revised July 17, 1995

Jacobian of the Picard Curve

J. R. Quine

1. Introduction

This paper deals with the problem of finding the compact genus three Riemann Surface such that the minimum non-zero norm of vectors in the Jacobian is maximal. The problem was suggested by the paper of Buser and Sarnak [BS]. At that time it was thought that Klein's surface $x^3y + y^3z + z^3x = 0$ would be the extremal surface. The purpose of this paper is to show that the Picard curve $y^3z = x^4 - z^4$ is the more likely extremal. It's minimum norm is $1 + 1/\sqrt{3}$ which is larger than $4/\sqrt{7}$ which is the norm for Klein's curve.

The above question is part of a more general systole problem as discussed in Gromov [G], which includes the classical Minkowski problem for lattices. In fact, the Jacobian of a genus g surface can be thought of as a $2g$ dimensional lattice, and so we are concerned with Minkowski's problem for this particular subset. Other classes of lattices that can be studied are the symplectic lattices [BS] and isodual lattices [CS2].

Systoles for the length spectrum were studied by P. Schmutz [S]. Of the curves that he studied, the genus three curve with the largest minimum non-zero length in the length spectrum is a curve he calls $M(3)$. It is the Picard curve mentioned above, and this led us to investigate the Jacobian of this curve. For extremal surfaces studied so far in genus less than or equal to 3, systoles for the length spectrum and for the Jacobian are in one-to-one correspondence [Q]. As we show below, this is the case for the Picard curve.

We show that the Jacobian of the Picard curve is the isodual lattice $M(E_6)$ discovered by Conway and Sloane [CS2]. Our analysis shows not only that the lattice is a Jacobian, but also gives an interesting cyclotomic construction of it. This type of construction of extremal lattices has its origins in the work of M. Craig [C]. Algebraic constructions of lattices are discussed in Chapter 8 of the book of Conway and Sloane [CS1].

We use standard techniques to compute the Jacobian of a surface from the automorphism group ([RR],[RL],[E],[B]). The action of the group on the cycles and holomorphic 1-forms can be given matrix representations. These give automorphisms of the symplectic

Jacobian lattice, and if the group is large enough, the lattice can be determined. Sometimes the lattice can be determined from the action on cycles alone, but it is helpful to use the action on forms also.

The Picard curve discussed here is part of a family of Picard curves $y^3 = p(x)$ where p is a quartic with distinct roots. This family is studied in the book of Holzapfel [H]. The choice of $p(x) = x^4 - 1$ gives the maximum number of automorphisms, 48, and we will refer to this curve in the family as the Picard curve.

Further investigations [QZ] have shown that the Picard curve gives a local maximum for symplectic lattices. It is not known whether it is the extremal genus 3 symplectic lattice or Jacobian. Natural candidates for extremals are surfaces with large automorphism groups. Tables for genus 3 ([K],[V]) indicate three likely extremals, Klein's curve with automorphism group of order 168, Fermat's $(y^4 + x^4 + z^4)$ with order 96, and the Picard curve with order 48. It can be shown that the minimum norm for Fermat's curve is 3/2, so the Picard curve is the best of these. As we can see by looking at these examples, however, the relationship between the order of the automorphism group and extremality of the Jacobian is not clear.

2. The Picard curve

We describe the hyperbolic geometry of the Picard curve. In the hyperbolic plane, the Poincaré disk model, take a triangle with angles $\pi/3, \pi/4$, and $\pi/12$. We will refer to this as a (3,4,12) triangle. Place it with the $\pi/12$ angle at the origin and reflect about the origin to get 24 copies. A 24 sided polygon with interior angles alternating $2\pi/3$ and $\pi/12$ is obtained. Figure 1 shows the polygon, where each (3,4,12) triangle is divided into 4 (2,3,12) triangles. This gives the fundamental region for the Picard curve. Number the sides from 0 to 23 as shown. Taking side pairing maps that identify each odd side k with the side $k + 7$ gives a genus 3 surface, the Picard surface, which we will denote S_P.

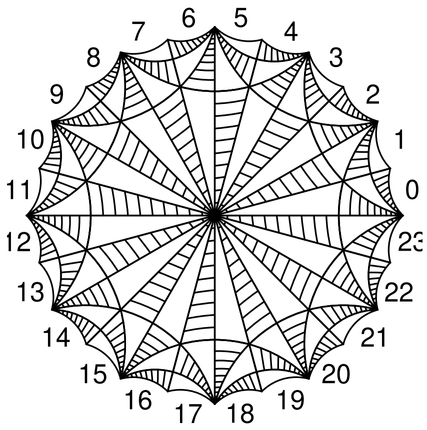

Figure 1

There are many possible equations for S_P. We can conformally map the fundamental region above onto the compact Riemann surface whose affine equation is $y^3 = x^4 - 1$. Take a (3,4,12) triangle oriented so that we go around it counterclockwise as we go from the $\pi/3$ to $\pi/4$ to $\pi/12$ vertex. Map it conformally onto the sector $-\pi/4 \leq \arg x \leq 0$ sending the $\pi/3$ vertex to 1, the $\pi/4$ vertex to 0 and the $\pi/12$ vertex to ∞. Note that after this mapping, the angles at the $\pi/3$ and $\pi/12$ vertices are tripled while the angle at the $\pi/4$ vertex remains the same. Considering the sector as part of one sheet of the surface $y^3 = x^4 - 1$, we see that the map can be extended by reflection to a map from the fundamental region to the branched covering surface of the x plane (considered as the Riemann sphere) defined by $y^3 = x^4 - 1$. It can be seen that the side pairing maps are preserved, so that the mapping extends to a conformal mapping from S_P to the three sheeted covering surface.

The curve also has equation $y^4 = x^4 - x$. The argument is similar to the one above except that we send the $\pi/3$ vertex to ∞, the $\pi/12$ vertex to 0 and the $\pi/4$ vertex to 1. In this case, the angles at the $\pi/4$ and $\pi/12$ vertices are multiplied by 4 and the angle at the $\pi/3$ vertex remains the same.

3. The automorphism group of the Picard curve

From the above description and from figure 1, we see that the side pairing maps are elements of the oriented triangle group $T(2,3,12)$. We take the generators of the triangle group to be T, the rotation about the origin by an angle $\pi/6$, and U, the rotation about the $\pi/2$ vertex on the positive real axis an angle π. The relations are $T^{12} = U^2 = (TU)^3 = I$ Let A_k be the side pairing map sending side $2k-1$ to side $2k+6$. We take the subscripts on the maps to be mod 12 and the sides to be mod 24. We can compute from the figure that

$$A_k = T^k(T^3UT^9U)T^{12-k}.$$

The automorphism group of S_P can be described in terms of $T(2,3,12)$. Let Γ be the group of hyperbolic isometries generated by the side pairing maps A_k. Then $S_P = \Gamma\backslash\mathbb{H}$ where $\mathbb{H}$ is the hyperbolic plane. One can check using the generators and relations above that Γ is a normal subgroup of $T(2,3,12)$. Thus the automorphism group is $T(2,3,12)/\Gamma$, a group of order 48. The cosets can be identified with the shaded triangles in figure 1.

It is helpful to have the automorphism group in terms of an equation of S_P. We use the equation $y^4 = x^4 - x$, or in projective coordinates, $y^4 = x^4 - xz^3$. The mapping above from the fundamental region onto the curve can be chosen so that the automorphisms take the form

$$T(x,y,z) = (\zeta^4 x, \zeta y, z) \qquad U(x,y,z) = (z-x, \sqrt{3}y, z+2x). \tag{1}$$

4. Jacobians

We recall the definition of the Jacobian. Let $\Omega(S)$ be the holomorphic 1-forms on S, or differentials of the first kind. It is a complex vector space of dimension g. Let $\Omega^*(S)$ be its complex dual. Since cycles in $H_1(S,\mathbb{Z})$ act complex linearly on $\Omega(S)$ by integration, we may consider $H_1(S,\mathbb{Z})$ as a subset of $\Omega^*(S)$, i.e., identify a cycle γ with $\int_\gamma$. A Hermitian inner product is defined on $\Omega(S)$ by

$$\langle\omega,\sigma\rangle = \frac{i}{2}\int_S \omega\wedge\overline{\sigma}\,.$$

By duality this defines a Hermitian inner product $\langle\ ,\ \rangle$ on $\Omega^*(S)$, and $H_1(S,\mathbb{Z})$ is a $2g$ dimensional sublattice of $\Omega^*(S)$.

For a more explicit description of the lattice, suppose that $\{\omega_1,\ldots,\omega_g\}$ is a basis for $\Omega(S)$, and let $\{\omega_1^*,\ldots,\omega_g^*\}$ be the dual basis. Suppose that $\{\gamma_1,\ldots,\gamma_n\}$ generates $H_1(S,\mathbb{Z})$. Then we have

$$\int_{\gamma_k} = \sum_j \left[\int_{\gamma_k} \omega_g\right]\omega_j^*$$

The matrix of coefficients

$$\Lambda = \left(\int_{\gamma_k} \omega_j\right)$$

is called a period matrix with respect to the above choices of generators. It is uniquely determined if $\{\gamma_1,\ldots,\gamma_n\}$ is a basis. Let $G = (\langle\omega_j,\omega_k\rangle)$ be the matrix of inner products (gram matrix) with respect to the given basis. The gram matrix for the dual basis is G^{-1} and the gram matrix of $H_1(S,\mathbb{Z})$ with respect to the generators $\int_{\gamma_k}$ is

$$\Lambda^* G^{-1} \Lambda. \tag{2}$$

In the case that the γ_k form a canonical basis, we write $\gamma_k = a_k$ for $k = 1\ldots g$ and $\gamma_k = b_k$ for $k = g+1\ldots 2g$. If we choose the basis $\{\omega_k\}$ to be dual to the a cycles and define a matrix Z by

$$\int_{b_k} \omega_j = z_{jk}$$

then the bilinear relations ([FK]) tell us that Z is symmetric with positive definite imaginary part. Writing $Z = X + iY$, the bilinear relations also tell us that $G = Y$. Thus we have $\Lambda = (I, Z)$. and by (2) we have that the gram matrix of the cycles γ_k is $P + iJ$ where

$$P = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}\begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It follows that the real part of the gram matrix has determinant 1, and the imaginary part of the Hermitian inner product is the intersection form on $H_1(S,\mathbb{Z})$, thus skew symmetric and integer valued. A quadratic form of determinant 1 which comes as the real part of a Hermitian form whose imaginary part is skew symmetric, non-singular, and integer valued, is called ***symplectic***. Likewise the corresponding lattice is called symplectic, hence $H_1(S,\mathbb{Z})$ has the structure of a symplectic lattice. A symplectic lattice is similar to its dual (isodual) [BS, appendix]. We call $H_1(S,\mathbb{Z})$ with its inner product the ***Jacobian lattice***. The torus $\Omega^*(S)/H_1(S,\mathbb{Z})$ is called the ***Jacobian***.

The Jacobian can be realized as a lattice in $\mathbb{C}^g$ with the standard Hermitian inner product. If $G^{-1} = G_1^* G_1$ then the gram matrix of the cycles becomes $\Lambda^* G_1^* G_1 \Lambda = (G_1\Lambda)^*(G_1\Lambda)$. thus the columns of $G_1\Lambda$ generate a lattice in $\mathbb{C}^g$ isomorphic to the Jacobian lattice.

5. Jacobians and automorphisms

Suppose $\alpha : S \to S$ is an automorphism of the surface and that $\{\omega_k\}$ and $\{\gamma_k\}$ are as in the previous section. With respect to these generators, suppose $A = A(\alpha)$ is the matrix of $\alpha^* : \Omega(S) \to \Omega(S)$ and $B = B(\alpha)$ is a matrix of $\alpha_* : H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z})$. Since for any cycle γ and any one form ω we have

$$\int_{\alpha_*\gamma} \omega = \int_\gamma \alpha^*\omega,$$

we have the matrix equation

$$A\Lambda = \Lambda B. \tag{3}$$

Likewise for one forms ω and σ

$$\int_S \omega \wedge \bar{\sigma} = \int_S \alpha^*\omega \wedge \overline{\alpha^*\sigma},$$

and this implies the matrix equation

$$AGA^* = G. \tag{4}$$

If the Jacobian is realized as a lattice in $\mathbb{C}^g$ as above, then the automorphism α is given by the unitary matrix $G_1 A G_1{}^{-1}$.

6. Jacobian of the Picard curve

We compute the Jacobian of the Picard curve from equations (3) and (4) above, that is, we determine the lattice from a subgroup of the automorphism group. First we determine the action of the automorphisms T and U (section 3) on $\Omega(S)$. Take holomorphic one forms $\omega_1 = z(zdx - xdz)/y^3$, $\omega_2 = (x/z)\omega_1$ and $\omega_3 = (y/z)\omega_1$. We compute that $T^*\omega_1 = \zeta\omega_1$ where $\zeta = \exp(\pi i/6)$. Thus by (1), the matrix of T with respect to these holomorphic one forms is

$$A(T) = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^5 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}.$$

Letting $\sigma = (zdx - xdz)/y^2$, we see that $U^*(\sigma) = -\sigma$. By (1) we compute that

$$A(U) = \frac{-1}{\sqrt{3}} \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}.$$

The action of T and U on $H_1(S, \mathbb{Z})$ can be determined from figure 1 and from the geometric description of these automorphisms. Let γ_k, $k = 0, \ldots, 11$, be defined as the cycle given by a closed curve on the surface from side $2k-1$ to $2k+6$, i.e., corresponding to the generator A_k of the fundamental group Γ. Since T is rotation about the origin an angle $\pi/6$, we easily see that $T_*\gamma_k = \gamma_{k+1}$ for $k = 0 \ldots 11$. Thus $B(T)$ is the corresponding permutation matrix. For the action of U, we need only that $U_*\gamma_0 = -\gamma_0$, i.e., the first column of $B(U)$ is the transpose of $(-1, 0, \ldots, 0)$. This can be most easily seen by noting that γ_0 can be taken to be a geodesic passing through the right angle vertex on the real axis (see figure 2).

Now (3) for $A = A(T)$ and $B = B(T)$ implies that the k^{th} column of Λ is the transpose of $(c_1\zeta^k, c_2\zeta^{5k}, c_3\zeta^{2k})$ for $k = 0, \ldots, 11$, where

$$c_k = \int_{\gamma_0} \omega_k$$

for $k = 1, 2, 3$. The entry in the first row, first column of equation (3) for $A = A(U)$, $B = B(U)$ gives the equation $c_2 = c_1(\sqrt{3} - 1)/2$.

Equation (4) for $A = A(T)$ implies that G is a diagonal matrix. If g_j, $j = 1, 2, 3$ are the entries on the diagonal, then (4) for $A = A(U)$ implies that $g_1 = 2g_2$. It follows that the columns of $G_1\Lambda$ can be taken to be transposes of vectors of the form

$$v_k = \left(\alpha\zeta^k, \alpha\frac{(\sqrt{3}-1)}{\sqrt{2}}\zeta^{5k}, \beta\zeta^{2k}\right),$$

for real constants α and β.

The constants α and β can be determined from the fact that the imaginary part of the Hermitian inner product is the intersection form. Since the vectors v_k correspond to the cycles γ_k, we can determine from figure 1 that $\overline{v_0}\cdot v_1 = 1$ and $\overline{v_0}\cdot v_2 = 1$. Solving these two equations for α and β, we get $\alpha = \sqrt{(3+\sqrt{3})/6}$ and $\beta = \sqrt{1/\sqrt{3}}$. Thus the Jacobian of the Picard curve is equivalent to the lattice in $\mathbb{C}^3$ spanned by the vectors

$$v_k = \left(\sqrt{\frac{3+\sqrt{3}}{6}}\zeta^k, \sqrt{\frac{3-\sqrt{3}}{6}}\zeta^{5k}, \sqrt{\frac{1}{\sqrt{3}}}\zeta^{2k}\right),$$

for $k = 0,\ldots,11$, where the standard Hermitian inner product is used.

7. Algebraic description of the Jacobian

Let α and β be as above and denote by L the lattice described above generated by the vectors $(\alpha\zeta^k, \alpha^{-1}\zeta^{5k}, \beta\zeta^{2k})$. It is useful to have an algebraic construction for L. In $\mathbb{Z}[x]$ let $\phi_1(x) = x^2 - x + 1$ and $\phi_2(x) = x^4 - x^2 + 1$. If $(\phi_1\phi_2)$ is the principal ideal generated by $\phi_1\phi_2$, then $\mathbb{Z}[x]/(\phi_1\phi_2)$ has the structure of a six dimensional lattice in one to one correspondence with polynomials p in $\mathbb{Z}[x]$ of degree ≤ 5. We describe a linear map from $\mathbb{Z}[x]/(\phi_1\phi_2)$ to L by sending a polynomial p to the vector $v(p) = (\alpha p(\zeta), \alpha^{-1}p(\zeta^5), \beta p(\zeta^2))$. Note that for p, q in $\mathbb{Z}[x]$, $p - q$ in $(\phi_1\phi_2)$ implies $v(p) = v(q)$. Now $v(p) = 0$ if and only if $p(\zeta) = 0$ and $p(\omega) = 0$, where $\omega = \zeta^2$. This implies $p \in (\phi_1)\cap(\phi_2) = (\phi_1\phi_2)$. Thus the linear map is an isomorphism, and we may identify element of L with elements in $\mathbb{Z}[x]/(\phi_1\phi_2)$. If v_k are the vectors defined in the previous section, then $v(x^k) = v_k$. The norm is given by $N(v(p)) = (1/6)[(3+\sqrt{3})|p(\zeta)|^2 + (3-\sqrt{3})|p(\zeta^5)|^2] + (1/\sqrt{3})|p(\zeta^2)|^2$.

For $v = v(p)$ as above, let $v^{(1)} = (\alpha p(\zeta), \alpha^{-1}p(\zeta^5), 0)$ and $v^{(2)} = (0, 0, \beta p(\zeta^2))$. Let $L^{(j)} = \{v^{(j)} | v \in L\}$ for $j = 1, 2$. As above, we see that $L^{(1)}$ is a lattice corresponding to $\mathbb{Z}[x]/(\phi_2) = \mathbb{Z}[\zeta]$ and $L^{(2)}$ a lattice corresponding to $\mathbb{Z}[x]/(\phi_1) = \mathbb{Z}[\omega]$.

8. Vectors of minimum norm and systoles

We will discuss further the lattice L with the purpose of finding the vectors of minimum norm. The lattices $L^{(j)}$, $j = 1, 2$ can be identified with classical lattices. Recall the definitions of the lattices $A_2 = \{(x_0, x_1, x_2) \in \mathbb{Z}^3 | \, x_0 + x_1 + x_2 = 0\}$ and $D_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 | \, x_1 + x_2 + x_3 + x_4 \text{ is even}\}$ [CS1]. The lattice A_2 is called the hexagonal lattice. If we restrict the real part of the Hermitian inner product to the lattices above, we see immediately that $L^{(2)}$ is the lattice $2^{-1/2}3^{-1/4}A_2$. To identify $L^{(1)}$, we compute that the real part of gram matrix with respect to the vectors $1, \zeta, \zeta^2, \zeta^3$ is

$$\frac{1}{2}\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

This is $1/2$ times the gram matrix of D_4 with respect to the vectors $(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 0, 1, 1)$. So $L^{(1)}$ as a real lattice can be identified with $(1/\sqrt{2})D_4$.

All vectors in $L^{(1)}$ have integer norm and all vectors in $L^{(2)}$ have norm an integer times $1/\sqrt{3}$. There are six vectors of minimum (non-zero) norm in $L^{(2)}$, corresponding to $1, \omega, \omega^2$ and negatives. These vectors are all of norm $1/\sqrt{3}$. In $L^{(1)}$, the 24 vectors of minimum norm correspond to ζ^k and $\zeta^{k+1} - \zeta^k$, for $k = 0, \ldots, 11$. These vectors are of norm 1.

We can now determine the vectors of minimum norm in L. First note that the vectors $v(x^k)$ and $v(x^{k+1} - x^k)$, $k = 0, \ldots, 11$ and negatives have norm $1 + 1/\sqrt{3}$. We claim that these are the 48 vectors of minimum norm. Given $v = v(p) \in L$ then $N(v) = N(v^{(1)}) + N(v^{(2)})$. If $v^{(1)} \neq 0$ and $v^{(2)} \neq 0$, then by the previous paragraph $N(v) \geq 1 + 1/\sqrt{3}$. If $v^{(1)} = 0$ and $v^{(2)} \neq 0$, then $p(\zeta) = 0$ so $v = v^{(2)}$ is not one of the vectors of minimum norm in $L^{(2)}$ and $N(v) \geq 2/\sqrt{3}$. Likewise if $v^{(2)} = 0$ and $v^{(1)} \neq 0$ we have $p(\omega) = 0$ and so $v = v^{(1)}$ is not one of the vectors of minimum norm in $L^{(1)}$. Hence $N(v) \geq 2$. Therefore, as claimed, there are 48 vectors of minimum norm. They correspond to the 24 cycles indicated by the geodesics in figure 2 where the cycles are taken once in each direction. These correspond to elements in the fundamental group determined by A_k and $A_k^{-1}A_{k+1}$, $k = 0, \ldots, 11$, and the length of these is the shortest length in the length spectrum (systole) as shown in [S]. Note that Schmutz refers to this curve as $M(3)$.

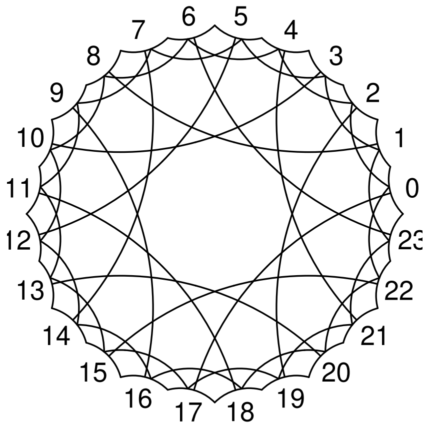

Figure 2

9. The lattice $M(E_6)$

Further investigation shows that the lattice L is the same as the lattice $M(E_6)$ described in [CS2]. Let $\mathcal{A}$ be the ring $\mathbb{Z}[x]/(\phi_1\phi_2)$ as above, and consider the ideals $\mathcal{A}_1 = (\phi_2)$ and $\mathcal{A}_2 = (x^3+1) = ((x+1)\phi_1)$ in $\mathcal{A}$. Note that for $p \in \mathcal{A}$ we have $v((x^3+1)p) = (1+i)v^{(1)}(p)$ and $v(\phi_2 p) = 2(1-\omega)v^{(2)}(p)$. Hence $v(\mathcal{A}_1) = (1+i)L^{(1)}$ is a sublattice of L which as a real lattice is isomorphic to D_4. Likewise $v(\mathcal{A}_2) = 2L^{(2)}$ is a sublattice which as a real lattice is isomorphic to $2^{1/2}3^{-1/4}\mathcal{A}_2$. These lattices are orthogonal and $v(\mathcal{A}_1) \oplus v(\mathcal{A}_2) \subset L$.

Now consider the group $G = L/v(\mathcal{A}_1) \oplus v(\mathcal{A}_2)$. Note that $2 = (1-x^3)(1+x^3) + (x^2+1)(x^4-x^2+1)$ and $-x^2-x+1 = (x^4-x^2+1) - x(x^3+1)$ are in $\mathcal{A}_1 \oplus \mathcal{A}_2$. From this we see that G is a group of order 4 consisting of equivalence classes of the vectors $0, v(1), v(x), v(x^2)$ and is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus we may write

$$L = \bigcup_{w \in G} (w + v(\mathcal{A}_1) \oplus v(\mathcal{A}_2)).$$

This is the description of $M(E_6)$ in [CS2]. The lattice L is obtained by gluing D_4 to a copy of A_2 with norms scaled by the factor $(4/3)^{1/2}$. If the latter is replaced by 2, one gets E_6 and if by $2/3$ one gets E_6^*. So, in a sense, L is the geometric mean of E_6 and E_6^*, thus the name $M(E_6)$.

Acknowledgements

Thanks to Joe Mott for helpful discussions on cyclotomic integers, to Neil Sloane for identifying the lattice, and to Paolo Aluffi and Carel Faber for helping to find the equation of the curve and its automorphisms. Thanks to Peter Sarnak for the encouragement.

References

[B] H. F. Baker, An Introduction to the Theory of Multiply Periodic Functions, Cambridge University Press, N. Y. 1907.

[BS] P. Buser and P. Sarnak, On the period matrix of a Riemann surface of large genus (with an appendix by J. H. Conway and N. J. A. Sloane), Invent. Math. 117(1994), 27-56.

[C] M. Craig, Extreme forms and cyclotomy, Mathematika 25(1978), 44-56.

[CS1] J. H. Conway and N. J. A. Sloane, Sphere Packings, Lattices and Groups, Springer Verlag, 1988.

[CS2] On lattices equivalent to their duals, Journal of Number Theory, to appear.

[E] C. J. Earle, H. E. Rauch, Function Theorist, in Differential Geometry and Complex Analysis, I, Chavel and H. M. Farkas eds.. Springer-Verlag, New York, 1985.

[FK] H. Farkas and I. Kra, Riemann Surfaces, Springer Verlag, 1991.

[G] M. Gromov, Systoles and intersystolic inequalities, preprint.

[H] R. P. Holzapfel, Geometry and Arithmetic Around Euler Partial Differential Equations, Reidel, 1986.

[K] A. Kuribayashi, I. Kuribayashi, Automorphism groups of compact Riemann surfaces of genera three and four, J. of Pure and Applied Algebra 65(1990), 277-292.

[Q] J. R. Quine, Systoles of two extremal Riemann surfaces, Journal of Geometric Analysis, to appear.

[QZ] J. R. Quine and P. Zhang, Extremal symplectic lattices, preprint.

[RL] H. E. Rauch and J. Lewittes, The Riemann surface of Klein with 168 automorphisms, in Problems in Analysis, a symposium in honor of Solomon Bochner, Princeton

University Press, Princeton, N. J.. 1970.

[RR] G. Riera and R. Rodrigues, The period matrix of Bring's curve, Pacific Journal of Mathematics, 154(1992),179-200.

[S] P. Schmutz, Riemann surfaces with shortest geodesic of maximal length, Geometric and Functional Analysis, 3(1993), 564-631.

[V] A. M. Vermeulen, Weierstrass Points of Weight Two on Curves of Genus Three, dissertation, University of Amsterdam, 1983.

J.R. Quine

Department of Mathematics

Florida State University

Tallahassee, FL 32306-3027

quine@math.fsu.edu