'(y) = \sum \gamma_k \lambda_k y^{\lambda_k - 1}$ etc.

By 'dominant term' we mean the coefficient of the lowest power of t after cancellations. This coefficient is a polynomial in y and z, giving the limit of the branch according to our definition.

Of course we need to verify that this definition behaves as expected, that is, that the limit of $\mathcal{C}$ is the union of the limits of its individual branches. We do so in the following lemma.

Lemma 3.5. *Let $\Phi(y,z)\in\mathbb{C}[[y]][z]$ be a monic polynomial,*

$$\Phi(y,z)=\prod_i(z-f_i(y))$$

a decomposition over $\mathbb{C}[[y^*]]$, and let $\alpha(t)$ be as above. Then the dominant term in $\Phi\circ\alpha(t)$ is the product of the limits of the branches $z=f_i(y)$ along α, defined as in Definition 3.4.

Proof. We can 'clear the denominators' in the exponents in f_i, by writing

$$\Phi(T^m,z)=\prod_i(z-\varphi_i(T))$$

where $\varphi_i(T)\in\mathbb{C}[[T]]$ and $\varphi_i(T)=f_i(T^m)$. For an integer ℓ such that $\ell a/m$ is integer, we may write

$$t=S^\ell\quad,\quad T^m=T(S)^m=S^{\ell a}+S^{\ell b}y=S^{\ell a}(1+S^{\ell(b-a)}y)$$

with $T(S)\in\mathbb{C}[[S,y]]$: explicitly,

$$T(S)=S^{\frac{\ell a}{m}}\left(1+\frac{1}{m}S^{\ell(b-a)}y+\frac{1}{m}\left(\frac{1}{m}-1\right)S^{\ell 2(b-a)}\frac{y^2}{2}+\cdots\right)$$

The dominant term (w.r.t. t) in

$$\Phi\circ\alpha(t)=\Phi(t^a+t^by,r(t)+s(t)t^by+t^cz)$$

equals the dominant term (w.r.t. S) in

$$\begin{aligned}\Phi(S^{\ell a}+S^{\ell b}y,r(S^\ell)+s(S^\ell)S^{\ell b}y+S^{\ell c}z)&=\Phi(T(S)^m,r(S^\ell)+s(S^\ell)S^{\ell b}y+S^{\ell c}z)\\&=\prod_i\left((r(S^\ell)+s(S^\ell)S^{\ell b}y+S^{\ell c}z)-\varphi_i(T(S))\right)\end{aligned}$$

Thus the dominant term in $\Phi\circ\alpha(t)$ is the product of the dominant terms in the factors

$$(r(S^\ell)+s(S^\ell)S^{\ell b}y+S^{\ell c}z)-\varphi_i(T(S))$$

and we have to verify that the dominant term here agrees with the one in Definition 3.4.

For this, we use a 'Taylor expansion' of φ_i. Write

$$\varphi_i(T(S))=\sum_{k\ge0}\frac{\partial^k\varphi_i(T(S))}{\partial y^k}|_{y=0}\frac{y^k}{k!}\quad.$$

We claim that

$$\frac{\partial^k\varphi_i(T(S))}{\partial y^k}|_{y=0}=f_i^{(k)}(S^{\ell a})S^{\ell bk}\quad:$$

indeed, this is immediately checked for $f_i(y)=y^\lambda$, hence holds for any f_i.

Therefore we have

$$\varphi_i(T(S))=\sum_{k\ge0}f_i^{(k)}(S^{\ell a})\frac{S^{\ell bk}y^k}{k!}$$

or, recalling $t = S^\ell$:

$$\varphi_i(T(S)) = f_i(t^a) + f_i'(t^a)t^b y + f_i''(t^a)t^{2b}\frac{y^2}{2} + \dots$$

This shows that

$$(r(S^\ell) + s(S^\ell)S^{\ell b}y + S^{\ell c}z) - \varphi_i(T(S))$$

is in fact given by

$$(r(t) + s(t)t^b y + t^c z) - \left(f_i(t^a) + f_i'(t^a)t^b y + f_i''(t^a)t^{2b}\frac{y^2}{2} + \dots\right) \quad ,$$

and in particular the dominant terms in the two expressions must match, as needed. □

The gist of this subsection is that we may use formal branches for $\mathcal{C}$ at p in order to compute the limit of $\mathcal{C}$ along germs $\alpha(t)$ of the type used above, provided that the limit of a branch is computed by using the formal Taylor expansion given in Definition 3.4. This fact will be used several times in §3.16 and ff.

3.7. We are finally ready to begin the discussion leading to the list of components given in §3.2.

Applying Lemma 2.1 amounts to studying germs $\alpha(t)$ in $\mathbb{P}^8$, centered at matrices α with image contained in $\mathcal{C}$—these are precisely the matrices in the base locus S of the rational map c introduced in §2. The corresponding component of E is determined by the center of the germ, and the limit.

As we may assume that $\alpha(0)$ is contained in $\mathcal{C}$, we may assume that $\operatorname{rk}\alpha(0) = 1$ or 2. We first consider the case of germs centered at a rank-2 matrix α, hence with image equal to a linear component of $\mathcal{C}$. We will show that any germ centered at such a matrix leads to a point in the component of type I listed in §3.2.

Write

$$\alpha(t) = \alpha(0) + t\beta(t) \quad ,$$

where $\alpha(0)$ has rank 2 and image defined by the linear polynomial L; thus, we may write the generator of the ideal of $\mathcal{C}$ as

$$F(x, y, z) = L(x, y, z)^m G(x, y, z)$$

with L not a factor of G. The curve defined by G is the 'residual of L in $\mathcal{C}$'.

Proposition 3.6. *The limit* $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ *is a fan consisting of an* m*-fold line* ℓ*, supported on* $\lim_{t\to 0} L \circ \beta(t)$*, and a star of lines through the point* $\ker\alpha$*. This star reproduces projectively the tuple cut out on* L *by the residual of* L *in* $\mathcal{C}$*.*

The terminology of stars and fans was introduced in [AF00c], §2.1. Here the m-fold line ℓ may contain the point $\ker\alpha$, in which case the fan degenerates to a star.

Proof. Write

$$\alpha(t) = \alpha(0) + t\beta(t) \quad ,$$

then the ideal of $\mathcal{C} \circ \alpha(t)$ is generated by

$$L(\alpha(t))^m G(\alpha(t)) = t^m L(\beta(t))^m G(\alpha(0) + t\beta(t)) \quad ,$$

since L is linear and vanishes along the image of $\alpha(0)$. As t approaches 0, $L(\beta(t))^m$ converges to an m-fold line, while the other factor converges to $G(\alpha(0))$, yielding the statement. □

A simple dimension count shows that the limits arising as in Proposition 3.6 *do* produce components of the projective normal cone. Indeed, matrices with image contained in a given line form a $\mathbb{P}^5$; for any given such matrix, the limits obtained consist of a fixed star through the kernel, plus a (multiple) line varying freely, accounting for 2 extra dimensions. These components are the components of type I described in §3.2 (also cf. [AF00a], §2, Fact 2(i)).

3.8. Having taken into account the case in which the center $\alpha(0)$ may be a rank-2 matrix, we are reduced to considering germs $\alpha(t)$ with $\alpha(0)$ of rank 1. Proposition 3.7 below will allow us to further assume that $\alpha(t)$ has a particularly simple (and polynomial) expression.

We define the degree of the zero polynomial to be $-\infty$. We denote by v the 'valuation' of a power series or polynomial, that is, its order of vanishing at 0; we define $v(0)$ to be $+\infty$.

Proposition 3.7. *With a suitable choice of coordinates, any germ α is equivalent to a product*

$$\begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & s & 1 \end{pmatrix} \cdot \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

with

- *$a \le b \le c$ integers, q, r, s polynomials;*
- *$\deg(q) < b - a$, $\deg(r) < c - a$, $\deg(s) < c - b$;*
- *$q(0) = r(0) = s(0) = 0$.*

If further $b = c$ and q, r are not both zero, then we may assume that $v(q) < v(r)$.

The proof of this proposition requires a few preliminary considerations.

3.9. The space $\mathbb{P}^8$ of 3×3 matrices considered above is, more intrinsically, the projective space $\mathbb{P}\operatorname{Hom}(V, W)$, where V and W are vector spaces of dimension 3. The generator F of the ideal of a plane curve of degree d is then an element of $\operatorname{Sym}^d W^*$; for $\varphi \in \operatorname{Hom}(V, W)$, the composition $F \circ \varphi$ (if nonzero) is the element of $\operatorname{Sym}^d V^*$ generating the ideal of $\mathcal{C} \circ \varphi$. We denote by $\mathrm{PGL}(V, W)$ the Zariski open subset of $\mathbb{P}\operatorname{Hom}(V, W)$ consisting of invertible transformations, and write $\mathrm{PGL}(V)$ for $\mathrm{PGL}(V, V)$ (which is a group under composition).

Homomorphisms λ of $\mathbb{C}^*$ to $\mathrm{PGL}(V)$ will be called '1-PS' (as in: '1-parameter subgroups'), as will be called their extensions $\mathbb{C} \to \mathbb{P}\operatorname{Hom}(V, V)$. Recall that every 1-PS can be written as

$$t \mapsto \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

after a suitable choice of coordinates in V, where $a \le b \le c$ are integers (and a may in fact be chosen to equal 0). Thus we may view a 1-PS as a $\mathbb{C}((t))$-valued point of $\mathrm{PGL}(V) \subset \mathbb{P}\mathrm{Hom}(V,V)$. The following lemma shows that these are the basic constituents of every germ $\alpha(t)$.

Lemma 3.8. *Every germ $\alpha(t)$ in $\mathbb{P}\mathrm{Hom}(V,W)$ is equivalent to a germ*

$$H \circ h_1 \circ \lambda \quad ,$$

where:

- *H is a constant invertible linear transformation $V \to W$;*
- *h_1 is a $\mathbb{C}[[t]]$-valued point of* $\mathrm{PGL}(V)$ *with $h_1(0) = Id_V$; and*
- *λ is a 1-PS.*

Proof. Every germ α can be written (cf. [MF82], p.53) as a composition:

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[rrr, bend right, "\alpha"'] & U \arrow[r, "\lambda'"] & U \arrow[r, "h'"] & W \end{tikzcd}$$

where k and h' are $\mathbb{C}[[t]]$-valued points of $\mathrm{PGL}(V,U)$, $\mathrm{PGL}(U,W)$ respectively and λ' is a 1-PS. In particular, $K = k(0)$ is an invertible linear transformation; by Lemma 3.3, the composition is equivalent to the composition

$$V \xrightarrow{K} U \xrightarrow{\lambda'} U \xrightarrow{h'} W \quad ,$$

which can be written as

$$\begin{tikzcd} V \arrow[r, "K"] \arrow[rrr, bend right, "\lambda"'] & U \arrow[r, "\lambda'"] & U \arrow[r, "K^{-1}"] & V \arrow[r, "K"] & U \arrow[r, "h'"] & W \end{tikzcd} \quad .$$

Here λ is again a 1-PS, as a conjugate of a 1-PS by a constant transformation. The statement follows by writing $h' \circ K = H \circ h_1$ as prescribed. $\square$

Now we choose coordinates in V so that λ is diagonal:

$$\lambda = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

with $a \le b \le c$ integers; thus we may view h_1 and λ as matrices, and we are interested in putting h_1 in a 'standard' form.

Lemma 3.9. *Let*

$$h_1 = \begin{pmatrix} u_1 & b_1 & c_1 \\ a_2 & u_2 & c_2 \\ a_3 & b_3 & u_3 \end{pmatrix}$$

be a $\mathbb{C}[[t]]$-valued point of $\mathrm{PGL}(V)$*, such that $h_1(0) = I_3$. Then h_1 can be written as a product $h_1 = h \cdot j$ with*

$$h = \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & s & 1 \end{pmatrix} \quad , \quad j = \begin{pmatrix} v_1 & e_1 & f_1 \\ d_2 & v_2 & f_2 \\ d_3 & e_3 & v_3 \end{pmatrix}$$

with q, r, s polynomials, *satisfying*

1. *$h(0) = j(0) = I_3$;*

2. $\deg(q) < b - a$, $\deg(r) < c - a$, $\deg(s) < c - b$;
3. $d_2 \equiv 0 \pmod{t^{b-a}}$, $d_3 \equiv 0 \pmod{t^{c-a}}$, $e_3 \equiv 0 \pmod{t^{c-b}}$.

Proof. Obviously $v_1 = u_1, e_1 = b_1$ and $f_1 = c_1$. Use division with remainder to write

$$v_1^{-1} a_2 = D_2 t^{b-a} + q$$

with $\deg(q) < b - a$, and let $d_2 = v_1 D_2 t^{b-a}$ (so that $qv_1 + d_2 = a_2$). This defines q and d_2, and uniquely determines v_2 and f_2. (Note that $q(0) = d_2(0) = f_2(0) = 0$ and that $v_2(0) = 1$.)

Similarly, we let r be the remainder of

$$(v_1 v_2 - e_1 d_2)^{-1}(v_2 a_3 - d_2 b_3)$$

under division by t^{c-a}; and s be the remainder of

$$(v_1 v_2 - e_1 d_2)^{-1}(v_1 b_3 - e_1 a_3)$$

under division by t^{c-b}.

Then $\deg(r) < c - a$, $\deg(s) < c - b$ and $r(0) = s(0) = 0$; moreover, we have

$$v_1 r + d_2 s \equiv a_3 \pmod{t^{c-a}}, \qquad e_1 r + v_2 s \equiv b_3 \pmod{t^{c-b}},$$

so we take $d_3 = a_3 - v_1 r - d_2 s$, $e_3 = b_3 - e_1 r - v_2 s$. This defines r, s, d_3 and e_3, and uniquely determines v_3. □

3.10. We are now ready to prove Proposition 3.7. We have written a germ equivalent to α as

$$H \cdot h \cdot j \cdot \lambda$$

with notations as above. Now, by (3) in Lemma 3.9 we have $j \cdot \lambda = \lambda \cdot \ell$ for ℓ with entries in $\mathbb{C}[[t]]$, and $L = \ell(0)$ lower triangular, with 1's on the diagonal. By Lemma 3.3 this germ is equivalent to

$$H \cdot h \cdot \lambda \cdot L = (H \cdot L) \cdot L^{-1} \cdot (h \cdot \lambda) \cdot L \quad .$$

We change coordinates in V by L^{-1}, so that $L^{-1} \cdot (h \cdot \lambda) \cdot L$ has matrix representation $h \cdot \lambda$. Finally, we choose coordinates in W so that $H \cdot L = I_3$, completing the proof of the first part of Proposition 3.7.

If $b = c$, then the condition that $\deg s < c - b = 0$ forces $s = 0$. Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

interchanges q and r; so we may assume $v(q) \le v(r)$ if q and r are not both 0. Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix}$$

replaces r by $uq + r$, allowing us to force $v(q) < v(r)$, and completing the proof of Proposition 3.7. □

3.11. By Proposition 3.7, and scaling the entries in the 1-PS so that $a = 0$, an arbitrary germ α is equivalent to one that, with a suitable choice of coordinates, can be written as

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $0 \leq b \leq c$ and q, r, and s polynomials satisfying certain conditions. We may in fact assume that $b > 0$, since we are already reduced to the case in which $\alpha(0)$ is a rank-1 matrix. If $b > 0$, then the center $\alpha(0)$ is the matrix

$$\alpha(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with image the point $(1 : 0 : 0)$ and kernel the line $x = 0$.

Further, if $(1 : 0 : 0)$ is not a point of the curve $\mathcal{C}$ then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is simply a multiple kernel line with ideal $(x^{\deg \mathcal{C}})$. Thus we may assume that $p = (1 : 0 : 0)$ is a point of $\mathcal{C}$. *In what follows, we will assume that α is a germ in the standard form given above, and all these conditions are satisfied.*

One last remark will be needed later in the section: if the polynomial $q(t)$ is known to be nonzero, then Proposition 3.7 admits the following refinement.

Lemma 3.10. *If $q \not\equiv 0$ in $\alpha(t)$, then $\alpha(t)$ is equivalent to a germ*

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r_1(t) & s_1(t)t^b & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ r & s & v \end{pmatrix} \quad ,$$

with

- *$a < b \leq c$ positive integers;*
- *$r_1(t)$ and $s_1(t)$ polynomials of degree $< c$, $< (c-b)$ respectively and vanishing at $t = 0$; and*
- *$u, r, s, v \in \mathbb{C}$, with $uv \neq 0$.*

If further $b = c$, then we may assume $a < v(r_1)$.

Proof. As $q \not\equiv 0$, we may write $q(t) = \tau(t)^a$ for $a = v(q)$ (so $0 < a < b$) and with $\tau(t) \in \mathbb{C}[[t]]$, such that $\tau(t)/t$ is a unit in $\mathbb{C}[[t]]$. Expressing t in terms of τ, we can set

$$\beta(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ \tau^a & u(\tau)\tau^b & 0 \\ \overline{r}(\tau) & \overline{s}(\tau)u(\tau)\tau^b & v(\tau)\tau^c \end{pmatrix}$$

so that $\alpha(t) = \beta(\tau(t))$, for suitable $\overline{r}(\tau)$, $\overline{s}(\tau)$, and invertible $u(\tau)$, $v(\tau)$ in $\mathbb{C}[[\tau]]$.

Since $\alpha(t)$ and $\beta(t)$ only differ by a change of parameter, they are equivalent in the sense of Definition 3.2.

Next, define $\rho(t), \sigma(t) \in \mathbb{C}[[t]]$ so that

$$\overline{r}(t) = \underline{\overline{r}(t)} + \rho(t)t^c \quad , \quad \overline{s}(t)t^b = \underline{\overline{s}(t)t^b} + \sigma(t)t^c$$

with $\underline{\overline{r}(t)}$, $\underline{\overline{s}(t)t^b}$ polynomials of degree less than c, and observe that then

$$\beta(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & u(t)t^b & 0 \\ \overline{r}(t) & \overline{s}(t)u(t)t^b & v(t)t^c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{\overline{r}(t)} & \underline{\overline{s}(t)t^b} & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & u(t) & 0 \\ \rho(t) & \sigma(t)u(t) & v(t) \end{pmatrix} \quad .$$

The rightmost matrix is invertible at 0, so by Lemma 3.3 $\beta(t)$ (and hence $\alpha(t)$) is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{\overline{r}(t)} & \underline{\overline{s}(t)t^b} & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ r & s & v \end{pmatrix}$$

where $u = u(0)$, $r = \rho(0)$, $s = \sigma(0)u(0)$, and $v = v(0)$. We have $uv \neq 0$ as both $u(t)$ and $v(t)$ are invertible.

To obtain the stated form, let $r_1(t) = \underline{\overline{r}(t)}$ and $s_1(t)$ so that $s_1(t)t^b = \underline{\overline{s}(t)t^b}$. Then $r_1(t)$ and $s_1(t)$ are polynomials of degree $< c$, $< (c-b)$ respectively, and $r_1(0) = s_1(0) = 0$ as an immediate consequence of $r(0) = s(0) = 0$.

Finally, note that $a = v(q)$ and $v(r_1) = v(r)$; if $b = c$, then we may assume $v(q) < v(r)$ by Proposition 3.7, and hence $a < v(r_1)$ as needed. □

The form obtained in Lemma 3.10 will be needed in a key reduction (Proposition 3.15) later in the section. The effect of the constant factor on the right in the germ appearing in the statement of Lemma 3.10 is simply to translate the limit (by an invertible transformation fixing the flag consisting of the line $x = 0$ and the point $(0:0:1)$). Thus this factor will essentially be immaterial in the considerations in this section.

3.12. In the following, it will be convenient to switch to affine coordinates centered at the point $(1:0:0)$: we will denote by (y,z) the point $(1:y:z)$; as we just argued, we may assume that the curve $\mathcal{C}$ contains the origin $p = (0,0)$. We write

$$F(1:y:z) = F_m(y,z) + F_{m+1}(y,z) + \cdots + F_d(y,z) \quad ,$$

with $d = \deg \mathcal{C}$, F_i homogeneous of degree i, and $F_m \neq 0$. Thus, $F_m(y,z)$ generates the ideal of the *tangent cone* of $\mathcal{C}$ at p.

3.13. In the next three subsections we consider the case in which $q = r = s = 0$, that is, in which $\alpha(t)$ is itself a 1-PS:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

with $1 \leq b \leq c$. Also, we may assume that b and c are coprime: this only amounts to a reparametrization of the germ by $t \mapsto t^{1/d}$, with $d = gcd(b,c)$; the new germ is not equivalent to the old one in terms of Definition 3.2, but clearly achieves the same limit.

Germs with $b = c (= 1)$ lead to components of type III:

Proposition 3.11. *If $q = r = s = 0$ and $b = c$, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a fan consisting of a star projectively equivalent to the tangent cone to $\mathcal{C}$ at p, and of a residual $(d-m)$-fold line supported on $\ker \alpha$.*

Proof. The composition $F \circ \alpha(t)$ is

$$F(x : t^b y : t^b z) = t^{bm} x^{d-m} F_m(y, z) + t^{b(m+1)} x^{d-(m+1)} F_{m+1}(y, z) + \cdots + t^{dm} F_d(y, z) \quad .$$

By definition of limit, $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ has ideal $(x^{d-m} F_m(y, z))$, proving the assertion. □

A dimension count (analogous to the one in §3.7) shows that the limits found in Proposition 3.11 contribute a component to the projective normal cone when the star is supported on three or more lines. These are the components 'of type III' in the terminology of §3.2; also cf. [AF00a], §2, Fact 4(i).

3.14. More components may arise due to 1-PS with $b < c$, but only if $\mathcal{C}$ is in a particularly special position relative to α.

Lemma 3.12. *If $q = r = s = 0$ and $b < c$, and $z = 0$ is not contained in the tangent cone to $\mathcal{C}$ at p, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is supported on a pair of lines.*

Proof. The condition regarding $z = 0$ translates into $F_m(1, 0) \neq 0$. Applying $\alpha(t)$ to F, we find:

$$F(x : t^b y : t^c z) = t^{bm} x^{d-m} F_m(y, t^{c-b} z) + t^{b(m+1)} x^{d-(m+1)} F_{m+1}(y, t^{c-b} z) + \cdots$$

Since $F_m(1, 0) \neq 0$, the dominant term on the right-hand-side is $x^{d-m} y^m$, proving the assertion. □

By Lemma 3.1, these limits do not contribute components to the projective normal cone.

Components that do arise due to 1-PS with $b < c$ may be described in terms of the *Newton polygon* for $\mathcal{C}$ at $(0, 0)$, relative to the line $z = 0$, which we may now assume (by the preceding lemma) is part of the tangent cone to $\mathcal{C}$ at p. The Newton polygon for $\mathcal{C}$ in the chosen coordinates is the boundary of the convex hull of the union of the positive quadrants with origin at the points (j, k) for which the coefficient of $x^i y^j z^k$ in the equation for $\mathcal{C}$ is nonzero (see [BK86], p.380). The part of the Newton polygon consisting of line segments with slope strictly between -1 and 0 does not depend on the choice of coordinates fixing the flag $z = 0$, $p = (0, 0)$.

Proposition 3.13. *Assume $q = r = s = 0$ and $b < c$.*

- *If $-b/c$ is not a slope of the Newton polygon for $\mathcal{C}$, then the limit $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is supported on (at most) three lines. Such limits do not contribute components to the projective normal cone.*
- *If $-b/c$ is a slope of a side of the Newton polygon for $\mathcal{C}$, then the ideal of the limit $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is generated by the polynomial obtained by setting to 0 the coefficients of the monomials in F* not *on that side. Such polynomials are in the form*

$$G = x^{\overline{q}} y^r z^q \prod_{j=1}^{S} (y^c + \rho_j x^{c-b} z^b)$$

Proof. For the first assertion, simply note that under the stated hypotheses only one monomial in F is dominant in $F \circ \alpha(t)$; hence, the limit is supported on the union of the coordinate axes. A simple dimension count shows that such limits may only span a 6-dimensional locus in $\mathbb{P}^8 \times \mathbb{P}^N$, so they do not determine a component of the projective normal cone.

The second assertion is analogous: the dominant terms in $F \circ \alpha(t)$ are precisely those on the side of the Newton polygon with slope equal to $-b/c$. It is immediate that the resulting polynomial can be factored as stated. □

Limits arising as in the second part of Proposition 3.13 are the curves studied in [AF00b], and appear as items (6) through (11) in the classification in §1 of [AF00c]. The number S of 'cuspidal' factors in G is the number of segments cut out by the integer lattice on the selected side of the Newton polygon.

Assume the point $p = (1 : 0 : 0)$ is a singular or an inflection point of the support of $\mathcal{C}$. If $b/c \neq 1/2$, then the corresponding limit will contribute a component to the PNC: indeed, the orbit of the corresponding limit curve has dimension 7. If $b/c = 1/2$, then a dimension count shows that the corresponding limit will contribute a component to the PNC unless it is supported on a conic union (possibly) the kernel line.

These are the components of type IV in §3.2, also cf. [AF00a], §2, Fact 4(ii).

3.15. If p is a *nonsingular, non-inflectional* point of the support of $\mathcal{C}$, then the Newton polygon consists of a single side with slope $-1/2$, and the polynomial G in the statement of Proposition 3.13 reduces to

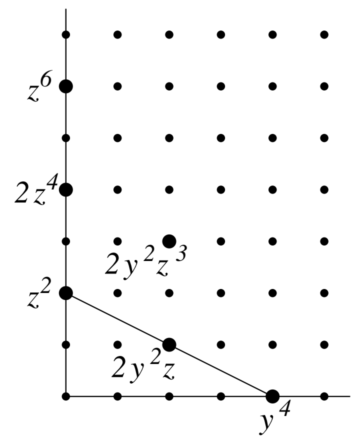
$$x^{d-2S}(y^2 + \rho xz)^S \quad ,$$

that is, a (multiple) conic union a (multiple) tangent line supported on $\ker \alpha$; here S is the multiplicity of the corresponding component of $\mathcal{C}$. The orbit of this limit curve has dimension 6; but as there is one such limit at almost all points of the support of every nonlinear component of $\mathcal{C}$, the collection of these limits span one component of the projective normal cone for each nonlinear component of $\mathcal{C}$. These components are the components of type II in §3.2, also cf. [AF00a], Fact 2(ii).

Example 3.14. Consider the 'double cubic' with (affine) ideal generated by

$$(y^2 + z^3 + z)^2 = y^4 + 2y^2z^3 + 2y^2z + z^6 + 2z^4 + z^2 \quad .$$

Its Newton polygon consists of one side, with slope $-1/2$:



The limit by the 1-PS

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}$$

consists, according to the preceding discussion, of the part of the above polynomial supported on the side. This may be checked directly:

$$(x(ty)^2 + (t^2z)^3 + x^2(t^2z))^2 = x^2t^4(y^4 + 2xy^2z + x^2z^2) + xt^8(2y^2z^3 + 2xz^4) + t^{12}z^6$$

the dominant terms in this expression are $x^2y^4 + 2x^3y^2z + x^4z^2 = x^2(y^2 + xz)^2$. The support of the limit is a conic union a tangent kernel line, as promised.

3.16. Having dealt with the 1-PS case in the previous sections, we may now assume that

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with the conditions listed in Proposition 3.7, and further *such that $q, r,$ and s do not all vanish identically.* As four of the five types of components listed in §3.2 have been identified, we are left with the task of showing that the only remaining components of the PNC to which such germs may lead are the ones 'of type V'. This will take the rest of the section.

The key to the argument will be a further restriction on the germs we need to consider. We are going to argue that the curve has a rank-2 limit unless $\alpha(t)$ and certain formal branches of the curve are closely related.

Work in affine coordinates $(y, z) = (1 : y : z)$. If $\mathcal{C}$ has multiplicity m at $p = (0, 0)$, then we can write the generator F for the ideal of $\mathcal{C}$ as a product of formal branches (cf. §3.6)

$$F = f_1 \cdot \cdots \cdot f_m$$

where each f_i is expressed as a power series with fractional exponents. Among these branches, we will especially focus on the ones that are tangent to the line $z = 0$, which may be written explicitly as

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_i} \neq 0$.

Notation. For $C \in \mathbb{Q}$, we will denote by $f_{(C)}(y)$ the finite sum ('truncation')

$$f_{(C)}(y) = \sum_{\lambda_i < C} \gamma_{\lambda_i} y^{\lambda_i} \quad .$$

For $c \in \mathbb{Z}$, we will also write $\underline{g(t)}$ for the truncation of $g(t)$ to t^c, so that $\underline{f(t^a)} = f_{(C)}(t^a)$ when $C = \frac{c}{a}$. Note that for all $b > a$ the truncation $\underline{f'(t^a)t^b}$ is determined by b and $\underline{f(t^a)}$ (and hence by $f_{(C)}(y)$ and a, b).

Proposition 3.15. *Let $\alpha(t)$ be as above, and assume that $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is not a rank-2 limit. Then $\mathcal{C}$ has a formal branch $z = f(y)$, tangent to $z = 0$, such that α is equivalent to a germ*

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ r & s & v \end{pmatrix} \quad ,$$

with $a < b < c$ positive integers, $u, r, s, v \in \mathbb{C}$, and $uv \neq 0$.

Further, it is necessary that $\frac{c}{a} \leq \lambda_0 + 2(\frac{b}{a} - 1)$.

The proof of this key reduction requires the study of several distinct cases. We will first show that under the hypothesis that $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is not a rank-2 limit we may assume that $q(t) \neq 0$, and this will allow us to replace it with a power of t; next, we will deal with the $b = c$ case; and finally we will see that if $b < c$ and $\alpha(t)$ is not in the stated form, then the limit of *every* branch of $\mathcal{C}$ is a $(0 : 0 : 1)$-star. This will imply that the limit of $\mathcal{C}$ is a kernel star in this case, proving the assertion by Lemma 3.1.

3.17. The first remark is that, under the assumptions that q, r, and s do not vanish, we may in fact assume that $q(t)$ is not zero.

Lemma 3.16. *If $\alpha(t)$ is as above, and $q \equiv 0$, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. This is a case-by-case analysis. Assume $q \equiv 0$; thus r and s are not both zero. For F the generator of the ideal of $\mathcal{C}$, consider the fate of an individual monomial $x^A y^B z^C$ under $\alpha(t)$:

$$m_{ABC} = x^A y^B (r(t)x + s(t)t^b y + t^c z)^C t^{bB}$$

If $r \equiv 0$ and $s \not\equiv 0$, note that $v(s) \leq \deg s < c - b$. Therefore, the dominating term in m_{ABC} is

$$x^A y^{B+C} t^{bB + (b+v(s))C} \quad .$$

Since $v(s) > 0$, the weights with which a fixed limit monomial $x^A y^{B+C}$ arises are mutually distinct, hence the limit monomial with minimum weight cannot be cancelled. Thus $\lim_{t\to 0} F \circ \alpha(t)$ is the sum of all the limit monomials $x^A y^{B+C}$ with minimum weight. Thus $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a kernel star, and hence a rank-2 limit by Lemma 3.1.

If $r \not\equiv 0$ and $s \not\equiv 0$, but $v(r) > b + v(s)$, the same discussion applies, with the same conclusion.

If $r \not\equiv 0$, and $s \equiv 0$ or $v(r) < b + v(s)(< c)$, then the dominating term in m_{ABC} is

$$x^{A+C} y^B t^{bB+v(r)C} \quad ;$$

as $v(r) > 0$ these limit monomials again have different weights, so $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is again a kernel star.

Finally, assume $r \not\equiv 0$, $s \not\equiv 0$, and $v(r) = b + v(s)$ (in particular, $v(r) \neq b$). The dominating terms are then those

$$x^A y^B (r_0 x + s_0 y)^C = r_0^C x^{A+C} y^B + \cdots + s_0^C x^A y^{B+C}$$

with minimal $bB + v(r)C$, where r_0, s_0 are the leading coefficients in $r(t)$, $s(t)$. These terms cannot all cancel: as $b \neq v(r)$, there must be exactly one term with maximum $B + C$, and the corresponding term $x^A y^{B+C}$ cannot be cancelled by other terms. As the limit is again a kernel star, hence a rank-2 limit, the assertion is proved. □

3.18. By Lemma 3.16 we may now assume that $q(t) \neq 0$. By Lemma 3.10 we may then replace $\alpha(t)$ with an equivalent germ

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r_1(t) & s_1(t)t^b & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ r & s & v \end{pmatrix}$$

with $a < b \leq c$, $r_1(t)$, $s_1(t)$ polynomials, and an invertible constant factor on the right. This constant factor is the factor appearing in the statement of Proposition 3.15. The limit of any curve under this germ is a rank-2 limit if and only if the limit by

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r_1(t) & s_1(t)t^b & t^c \end{pmatrix}$$

is a rank-2 limit, so we may ignore the constant matrix on the right in the rest of the proof of Proposition 3.15.

Renaming $r_1(t)$, $s_1(t)$ by $r(t)$, $s(t)$ respectively, we are reduced to studying germs

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $a < b \leq c$ positive integers and $r(t)$, $s(t)$ polynomials of degree $< c$, $< (c - b)$ respectively and vanishing at $t = 0$.

In order to complete the proof of Proposition 3.15, we have to show that if $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is not a rank-2 limit then $b < c$ and $r(t)$, $s(t)$ are as stated.

3.19. We first deal with the case $b = c$.

Lemma 3.17. *Let $\alpha(t)$ be as above. If $b = c$, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. If $b = c$, then $s = 0$ necessarily:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & 0 & t^b \end{pmatrix} \quad ,$$

and further $a < v(r)$ (by Lemma 3.10).

Decompose $F(1 : y : z)$ in $\mathbb{C}[[y, z]]$: $F(1 : y : z) = G(y, z) \cdot H(y, z)$, where $G(y, z)$ collects the branches that are *not* tangent to $z = 0$. Writing $G(y, z)$ as a sum of homogeneous terms in y, z:

$$G(y, z) = G_{m'}(y, z) + \text{higher order terms}$$

with $G_{m'}(1, 0) \neq 0$, and applying $\alpha(t)$ gives

$$G_{m'}(t^a x + t^b y, r(t)x + t^b z) + \text{higher order terms} \quad .$$

As $a < v(r)$ and $a < b$, the dominant term in this expression is $t^{m'a}x^{m'}$: that is, the limit of these branches is supported on the kernel line $x = 0$.

The (formal) branches collected in $H(y, z)$ are tangent to $z = 0$, and we can write such branches as power series with fractional coefficients (cf. §3.6):

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_i} \neq 0$. We will be done if we show that the limit of such a branch (in the sense of Definition 3.4) is given by an equation in x and z: the limit $\lim_{t \to 0} \mathcal{C} \circ \alpha(t)$ will then (cf. Lemma 3.5) be a $(0 : 1 : 0)$-star, hence a rank-2 limit by Lemma 3.1.

The affine equation of the limit of $z = f(y)$ is given by the dominant terms in

$$r(t) + t^b z = f(t^a) + f'(t^a)t^b y + \cdots$$

we observe that y appears on the right-hand-side with weight larger than b (as $\lambda_0 > 1$). On the other hand, z only appears on the left-hand-side, so it cannot be cancelled by other parts in the expression. It follows that the weight of the dominant terms is $\leq b$, and in particular that y does not appear in these dominant terms. This shows that the equation of the limit does not depend on y, and we are done. □

3.20. Next, assume that α is parametrized by

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with the usual conditions on $r(t)$ and $s(t)$, and further $b < c$.

We want to study the limits of individual branches of $\mathcal{C}$ under such a germ. We first deal with branches that are not tangent to $z = 0$:

Lemma 3.18. *Under these assumptions on α, the limits of branches that are not tangent to the line $z = 0$ are necessarily $(0 : 0 : 1)$-stars. Further, if $a < v(r)$ then the limit of such branches is the kernel line $x = 0$.*

Proof. Formal branches that are not tangent to the line $y = 0$ may be written (cf. §3.6)

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 \leq \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_i} \neq 0$, and have limit along $\alpha(t)$ given by the dominant terms in

$$r(t) + s(t)t^b y + t^c z = f(t^a) + f'(t^a)t^b y + \dots \quad .$$

Branches that are *not* tangent to $z = 0$ have $\lambda_0 = 1$, hence $f'(y) = \gamma_1 + \dots$ with $\gamma_1 \neq 0$. Hence, the term $f'(t^a)t^b y$ on the right has weight t^b, and is not cancelled by other terms in the expression (since $v(s) > 0$). This implies that the dominant weight is $\leq b < c$, and in particular that the equation of the limit does not involve z. Hence the limit is a $(0:0:1)$ star, as needed.

If $a < v(r)$ and $\lambda_0 = 1$, then the dominant weight is $a < b$, hence the equation of the limit does not involve y either, so the limit is a kernel line, as claimed.

Analogous arguments can be used to treat formal branches that are tangent to $y = 0$. □

3.21. Next, consider a formal branch that is tangent to $z = 0$:

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $1 < \lambda_0 < \lambda_1 < \dots$

Lemma 3.19. *Under the same assumptions on α as in Lemma 3.18, the limit of $z = f(y)$ by α is a $(0:0:1)$-star unless*

- $r(t) \equiv f(t^a) \pmod{t^c}$*;*
- $s(t) \equiv f'(t^a) \pmod{t^{c-b}}$.

Proof. The limit of the branch is given by the dominant terms in

$$r(t) + s(t)t^b y + t^c z = f(t^a) + f'(t^a)t^b y + \dots$$

If $r(t) \not\equiv f(t^a) \pmod{t^c}$, then the weight of the branch is necessarily $< c$, so the ideal of the limit is generated by a polynomial in x and y, as needed. The same reasoning applies if $s(t) \not\equiv f'(t^a) \pmod{t^{c-b}}$. □

3.22. Proposition 3.15 is now essentially proved. We tie up here the loose ends of the argument.

Proof. Assume

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with the conditions listed in Proposition 3.7, and such that q, r, and s do not all vanish identically, and assume $\lim_{t \to 0} \mathcal{C} \circ \alpha(t)$ is not a rank-2 limit. By Lemma 3.16

we may assume $q(t) \neq 0$; hence by Lemma 3.10 $\alpha(t)$ is equivalent to a germ

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r_1(t) & s_1(t)t^b & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ r & s & v \end{pmatrix} \quad ,$$

with $a < b \le c$, $r_1(t)$, $s_1(t)$ polynomials, and $u, r, s, v \in \mathbb{C}$ with $uv \neq 0$. The limit along this germ is not a rank-2 limit if and only if the limit along

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r_1(t) & s_1(t)t^b & t^c \end{pmatrix}$$

is not a rank-2 limit. Assuming this is the case, necessarily $b < c$ by Lemma 3.17. Further, by Lemma 3.18 the limits of all branches that are not tangent to $z = 0$ are $(0 : 0 : 1)$-stars, hence rank-2 limits (by Lemma 3.1); the same holds for all formal branches $z = f(y)$ tangent to $z = 0$ unless $r_1(t) = \underline{f(t^a)}$ and $s_1(t)t^b = \underline{f'(t^a)t^b}$, by Lemma 3.19. Hence, if $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is not a rank-2 limit then $\alpha(t)$ must be equivalent to one in the form given in the statement of the proposition, for some formal branch $z = f(y)$ tangent to $z = 0$.

Finally, to see that the stated condition on $\frac{c}{a}$ must hold, look again at the limit of the formal branch $z = f(y)$, that is, the dominant term in

$$r(t) + s(t)t^b y + t^c z = f(t^a) + f'(t^a)t^b y + \frac{f''(t^a)t^{2b}y^2}{2} + \cdots :$$

the dominant weight will be less than c (causing the limit to be a $(0 : 0 : 1)$-star) if $c > 2b + v(f''(t^a)) = 2b + a(\lambda_0 - 2)$. The stated condition follows at once. $\square$

The effect of the constant factor on the right in the germ appearing in the statement of Proposition 3.15 is simply to translate the limit (by an invertible transformation fixing the flag consisting of the line $x = 0$ and the point $(0 : 0 : 1)$). Hence, for the remaining considerations in this section we may and will ignore this factor.

Also, we will replace t by $t^{1/d}$ in the germ obtained in Proposition 3.15 to ensure that the exponents appearing in its expression are relatively prime; the resulting germ determines the same component of the PNC.

3.23. The next reduction concerns the possible triples $a < b < c$ determining limits contributing to components of the PNC. This is best expressed in terms of $B = \frac{b}{a}$ and $C = \frac{c}{a}$. Let

$$z = f(y) = \sum_{i \ge 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_i} \neq 0$, be a formal branch tangent to $z = 0$. Every choice of such a branch and of a rational number $C = \frac{c}{a} > 1$ determines a truncation

$$f_{(C)}(y) = \sum_{\lambda_i < C} \gamma_{\lambda_i} y^{\lambda_i} \quad .$$

With this notation, the truncation $\underline{f(t^a)}$ equals $f_{(C)}(t^a)$.

The choice of a rational number $B = \frac{b}{a}$ satisfying $1 < B < C$ determines now a germ as prescribed by Proposition 3.15:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

(choosing the smallest positive integer a for which the entries of this matrix have integer exponents). Observe that the truncation $\underline{f(t^a)} = f_{(C)}(t^a)$ is identically 0 if and only if $C \le \lambda_0$. Also observe that $\underline{f'(t^a)t^b}$ is determined by $f_{(C)}(t^a)$, as it equals the truncation to t^c of $(f_{(C)})'(t^a)t^b$.

Proposition 3.20. *If $C \le \lambda_0$ or $B \neq \frac{C-\lambda_0}{2}+1$, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a rank-2 limit.*

We deal with the different cases separately.

Lemma 3.21. *If $C \le \lambda_0$, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a $(0:1:0)$-star.*

Proof. If $C = \frac{c}{a} \le \lambda_0$, then $f_{(C)}(y) = 0$, so

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad .$$

Collect the branches that are not tangent to $z = 0$ into $\Phi(y, z) \in \mathbb{C}[[y, z]]$, with initial form $\Phi_m(y, z)$. Applying $\alpha(t)$ to these branches gives

$$\Phi_m(t^a x + t^b y, t^c z) + \dots$$

with limit a kernel line since $a < c$ and $\Phi_m(1 : 0) \neq 0$.

As for the branches that are tangent to $z = 0$, let $z = f(y)$ be such a formal branch. The limit along $\alpha(t)$ is given by the dominant terms in

$$t^c z = f(t^a) + f'(t^a)t^b y + \dots$$

All terms on the right except the first one have weight larger than $a\lambda_0 \ge aC = c$, hence the dominant term does not involve y, concluding the proof. □

By Lemma 3.1, the limits obtained in Lemma 3.21 are rank-2 limits, so the first part of Proposition 3.20 is proved. As for the second part, if $B < \frac{C-\lambda_0}{2} + 1$ then $C > \lambda_0 + 2(B-1)$, and the limit is a rank-2 limit by the last assertion in Proposition 3.15.

For $B \ge \frac{C-\lambda_0}{2} + 1$, the limit of a branch tangent to $z = 0$ depends on whether the branch truncates to $f_{(C)}(y)$ or not. These cases are studied in the next two lemmas.

Lemma 3.22. *Assume $C > \lambda_0$ and $B \ge \frac{C-\lambda_0}{2} + 1$, and let $z = g(y)$ be a formal branch tangent to $z = 0$, such that $g_{(C)}(y) \neq f_{(C)}(y)$. Then the limit of the branch is supported on a kernel line.*

Proof. The limit of the branch is determined by the dominant terms in

$$\underline{f(t^a)} + \underline{f'(t^a)t^b} y + t^c z = g(t^a) + g'(t^a)t^b y + \dots$$

Assume the truncations $g_{(C)}$ and $f_{(C)}$ do not agree. If the first term at which they disagree has weight lower than $B + \lambda_0 - 1$, then the dominant terms in the expansion have weight lower than the weight of $\underline{f'(t^a)t^b y}$, and it follows that the limit is supported on $x = 0$. So we may assume that

$$g_{(C)}(y) = f_{(C)}(y) + y^{B+\lambda_0-1}\rho(y)$$

for some $\rho(y)$. We claim that then the terms $\underline{f'(t^a)t^b y}$ and $g'(t^a)t^b y$ agree modulo t^c: this implies that the dominant term is independent of y. As the dominant term is also independent of z (since the truncations $g_{(C)}$ and $f_{(C)}$ do not agree), the statement will follow from our claim.

In order to prove the claim, observe that

$$(g_{(C)})'(y) = (f_{(C)})'(y) + (B + \lambda_0 - 1)y^{B+\lambda_0-2}\rho(y) + y^{B+\lambda_0-1}\rho'(y) \quad :$$

thus, $(g_{(C)})'(y)y^B$ and $(f_{(C)})'(y)y^B$ must agree modulo $y^{2B+\lambda_0-2}$. Since $B \geq \frac{C-\lambda_0}{2} + 1$, we have

$$2B + \lambda_0 - 2 \geq (C - \lambda_0) + 2 + \lambda_0 - 2 = C \quad ;$$

hence $(g_{(C)})'(t^a)t^b$ and $(f_{(C)})'(t^a)t^b$ must agree modulo $t^{aC} = t^c$. It follows that $\underline{f'(t^a)t^b y}$ and $g'(t^a)t^b y$ agree modulo t^c, and we are done. $\square$

Lemma 3.23. *Assume $C > \lambda_0$ and $B \geq \frac{C-\lambda_0}{2} + 1$, and let $z = g(y)$ be a formal branch tangent to $z = 0$, such that $g_{(C)}(y) = f_{(C)}(y)$. Denote by $\gamma_C^{(g)}$ the coefficient of y^C in $g(y)$.*

- *If $B > \frac{C-\lambda_0}{2} + 1$, then the limit of the branch $z = g(y)$ by $\alpha(t)$ is the line*

$$z = (C - B + 1)\gamma_{C-B+1}y + \gamma_C^{(g)} \quad .$$

- *If $B = \frac{C-\lambda_0}{2} + 1$, then the limit of the branch $z = g(y)$ by $\alpha(t)$ is the conic*

$$z = \frac{\lambda_0(\lambda_0 - 1)}{2}\gamma_{\lambda_0}y^2 + \frac{\lambda_0 + C}{2}\gamma_{\frac{\lambda_0+C}{2}}y + \gamma_C^{(g)} \quad .$$

Proof. Rewrite the expansion whose dominant terms give the limit of the branch as:

$$t^c z = (g(t^a) - \underline{f(t^a)}) + (g'(t^a)t^b - \underline{f'(t^a)t^b})y + \frac{g''(t^a)}{2}t^{2b}y^2 + \dots$$

The dominant term has weight $c = Ca$ by our choices; if $B > \frac{C-\lambda_0}{2} + 1$ then the weight of the coefficient of y^2 exceeds c, so it does not survive the limiting process, and the limit is a line. If $B = \frac{C-\lambda_0}{2} + 1$, the term in y^2 is dominant, and the limit is a conic.

The explicit expressions given in the statement are obtained by reading the coefficients of the dominant terms. $\square$

We can now complete the proof of Proposition 3.20:

Lemma 3.24. *If $B > \frac{C-\lambda_0}{2} + 1$, then the limit $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. We will show that the limit is necessarily a kernel star, which gives the statement by Lemma 3.1.

As $B > 1$, the coefficient γ_{C-B+1} is determined by the truncation $f_{(C)}$, and in particular it is the same for all formal branches with that truncation. If $B > \frac{C-\lambda_0}{2}+1$, by Lemma 3.23 the branches contributes a line through the fixed point $(0 : 1 : (C-B+1)\gamma_{C-B+1})$. We are done if we check that all other branches contribute a kernel line $x = 0$: and this is implied by Lemma 3.18 for branches that are not tangent to $z = 0$ (note $a < v(r)$ for the germs we are considering), and by Lemma 3.22 for formal branches $z = g(y)$ tangent to $z = 0$ but whose truncation $g_{(C)}$ does not agree with $f_{(C)}$. □

3.24. Finally we are ready to complete the description of the components given in §3.2. By Propositions 3.15 and 3.20, germs leading to components of the PNC that have not yet been accounted for must be (up to a constant translation, and up to replacing t by $t^{1/d}$) in the form

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

for some branch $z = f(y) = \gamma_{\lambda_0} y^{\lambda_0} + \dots$ of $\mathcal{C}$ tangent to $z = 0$ at $p = (0, 0)$, and further satisfying $C > \lambda_0$ and $B = \frac{C-\lambda_0}{2} + 1$ for $B = \frac{b}{a}$, $C = \frac{c}{a}$. New components of the PNC will arise depending on the limit $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$, which we now determine.

Lemma 3.25. *If $C > \lambda_0$ and $B = \frac{C-\lambda_0}{2} + 1$, then the limit $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ consists of a union of quadritangent conics, with distinguished tangent equal to the kernel line $x = 0$, and of a multiple of the distinguished tangent line.*

Proof. Both γ_{λ_0} and $\gamma_{\frac{\lambda_0+C}{2}}$ are determined by the truncation $f_{(C)}$ (since $C > \lambda_0$); hence the equations of the conics

$$z = \frac{\lambda_0(\lambda_0 - 1)}{2}\gamma_{\lambda_0} y^2 + \frac{\lambda_0 + C}{2}\gamma_{\frac{\lambda_0+C}{2}} y + \gamma_C$$

contributed (according to Lemma 3.23) by different branches with truncation $f_{(C)}$ may only differ in the γ_C coefficient.

It is immediately verified that all such conics are tangent to the kernel line $x = 0$, at the point $(0 : 0 : 1)$, and that any two such conics meet only at the point $(0 : 0 : 1)$; thus they are necessarily quadritangent.

Finally, the branches that do not truncate to $f_{(C)}(y)$ must contribute kernel lines, by Lemmas 3.18 and 3.22. □

The type of curves arising as the limits described in Lemma 3.25 are studied in [AF00b], §4.1; also see item (12) in the classification of curves with small orbit in [AF00c], §1. The degenerate case in which only one conic appears does not lead to a component of the projective normal cone, by the usual dimension considerations. A component is present as soon as there are two or more conics, that is, as soon as two branches contribute distinct conics to the limit.

This leads to the description given in §3.2. We say that a rational number C is 'characteristic' for $\mathcal{C}$ (with respect to $z=0$) if at least two formal branches of $\mathcal{C}$ (tangent to $z=0$) have the same nonzero truncation, but different coefficients for y^C.

Proposition 3.26. *The set of characteristic rationals is finite.*

The limit $\lim_{t\to 0}\mathcal{C}\circ\alpha(t)$ *obtained in Lemma 3.25 determines a component of the projective normal cone precisely when C is characteristic.*

Proof. If $C\gg 0$, then branches with the same truncation must in fact be identical, hence they cannot differ at y^C, hence C is not characteristic. Since the set of exponents of any branch is discrete, the first assertion follows.

The second assertion follows from Lemma 3.25: if $C>\lambda_0$ and $B=\frac{C-\lambda_0}{2}+1$, then the limit is a union of a multiple kernel line and conics with equation

$$z=\frac{\lambda_0(\lambda_0-1)}{2}\gamma_{\lambda_0}y^2+\frac{\lambda_0+C}{2}\gamma_{\frac{\lambda_0+C}{2}}y+\gamma_C \quad :$$

these conics are different precisely when the coefficients γ_C are different, and the statement follows. □

Proposition 3.26 leads to the procedure giving components of type V explained in §3.2 (also cf. [AF00a], §2, Fact 5), concluding the set-theoretic description of the projective normal cone given there.

3.25. Rather than reproducing from §3.2 the procedure leading to components of type V, we offer another explicit example by applying it to the curve considered in §2.2.

Example 3.27. We obtain the components of type V due to the singularity at the point $p=(1:0:0)$ on the curve

$$x^3z^4-2x^2y^3z^2+xy^6-4xy^5z-y^7=0$$

The ideal of the tangent cone at p is (z^4), so that this curve has four formal branches, all tangent to the line $z=0$. These can be computed with ease:

$$\begin{cases} z= & y^{3/2}- \; y^{7/4}\\ z= & y^{3/2}+\; y^{7/4}\\ z= & -y^{3/2}+iy^{7/4}\\ z= & -y^{3/2}-iy^{7/4}\end{cases}.$$

We find a single characteristic $C=\frac{7}{4}$, and two truncations $y^{3/2}$, $-y^{3/2}$. In both cases $\lambda_0=\frac{3}{2}$, so $B=\frac{\frac{7}{4}-\frac{3}{2}}{2}+1=\frac{9}{8}$. The lowest integer a clearing all denominators is 8, and we find the two germs

$$\alpha_1(t)=\begin{pmatrix}1&0&0\\ t^8&t^9&0\\ t^{12}&\frac{3}{2}t^{13}&t^{14}\end{pmatrix}\quad,\quad \alpha_2(t)=\begin{pmatrix}1&0&0\\ t^8&t^9&0\\ -t^{12}&-\frac{3}{2}t^{13}&t^{14}\end{pmatrix}\quad.$$

As in each case there are two branches (with different $y^{7/4}$-coefficients), the limits must in both cases be pairs of quadritangent conics, union a triple kernel line. The limit along α_1 is listed in §2.2; the limit along α_2 must be, according to the formula given above,

$$x^3\left(zx-\frac{\frac{3}{2}\cdot\frac{1}{2}}{2}(-1)y^2-\frac{\frac{3}{2}+\frac{7}{4}}{2}\cdot 0\cdot yx-ix^2\right)\left(zx-\frac{\frac{3}{2}\cdot\frac{1}{2}}{2}(-1)y^2-\frac{\frac{3}{2}+\frac{7}{4}}{2}\cdot 0\cdot yx+ix^2\right)$$

that is (up to a constant factor),

$$x^3\left(64x^2z^2+48xy^2z+9y^4+64x^4\right) \quad ,$$

as may also be checked directly.

3.26. In this subsection we briefly describe the contents of Aldo Ghizzetti's first paper [Ghi36b]. Following this paper, Ghizzetti turned to analysis under the mentorship of Guido Fubini and Mauro Picone. His 50 year career was crowned by the election into the *Accademia Nazionale dei Lincei*. See [Fic94].

In [Ghi36b], Ghizzetti reports the results of his 1930 thesis under Alessandro Terracini. He determines here the limits of one-parameter families of 'homographic' plane curves (that is, of curves in the same orbit under the PGL(3) action).

A main feature of loc. cit. is the classification of one-dimensional systems of 'homographies' approaching a degenerate homography. Let $\Omega_t=(a_{ik}(t))$ be such a system, where the coefficients $a_{ik}(t)$ are power series in t in a neighborhood of $t=0$; we assume that $\det(a_{ik}(t))$ vanishes at $t=0$ but is not identically zero. The Ω_t are viewed as transformations from a projective plane π to another plane π':

$$x'_i=\sum_{k=1}^{3}a_{ik}(t)x_k \quad , \qquad (i=1,2,3).$$

A plane curve C' of degree d in π' with equation $F(x'_1,x'_2,x'_3)=0$ is given; transforming it by Ω_t gives a curve C_t in π and the goal is to determine the limiting curve C_0. The coordinates x_k may be modified by a system of non-degenerate homographies of π to reduce Ω_t to a simpler form.

When Ω_0 has rank 2, it defines a kernel point S in π and an image line s' in π'. When C' does not contain s', it is easy to see that C_0 consists of d lines through S, the d-tuple being projectively equivalent to the d points that C' cuts out on s'. Let s_0 be the limiting position of the line s_t that results from transforming s' by Ω_t. When C' contains s' with multiplicity m, the limiting curve C_0 consists of a $(d-m)$-tuple of lines through S (projectively equivalent to the $(d-m)$-tuple of points that the residual of C' cuts out on s') and the line s_0 with multiplicity m. These are the limit curves of type I.

When Ω_0 has rank 1, it defines a kernel line s in π and an image point S' in π'. When C' doesn't pass through S', the limiting curve C_0 consists of the line s with multiplicity d. Assume from now on that C' passes through S'. For $t\neq 0$, let S_t be the inverse image point under Ω_t of S' and let S_0 be the limiting point. The author considers two cases: S_0 is not contained in s (case I) or it is (case II). (We reserve the word 'cases' for Ghizzetti's classification and continue to use 'types' for ours.) After a change of coordinates in π it may be assumed that S_t is fixed and coincides with S_0. Then Ω_t induces a linear map ω_t on the pencils through S_0 and S' and one obtains a limiting transformation ω_0, which is either non-degenerate (cases I_1 and II_1) or degenerate (cases I_2 and II_2).

Case I_1 yields one-parameter subgroups with two equal weights; this gives the limit curves of type III, the fans of Proposition 3.11. In case II_1 the kernel line s contains S_0 and the limit curves become kernel stars. Ghizzetti explicitly shows that they are rank-2 limits.

In cases I_2 and II_2, the degenerate map ω_0 defines a kernel line $\overline{s}$ through S_0 and an image line $\overline{s}'$ through S'. Let $\overline{s}_0$ be the limiting position of the line $\overline{s}_t$ that results from transforming $\overline{s}'$ by ω_t. The author distinguishes two subcases: $\overline{s}$ and $\overline{s}_0$ are distinct (cases I_{21} and II_{21}) or they coincide (cases I_{22} and II_{22}). Moreover, in case II_{22} it is necessary to distinguish whether s differs from $\overline{s} = \overline{s}_0$ (case II'_{22}) or coincides with it (case II''_{22}).

Case I_{21} yields one-parameter subgroups with three distinct weights: in the notation of §3.13 (with $b < c$), s is $x = 0$, $\overline{s}$ is $y = 0$, and $\overline{s}_0$ is $z = 0$. This gives the limit curves of types II and IV, cf. §§3.14 and 3.15.

Each of the cases I_{22}, II_{21}, and II'_{22} leads to limit curves that are kernel stars; again, Ghizzetti shows explicitly that they are rank-2 limits.

The case II''_{22} remains: $s = \overline{s} = \overline{s}_0$. Assume that C' is tangent to $\overline{s}'$ in S'; if not, C_0 consists of the line s with multiplicity d. After a change of coordinates in π and a change of parameter (similar to the one in Lemma 3.10), the matrix of Ω_t has three zero entries, one entry equal to 1, and one entry a positive power of t; each of the remaining entries vanishes at $t = 0$ but is not identically zero. (The matrix is *not* in triangular form.) The vanishing orders of two of the entries (m and $m + n$ in Ghizzetti's notation) arose naturally in the classification leading to the present case. The other three vanishing orders are called $m + p$, q, and r, and it is necessary to analyze the various possibilities for the triple p, q, and r. (In the last section of loc. cit. the author remarks that m and n are the degree and class of the curve of image points under Ω_t of a fixed point in π and that p, q, and r are similarly related to the curve of inverse image points of a fixed point in π'.)

Ghizzetti distinguishes five cases in his analysis of the triple p, q, and r. After simplifying coordinate changes, the Ω_t are applied to the branches of C' at S' that are tangent to $\overline{s}'$. The first four cases lead to limit curves that are kernel stars. In the study of the fifth case, it is necessary to distinguish five subcases. The first three lead to limit curves that are kernel stars. The fourth case is not worked out in detail. The limit curve consists of lines; in general, not all of these lines belong to the same pencil, according to Ghizzetti, but in fact they form a kernel star (cf. Lemma 3.24). The fifth and final case leads to a union of quadritangent conics and a multiple of the kernel line, as in Lemma 3.25. These are the limit curves of type V; with them, Ghizzetti concludes his analysis of the possible limit curves arising from a system of non-degenerate homographies approaching a degenerate homography.

One of the consequences of Ghizzetti's work is that the irreducible components of limit curves are very special from the projective standpoint; for example, they are necessarily isomorphic to their own dual. It is clear that such curves have 'small' linear orbit; Ghizzetti's characterization was extended to curves with small orbit in projective spaces of arbitrary dimension in one of Ciro Ciliberto's first papers, [Cil77]. Plane curves with small linear orbits are classified in [AF00c].

It will be obvious from the above that Ghizzetti's approach and ours are quite similar. Let us then conclude this section by indicating some of the differences. From a technical viewpoint, Proposition 3.7 and its refinement Lemma 3.10 lead us to the key reduction of Proposition 3.15; this result allows us to restrict our attention to germs that are essentially determined by a branch of $\mathcal{C}$. Although this reduction is perhaps not as geometrically meaningful as Ghizzetti's approach, it appears to lead to a considerable simplification. Also, the fact that the limit curves necessarily have infinite stabilizer plays a less prominent role in [Ghi36b]. Most importantly, Ghizzetti's goal was essentially the set-theoretic description of the boundary components of the linear orbit of a curve, while our enumerative applications in [AF00a] require the more refined information carried by the projective normal cone dominating the boundary. This forces us to be more explicit concerning equivalence of germs, and leads us to a rather different classification than the one considered by Ghizzetti. In fact, the set-theoretic description alone of (even) the projective normal cone does not suffice for our broader goals, so that we need to refine our analysis considerably in order to determine the projective normal cone *as a cycle,* in the next section. For this, the notion of equivalence introduced in Definition 3.2 will play a crucial role, cf. Proposition 4.5 and Lemma 4.10.

In any case, Ghizzetti's contribution remains outstanding for its technical prowess, and it is an excellent example of concreteness and concision in the exposition of very challenging material. The fact that this is his first paper makes our admiration of it only stronger.

4. The projective normal cone as a cycle

4.1. As mentioned in §1 and §2, the enumerative computations in [AF00a] require the knowledge of the *cycle* supported on the projective normal cone; that is, we need to compute the *multiplicities* with which the components identified in §3 appear in the PNC. This is what we do in this section.

Our general strategy will be the following. By normalizing the graph of the basic rational map $\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$, we will distinguish different 'ways' in which a component may arise, and compute a contribution to the multiplicity due to each way. This contribution will be obtained by carefully evaluating different ingredients: the order of contact of certain germs with the base scheme of the rational map, the number of components in the normalization dominating a given component of the PNC, and the degree of the restriction of the normalization map to these components.

4.2. Here is the result. The 'multiplicities' in the following list are contributions to the multiplicity of each individual component from the possibly different ways to obtain it. This list should be compared with [AF00a], §2, Facts 1 through 5.

- *Type I.* The multiplicity of the component determined by a line $\ell \subset \mathcal{C}$ equals the multiplicity of ℓ in $\mathcal{C}$.
- *Type II.* The multiplicity of the component determined by a nonlinear component $\mathcal{C}'$ of $\mathcal{C}$ equals $2m$, where m is the multiplicity of $\mathcal{C}'$ in $\mathcal{C}$.

- *Type III.* The multiplicity of the component determined by a singular point p of $\mathcal{C}$ such that the tangent cone λ to $\mathcal{C}$ at p is supported on three or more lines equals mA, where m is the multiplicity of $\mathcal{C}$ at p and A equals the number of automorphisms of λ as a tuple in the pencil of lines through p.
- *Type IV.* The multiplicity of the component determined by one side of a Newton polygon for $\mathcal{C}$, with vertices (j_0, k_0), (j_1, k_1) (where $j_0 < j_1$) and limit
$$x^{\overline{q}} y^r z^q \prod_{j=1}^{S} \left(y^c + \rho_j x^{c-b} z^b\right) \quad ,$$
(with b and c relatively prime) equals
$$\frac{j_1 k_0 - j_0 k_1}{S} A \quad ,$$
where A is the number of automorphisms $\mathbb{A}^1 \to \mathbb{A}^1$, $\rho \mapsto u\rho$ (with u a root of unity) preserving the S-tuple $\{\rho_1, \ldots, \rho_S\}$.[1]
- *Type V.* The multiplicity of the component corresponding to the choice of a characteristic C and a truncation $f_{(C)}(y)$ at a point p, with limit
$$x^{d-2S} \prod_{i=1}^{S} \left(zx - \frac{\lambda_0(\lambda_0 - 1)}{2} \gamma_{\lambda_0} y^2 - \frac{\lambda_0 + C}{2} \gamma_{\frac{\lambda_0 + C}{2}} yx - \gamma_C^{(i)} x^2 \right)$$
is $\ell W A$, where:
  - ℓ is the least positive integer μ such that $f_{(C)}(y^\mu)$ has integer exponents.
  - W is defined as follows. For each formal branch β of $\mathcal{C}$ at p, let v_β be the first exponent at which β and $f_{(C)}(y)$ differ, and let w_β be the minimum of C and v_β. Then W is the sum $\sum w_\beta$.
  - A is twice the number of automorphisms $\gamma \to u\gamma + v$ preserving the S-tuple $\{\gamma_C^{(1)}, \ldots, \gamma_C^{(S)}\}$.

Concerning the 'different ways' in which a component may be obtained (each producing a multiplicity computed by the above recipe), no subtleties are involved for components of type I, II, or III: there is only one contribution for each of the specified data—that is, exactly one contribution of type I from each line contained in $\mathcal{C}$, one contribution of type II from each nonlinear component of $\mathcal{C}$, and one of type III from each singular point of $\mathcal{C}$ at which the tangent cone is supported on three or more distinct lines.

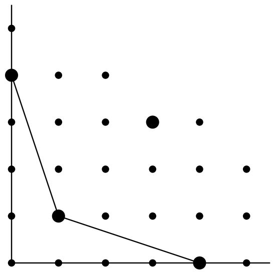
As usual, the situation is a little more complex for components of type IV and V.

Components of type IV correspond to sides of Newton polygons; one polygon is obtained for each line in the tangent cone at a fixed singular point p of $\mathcal{C}$, and each of these polygons provides a set of sides (with slope strictly between -1 and 0). Exactly one contribution has to be counted for each side obtained in this fashion. Note that sides of different Newton polygons may lead to the same limits, hence to the same component of the PNC.

Example 4.1. The curve
$$(y - z^3)(z - y^3) = 0$$

[1] Note: the number A given here is denoted A/δ in [AF00a].

has a node at the origin p: $(y,z)=(0,0)$; the two lines in the tangent cone both give the Newton polygon



each with one side with slope $-1/3$. The corresponding limits, obtained (as prescribed in §3.14) via the germs

$$\begin{pmatrix}1&0&0\\0&t&0\\0&0&t^3\end{pmatrix}\quad,\quad\begin{pmatrix}1&0&0\\0&t^3&0\\0&0&t\end{pmatrix}$$

are respectively

$$y(z-y^3)\quad,\quad z(y-z^3)\quad.$$

These limits belong to the same PGL(3) orbit; thus the two sides determine the same component of the PNC. According to the result given above, the multiplicity of this component receives a contribution of 4 from each of the sides, so the component appears with multiplicity 8 in the PNC.

Components of type V are determined by a choice of a singular point p of $\mathcal{C}$, a line L in the tangent cone to $\mathcal{C}$ at p, a characteristic C and a truncation $f_{(C)}(y)$ of a formal branch of $\mathcal{C}$ tangent to L. Recall that this data determines a triple of positive integers $a<b<c$ with $C=c/a$: the number C and the truncation $f_{(C)}(y)$ determine B as in the beginning of §3.24, and a is the smallest integer clearing denominators of all exponents in the corresponding germ $\alpha(t)$. Again, different choices may lead to the same component of the PNC, and we have to specify when choices should be counted as giving separate contributions. Of course different points p or different lines in the tangent cone at p give separate contributions; the question is when two sets of data $(C, f_{(C)}(y))$ for the same point, with respect to the same tangent line, should be counted separately.

To state the result, we say that $(C, f_{(C)}(y))$, $(C', g_{(C')}(y))$ (or the truncations $f_{(C)}$, $g_{(C')}$ for short) are *sibling* data if $C=C'$ and $f_{(C)}(t^a)=g_{(C)}((\xi t)^a)$ for an a-th root ξ of 1.

Example 4.2. If $z=f(y)$, $z=g(y)$ are formal branches belonging to the same irreducible branch of $\mathcal{C}$ at p, then the corresponding truncations $f_{(C)}(y)$, $g_{(C)}(y)$ are siblings for all C.

Indeed, if the branch has multiplicity m at p then $f(\tau^m)=\varphi(\tau)$ and $g(\tau^m)=\psi(\tau)$, with $\psi(\tau)=\varphi(\zeta\tau)$ for an m-th root ζ of 1 ([Fis01], §7.10). That is,

$$\text{if}\quad f(y)=\sum\gamma_{\lambda_i}y^{\lambda_i}\quad,\quad\text{then}\quad g(y)=\sum\zeta^{m\lambda_i}\gamma_{\lambda_i}y^{\lambda_i}$$

for ζ an m-th root of 1. Note that the positive integer a determined by C is such that $a\lambda_i$ is integer for all the exponents λ_i lower than C.

Now let ρ be an (am)-th root of 1 such that $\rho^a = \zeta$, and set $\xi = \rho^m$; since the exponents $a\lambda_i$ in the truncations are integers, as well as all exponents $m\lambda_i$, we have

$$\zeta^{m\lambda_i} = \rho^{ma\lambda_i} = \xi^{a\lambda_i}$$

for all exponents $\lambda_i < C$, and this shows that the truncations are siblings.

With this notion, we can state precisely when two truncations $(C, f_{(C)}(y))$ at the same point, with respect to the same tangent line, yield separate contributions: they do if and only if they are *not* siblings.

The proof that the formulas presented above hold occupies the rest of this paper. Rather direct arguments can be given for the 'global' components, of type I and type II, cf. §§4.8 and 4.9. The 'local' types III, IV, and V require some preliminaries, covered in the next several sections.

4.3. The main character in the story will be the *normalization* $\overline{\mathbb{P}}$ of the closure $\widetilde{\mathbb{P}}^8$ of the graph of the basic rational map $\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$ introduced in §2. We denote by

$$n : \overline{\mathbb{P}} \to \widetilde{\mathbb{P}}^8$$

the normalization map, and by $\overline{n}$ the composition $\overline{\mathbb{P}} \to \widetilde{\mathbb{P}}^8 \to \mathbb{P}^8$.

In §2, Lemma 2.1 we have realized the PNC as a subset of $\widetilde{\mathbb{P}}^8 \subset \mathbb{P}^8 \times \mathbb{P}^N$. Recall that the PNC is in fact the exceptional divisor E in $\widetilde{\mathbb{P}}^8$, where the latter is viewed as the blow-up of $\mathbb{P}^8$ along the base scheme S of the basic rational map. If $F \in \mathbb{C}[x, y, z]$ generates the ideal of $\mathcal{C}$ in $\mathbb{P}^2$, then the ideal of S in $\mathbb{P}^8$ is generated by all

$$F(\varphi(x_0, y_0, z_0))$$

viewed as polynomials in $\varphi \in \mathbb{P}^8$, as (x_0, y_0, z_0) ranges over $\mathbb{P}^2$. We denote by E_i the supports of the components of E, and by m_i the multiplicity of E_i in E.

We also denote by $\overline{E}$ the Cartier divisor $n^{-1}(E) = \overline{n}^{-1}(S)$ in $\overline{\mathbb{P}}$, and by

$$\overline{E}_{i1}, \dots, \overline{E}_{ir_i}$$

the supports of the components of $\overline{E}$ lying above a given component E_i of E. Finally, we let m_{ij} be the multiplicity of $\overline{E}_{ij}$ in $\overline{E}$. That is:

$$[E] = \sum m_i[E_i] \quad , \quad [\overline{E}] = \sum m_{ij}[\overline{E}_{ij}] \quad .$$

Proposition 4.3. *We have*

$$m_i = \sum_{j=1}^{r_i} e_{ij} m_{ij}$$

where e_{ij} is the degree of $n|_{\overline{E}_{ij}} : \overline{E}_{ij} \to E_i$.

Proof. This follows from the projection formula and $(n|_{\overline{E}_{ij}})_*[\overline{E}_{ij}] = e_{ij}[E_i]$. □

4.4. In order to apply Proposition 4.3, we have to develop tools to evaluate the multiplicities m_{ij} and the degrees e_{ij}. For 'local' components, we will obtain this information by describing $\overline{E}_{ij}$ in terms of lifts of certain germs from $\mathbb{P}^8$. Until the end of §4.7 we focus on components of type III, IV, and V.

Every germ $\alpha(t)$ in $\mathbb{P}^8$, whose general element is invertible, and such that $\alpha(0) \in S$, lifts to a unique germ in $\widetilde{\mathbb{P}}^8$ centered at a point $(\alpha(0), \mathcal{X})$ of the PNC. The germ lifts to a unique germ in $\overline{\mathbb{P}}$, centered at a point of $\overline{E}$.

Definition 4.4. We denote by $\overline{\alpha}$ the center of the lift of $\alpha(t)$ to $\overline{\mathbb{P}}$. We will say that $\alpha(t)$ is a 'marker' germ if

- $\overline{\alpha}$ belongs to exactly one component of $\overline{E}$, and the lift of $\alpha(t)$ is transversal to (the support of) $\overline{E}$ at $\overline{\alpha}$;
- $\overline{\mathbb{P}}$ is nonsingular at $\overline{\alpha}$;
- $\lim \mathcal{C} \circ \alpha(t)$ is a curve of the type described in §3.2.

Thus, marker germs 'mark' one component of $\overline{E}$, and $\overline{\mathbb{P}}$ is particularly well-behaved around marker germs. Note that since $\overline{\mathbb{P}}$ is normal, it is nonsingular along a dense open set in each component of $\overline{E}$: so the second requirement in the definition of marker germ is satisfied for a general $\overline{\alpha}$ on every component of $\overline{E}$. It follows that every component of $\overline{E}$ admits marker germs. Our next result is that, for marker germs, our notion of 'equivalence' (Definition 3.2) translates nicely in terms of the lifts to $\overline{E}$.

Proposition 4.5. *Let $\alpha_0(t)$, $\alpha_1(t)$ be germs, and assume that $\alpha_1(t)$ is a marker germ. Then $\alpha_0(t)$ is equivalent to $\alpha_1(t)$ if and only if $\overline{\alpha}_0 = \overline{\alpha}_1$ and the lift of $\alpha_0(t)$ is transversal to $\overline{E}$. In particular, $\alpha_0(t)$ is then a marker germ as well.*

Proof. Assume first that the germs $\alpha_0(t)$ and $\alpha_1(t)$ are equivalent. Then α_0, α_1 are fibers of a family $\alpha_h(t) = A(h,t)$ where $A : \mathcal{H} \times \operatorname{Spec} \mathbb{C}[[t]] \to \mathbb{P}^8$ is as specified in Definition 3.2. In particular, the map $\mathcal{H} \to \widetilde{\mathbb{P}}^8$, $h \mapsto (\alpha_h(0), \lim_{t\to 0} \mathcal{C} \circ \alpha_h(t))$ is constant. This map factors

$$h \mapsto \overline{\alpha}_h \overset{n}{\mapsto} (\alpha_h(0), \lim_{t\to 0} \mathcal{C} \circ \alpha_h(t))$$

and n is finite over $(\alpha_h(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$, so $h \mapsto \overline{\alpha}_h$ is constant; in particular, $\overline{\alpha}_0 = \overline{\alpha}_1$.

Further, the weight of $F(\alpha_h(t))$ is constant as h varies; this implies that the intersection numbers of all lifts of the germs $\alpha_h(t)$ with $\overline{E}$ are equal. In particular, the lift of $\alpha_0(t)$ is transversal to $\overline{E}$ if and only if the lift of $\alpha_1(t)$ is. Since $\alpha_1(t)$ is a marker germ, it follows that the lift of $\alpha_0(t)$ is transversal to $\overline{E}$, as needed.

For the other implication, $\overline{\mathbb{P}}$ is nonsingular at $\overline{\alpha} := \overline{\alpha}_0 = \overline{\alpha}_1$ since $\alpha_1(t)$ is a marker germ. Let $(\overline{z}) = (z_1, \dots, z_8)$ be a system of local parameters for $\overline{\mathbb{P}}$ centered at $\overline{\alpha}$, and write the lifts $\overline{\alpha}_i(t)$ as

$$t \to (\overline{z}^{(i)}(t)) = (z_1^{(i)}(t), \dots, z_8^{(i)}(t)) \quad .$$

Now

$$A(h,t) := \overline{n}\left((1-h)\overline{z}^{(0)}(t) + h\overline{z}^{(1)}(t)\right)$$

defines a map $\mathbb{A}^1 \times \operatorname{Spec}\mathbb{C}[[t]] \to \mathbb{P}^8$, so that $\alpha_h(t) := A(h,t)$ interpolates between $\alpha_0(t)$ and $\alpha_1(t)$. The map $A(\_,0)$ is constant since all lifts $\overline{\alpha}_h(t)$ meet $\overline{E}$ at $\overline{\alpha}$. By the same token, writing

$$F(\alpha_h(t)) = G(h)t^w + \text{higher order terms} \quad ,$$

necessarily $G(h) = \rho(h)G$ for $\rho(h) \in \mathbb{C}$ and G a polynomial in x, y, z independent of h. Since both $\overline{\alpha}_0(t)$ and $\overline{\alpha}_1(t)$ are transversal to $\overline{E}$, we have that $\rho(0)$ and $\rho(1)$ are both nonzero. Taking $\mathcal{H}$ to be the complement of the zero-set of ρ in $\mathbb{A}^1$ and restricting A to $\mathcal{H} \times \operatorname{Spec}\mathbb{C}[[t]]$ we obtain a map as prescribed in Definition 3.2, showing that $\alpha_0(t)$ and $\alpha_1(t)$ are equivalent. □

Corollary 4.6. *The germs $\alpha(t)$ obtained in section 3 in order to identify components of the PNC are marker germs.*

Proof. Indeed, we have shown in §3 that *every* germ determining a component of the PNC is equivalent to one such germ $\alpha(t^d)$, with d a positive integer; in particular this holds with $d = 1$ for marker germs, showing that such $\alpha(t)$ lift to germs that are transversal to $\overline{E}$, and meet it at points at which $\overline{\mathbb{P}}$ is nonsingular. □

4.5. Another important ingredient is the PGL(3) action on $\overline{\mathbb{P}}$.

The PGL(3) action on $\mathbb{P}^8$ given by multiplication on the right makes the basic rational map

$$\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$$

equivariant, and hence induces a right PGL(3) action on $\widetilde{\mathbb{P}}^8$ and $\overline{\mathbb{P}}$, fixing each component of $\overline{E}$. Explicitly, the action is realized by setting $\overline{\alpha} \cdot N$ to be the center of the lift of the germ $\alpha(t) \cdot N$, for $N \in \mathrm{PGL}(3)$. We record the following trivial but useful remark:

Lemma 4.7. *If $\beta(t) = \alpha(t) \cdot N$ for $N \in \mathrm{PGL}(3)$, then $\overline{\alpha}$ and $\overline{\beta}$ belong to the same component of $\overline{E}$.*

Proof. Indeed, then $\overline{\beta}$ belongs to the PGL(3) orbit of $\overline{\alpha}$. □

Lemma 4.8. *Let $\overline{D}$ be a component of $\overline{E}$ of type III, IV, or V. The orbit of a general $\overline{\alpha}$ in $\overline{D}$ is dense in $\overline{D}$.*

Proof. This follows immediately from the description of the general elements in the components of the PNC, given in §3.2. □

Combining this observation with Lemma 4.7 and Proposition 4.5 gives a precise description of the fibers of $\overline{E}$ over S:

Corollary 4.9. *Let $\alpha(t)$, $\beta(t)$ be marker germs such that $\alpha(0) = \beta(0)$. Then $\overline{\alpha}$ and $\overline{\beta}$ belong to the same component of $\overline{E}$ if and only if $\beta(t)$ is equivalent to $\alpha(t) \cdot N$ for some constant invertible matrix N such that $\alpha(0) \cdot N = \alpha(0)$.*

Proof. Let $\alpha(t)$, $\beta(t)$ be marker germs such that $\alpha(0) = \beta(0)$. If $\overline{\alpha}$ and $\overline{\beta}$ belong to the same component of $\overline{E}$, then by Lemma 4.8 there exists $N \in \mathrm{PGL}(3)$ such that $\overline{\beta} = \overline{\alpha} \cdot N$; by Proposition 4.5, $\beta(t)$ is equivalent to $\alpha(t) \cdot N$. Further, $\alpha(0) = \beta(0) =$

$\alpha(0) \cdot N$ (by definition of equivalent germs), hence the stated condition on N must hold.

The other implication is immediate from Lemma 4.7. $\square$

This description yields our main tool for computing the degrees e_{ij}, Proposition 4.12 below.

First, equivalence of marker germs can be recast in the following apparently stronger form.

Lemma 4.10. *Two marker germs $\alpha_0(t)$ and $\alpha_1(t)$ are equivalent (w.r.t. $\mathcal{C}$) if and only if there exists a unit $\nu(t) \in \mathbb{C}[[t]]$ and a $\mathbb{C}[h][[t]]$-valued point $N(h,t)$ of* PGL(3)*, such that*

- *$N(0,t)$ is the identity;*
- *$N(h,0)$ is the identity; and*
- *$\alpha_1(t\nu(t)) = \alpha_0(t) \cdot N(1,t)$.*

Proof. If a matrix $N(h,t)$ exists as in the statement, define $A : \mathbb{A}^1 \times \operatorname{Spec} \mathbb{C}[[t]] \to \mathbb{P}^8$ by setting $A(h,t) := \alpha_0(t) \cdot N(h,t)$. Then the conditions prescribed by Definition 3.2 are satisfied with $h_0 = 0$, $h_1 = 1$, showing that $\alpha_0(t)$ is equivalent to $\alpha_1(t)$.

For the converse: since $\alpha_0(t)$ and $\alpha_1(t)$ are equivalent, $\overline{\alpha}_0 = \overline{\alpha}_1$ by Proposition 4.5, and $(\alpha_0(0), \lim_{t\to 0} \mathcal{C} \circ \alpha_0(t)) = (\alpha_1(0), \lim_{t\to 0} \mathcal{C} \circ \alpha_1(t))$. Let $(\alpha, \mathcal{X})$ be this point of $\widetilde{\mathbb{P}}^8$, and let D be the (unique) component of E which contains it. For the remainder of the argument, we use the standing assumption that D is a component of type III, IV, or V.

Under this assumption, the stabilizer of $(\alpha, \mathcal{X})$ has dimension 1; consider an $\mathbb{A}^7$ transversal to the stabilizer at the identity I, let $U = \mathbb{A}^7 \cap \mathrm{PGL}(3)$, and consider the action map $U \times \operatorname{Spec} \mathbb{C}[[t]] \to \overline{\mathbb{P}}$:

$$(\varphi, t) \mapsto \overline{\alpha}_0(t) \circ \varphi$$

where $\overline{\alpha}_0(t)$ is the lift of $\alpha_0(t)$ to $\overline{\mathbb{P}}$.

Note that $\alpha_0(t)$ factors through this map:

$$\operatorname{Spec} \mathbb{C}[[t]] \to U \times \operatorname{Spec} \mathbb{C}[[t]] \to \overline{\mathbb{P}} \to \mathbb{P}^8$$

by

$$t \mapsto (I, t) \mapsto \overline{\alpha}_0(t) \mapsto \alpha_0(t) \quad .$$

Lifting $\alpha_1(t)$ we likewise get a factorization

$$t \mapsto (M(t), z(t)) \mapsto \overline{\alpha}_0(z(t)) \circ M(t) = \overline{\alpha}_1(t) \mapsto \alpha_1(t)$$

for suitable $M(t)$, $z(t)$. We may assume that the center $(M(0), z(0))$ of the lift of $\alpha_1(t)$ equals the center $(I, 0)$ of the lift of $\alpha_0(t)$; also, $z(t)$ vanishes to order 1 at $t = 0$, since the lift of $\alpha_1(t)$ is transversal to $\overline{E}$. Hence there exists a unit $\nu(t)$ such that $z(t\nu(t)) = t$, and we can effect the parameter change

$$\alpha_1(t\nu(t)) = \alpha_0(t) \circ M(t\nu(t)) = \alpha_0(t) \circ N(t) \quad ,$$

where we have set $N(t) = M(t\nu(t))$, a $\mathbb{C}[[t]]$-valued point of PGL(3).

Finally, interpolate between (I, t) and $(N(t), t)$ in $U \times \operatorname{Spec} \mathbb{C}[[t]] \subset \mathbb{A}^7 \times \operatorname{Spec} \mathbb{C}[[t]]$ (much as we did already in the proof of Proposition 4.5), by

$$(h,t) \mapsto ((1-h)I + hN(t), t) \quad .$$

Setting $N(h,t) := (1-h)I + hN(t)$, we obtain the sought $\mathbb{C}[h][[t]]$-valued point of PGL(3). Indeed: $N(0,t) = I$ for all t; $N(h,0) = I$ for all h; and $\alpha_0(t) \cdot N(1,t) = \alpha_0(t) \cdot N(t) = \alpha_1(t\nu(t))$. □

4.6. The dependence of this observation on a parameter change prompts the following definition. Let $\alpha(t)$ be a marker germ for a component $\overline{D}$, and consider the $\mathbb{C}((t))$-valued points of PGL(3) obtained as products

$$M_\nu(t) := \alpha(t)^{-1} \cdot \alpha(t\nu(t))$$

as $\nu(t)$ ranges over all units in $\mathbb{C}[[t]]$. Among all the $M_\nu(t)$, consider those that are in fact $\mathbb{C}[[t]]$-valued points of PGL(3), and in that case let

$$M_\nu := M_\nu(0) \quad .$$

Let $\alpha = \alpha(0)$, and $\mathcal{X} = \lim_{t\to 0} \mathcal{C} \circ \alpha(t)$.

Lemma 4.11. *The set of all M_ν so obtained is a subgroup of the stabilizer of $(\alpha, \mathcal{X})$.*

Proof. Let M_ν be as above; that is, $M_\nu = M_\nu(0)$, where $M_\nu(t) = \alpha(t)^{-1} \cdot \alpha(t\nu(t))$ is a $\mathbb{C}[[t]]$-valued point of PGL(3). Since

$$\alpha(t\nu(t)) = \alpha(t) \cdot M_\nu(t) \quad ,$$

we have

$$\alpha = \alpha(0\nu(0)) = \alpha(0) \cdot M_\nu(0) = \alpha \cdot M_\nu \quad ,$$

showing that M_ν stabilizes α. Further

$$\mathcal{C} \circ \alpha(t\nu(t)) = \mathcal{C} \circ \alpha(t) \circ M_\nu(t) \quad ,$$

and taking the limit

$$\lim_{t\to 0} C \circ \alpha(t\nu(t)) = (\lim_{t\to 0} C \circ \alpha(t)) \circ M_\nu(0) \quad ,$$

that is,

$$\mathcal{X} = \mathcal{X} \circ M_\nu$$

since the limits along $\alpha(t)$ and $\alpha(t\nu(t))$ must agree as the two germs only differ by a change of parameter. Thus M_ν stabilizes $\mathcal{X}$ as well, as needed.

To verify that the set $\{M_\nu\}_\nu$ forms a group, let $\overline{\nu}(t)$ be the unit such that

$$\overline{\nu}(t\nu(t)) = \nu(t)^{-1} \quad .$$

Then we find

$$\begin{aligned} M_\nu(t) \cdot M_{\overline{\nu}}(t\nu(t)) &= \alpha(t)^{-1}\alpha(t\nu(t))\,\alpha(t\nu(t))^{-1}\alpha(t\nu(t)\overline{\nu}(t\nu(t))) \\ &= \alpha(t)^{-1}\alpha(t\nu(t)\nu(t)^{-1}) \\ &= I \quad , \end{aligned}$$

the identity matrix. That is, $M_{\overline{\nu}}(t\nu(t)) = M_\nu(t)^{-1}$, and hence

$$M_{\overline{\nu}} = M_{\overline{\nu}}(0) = M_\nu(0)^{-1} = M_\nu^{-1} \quad .$$

Similarly, for $\nu_1(t)$, $\nu_2(t)$ units, let

$$\nu_3(t) = \nu_1(t)\nu_2(t\nu_1(t)) \quad .$$

Then we find

$$\begin{aligned} M_{\nu_3}(t) &= \alpha(t)^{-1}\alpha(t\nu_3(t)) \\ &= \alpha(t)^{-1}\alpha(t\nu_1(t))\alpha(t\nu_1(t))^{-1}\alpha(t\nu_1(t)\nu_2(t\nu_1(t))) \\ &= M_{\nu_1}(t)M_{\nu_2}(t\nu_1(t)) \quad , \end{aligned}$$

and hence $M_{\nu_3} = M_{\nu_3}(0) = M_{\nu_1}(0)M_{\nu_2}(0) = M_{\nu_1}M_{\nu_2}$ as needed. □

We call *inessential* the components of the stabilizer of $(\alpha, \mathcal{X})$ containing elements of the subgroup identified in Lemma 4.11. These components form the *inessential subgroup* of the stabilizer of $(\alpha, \mathcal{X})$, corresponding to the choice of $\alpha(t)$.

This apparently elusive notion is crucial for our tool to compute the degrees e_{ij}.

Proposition 4.12. *Let D be a component of E, let $\overline{D}$ be any component of $\overline{E}$ dominating D. Then the degree of $\overline{D}$ over D is the index of the inessential subgroup in the stabilizer of a general point of D.*

Proof. Let $(\alpha, \mathcal{X})$ be a general point of D, and let $\overline{\alpha}_1, \dots, \overline{\alpha}_r$ be its preimages in $\overline{D}$; so the degree of $\overline{D}$ over D equals r. We may assume that $\mathcal{X}$ is a limit curve of the type described in §3.2, and that $\overline{\mathbb{P}}$ is nonsingular at all $\overline{\alpha}_i$, hence for $i = 1, \dots, r$ we may choose a *marker* germ $\alpha_i(t)$ whose lift is centered at $\overline{\alpha}_i$. Let $\alpha(t) = \alpha_1(t)$. By Corollary 4.9, every $\alpha_i(t)$ is equivalent to $\alpha(t) \cdot N_i$ for some $N_i \in \mathrm{PGL}(3)$ stabilizing $(\alpha, \mathcal{X})$; conversely, if $N \in PGL(3)$ fixes $(\alpha, \mathcal{X})$ then $\alpha(t) \cdot N$ is equivalent to one of the $\alpha_i(t)$. Thus, the action $N \mapsto \overline{\alpha} \cdot N$ maps the stabilizer of $(\alpha, \mathcal{X})$ onto the fiber of $\overline{D}$ over $(\alpha, \mathcal{X})$.

Therefore we simply have to check that two elements of the stabilizer map to the same point in $\overline{E}$ if and only if they are in the same coset w.r.t. the inessential subgroup; that is, it is enough to check that $\overline{\alpha} = \overline{\alpha} \cdot N$ if and only if N is in the inessential subgroup.

First assume that N is in the inessential subgroup, that is, N is in a component containing an element M_ν as above. Since the fiber of $\overline{E}$ over $(\alpha, \mathcal{X})$ is finite, then $\overline{\alpha} \cdot N = \overline{\alpha} \cdot M_\nu$. Now $\alpha(t) \cdot M_\nu$ and $\alpha(t) \cdot M_\nu(t)$ are equivalent by Lemma 3.3; since $\alpha(t) \cdot M_\nu(t) = \alpha(t\nu(t))$, we have that $\overline{\alpha} \cdot M_\nu = \overline{\alpha}$. Thus $\overline{\alpha} \cdot N = \overline{\alpha}$ as needed.

For the converse, assume $\overline{\alpha} = \overline{\alpha} \cdot N$. By Proposition 4.5 and Lemma 4.10, if $\overline{\alpha} = \overline{\alpha} \cdot N$ then there is a $\mathbb{C}[h][[t]]$-valued point $N(h,t)$ of PGL(3) and a unit $\nu(t)$ in $\mathbb{C}[[t]]$ such that $N(0,t) = N(h,0) = I$ and

$$\alpha(t) \cdot N(1,t) = \alpha(t\nu(t)) \cdot N.$$

Now $M_\nu(t) := \alpha(t)^{-1}\alpha(t\nu(t)) = N(1,t)N^{-1}$ is a $\mathbb{C}[[t]]$-valued point of PGL(3), thus $M_\nu = N(1,0)N^{-1} = N^{-1}$ is in the inessential subgroup. This shows that N is in the inessential subgroup, completing the proof. □

4.7. Our work in §3 has produced a list of marker germs; these can be used to evaluate the multiplicities m_{ij}.

Definition 4.13. The *weight* of a germ $\alpha(t)$ is the order of vanishing in t of $F \circ \alpha(t)$.

Note that the weight of $\alpha(t)$ is the 'order of contact' of $\alpha(t)$ with S: indeed, it is the minimum intersection multiplicity of $\alpha(t)$ and generators $F \circ \varphi(x_0, y_0, z_0)$ of the ideal of S, at $\alpha(0)$.

Lemma 4.14. *The multiplicity m_{ij} is the minimum weight of a germ $\alpha(t)$ such that $\overline{\alpha} \in \overline{E}_{ij}$. This weight is achieved by a marker germ for $\overline{E}_{ij}$.*

Proof. Let $\overline{\alpha}$ be a general point of $\overline{E}_{ij}$. Since $\overline{\mathbb{P}}$ is normal, we may assume that it is nonsingular at $\overline{\alpha}$. Let $(\overline{z}) = (z_1, \dots, z_8)$ be a system of local parameters for $\overline{\mathbb{P}}$ centered at $\overline{\alpha}$, and such that the ideal of $\overline{E}_{ij}$ is (z_1) near $\overline{\alpha}$; thus the ideal of $\overline{E}$ is $(z_1^{m_{ij}})$ near $\overline{\alpha}$. Consider the germ $\overline{\alpha}(t)$ in $\overline{\mathbb{P}}$ defined by

$$\overline{\alpha}(t) = (t, 0, \dots, 0) \quad ,$$

and its push-forward $\alpha(t) = \overline{n}(\overline{\alpha}(t))$ in $\mathbb{P}^8$.

The weight of $\alpha(t)$ is the order of contact of $\alpha(t)$ with S; hence it equals the order of contact of $\overline{\alpha}(t)$ with $\overline{n}^{-1}(S) = \overline{E}$; pulling back the ideal of $\overline{E}$ to $\overline{\alpha}(t)$, we see that this equals m_{ij}.

In fact, this argument shows that the weight of any germ in $\mathbb{P}^8$ lifting to a germ in $\overline{\mathbb{P}}$ meeting the support of $\overline{E}$ *transversally* at a general point of $\overline{E}_{ij}$ is m_{ij}. It follows that the weight of *any* germ lifting to one meeting $\overline{E}_{ij}$ must be $\geq m_{ij}$, completing the proof of the first assertion.

The second assertion is immediate, as the germ $\alpha(t)$ constructed above is a marker germ for $\overline{E}_{ij}$. □

Applying Proposition 4.3 requires the list of the components $\overline{E}_{ij}$ of $\overline{E}$ dominating a given component E_i of E, and for each $\overline{E}_{ij}$ the two numbers e_{ij} and m_{ij}. These last two elements of information will be obtained by applying Proposition 4.12 and Lemma 4.14. Obtaining the list $\overline{E}_{ij}$ and a local description of $\overline{\mathbb{P}}$ requires a case-by-case analysis.

4.8. We start with components of type I in this subsection. As it happens, $\overline{\mathbb{P}} \to \widetilde{\mathbb{P}}^8$ is an isomorphism near the general point of such a component, and we can perform the multiplicity computation directly on $\widetilde{\mathbb{P}}^8$.

Proposition 4.15. *Assume $\mathcal{C}$ contains a line ℓ with multiplicity m, and let $(\alpha, \mathcal{X})$ be a general point of the corresponding component D of E. Then $\widetilde{\mathbb{P}}^8$ is nonsingular near $(\alpha, \mathcal{X})$, and D appears with multiplicity m in E.*

Proof. We are going to show that, in a neighborhood of $(\alpha, \mathcal{X})$, $\widetilde{\mathbb{P}}^8$ is isomorphic to the blow-up of $\mathbb{P}^8$ along the $\mathbb{P}^5$ of matrices whose image is contained in ℓ. The nonsingularity of $\widetilde{\mathbb{P}}^8$ near $(\alpha, \mathcal{X})$ follows from this.

Choose coordinates so that ℓ is the line $z = 0$, and the affine open set U in $\mathbb{P}^8$ with coordinates

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix}$$

contains α. The $\mathbb{P}^5$ of matrices with image contained in ℓ intersects this open set along $p_6 = p_7 = p_8 = 0$, so we can choose coordinates $q_1, \dots, q_8$ in an affine open subset V of the blow-up of $\mathbb{P}^8$ along $\mathbb{P}^5$ so that the blow-up map is given by

$$\begin{cases} p_i = q_i \quad i = 1, \dots, 5 \\ p_6 = q_6 \\ p_7 = q_6 q_7 \\ p_8 = q_6 q_8 \end{cases}$$

(the part of the blow-up over U is covered by three such open sets; it should be clear from the argument that the choice made here is immaterial).

Under the hypotheses of the statement, the ideal of $\mathcal{C}$ is generated by $z^m G(x, y, z)$, where z does not divide G; that is, $G(x, y, 0) \neq 0$. The rational map $\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$ acts on U by sending $(p_1, \dots, p_8)$ to the curve with ideal generated by

$$(p_6 x + p_7 y + p_8 z)^m G(x + p_1 y + p_2 z, p_3 x + p_4 y + p_5 z, p_6 x + p_7 y + p_8 z) \quad .$$

Composing with the blow-up map:

$$V \longrightarrow U \dashrightarrow \mathbb{P}^N$$

we find that $(q_1, \dots, q_8)$ is mapped to the curve with ideal generated by

$$(x + q_7 y + q_8 z)^m G(x + q_1 y + q_2 z, q_3 x + q_4 y + q_5 z, q_6 x + q_6 q_7 y + q_6 q_8 z) \quad ,$$

where a factor of q_6^m has been eliminated. Note that no other factor of q_6 can be extracted, by the hypothesis on G.

A coordinate verification shows that the induced map

$$V \longrightarrow U \times \mathbb{P}^N \subset \mathbb{P}^8 \times \mathbb{P}^N \quad ,$$

which clearly maps V to $\widetilde{\mathbb{P}}^8 \subset \mathbb{P}^8 \times \mathbb{P}^N$ and the exceptional divisor $q_6 = 0$ to D, is an isomorphism onto the image in a neighborhood of a general point of the exceptional divisor, proving that $\widetilde{\mathbb{P}}^8$ is nonsingular in a neighborhood of the general $(\alpha, \mathcal{X})$ in D.

For the last assertion in the statement, pull-back the generators of the ideal of S to V:

$$q_6^m (x + q_7 y + q_8 z)^m G(x + q_1 y + q_2 z, q_3 x + q_4 y + q_5 z, q_6 x + q_6 q_7 y + q_6 q_8 z) \quad ,$$

as $(x : y : z)$ ranges over $\mathbb{P}^2$. For $(x : y : z) = (1 : 0 : 0)$ this gives the generator

$$q_6^m G(1, q_3, q_6) \quad .$$

At a general $(\alpha, \mathcal{X})$ in D we may assume that $G(1, q_3, 0) \neq 0$ (again by the hypothesis on G), and we find that the ideal of E is (q_6^m) near $(\alpha, \mathcal{X})$. This shows that the multiplicity of the component is m, as stated. □

Proposition 4.15 yields the multiplicity statement concerning type I components in §4.2; also cf. Fact 2 (i) in §2 of [AF00a].

4.9. Components of type II can be analyzed almost as explicitly as components of type I, without employing the tools developed in §§4.3–4.7.

Recall from §3.15 that every nonlinear component $\mathcal{C}'$ of a curve $\mathcal{C}$ determines a component D of the exceptional divisor E.

Proposition 4.16. *Assume $\mathcal{C}$ contains a nonlinear component $\mathcal{C}'$, with multiplicity m, and let D be the corresponding component of E. Then D appears with multiplicity $2m$ in E.*

This can be proved by using the blow-ups described in [AF93], which resolve the indeterminacies of the basic rational map $\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$ over nonsingular non-inflectional points of $\mathcal{C}$. We sketch the argument here, leaving detailed verifications to the reader.

Proof. In [AF93] it is shown that two blow-ups at smooth centers suffice over nonsingular, non-inflectional points of $\mathcal{C}$. While the curve was assumed to be reduced and irreducible in loc. cit., the reader may check that the same blow-ups resolve the indeterminacies over a possibly multiple component $\mathcal{C}'$, near nonsingular, non-inflectional points of the support of $\mathcal{C}'$. Let V be the variety obtained after these two blow-ups.

Since the basic rational map is resolved by V over a general point of $\mathcal{C}'$, the inverse image of the base scheme S is locally principal in V over such points. By the universal property of blow-ups, the map $V \to \mathbb{P}^8$ factors through $\widetilde{\mathbb{P}}^8$ over a neighborhood of a general point of $\mathcal{C}'$. It may then be checked that the second exceptional divisor obtained in the sequence maps birationally onto D, and appears with a multiplicity of $2m$.

The statement follows. □

Proposition 4.16 yields the multiplicity statement concerning type II components in §4.2; also cf. Fact 2 (ii) in §2 of [AF00a].

4.10. Next, we consider components of type III. Recall that there is one such component for every singular point p of $\mathcal{C}$ at which the tangent cone to $\mathcal{C}$ consists of at least three lines, and that we have shown (cf. Propositions 3.7 and 3.11) that every marker germ $\alpha(t)$ leading to one of these components is equivalent to one which, in suitable coordinates $(x : y : z)$, may be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \quad ,$$

where p has coordinates $(1 : 0 : 0)$ and the kernel line has equation $x = 0$. Call D the component of E corresponding to one such point; note that D dominates the subset of S consisting of matrices whose image is the point $(1 : 0 : 0)$.

Proposition 4.17. *There is exactly one component $\overline{D}$ of $\overline{E}$ dominating D, and the degree of the map $\overline{D} \to D$ equals the number of linear automorphisms of the tuple determined by the tangent cone to $\mathcal{C}$ at p. The minimal weight of a germ leading to D equals the multiplicity of p on $\mathcal{C}$.*

In this statement, the *linear automorphisms* of a tuple of points in $\mathbb{P}^1$ are the elements of its PGL(2)-stabilizer; this is a finite set if and only if the tuple is supported on at least three points.

Proof. First we will verify that if $\alpha(t)$, $\beta(t)$ are marker germs leading to D, then $\overline{\alpha}$ and $\overline{\beta}$ belong to the same component of $\overline{E}$. This will show that there is only one component of $\overline{E}$ over D. By Proposition 4.5, we may replace $\alpha(t)$ and $\beta(t)$ by equivalent germs; as recalled above, in suitable coordinates we may then assume

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \quad ,$$

and $\beta(t)$ may be assumed to be in the same form after a change of coordinates. That is, we may assume

$$\beta(t) = M \cdot \alpha(t) \cdot N$$

for constant matrices M, N. As D dominates the subset of S consisting of matrices with image $(1:0:0)$, the image of $\alpha(0)$ and $\beta(0)$ is necessarily $(1:0:0)$; this implies that M is in the form

$$\begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & H & I \end{pmatrix}$$

with $A(EI - FH) \neq 0$.

Now we claim that $M \cdot \alpha(t) \cdot N$ is equivalent to $\alpha(t) \cdot N'$ for another constant matrix N'. Indeed, note that

$$M \cdot \alpha(t) \cdot N = \alpha(t) \cdot \begin{pmatrix} A & Bt & Ct \\ 0 & E & F \\ 0 & H & I \end{pmatrix} \cdot N =: \alpha(t) \cdot N'(t) \quad .$$

Writing $N' = N'(0)$ and applying Lemma 3.3, we establish the claim.

By Proposition 4.5, we may thus assume that $\beta(t) = \alpha(t) \cdot N'$. It follows that $\overline{\beta}$ and $\overline{\alpha}$ are on the same component of $\overline{E}$, by Lemma 4.7.

The degree of $\overline{D} \to D$ is evaluated by using Proposition 4.12. By the description in §3.2, the limit of $\mathcal{C}$ along a marker germ $\alpha(t)$ as above consists of a fan $\mathcal{X}$ whose star reproduces the tangent cone to $\mathcal{C}$ at p, and whose free line is supported on the kernel line $x = 0$. It is easily checked that the stabilizer of $(\alpha(0), \mathcal{X})$ has one component for each element of PGL(2) fixing the tuple determined by the tangent cone to $\mathcal{C}$ at p and that the inessential subgroup equals the identity component.

Finally, the foregoing considerations show that the minimal weight of a germ leading to D is achieved by $\alpha(t)$, and this is immediately computed to be the multiplicity of $\mathcal{C}$ at p. $\square$

By Proposition 4.3, Proposition 4.17 implies the multiplicity statement for type III components in §4.2; also cf. Fact 4 (i) in §2 of [AF00a].

4.11. Recall from §3 that components of type IV arise from certain sides of the Newton polygon determined by the choice of a point p on $\mathcal{C}$ and of a line in the tangent cone to $\mathcal{C}$ at p. If coordinates $(x:y:z)$ are chosen so that $p=(1:0:0)$, and the tangent line is the line $z=0$, then the Newton polygon (see §3.14) consists of the convex hull of the union of the positive quadrants with origin at the points (j,k) for which the coefficient of $x^iy^jz^k$ in the equation for $\mathcal{C}$ is nonzero. The part of the Newton polygon consisting of line segments with slope strictly between -1 and 0 does not depend on the choice of coordinates fixing the flag $z=0$, $p=(0,0)$. We have found (see Proposition 3.13 and ff.) that if $-b/c$ is a slope of the Newton polygon, with b, c relatively prime, then

$$\alpha(t)=\begin{pmatrix}1 & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix}$$

is a marker germ for a component of type IV if p is a singular or inflection point of the support of $\mathcal{C}$, and the limit

$$x^{\overline{q}}y^rz^q\prod_{j=1}^{S}(y^c+\rho_jx^{c-b}z^b)$$

is not supported on a conic union (possibly) the kernel line.

It is clear that germs arising from sides of Newton polygons corresponding to different lines in the tangent cone at p cannot be equivalent, so we concentrate on one side of one polygon.

Lemma 4.18. *Let D be the component of E of type IV corresponding to one side of the Newton polygon of slope $-b/c$, as above. Then there is exactly one component $\overline{D}$ of $\overline{E}$ over D corresponding to this side, and the degree of the map $\overline{D}\to D$ equals the number of automorphisms $\mathbb{A}^1\to\mathbb{A}^1$, $\rho\mapsto u\rho$ (with u a root of unity) preserving the S-tuple $\{\rho_1,\dots,\rho_S\}$.*

Proof. As in the proof of Proposition 4.17, we begin by verifying that if $\alpha(t)$, $\beta(t)$ are marker germs leading to D, then $\overline{\alpha}$ and $\overline{\beta}$ belong to the same component, by essentially the same strategy.

By Proposition 4.5, we may replace $\alpha(t)$ and $\beta(t)$ with equivalent germs; thus we may choose

$$\alpha(t)=\begin{pmatrix}1 & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix}$$

and $\beta(t)=M\cdot\alpha(t)\cdot N$ for constant invertible matrices M and N. As $\beta(t)$ leads to D, the matrix M must preserve the flag consisting of $p=(1:0:0)$ and the line $z=0$ (as this is the data which determines the Newton polygon). This implies that

$$M=\begin{pmatrix}A & B & C\\ 0 & E & F\\ 0 & 0 & I\end{pmatrix}$$

with $AEI \neq 0$. Note that

$$M \cdot \alpha(t) = \alpha(t) \cdot \begin{pmatrix} A & t^b & t^c \\ 0 & E & t^{c-b} \\ 0 & 0 & I \end{pmatrix} \quad ;$$

applying Lemma 3.3, we find that $M \cdot \alpha(t) \cdot N$ is equivalent to

$$\alpha(t) \cdot \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I \end{pmatrix} \cdot N \quad ,$$

since $c > b > 0$.

We conclude that $\beta(t)$ is equivalent to $\alpha(t) \cdot N'$ for some constant invertible matrix N', and Lemma 4.7 then implies that $\overline{\alpha}$, $\overline{\beta}$ belong to the same component of $\overline{D}$, as needed.

Next, we claim that the inessential subgroup of the stabilizer consists of the component of the identity; by Proposition 4.12, it follows that the degree of the map $\overline{D} \to D$ equals the number of components of the stabilizer of $(\alpha, \mathcal{X}) := (\alpha(0), \lim_{t\to 0} \mathcal{C} \circ \alpha(t))$.

To verify our claim, note that for all units $\nu(t)$

$$\alpha(t)^{-1} \cdot \alpha(t\nu(t)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b\nu(t)^b & 0 \\ 0 & 0 & t^c\nu(t)^c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu(t)^b & 0 \\ 0 & 0 & \nu(t)^c \end{pmatrix}$$

is a $\mathbb{C}[[t]]$-valued point of PGL(3). Thus the inessential components of the stabilizer are those containing elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu^b & 0 \\ 0 & 0 & \nu^c \end{pmatrix} \quad ,$$

for $\nu \in \mathbb{C}$, $\nu \neq 0$. This is the component containing the identity, as claimed.

The number of components of the stabilizer of $(\alpha, \mathcal{X})$ is determined as follows. The limit is

$$x^{\overline{q}} y^r z^q \prod_{j=1}^{S} (y^c + \rho_j x^{c-b} z^b) \quad ,$$

and the orbit of $(\alpha, \mathcal{X})$ has dimension 7. The degree of $\overline{D} \to D$ equals the number of components of the stabilizer of $(\alpha, \mathcal{X})$, that is, the subset of the stabilizer of $\mathcal{X}$ fixing the kernel line $x = 0$. If the orbit of $\mathcal{X}$ has dimension 7, this number equals the number of components of the stabilizer of $\mathcal{X}$, or the same number divided by 2, according to whether the kernel line is identified by $\mathcal{X}$ or not. The latter eventuality occurs precisely when $c = 2$ and $q = \overline{q}$; the stated conclusion follows then from Lemma 3.1 in [AF00b]. Analogous arguments apply when the orbit of $\mathcal{X}$ has dimension less than 7. □

In order to complete the proof of the multiplicity statement for type IV components we just need a weight computation.

Lemma 4.19. *The minimum weight of a marker germ for the component corresponding to a side of the Newton polygon with vertices* (j_0, k_0), (j_1, k_1), $j_0 < j_1$, *is*

$$\frac{j_1 k_0 - j_0 k_1}{S} \quad .$$

Proof. With notations as above, we have seen that every marker germ leading to the component is equivalent to $\alpha(t) \cdot N$, where

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad ;$$

thus the minimum weight is achieved by this germ.

The limit

$$x^{\overline{q}} y^r z^q \prod_{j=1}^{S} (y^c + \rho_j x^{c-b} z^b)$$

appears with weight $br + cq + Sbc$, so we just have to show that

$$Sbc + br + cq = \frac{j_1 k_0 - j_0 k_1}{S} \quad .$$

This is immediate, as $(j_0, k_0) = (r, q + Sb)$, $(j_1, k_1) = (r + Sc, q)$. □

The prescription for type IV components now follows from Lemmas 4.18 and 4.19, Proposition 4.3, and Lemma 4.14.

We note that the same prescription yields the correct multiplicity for type II limits as well: indeed, the side of the Newton polygon corresponding to type II limits (as in §3.15) has vertices $(0, m)$ and $(2m, 0)$, where m is the multiplicity of the corresponding nonlinear component of $\mathcal{C}$; so $S = m$ and $(j_1 k_0 - j_0 k_1)/S = 2m^2/m = 2m$, in agreement with Proposition 4.16.

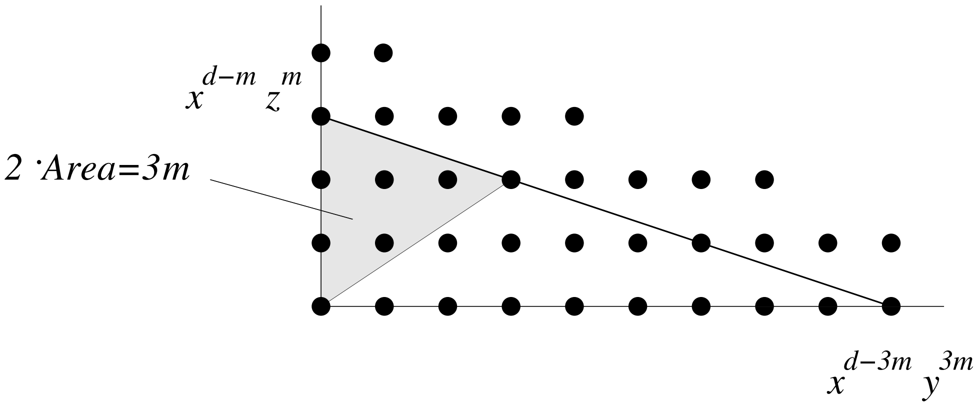
Fact 4(ii) in [AF00a], §2, reproduces the result proved here; the reader should note that the number denoted A here is denoted A/δ in loc. cit.

The weight computed in Lemma 4.19:

$$\frac{j_1 k_0 - j_0 k_1}{S} \quad ,$$

happens to equal $2/S$ times the area of the triangle with vertices $(0, 0)$, (j_0, k_0), and (j_1, k_1).

An example illustrating this for $b/c = 1/3$:



We do not have a conceptual explanation for this observation.

4.12. We are left with components of type V, whose analysis is predictably subtler. In §3 we have found that such components arise from suitable truncations of the Puiseux expansion of the branches of $\mathcal{C}$ at a singular point p of its support. Choosing coordinates $(x : y : z)$ so that $p = (1 : 0 : 0)$ and the branch has tangent cone supported on the line $z = 0$, we have found a marker germ

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

where

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

is the Puiseux expansion of a corresponding formal branch, $a < b < c$ are positive integers, $C = \frac{c}{a}$ is 'characteristic' in the sense explained in 3.2, underlining denotes truncation to t^c, and a is the smallest positive integer for which all entries in $\alpha(t)$ are polynomials. The limit obtained along this germ is:

$$x^{d-2S} \prod_{i=1}^{S} \left(zx - \frac{\lambda_0(\lambda_0 - 1)}{2} \gamma_{\lambda_0} y^2 - \frac{\lambda_0 + C}{2} \gamma_{\frac{\lambda_0+C}{2}} yx - \gamma_C^{(i)} x^2 \right) \quad ,$$

where $\gamma_C^{(i)}$ are the coefficients of y^C for all formal branches sharing the truncation $f_{(C)}(y) = \sum_{\lambda_i < C} \gamma_{\lambda_i} y^{\lambda_i}$.

As the situation is more complex than for other components, we proceed through the proof of the multiplicity statement given in §4.2 one step at the time.

4.13. Through the procedure recalled above, the choice of a characteristic C and of a truncation $f_{(C)}(y)$ of a formal branch determines a germ, and hence (by lifting to the normalization) a component of $\overline{E}$ over a fixed type V component D of E. In fact, by Proposition 3.15 and Lemma 4.7, every component $\overline{D}$ over D is marked in this fashion.

Clearly different points or different lines in the tangent cone yield different components $\overline{D}$ over D. As stated in §4.2, for a fixed point and line there are different contributions for truncations that are not 'siblings'. In other words, we must show that there is a bijection between the set of components $\overline{D}$ over D corresponding to a given point and line, and set of data $(C, f_{(C)}(y))$ as above, modulo the sibling relation. We will now recall this notion, and prove this fact in Proposition 4.21 below.

We say that $(C, f_{(C)}(y))$, $(C', g_{(C')}(y))$ (or the truncations $f_{(C)}$, $g_{(C')}$ for short) are *sibling* data if the corresponding integers $a < b < c$, $a' < b' < c'$ are the same (so in particular $C = C'$) and further

$$g_{(C)}(y) = \sum_{\lambda_i < C} \xi^{a\lambda_i} \gamma_{\lambda_i} y^{\lambda_i}$$

(that is, $f_{(C)}(t^a) = g_{(C)}((\xi t)^a)$) for an a-th root ξ of 1.

Both Proposition 4.21 and the determination of the inessential subgroup rely on the following technical lemma.

Lemma 4.20. *Let*

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \quad , \quad \beta(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^{a'} & t^{b'} & 0 \\ \underline{g(t^{a'})} & \underline{g'(t^{a'})t^{b'}} & t^{c'} \end{pmatrix}$$

be two marker germs of the type considered above, and assume that $\alpha(t)^{-1}\beta(\tau(t))$ is a $\mathbb{C}[[t]]$-valued point of $\mathrm{PGL}(3)$*, for a change of parameter $\tau(t) = t\nu(t)$ with $\nu(t) \in \mathbb{C}[[t]]$ a unit. Then $a' = a$, $b' = b$, $c' = c$, and $\nu(t) = \xi(1 + t^{b-a}\mu(t))$, where ξ is an a-th root of* 1 *and $\mu(t) \in \mathbb{C}[[t]]$; further, $\underline{g((\xi t)^a)} = \underline{f(t^a)}$.*

Proof. Write $\varphi(t) = \underline{f(t^a)}$ and $\psi(t)t^b = \underline{f'(t^a)t^b}$. The hypothesis is that

$$\alpha(t)^{-1} \cdot \beta(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\tau^{a'} - t^a}{t^b} & \frac{\tau^{b'}}{t^b} & 0 \\ \frac{\underline{\underline{g(\tau^{a'})}} - \varphi(t) - (\tau^{a'} - t^a)\psi(t)}{t^c} & \frac{\underline{\underline{g'(\tau^{a'})\tau^{b'}}} - \psi(t)\tau^{b'}}{t^c} & \frac{\tau^{c'}}{t^c} \end{pmatrix}$$

has entries in $\mathbb{C}[[t]]$, and its determinant is a unit in $\mathbb{C}[[t]]$. The latter condition implies $b' = b$ and $c' = c$. As

$$\frac{\tau^{a'} - t^a}{t^b} \in \mathbb{C}[[t]] \quad ,$$

necessarily $a' = a$ and $t^a(\nu(t)^a - 1) = (\tau^a - t^a) \equiv 0 \mod t^b$. Since $b > a$, this implies

$$\nu(t) = \xi(1 + t^{b-a}\mu(t))$$

for ξ an a-th root of 1 and $\mu(t) \in \mathbb{C}[[t]]$. Also note that since the triples (a, b, c) and (a', b', c') coincide, necessarily the dominant term in $g(y)$ has the same exponent λ_0 as in $f(y)$, since $a\lambda_0 = 2a - 2b + c$.

Now we claim that

$$g(\tau^a) - (\tau^a - t^a)\psi(t) \equiv g((\xi t)^a) \mod t^c \quad .$$

Granting this for a moment, it follows that

$$\underline{g(\tau^{a'})} - \varphi(t) - (\tau^{a'} - t^a)\psi(t) \equiv g((\xi t)^a) - f(t^a) \mod t^c \quad ;$$

hence, the fact that the $(3,1)$ entry is in $\mathbb{C}[[t]]$ implies that

$$g((\xi t)^a) \equiv f(t^a) \mod t^c \quad ,$$

which is what we need to show in order to complete the proof.

Since the $(3,2)$ entry is in $\mathbb{C}[[t]]$, necessarily

$$g'(\tau^a) \equiv \psi(t) \mod t^{c-b} \quad ;$$

so our claim is equivalent to the assertion that

$$g(\tau^a) - (\tau^a - t^a) g'(\tau^a) \equiv g((\xi t)^a) \mod t^c \quad .$$

By linearity, in order to prove this it is enough to verify the stated congruence for $g(y) = y^\lambda$, with $\lambda \geq \lambda_0$. That is, we have to verify that if $\lambda \geq \lambda_0$ then

$$\tau^{a\lambda} - (\tau^a - t^a)\lambda\tau^{a\lambda - a} \equiv (\xi t)^{a\lambda} \mod t^c \quad .$$

For this, observe

$$\tau^{a\lambda} = (\xi t)^{a\lambda}(1 + t^{b-a}\mu(t))^{a\lambda} \equiv (\xi t)^{a\lambda}(1 + a\lambda t^{b-a}\mu(t)) \mod t^{a\lambda + 2(b-a)}$$

and similarly

$$\tau^{a\lambda - a} = (\xi t)^{a\lambda - a}(1 + t^{b-a}\mu(t))^{a\lambda - a} \equiv t^{-a}(\xi t)^{a\lambda} \mod t^{a\lambda - a + (b-a)} \quad ,$$

$$(\tau^a - t^a) = (\xi t)^a (1 + t^{b-a}\mu(t))^a - t^a \equiv a t^b \mu(t) \mod t^{a + 2(b-a)} \quad .$$

Thus

$$(\tau^a - t^a)\lambda\tau^{a\lambda - a} \equiv (\xi t)^{a\lambda} a\lambda t^{b-a}\mu(t) \mod t^{a\lambda + 2(b-a)}$$

and

$$\tau^{a\lambda} - (\tau^a - t^a)\lambda\tau^{a\lambda - a} \equiv (\xi t)^{a\lambda} \mod t^{a\lambda + 2(b-a)} \quad .$$

Since

$$a\lambda + 2(b-a) \geq a\lambda_0 + 2b - 2a = c \quad ,$$

our claim follows. □

4.14. The first use of this observation is in the following result.

Proposition 4.21. *Two truncations $f_{(C)}(y)$, $g_{(C')}(y)$ determine the same component $\overline{D}$ over D if and only if they are siblings.*

Proof. Assume that $f_{(C)}(y)$, $g_{(C')}(y)$ are siblings. Then $C = C'$, and for an a-th root ξ of 1 the corresponding germs

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \quad , \quad \beta(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{g(t^a)} & \underline{g'(t^a)t^b} & t^c \end{pmatrix}$$

satisfy

$$\alpha(\xi t) = \begin{pmatrix} 1 & 0 & 0 \\ (\xi t)^a & (\xi t)^b & 0 \\ \underline{f((\xi t)^a)} & \underline{f'((\xi t)^a)(\xi t)^b} & (\xi t)^c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b\xi^b & 0 \\ \underline{g(t^a)} & \underline{g'(t^a)t^b}\xi^b & t^c\xi^c \end{pmatrix}$$

$$= \beta(t) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^b & 0 \\ 0 & 0 & \xi^c \end{pmatrix} \quad .$$

By Lemma 4.7, the lifts of $\alpha(\xi t)$ and $\beta(t)$ belong to the same component of $\overline{E}$. As $\alpha(\xi t)$ only differs from $\alpha(t)$ by a reparametrization, this shows that $\overline{\alpha}$ and $\overline{\beta}$ belong to the same component of $\overline{E}$, as needed.

For the converse, assume

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \quad , \quad \beta(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^{a'} & t^{b'} & 0 \\ \underline{g(t^{a'})} & \underline{g'(t^{a'})t^{b'}} & t^{c'} \end{pmatrix}$$

mark the same component $\overline{D}$. Since the limit of $\mathcal{C}$ along $\beta(t)$ has 7-dimensional orbit, the action of PGL(3) on $\overline{D}$ is transitive on a dense open set. Therefore, we have that $\alpha(t) \cdot N$ and $\beta(t)$ are equivalent for some $N \in \mathrm{PGL}(3)$. By Lemma 4.10, there is a $\mathbb{C}[h][[t]]$-valued point $N(h,t)$ of PGL(3) such that $\beta(t\nu(t)) = \alpha(t) \cdot N(1,t)$, for a unit $\nu(t)$. That is,

$$N(1,t) = \alpha(t)^{-1} \cdot \beta(\tau(t))$$

is a $\mathbb{C}[[t]]$-valued point of PGL(3), for $\tau(t) = t\nu(t)$. By Lemma 4.20, this implies $a' = a$, $b' = b$, $c' = c$, and $\underline{g((\xi t)^a)} = \underline{f(t^a)}$, for an a-th root ξ of 1, showing that the truncations are siblings. □

4.15. By Proposition 4.3, the multiplicity of D is a sum over the distinct sibling classes of truncations producing a given limit. Evaluating the degree of the map $\overline{D} \to D$ requires the determination of the inessential subgroup of the stabilizer, which also makes crucial use of Lemma 4.20.

Lemma 4.22. *Let*

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

be the germ determined by C and the truncation $f_{(C)}(y) = \sum_{\lambda_i < C} \gamma_{\lambda_i} y^{\lambda_i}$ as above. Then the inessential subgroup corresponding to $\alpha(t)$ consists of the components of the stabilizer of $(\alpha(0), \lim_{t\to 0} C \circ \alpha(t))$ containing matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^c \end{pmatrix}$$

with η an h-th root of 1, where h is the greatest common divisor of a and all $a\lambda_i$ ($\lambda_i < C$).

Proof. For every h-th root η of 1, any component of the stabilizer containing a diagonal matrix of the given form is in the inessential subgroup: indeed, such a diagonal matrix can be realized as $\alpha(t)^{-1} \cdot \alpha(\eta t)$.

To see that, conversely, every component of the inessential subgroup is as stated, apply Lemma 4.20 with $\beta(t) = \alpha(t)$. We find that if $\alpha(t)^{-1} \cdot \alpha(t\nu(t))$ is a $\mathbb{C}[[t]]$-valued point of PGL(3), then $\nu(t) = \eta(1 + t^{b-a}\mu(t))$, with η an a-th root of 1, and further

$$\underline{f(t^a)} = \underline{f((\eta t)^a)} \quad ,$$

that is,

$$\sum_{\lambda_i<C} \gamma_{\lambda_i} y^{\lambda_i} = \sum_{\lambda_i<C} \eta^{a\lambda_i} \gamma_{\lambda_i} y^{\lambda_i} \quad .$$

Therefore $\eta^{a\lambda_i} = 1$ for all i such that $\lambda_i < C$, and η is an h-th root of 1.

For $\nu(t) = \eta(1 + t^{b-a}\mu(t))$, the matrix $\alpha(t)^{-1} \cdot \alpha(t\nu(t))|_{t=0}$ is lower triangular, of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a\mu_0 & \eta^b & 0 \\ \gamma_{\lambda_0}\binom{\lambda_0}{2}(a\mu_0)^2 + \gamma_{\frac{\lambda_0+C}{2}}\frac{\lambda_0+C}{2}(a\mu_0) & 2\gamma_{\lambda_0}\binom{\lambda_0}{2}(a\mu_0)\eta^b & \eta^c \end{pmatrix}$$

where $\mu_0 = \mu(0)$. If a component of the stabilizer contains this matrix for some $\nu(t)$, then it must contain all such matrices for all μ_0, and in particular that component must contain the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^c \end{pmatrix} \quad ;$$

the statement follows. □

Note that $\eta^c = (\eta^b)^2$ since $c - 2b = a\lambda_0 - 2a$ is divisible by h; this is in fact a necessary condition for the diagonal matrix above to belong to the stabilizer. Moreover, if $\gamma_{\frac{\lambda_0+C}{2}} \neq 0$, then necessarily $\eta^b = 1$; as the proof of the following proposition shows, this implies $h = 1$.

Proposition 4.23. *For the component $\overline{D}$ determined by the truncation $f_{(C)}(y)$ as above, let A be the number of components of the stabilizer of the limit*

$$x^{d-2S} \prod_{i=1}^{S} \left(zx - \frac{\lambda_0(\lambda_0 - 1)}{2}\gamma_{\lambda_0} y^2 - \frac{\lambda_0 + C}{2}\gamma_{\frac{\lambda_0+C}{2}} yx - \gamma_C^{(i)} x^2 \right)$$

(that is, by [AF00b], *§4.1, twice the number of automorphisms $\gamma \to u\gamma + v$ preserving the S-tuple $\{\gamma_C^{(1)}, \dots, \gamma_C^{(S)}\}$). Then the degree of the map $\overline{D} \to D$ equals $\frac{A}{h}$, where h is the number determined in Lemma 4.22.*

Proof. As the kernel line α must be supported on the distinguished tangent of the limit $\mathcal{X}$, the stabilizer of $(\alpha, \mathcal{X})$ equals the stabilizer of $\mathcal{X}$, and in particular it consists of A components.

Next, observe that for $\eta_1 \neq \eta_2$ two h-th roots of 1, the two matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta_1^b & 0 \\ 0 & 0 & \eta_1^c \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta_2^b & 0 \\ 0 & 0 & \eta_2^c \end{pmatrix}$$

are distinct: indeed, if $\eta^b = \eta^c = 1$, then the order of η divides every exponent of every entry of $\alpha(t)$, hence it equals 1 by the minimality of a. Further, the components of the stabilizer containing these two matrices must be distinct: indeed, the description of the identity component of the stabilizer of a curve consisting of quadritangent conics given in [AF00c], §1, shows that the only diagonal matrix in the component of the identity is in fact the identity itself.

Hence the index of the inessential subgroup equals A/h, and the statement follows then from Proposition 4.12. □

4.16. All that is left is the computation of the *weight* of the marker germ

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \quad .$$

For every formal branch β of $\mathcal{C}$ at p, define an integer w_β as follows:

- if the branch is not tangent to the line $z = 0$, then $w_\beta = 1$;
- if the branch is tangent to the line $z = 0$, but does not truncate to $f_{(C)}$, then $w_\beta =$ the first exponent at which β and the truncation differ;
- if the branch truncates to $f_{(C)}$, then $w_\beta = C$.

Lemma 4.24. *The weight of $\alpha(t)$ equals aW, with $W = \sum w_\beta$.*

Proof. It is immediately checked that aw_β equals the order of vanishing in t of the composition of each formal branch with $\alpha(t)$, so $a \sum w_\beta$ equals the order of vanishing of $F \circ \alpha(t)$, which is the claim. □

We are finally ready to conclude the verification of the multiplicity statement for components of type V given in §4.2.

The upshot of the foregoing discussion is that the multiplicity equals the sum of contributions from each sibling class of truncations $f_{(C)}(y)$.

Proposition 4.25. *Let D be the component of type V determined by the choice of C and of the truncation $f_{(C)}(y)$, and let ℓ be the minimum among the positive integers μ such that $f_{(C)}(y^\mu)$ has integer exponents. Then, with notations as above, the contribution of the sibling class of $f_{(C)}(y)$ to the multiplicity of D is ℓWA.*

Proof. By Proposition 4.23 and Lemma 4.24, the sibling class of $f_{(C)}(y)$ contributes $aW\frac{A}{h}$. So all we have to prove is that $\ell = \frac{a}{h}$, with ℓ as in the statement.

For this, let λ_i, $i = 1, \dots, r$ be the exponents appearing in $f_{(C)}(y)$. If h' is any divisor of a and all $a\lambda_i$, then as $\frac{a}{h'}\lambda_i$ are integers, necessarily $\frac{a}{h'}$ is a multiple of ℓ. That is, h' divides $\frac{a}{\ell}$. On the other hand, $\frac{a}{\ell}$ is a divisor of a and all $a\lambda_i$. Hence $\frac{a}{\ell}$ equals the greatest common divisor of a and all $a\lambda_i$, which is the claim. □

Proposition 4.25 completes the proof of the multiplicity statement for type V components; also cf. [AF00a], §2, Fact 5.

This concludes the proof of the multiplicity statement given in §4.2.

5. Examples

5.1. In this final section we collect several explicit examples of limits of translates of plane curves, obtained by applying the results presented in this paper. We will describe the limits corresponding to the different components of the PNC for the curves we will consider, and marker germs for these components. We will generally pass in silence degenerate limits such as multiple lines (obtained for example as $\lim \mathcal{C} \circ \alpha(t)$, for $\alpha(0)$ a rank 1 matrix with image not contained in $\mathcal{C}$), or rank-2 limits. Limits will often be described in terms of the geometry of the curve, and representative pictures will be superimposed on the curve to emphasize this relation; of course such pictures should not be taken too literally.

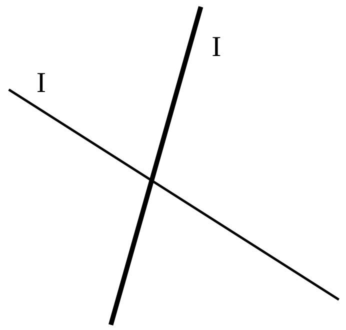
We will also compute the degrees of the orbit closures of the curves we will consider, as an illustration of the formulas in [AF00a] that we obtained as main application of the results presented here. Several other enumerative examples can be found in [AF00a]. Concerning these enumerative computations, we will freely use the terminology introduced in loc. cit., and in particular the notions of *predegree* and *adjusted predegree polynomial (a.p.p.)* (cf. §1 of [AF00a]).

5.2. Let d_1, d_2, m_1, m_2 be positive integers. Consider a curve $\mathcal{C}$ consisting of the union of two general curves $\mathcal{C}_1, \mathcal{C}_2$ of degrees $d_1 \leq d_2$, in general position and appearing with multiplicity m_1, m_2 respectively.

We distinguish three cases:

1. $1 = d_1 = d_2$;
2. $1 = d_1 < d_2$;
3. $1 < d_1 \leq d_2$.

In case (1) the curve is the union of two distinct lines with multiplicity m_1, m_2.



According to §3.2 and §4.2, the PNC consists of two components of type I, appearing with multiplicity m_1, m_2 (also cf. §3.7 and §4.8); and the limits attained by $\mathcal{C}$ are either translates of $\mathcal{C}$, or multiple lines.

Since the line $\mathcal{C}_1$ meets the rest of $\mathcal{C}$ at one point with multiplicity m_2, the contribution of $\mathcal{C}_1$ to the a.p.p. of $\mathcal{C}$ is the antiderivative w.r.t. H of

$$-\frac{m_1^3}{2}\exp(-(m_1+m_2)H)H^2\left(1+m_2H+\frac{m_2^2H^2}{2}\right)$$

(Proposition 3.1 in [AF00a]); and similarly for $\mathcal{C}_2$. The contribution of both together is therefore

$$-\frac{(m_1^3+m_2^3)H^3}{6}+\frac{(m_1^4+m_2^4)H^4}{8}-\frac{(m_1^5+m_2^5)H^5}{20}+\frac{(m_1^3+m_2^3)^2H^6}{72}-\dots \quad ,$$

so the a.p.p. of $\mathcal{C}$ is (Proposition 1.1 in [AF00a])

$$\exp((m_1+m_2)H)\left(1-\frac{(m_1^3+m_2^3)H^3}{6}+\frac{(m_1^4+m_2^4)H^4}{8}-\frac{(m_1^5+m_2^5)H^5}{20}+\dots\right)$$

$$=1+(m_1+m_2)H+\frac{(m_1+m_2)^2}{2}H^2+\frac{m_1m_2(m_1+m_2)}{2}H^3+\frac{m_1^2m_2^2}{4}H^4$$

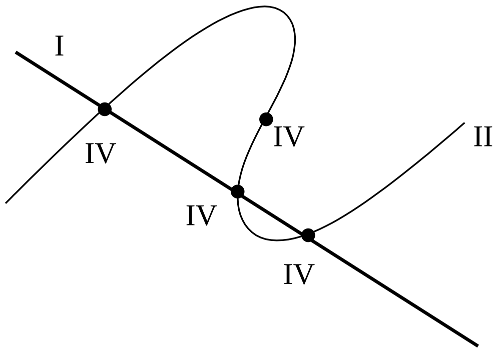
$$= \left(1 + m_1 H + \frac{m_1^2 H^2}{2}\right)\left(1 + m_2 H + \frac{m_2^2 H^2}{2}\right) \quad .$$

It follows (§1 in [AF00a]) that the orbit closure of $\mathcal{C}$ has dimension 4, and predegree $4!\frac{m_1^2 m_2^2}{4}$; that is, degree

$$\begin{cases} 6m_1^2 m_2^2 & \text{if } m_1 \neq m_2 \\ 3m^4 & \text{if } m_1 = m_2 = m \end{cases}$$

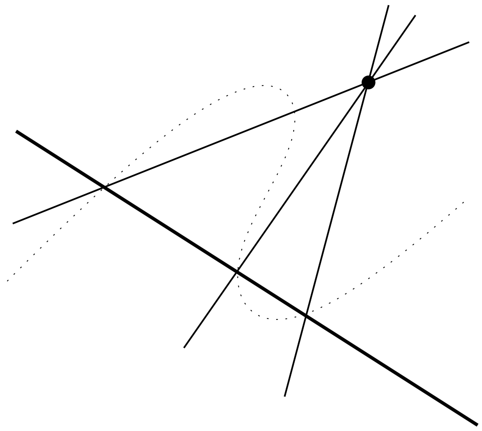
This of course agrees with the naive dimension count, and multiplicity and combinatorial considerations.

In case (2) the curve is the transversal union of a line, with multiplicity m_1, and a general nonsingular curve of degree d_2, with multiplicity m_2.



According to §3.2 and §4.2 the PNC has one component of type I, with multiplicity m_1, one component of type II, with multiplicity $2m_2$, and several 'local components' of type IV: one for each of the $3d_2(d_2 - 2)$ ordinary flexes of $\mathcal{C}_2$, and one for each of the d_2 points of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Limits corresponding to the type I component are obtained as $\lim \mathcal{C} \circ \alpha(t)$ for $\alpha(0)$ a rank-2 matrix whose image is the line $\mathcal{C}_1$. Such limits are fans determined by the intersection of $\mathcal{C}_1$ with the rest of the curve:

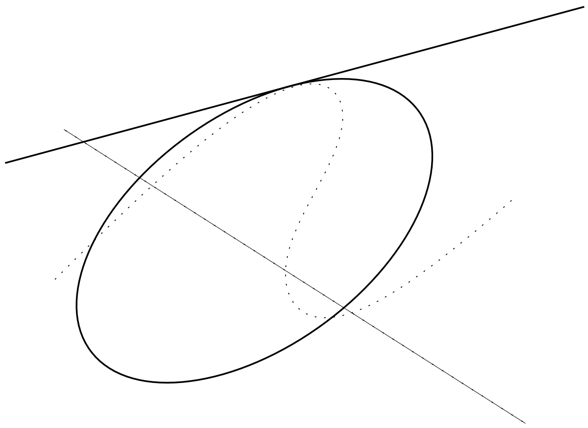


The multiplicity of the non-concurrent line in the fan is m_1, and the multiplicities of the star lines all equal m_2.

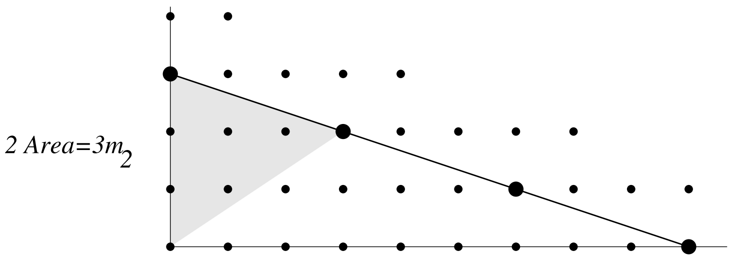
The type II component can be marked by a 1-PS with weights $(1,2)$:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}$$

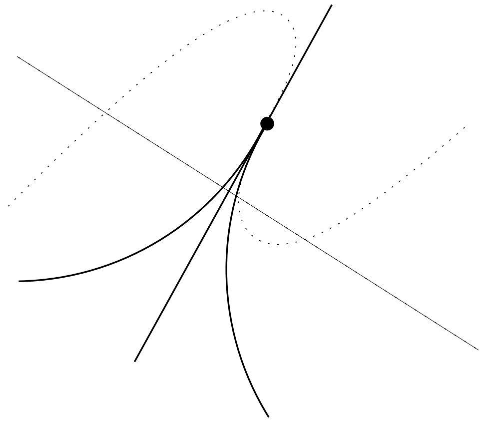
if $p = (1:0:0)$ is a general point of $\mathcal{C}_2$, and the highest weight line $z = 0$ is the tangent line to $\mathcal{C}_2$ at p (cf. §3.15). The corresponding limit consists of a conic, with multiplicity m_2, and tangent to the kernel line, union the kernel line with multiplicity $m_1 + m_2(d_2 - 2)$:



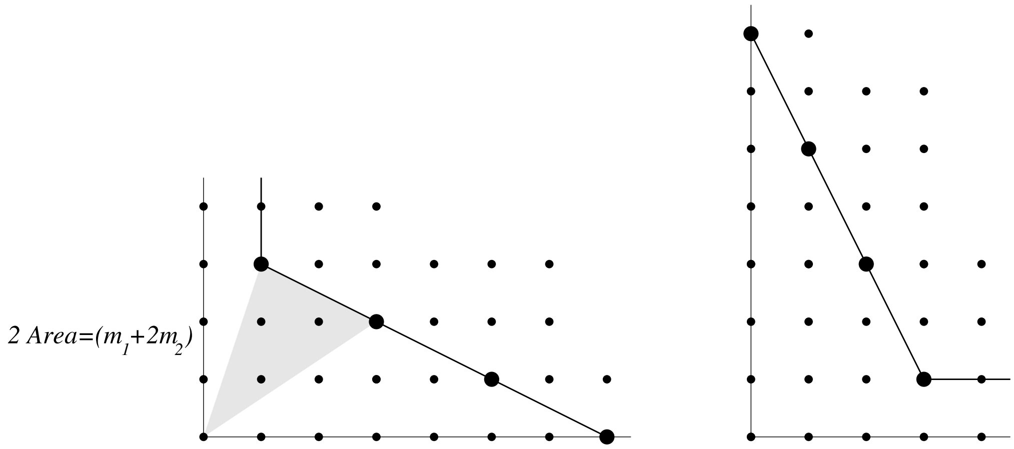
At each inflection point the relevant side of the Newton polygon is



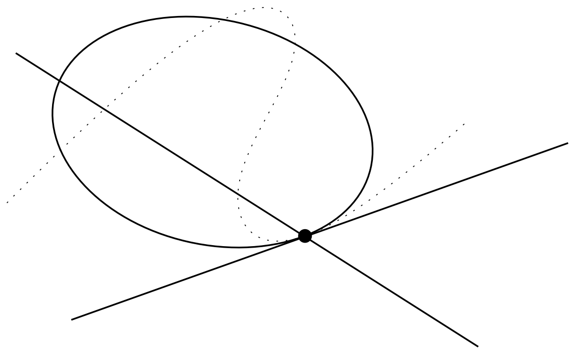
joining $(0, m_2)$ and $(3m_2, 0)$; according to §4.2 such type IV components appear with multiplicity $3m_2$ (cf. §4.11). Marker germs for one of these components can be chosen to be 1-PS $\alpha(t)$ with weights $(1,3)$, image of $\alpha(0)$ equal to the flex, and highest weight line on the inflectional tangent (§3.14). The limits consist of a cuspidal cubic with multiplicity m_2 and cuspidal tangent on the kernel line, union the kernel line with multiplicity $m_1 + m_2(d_2 - 3)$:



The d_2 points of intersection are nodes, with multiple branches, and one of whose branches is supported on a line. The Newton polygons corresponding to the two tangent directions are



and only one side (joining (m_1, m_2) and $(m_1 + 2m_2, 0)$) has slope strictly between -1 and 0. So each of these points contributes one component of type IV, appearing with multiplicity $m_1 + 2m_2$. This component is marked by a 1-PS $\alpha(t)$ with weights $(1, 2)$ with $\operatorname{im}\alpha(0)$ an intersection point, and highest weight line tangent to the non-linear branch. Limits consist of a conic with multiplicity m_2 and tangent to the kernel line, the kernel line with multiplicity $m_2(d_2 - 2)$, and a transversal line through the point of intersection, with multiplicity m_1:



No components of type III appear, since the tangent cone at each point of type $\mathcal{C}$ is supported on ≤ 2 lines. The curve $\mathcal{C}$ has no characteristics at its singularities, so the PNC has no components of type V in this case, cf. again §3.2.

Proposition 3.1 in [AF00a] gives the contribution due to the type I component as the antiderivative of

$$-\frac{m_1^3}{2}\exp(-(m_1+m_2d_2)H)H^2\left(1+m_2H+\frac{m_2^2H^2}{2}\right)^{d_2} \quad :$$

indeed, the line meets $\mathcal{C}_2$ at d_2 points, each with multiplicity m_2. Explicitly:

$$-\frac{m_1^3H^3}{6}+\frac{m_1^4H^4}{8}-\frac{m_1^5H^5}{20}+\frac{m_1^3(m_1^3+m_2^3d_2)H^6}{72}$$
$$-\frac{m_1^3(m_1^4+4m_1m_2^3d_2+3m_2^4d_2)H^7}{336}+\frac{m_1^3(m_1^5+10m_1^2m_2^3d_2+15m_1m_2^4d_2+6m_2^5d_2)H^8}{1920}$$

As for the type II component, its contribution is

$$-2m_2^5d_2\left(\frac{H^5}{20}-\frac{(5(m_1+m_2d_2)+18m_2)H^6}{360}+\frac{(9(m_1+m_2d_2)+8m_2)m_2H^7}{420}\right.$$
$$\left.-\frac{(m_1+m_2d_2)m_2^2H^8}{60}\right)$$

according to Proposition 3.2 in [AF00a].

'Local contributions' from type IV components are evaluated using Proposition 3.4 in [AF00a]. The data needed in order to apply this formula consists of the vertices of the corresponding side of the Newton polygon, and the multiplicities of the curvilinear components in the limit. We obtain a contribution of

$$-\frac{m_2^6H^6}{48}+\frac{3m_2^7H^7}{70}-\frac{197m_2^8H^8}{4480}$$

from each of the $3d_2(d_2-2)$ inflection points, and of

$$m_1m_2^3(m_1+2m_2)\left(-\frac{(m_1+m_2)H^6}{72}+\frac{(20m_1^2+45m_1m_2+36m_2^2)H^7}{1680}\right.$$
$$\left.-\frac{(10m_1^3+35m_1^2m_2+48m_1m_2^2+32m_2^3)H^8}{1920}\right)$$

from each of the d_2 nodes. Combining all these contributions and applying Proposition 1.1 in [AF00a] yields the a.p.p. for $\mathcal{C}$. The coefficient of H^8 in this polynomial, multiplied by 8!, gives the predegree of $\mathcal{C}$ for $d_2 > 2$:

$$\begin{aligned}(d_2-2)d_2m_2^6\bigl(28(d_2^4+2d_2^3+4d_2^2-22d_2-33)m_1^2 \\ +8(d_2^5+2d_2^4+4d_2^3+8d_2^2-411d_2+744)m_1m_2 \\ +(d_2^6+2d_2^5+4d_2^4+8d_2^3-1356d_2^2+5280d_2-5319)m_2^2\bigr)\end{aligned}$$

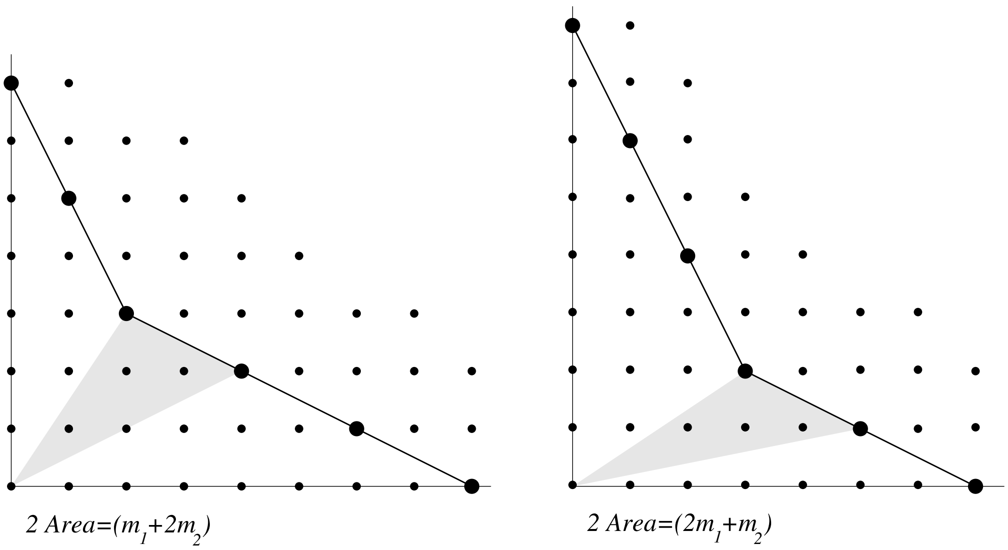
This expression vanishes for $d_2 = 2$ because in that case the orbit closure has dimension < 8; the predegree can then be computed from the coefficient of H^7 in the a.p.p., giving $84m_1^2m_2^5$. Accounting for the stabilizer, this gives $21m_1^2m_2^5$ for the *degree* of the orbit closure in this case, again agreeing with naive combinatorial considerations.

For $d_2 > 2$ and $m_1 = m_2 = 1$ the expression given above counts the number of configurations containing 8 points in general position. The individual sub-expressions in this formula can also be given a concrete enumerative interpretation. For example,

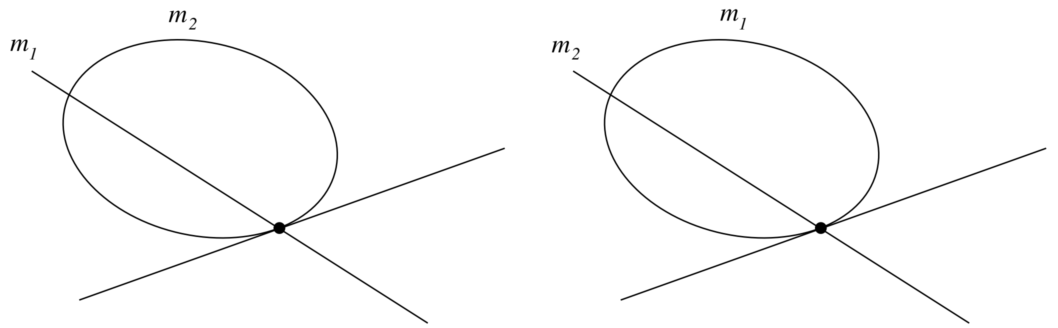
$$(d_2-2)d_2\left(d_2^4+2d_2^3+4d_2^2-22d_2-33\right) = d_2^6-30d_2^3+11d_2^2+66d_2$$

is (for $d_2 > 2$) the number of embeddings of a given general plane curve of degree d_2 containing 6 general points, and satisfying the constraint of having a given general section of $\mathcal{O}(1)$ contained in a prescribed line.

In case (3), $\mathcal{C}$ is the union of two general curves of degrees ≥ 2, in general position. The discussion is analogous to that given in case (2); in this case the PNC will have no component of type I, but a second component of type II, with multiplicity $2m_1$; and there will be $3d_1(d_1-2)$ new components of type IV corresponding to the flexes on $\mathcal{C}_1$, each with multiplicity $3m_1$. A new phenomenon concerns the components of type IV due to the points of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$. At such points the relevant Newton polygons:



have *two* sides with slope between -1 and 0 (these will join the points (m_1, m_2) and $(m_1+2m_2, 0)$, respectively (m_2, m_1) and $(2m_1+m_2, 0)$). Thus, each of these d_1d_2 points can potentially contribute two components to the PNC. The components are marked, as above, by 1-PS germs $\alpha(t)$ with weights $(1,2)$, $\operatorname{im}\alpha(0)$ the intersection point, and highest weight line tangent to one of the branches. The corresponding limits are schematically represented by



indicating multiplicities. The components are distinct if and only if $m_1 \neq m_2$ (according to §3.2). If $m_1 = m_2 = m$, the single type IV component determined by such a point has multiplicity $6m$ in the PNC.

The a.p.p. for $\mathcal{C}$ can be determined by using the results in [AF00a], similarly to case (2), using the Newton polygon data listed above. The predegree of $\mathcal{C}$ turns out to be

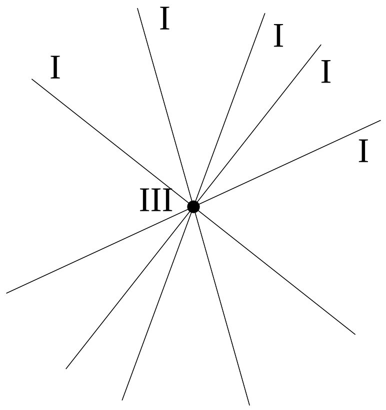
$$\begin{aligned}(m_1d_1+m_2d_2)^8 &- 28(m_1d_1+m_2d_2)^2(49d_1^2m_1^6+24d_1d_2m_1^5m_2+30d_1d_2m_1^4m_2^2+20d_1d_2m_1^3m_2^3\\ &+ 30d_1d_2m_1^2m_2^4 + 24d_1d_2m_1m_2^5 + 49d_2^2m_2^6) + 72(d_1m_1+d_2m_2)(111d_1^2m_1^7 + 63d_1d_2m_1^6m_2\\ &+ 42d_1d_2m_1^5m_2^2 + 35d_1d_2m_1^4m_2^3 + 35d_1d_2m_1^3m_2^4 + 42d_1d_2m_1^2m_2^5 + 63d_1d_2m_1m_2^6 + 111d_2^2m_2^7)\\ &- (15879d_1^2m_1^8 + 11904d_1d_2m_1^7m_2 + 2688d_1d_2m_1^6m_2^2 + 2688d_1d_2m_1^5m_2^3 + 2310d_1d_2m_1^4m_2^4\\ &+2688d_1d_2m_1^3m_2^5+2688d_1d_2m_1^2m_2^6+11904d_1d_2m_1m_2^7+15879d_2^2m_2^8)+10638(d_1m_1^8+d_2m_2^8).\end{aligned}$$

In the reduced case, let $d = d_1 + d_2$ be the degree of the curve and $n = d_1d_2$ be the number of points of intersection; then this formula evaluates the predegree of $\mathcal{C}$ as

$$d^8 - 1372d^4 + 7992d^3 - 15879d^2 + 10638d - 24n(35d^2 - 174d + 213) \quad .$$

In fact, this is the predegree of a general plane curve of degree d with n ordinary nodes, cf. Example 4.1 in [AF00a].

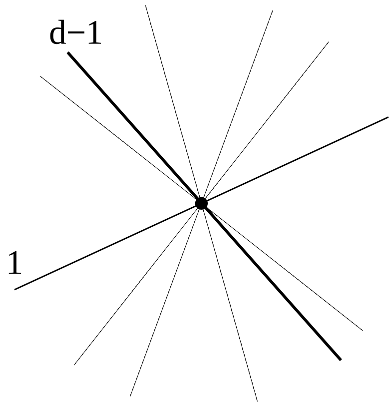
5.3. Let $\mathcal{C}$ be a star consisting of $d \geq 3$ distinct reduced lines through a point:



and let A be the number of automorphisms of the tuple of lines. Thus $A = 6$ for $d = 3$, $A = 4$ for general stars with $d = 4$, and $A = 1$ for general stars with $d > 4$. According to §3.2 and §4.2, the PNC for this curve has one reduced component of type I for each line, and one component of type III, appearing with multiplicity dA. If coordinates are chosen so that one of the lines is $z = 0$, the germ

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}$$

marks the corresponding type I component. In this case the limit consists of a pair of lines, with multiplicities 1 and $d-1$ respectively (Proposition 3.6):



The component of type III is marked by 1-PS with image the multiple point, and equal weights (Proposition 3.11):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$$

if the center of the star is at $(1 : 0 : 0)$. This example is somewhat atypical, in that *these limits are nothing but translates of the original curve.*

Proposition 3.1 in [AF00a] can again be invoked to evaluate the (additive) contribution to the a.p.p. due to the d type I components, as the antiderivative of

$$-\frac{d}{2}\exp(-dH)H^2\left(1 + (d-1)H + \frac{(d-1)^2H^2}{2}\right) \quad .$$

Proposition 3.3 in loc.cit. evaluates the contribution due to the type III component:

$$-\frac{d^2(d-1)(d-2)(d^2+3d-3)}{30}\left(\frac{H^6}{24} - \frac{dH^7}{28} + \frac{d^2H^8}{64}\right) \quad .$$

Note that the factor of A appearing in the multiplicity of the type III component is absorbed by other factors in computing this contribution, so that the result does not depend on A after all. Also note that both contributions from type I and III have nonzero coefficients for H^6, H^7, and H^8; however, the a.p.p. of $\mathcal{C}$ must have degree 5, because the dimension of the orbit closure of $\mathcal{C}$ has dimension 5. This means that cancellations must occur in the computation of the a.p.p., and indeed, applying Proposition 1.1 from [AF00a] gives

$$1 + dH + \frac{d^2H^2}{2} + \frac{(d-1)d(d+1)H^3}{3!} + \frac{(d-1)d(d^2+d-3)H^4}{4!} + \frac{(d-2)(d-1)d(d^2+3d-3)H^5}{5!} \quad .$$

The conclusion is that the degree of the orbit closure of $\mathcal{C}$ is

$$\frac{(d-2)(d-1)d(d^2+3d-3)}{A} \quad .$$

In fact, the a.p.p. for $\mathcal{C}$ is seen to equal the truncation to H^5 of the power

$$\left(1 + H + \frac{H^2}{2}\right)^d \quad ,$$

a phenomenon that can be explained by 'multiplicativity' considerations such as those presented in §4.2 of [AF00a]. Theorem 2.5 (i) in [AF00c] shows how to modify this statement in order to account for possible multiplicities of the lines in the star.

5.4. As a final example, we consider the curve $\mathcal{C}$ of degree 7 from §2.2 with equation

$$x^3z^4 - 2x^2y^3z^2 + xy^6 - 4xy^5z - y^7 = 0.$$

Without difficulty, one finds that $\mathcal{C}$ has three singularities, at $P = (1 : 0 : 0)$, $Q = (0 : 0 : 1)$, and $R = (1 : -4 : -8)$. One sees that P and Q are irreducible singularities, while R is an ordinary node. It follows that $\mathcal{C}$ is irreducible. Thus the PNC has one global component of type II, with multiplicity 2. Clearly, there are no components of type III.

To describe the local components of the PNC, a closer analysis of the singularities of $\mathcal{C}$ is required. We begin at $P = (1 : 0 : 0)$. It turns out that $\mathcal{C}$ has a very simple Puiseux expansion there:

$$\begin{cases} z = t^6 + t^7, \\ y = t^4. \end{cases}$$

(In particular, $\mathcal{C}$ is a rational curve.) The singularity has two Puiseux pairs, $(2, 3)$ and $(2, 7)$. In the notation of §5 of [AF00a]: $m = 4$, $n = e_1 = 6$, $d_1 = 2$, $e_2 = 7$, $d_2 = 1$, $r = 2$, and the singularity absorbs 55 flexes.

The singularity $Q = (0 : 0 : 1)$ has one Puiseux pair $(3, 7)$. In the notation of loc. cit., $m = 3$, $n = e_1 = 7$, $d_1 = 1$, $r = 1$, and the singularity absorbs 43 flexes. The ordinary node R absorbs at least 6 flexes, which leaves at most 1 flex from the total number of $3d(d-2) = 105$ flexes. It turns out that R is not a flecnode and that the point $F = (7^7, 2^8 7^3, 2^{12} 3)$ is a simple flex. (In the given parametrization, F corresponds to $t = -4/7$ and R to $t = -1 \pm i$.)

At the simple flex F, the relevant side of the Newton polygon joins the points $(0, 1)$ and $(3, 0)$. The corresponding type IV component appears with multiplicity 3. It is marked by a 1-PS $\alpha(t)$ with weights $(1, 3)$, $\operatorname{im} \alpha(0) = F$, and highest weight line the tangent line to $\mathcal{C}$ at F.

At the ordinary node R, the two lines in the tangent cone both yield a side of the Newton polygon joining the points $(1,1)$ and $(3,0)$. The corresponding type IV component appears with multiplicity $3+3=6$. It is marked by 1-PS $\alpha(t)$ with weights $(1,2)$, $\operatorname{im}\alpha(0)=R$, and highest weight line one of the two tangent lines.

At the singular point $Q=(0:0:1)$, the relevant side of the Newton polygon joins the points $(0,3)$ and $(7,0)$. The corresponding type IV component appears with multiplicity 21 and is marked by the 1-PS

$$\alpha(t)=\begin{pmatrix} t^7 & 0 & 0\\ 0 & t^3 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The limit curve has equation $x^3z^4=y^7$. The 3 branches of $\mathcal{C}$ at Q do not possess a characteristic C, so there isn't a component of type V here.

Finally, consider the singular point $P=(1:0:0)$. The relevant side of the Newton polygon joins the points $(0,4)$, $(3,2)$, and $(6,0)$. The corresponding type IV component is marked by the 1-PS

$$\alpha(t)=\begin{pmatrix} 1 & 0 & 0\\ 0 & t^2 & 0\\ 0 & 0 & t^3\end{pmatrix}.$$

The limit curve has equation $x^3z^4-2x^2y^3z^2+xy^6=x(y^3-xz^2)^2=0$ (a double cuspidal cubic together with its unique inflectional tangent, which equals the kernel line). Thus the type IV component appears with multiplicity 12.

In Example 3.27 we obtained the components of type V due to P. The two truncations $y^{3/2}$ and $-y^{3/2}$ are siblings (cf. Example 4.2; take a primitive 8th root of 1 for ξ). Thus we get a single contribution. We have $\ell=2$, $W=2\cdot\frac{3}{2}+2\cdot\frac{7}{4}=\frac{13}{2}$, and $A=4$. Hence the multiplicity of this component equals 52.

We conclude by computing the a.p.p. for $\mathcal{C}$. The contribution of the type II component is

$$-\frac{7}{10}H^5+\frac{371}{180}H^6-\frac{71}{30}H^7+\frac{49}{30}H^8,$$

the contribution of the type IV component due to the flex F is

$$-\frac{1}{48}H^6+\frac{3}{70}H^7-\frac{197}{4480}H^8,$$

and the contribution of the type IV component due to the node R is

$$-\frac{1}{6}H^6+\frac{101}{280}H^7-\frac{25}{64}H^8;$$

all three are special cases of formulas stated earlier in this section.

The contributions of the irreducible singularities P and Q are evaluated using Theorem 5.1 in [AF00a]. In the notation used there, the (additive) contribution of P is

$$-\left\{(24P(4,6)+2P(2,4))\cdot\left(\frac{k^2H^6}{6!}+\frac{kH^7}{7!}+\frac{H^8}{8!}\right)\right\}_2=-\frac{577}{30}H^6+\frac{5779}{70}H^7-\frac{6353}{35}H^8,$$

while the contribution of Q equals

$$-\left\{21P(3,7)\cdot\left(\frac{k^2H^6}{6!}+\frac{kH^7}{7!}+\frac{H^8}{8!}\right)\right\}_2=-\frac{3059}{240}H^6+\frac{2199}{40}H^7-\frac{15775}{128}H^8$$

(note $P(1,2)=0$).

The a.p.p. for $\mathcal{C}$ equals therefore the truncation to H^8 of

$$\exp(7H)\cdot\left(1-\frac{7}{10}H^5-\frac{5419}{180}H^6+\frac{56939}{420}H^7-\frac{509977}{1680}H^8\right),$$

that is,

$$1+7H+\frac{49}{2}H^2+\frac{343}{6}H^3+\frac{2401}{24}H^4+\frac{16723}{120}H^5+\frac{6163}{48}H^6+\frac{119417}{1680}H^7+\frac{145139}{13440}H^8.$$

Since P, Q, R, and F form a frame, $\mathcal{C}$ has trivial stabilizer. Therefore the degree of its orbit closure equals

$$8!\cdot\frac{145139}{13440}=435417.$$

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