INFINITE PRODUCTS OF ZERO-DIMENSIONAL COMMUTATIVE RINGS

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1. Introduction. Suppose $\{R_a\}_{a\in A}$ is a family of zero-dimensional principal ideal rings (PIR's) and let $R = \prod_{a\in A} R_a$. If R is of positive dimension, it is shown in [GH6] that:

(1.1) If I is the direct sum ideal of R and $J \subseteq I$ is an ideal of R, then each extension ring of R/J is infinite-dimensional, or equivalently, R/J cannot be imbedded in a finite-dimensional ring. In particular, each extension ring of R is infinite-dimensional.

(1.2) If S is a nonzero homomorphic image of R, then S has dimension either 0, 1, or ∞, and all three of these values are, in fact, realized as the dimension of a homomorphic image of R.

In this paper we consider these assertions in the case where $R = \prod_{a\in A} R_a$, where $\{R_a\}_{a\in A}$ is a family of zero-dimensional commutative rings. We prove in Theorem 2.1 that statement (1.1) holds more generally for a product of zero-dimensional rings—that is, if $\{R_a\}_{a\in A}$ is a family of zero-dimensional rings and $R = \prod_{a\in A} R_a$ is infinite-dimensional, then each extension ring of R or of R/J, where J is contained in the direct sum ideal $I = \bigoplus_{a\in A} R_a$, is infinite-dimensional.

The situation with regard to (1.2), however, is different. In Section 3 we exhibit, for each positive integer d, a product R of zero-dimensional local rings such that the values realizable as the dimension of a homomorphic image of R are precisely $0, 1, 2, \ldots d, \infty$. We also establish existence of a product of zero-dimensional local rings for which there exists a homomorphic image of dimension d for each positive integer d.

All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity of S is contained in R, and hence is the unity of R. All allusions to the dimension, $\dim(R)$, of a ring R refer to its Krull dimension. Thus $\dim(R) = n$ if there is a chain $P_0 < P_1 < \cdots < P_n$ of prime ideals of $\mathrm{Spec}(R)$ and no chain of longer length; if there is no bound on the length of such chains, we say that R is *infinite-dimensional*. If x is a nilpotent element of a ring R, the *index of nilpotency of x*, denoted by $\eta(x)$, is the least positive integer n such that $x^n = 0$. The *index of nilpotency* of R, denoted

1991 *Mathematics Subject Classification.* 13B02, 13C15.

$\eta(R)$, is defined to be $\sup\{\eta(x) : x$ is a nilpotent element of $R\}$. Index of nilpotency arises in considering the dimension of the product R of a family $\{R_a\}_{a\in A}$ of zero-dimensional rings; for example, Theorem 3.4 of [GH2] shows that R is zero-dimensional if and only if there exists $k \in \mathbb{Z}^+$ such that $\{a \in A : \eta(R_a) > k\}$ is finite.

(1.3) Suppose $\{R_a\}_{a\in A}$ is a family of nonzero rings. In considering homomorphic images of $R = \prod_{a\in A} R_a$ in Section 3, it is convenient to regard R as the set of functions f from A to the union of the rings R_a with the property that $f(a) \in R_a$ for each $a \in A$. From this perspective, we use the following notation.

(i) For $f \in R$, let $\mathcal{Y}(f) = \{a \in A : f(a)$ is a nonunit of $R_a\}$. Note that f is a unit of R if and only if $\mathcal{Y}(f) = \emptyset$.

(ii) For an ideal J of R, let $\mathcal{Y}(J) = \{\mathcal{Y}(f) : f \in J\}$.

(iii) For $X \subseteq A$, let $e_X \in R$ denote the characteristic function of the set X; thus, $e_X(a) = 1 \in R_a$ if $a \in X$ and $e_X(a) = 0$ if $a \notin X$. For $a \in A$, we write e_a in place of $e_{\{a\}}$; the set $\{e_a : a \in A\}$ generates the direct sum ideal of R—that is, the ideal of R consisting of all functions f that are finitely nonzero. Note that if $f \in R$ and $X = \mathcal{Y}(f)$, then $(1 - e_X)R \subseteq fR$.

It is also useful in considering the ideal structure of $R = \prod_{a\in A} R_a$ to employ the terminology of filters and ultrafilters on A (cf. [B, Chapter I, Section 6], [GJ, Chapter 2], [BS, Chapter 1] or [LLS]). We recall that a *filter* on A is a nonempty collection $\mathcal{F}$ of subsets of A such that $\emptyset \notin \mathcal{F}$ and such that $\mathcal{F}$ is closed under finite intersection and taking supersets in A. An *ultrafilter* on A is a maximal filter. Each filter on A is contained in an ultrafilter on A and a filter $\mathcal{F}$ on A is an ultrafilter if and only if for each subset X of A, either X or its complement $X' = A - X$ is in $\mathcal{F}$. A filter $\mathcal{F}$ is said to be *fixed* if there exists $a \in A$ such that each $X \in \mathcal{F}$ contains a. If there does not exist such an a, then $\mathcal{F}$ is called a *free filter*. The set A admits a free ultrafilter if and only if A is infinite; if $\mathcal{F}$ is a free ultrafilter on A, then $\mathcal{F}$ contains the free filter consisting of the cofinite subsets of A. In the case where each R_a is quasilocal, Proposition 2.3 of [GH2] shows that there is a natural bijection between the set of maximal ideals of R and the set of ultrafilters on A. On the set of maximal ideals of R, this bijection is the map $M \to \mathcal{Y}(M)$; the inverse map is given by $U \to \{f \in R : \mathcal{Y}(f) \in U\}$. In particular, $M = \{f \in R : \mathcal{Y}(f) \in \mathcal{Y}(M)\}$ for each maximal ideal M of R if each R_a is quasilocal.

2. Infinite-dimensional extension rings.

Our purpose in this section is to prove the following theorem, which extends statement (1.1) to the case where R is a product of arbitrary zero-dimensional rings.

Theorem 2.1. Suppose $\{R_a\}_{a\in A}$ is a family of zero-dimensional rings, $R = \prod_{a\in A} R_a$, and $I = \bigoplus_{a\in A} R_a$ is the direct sum ideal of R. If R is infinite-dimensional and if J is an ideal of R contained in I, then each extension ring of R/J is infinite-dimensional.

The proof of Theorem 2.1 uses the concept of a universally contracted ideal, where an ideal I of a ring R is said to be *universally contracted* if I is contracted from every extension ring of R—that is, $IS \cap R = I$ for each extension ring S of R.

Remark 2.2. A principal ideal generated by an idempotent element is well known to be universally contracted (cf. (2.2) of [GH3]). Since a finitely generated idempotent ideal is principal, generated by an idempotent element, and since a directed union of universally contracted ideals is again universally contracted, an ideal generated by idempotent elements is universally contracted.

We also use the following observations about the dimension of extension rings.

(2.3) Suppose I is a universally contracted ideal of a ring R. If each extension ring of R/I is infinite-dimensional (resp., of dimension at least n), then each extension ring of R is infinite-dimensional (resp., of dimension at least n).

Proof. This is clear in view of the fact that if S is an extension ring of R, then there exists an ideal J of S such that $J \cap R = J$, and thus S/J is an extension ring of R/I.

(2.4) Suppose N is a multiplicative set in a ring R. If each extension ring of the localization R_N is infinite-dimensional (resp., of dimension at least n), then each extension ring S of R is infinite-dimensional (resp., of dimension at least n).

Proof. This is clear since S_N is an extension ring of R_N and $\dim(S_N) \leq \dim(S)$.

As a corollary of (2.3) we observe in (2.5) that it suffices to prove Theorem 2.1 in the case where $J = I$.

Corollary 2.5. Let $\{R_a\}_{a\in A}$ be a family of rings and let I denote the direct sum ideal of $R = \prod_{a\in A} R_a$. If every extension ring of R/I is infinite-dimensional (resp., of dimension at least n), then every extension ring of R/J is infinite-dimensional (resp., of dimension at least n) for each ideal J of R contained in I.

Proof. The ideal I/J of R/J is generated by idempotent elements, and hence is universally contracted. The assertion therefore follows from (2.3).

Lemma 2.6 is the last preliminary result used in the proof of Theorem 2.1; it is concerned with ideal-theoretic properties in infinite products.

Lemma 2.6. Let $\{S_a\}_{a\in A}$ be a family of nonzero rings, let $S = \prod_{a\in A} S_a$, and let I be the direct sum ideal of S.

(1) If N_a is a multiplicative set in S_a and if $N = \prod_{a\in A} N_a$, then S_N is canonically isomorphic to $\prod_{a\in A}(S_a)_{N_a}$ under the map μ taking $f/g \in S_N$ to $\{f(a)/g(a)\}_{a\in A} \in \prod_{a\in A}(S_a)_{N_a}$. Moreover, if $\phi : S \to \prod_{a\in A}(S_a)_{N_a}$ denotes the composition of the canonical map of $S \to S_N$ with the map μ, then $\phi(I)$ is the direct sum ideal of $\prod_{a\in A}(S_a)_{N_a}$.

(2) If J_a is an ideal in S_a and if $J = \prod_{a\in A} J_a$, then S/J is canonically isomorphic to $\prod_{a\in A}(S_a/J_a)$, and if $\psi : S \to \prod_{a\in A}(S_a/J_a)$ denotes the composite of the canonical homomorphism of $S \to S/J$ with this map of $S/J \to \prod_{a\in A}(S_a/J_a)$, then $\psi(I)$ is the direct sum ideal of $\prod_{a\in A}(S_a/J_a)$.

(3) If $f \in S$ and if $C = (0) : f$, then $I : f = I + C$.

Proof of Lemma 2.6. (1): Let I^* be the direct sum ideal of $\prod_{A\in A}(S_a)_{N_a}$. The inclusion $\phi(I) \subseteq I^*$ is clear. The reverse inclusion is also clear from the observation that for each $a \in A$, if e_a is the idempotent of S associated with a and f_a is the idempotent of $\prod_{A\in A}(S_a)_{N_a}$ associated with a, then $f_a = \phi(e_a)$.

The proof of (2) is similar to that of (1).

In (3), the inclusion $I + C \subseteq I : f$ is clear. Conversely, if $g \in I : f$, then fg has finite support $B \subseteq A$. Thus $e_B \in I$ and $fg = fge_B$. We have $g = ge_B + g(1 - e_B)$, where $ge_B \in I$ and $g(1 - e_B)f = fge_B(1 - e_B) = 0$, so $g(1 - e_B) \in C$. Therefore $g \in I + C$ and equality holds.

Proof of Theorem 2.1. The proof is obtained through a series of reductions. Since R is infinite-dimensional, there exists a sequence $a_1, a_2, \ldots$ of distinct elements of A such that $\eta(R_{a_i}) > i$ for each i. Let $B = \{a_i\}_{i=1}^{\infty}$. Then $Re_B \cong \prod_{i=1}^{\infty} R_{a_i}$ is a direct summand of R, Ie_B is the direct sum ideal of Re_B, and

$$R/I \cong (Re_B/Ie_B) \oplus (R(1 - e_B)/I(1 - e_B)).$$

Hence to show each extension ring of R/I is infinite-dimensional, it suffices to show that each extension ring of Re_B/Ie_B is infinite-dimensional. Therefore without loss of generality we assume that $A = \mathbb{Z}^+$ and $\eta(R_i) > i$ for each i.

For each i choose $x_i \in R_i$ to be a nilpotent element such that $x_i^i \neq 0$. There exists a maximal ideal M_i of R_i such that $x_i^i/1 \neq 0$ in $(R_i)_{M_i}$. Let $N_i = R_i - M_i$ and let $N = \prod_{i=1}^{\infty} N_i$. Using permutability of the operations of localization and residue class formation and applying Lemma 2.6 along with (2.4), it follows that we may assume without loss of generality that each R_i is quasilocal and $\eta(R_i) > i$ for each i.

Let $B = \{i \in \mathbb{Z}^+ : \operatorname{char}(R_i) = 0\}$. For each $i \in B$, the ring R_i contains the field $\mathbb{Q}$ up to isomorphism (cf. [GH5, Prop. 2.3]). Suppose B is infinite. Then $\prod_{i\in B} R_i$ is infinite-dimensional, and as in the first reduction above, it suffices in this case to prove Theorem 2.1 under the assumption that $A = B$. Corollary 3.7 of [GH6] shows, however, that each extension ring of R/I is infinite-dimensional in the case where $A = B$. Thus, assume B is finite. Then without loss of generality we may assume that B is empty.

Since R_i is quasilocal, the prime subring π_i of R_i is isomorphic to $\mathbb{Z}/p_i^{m_i}\mathbb{Z}$ for some prime p_i and positive integer m_i. Each π_i is a chained ring and the direct sum ideal I^* of $S = \prod_{i=1}^\infty \pi_i$ is contracted from the direct sum ideal I of R. If S is infinite-dimensional, Theorem 3.2 of [GH6] shows that each extension ring of S/I^*, and hence of R/I, is infinite-dimensional. Therefore we assume that S is finite-dimensional, and hence [GH2, Theorem 3.4] the set $\{m_i\}_{i=1}^\infty$ is finite. As in previous reductions, we may then assume that $|\{m_i\}_{i=1}^\infty| = 1$. We denote by m the common value of the integers m_i. We are almost to our final reduction. Choose $x_i \in R_i$ to be a nilpotent element such that $x_i^i \neq 0$ and let $T_i = \pi_i[x_i]$. By reasoning similar to that used in previous cases, it suffices to consider the case where $R_i = T_i$ for each i, and here we use induction on m.

If $m = 1$, each R_i is a chained ring, and the fact that each extension ring of R/I is infinite-dimensional follows from Theorem 3.2 of [GH6]. Assume the result for $m < k$ and consider the case $m = k$, where $k > 1$. Let $f \in R$ be such that $f(i) = p_i$ for each i and let $C = \operatorname{Ann}_R(f)$. Part (3) of Lemma 2.6 shows that $I + C = I : f$, so $I + C$ is a universally contracted ideal of R. We have $C = \prod_{i=1}^\infty C_i$, where $C_i = \operatorname{Ann}_{R_i}(p_i)$. Thus $T = \prod_{i=1}^\infty (R_i/C_i) \cong R/C$. By part(2) of Lemma 2.6, the image of I under the canonical map ϕ of R onto T is the direct sum ideal of T. The prime subring of R_i/C_i is isomorphic to $\mathbb{Z}/p_i^{k-1}\mathbb{Z}$ for each i, and R_i/C_i is a simple extension of its prime subring by the element $x_i + C_i$. Thus, if T is infinite-dimensional, the induction hypothesis implies that each extension ring of

$$T/\phi(I) \cong (R/C)/(I + C)/C \cong R/(I + C) = R/(I : f)$$

is infinite-dimensional. Since the ideal $(I : f)/I$ of R/I is universally contracted, (2.3) implies that each extension ring of R/I is infinite-dimensional. On the other hand, if T is finite-dimensional, then there is a positive integer s such that $x_i^s \in C_i$ for each i. In this case, let $g \in R$ be such that $g(i) = x_i^s$ for each i. If $E = \operatorname{Ann}_R(g)$, then as in the preceding case, $I + E = I : g$, etc., and $R/(I : g)$ is isomorphic to W/I^*, where $W = \prod_{i=1}^\infty (R_i/\operatorname{Ann}_{R_i}(x_i^s))$ and I^* is the direct sum ideal of W. Because $x^{i-s} \notin \operatorname{Ann}_{R_i}(x_i^s)$ for $i > s$, the ring W is infinite-dimensional; also, $R_i/\operatorname{Ann}_{R_i}(x_i^s)$ has prime subring isomorphic to the field $\mathbb{Z}/p_i\mathbb{Z}$ since, by assumption, $x_i^s \in \operatorname{Ann}_{R_i}(p_i)$ for each

i. Therefore each $R_i/\operatorname{Ann}_{R_i}(x_i^s)$ is a PIR, so each extension ring of W/I^* is infinite-dimensional. As in the preceding case, it then follows that each extension ring of R/I is infinite-dimensional. By induction, this completes the proof of Theorem 2. □

3. Finite-dimensional homomorphic images.

(3.1) In this section we consider finite-dimensional homomorphic images of a product of zero-dimensional rings. We exhibit, for each positive integer d, a product R of zero-dimensional local rings such that the values realizable as the dimension of a homomorphic image of R are $0, 1, \ldots, d, \infty$ (see Theorem 3.8). We also show in Example 3.10 that such an R may have a homomorphic image of dimension d for each positive integer d. For these purposes we consider a local (Noetherian) ring T with maximal ideal P. For each positive integer n, let $R_n = T/P^n$, and let $R = \prod_{n=1}^{\infty} R_n$. Recall that a sequence $\{a_n\}_{n=1}^{\infty} \in R$ is said to be *coherent* if $a_n \in R_n = T/P^n$ is the image of a_{n+1} under the canonical map of $T/P^{n+1} \to T/P^n$ for each positive integer n [AM, page 103]. The completion $\widehat{T}$ of T with respect to the P-adic topology on T may be defined to be the subring of R consisting of all coherent sequences [AM, pages 103-105].

Theorem 3.2. Suppose (T, P) is a local ring, and for each positive integer n, let $R_n = T/P^n$. Let $R = \prod_{n=1}^{\infty} R_n$ and identify the completion $(\widehat{T}, \widehat{P})$ of T under the P-adic topology with the subring of all coherent sequences in R. Let M be a free maximal ideal of R, let $S = R_M$, and let $Q = \bigcap_{n=1}^{\infty} M^n S$.

(1) The canonical homomorphism $\phi : R \to S$ restricts to an injection on $\widehat{T}$. Hence we consider $\widehat{T}$ as a subring of S.

(2) We have $MS = PS$ and $M^n S \cap \widehat{T} = \widehat{P}^n$ for each positive integer n. Therefore $\widehat{T} \cap Q = (0)$.

(3) If the residue field $T/P = K$ of T is finite, then each maximal ideal of R has residue field K. In this case $S = \widehat{T} + Q$, $S/Q \cong \widehat{T}$, and $\widehat{T}$ is a homomorphic image of R.

Proof. (1): Suppose $a = \{a_i\}_{i=1}^{\infty} \in \widehat{T}$ is such that $\phi(a) = 0$ in S. Then $H = \{i \in \mathbb{Z}^+ : a_i = 0\} \in \mathcal{U} = \mathcal{Y}(M)$, the free ultrafilter associated with M. If $a \neq 0$, then H is a finite set since, by definition of $\widehat{T}$, $a_i \neq 0$ implies $a_j \neq 0$ for each $j \geq i$. Because $\mathcal{U}$ contains no finite subset of $\mathbb{Z}^+$ (cf. [GH6, Section 2]), we conclude that $a = 0$, and hence that the restriction of ϕ to $\widehat{T}$ is an injection, as we wished to show.

(2): The Jacobson radical $J(R)$ of R is $\prod_{n=1}^{\infty} J(R_n) = \prod_{n=1}^{\infty}(P/P^n)$. Hence the canonical image of P in R (which we identify with P) is contained in M, so that $PS \subseteq MS$. For the reverse inclusion, take $g \in M$ and let $\{f_i\}_{i=1}^{r}$ be a finite set of generators for P. Then $\mathcal{Y}(g) \in \mathcal{U}$ and for each $j \in \mathcal{Y}(g)$, it follows from the definition of $\mathcal{Y}(g)$ that $g(j) \in P/P^j =$

$(f_1(j),\dots,f_r(j))R_j$. Since $G = \{j \in \mathbb{Z}^+ : g(j) \in (f_1(j),\dots,f_r(j))R_j\} \supseteq \mathcal{Y}(g)$, we have $G \in \mathcal{U}$ and therefore $gS \subseteq (f_1,\dots,f_r)S = PS$ [GH6, Prop. 2.4].

The inclusion $\widehat{P}^n \subseteq M^nS \cap \widehat{T}$ follows from the inclusion $P \subseteq MS$ and the fact that $\widehat{P} = P\widehat{T}$. For the other direction, let $\{g_i\}_{i=1}^s$ be a finite set of generators of P^n and suppose $f \in M^nS \cap \widehat{T}$. Since $M^nS = P^nS = (g_1,\dots,g_s)S$, it follows that $\{i \in \mathbb{Z}^+ : f(i) \in (g_1(i),\dots,g_s(i))R_i\} \in \mathcal{U}$ [GH6, Prop. 2.4]. Since $\mathcal{U}$ is free, there exists $k > n$ such that $f(k) \in (g_1(k),\dots,g_s(k))R_k$. We have $R_k = T/P^k \cong \widehat{T}/\widehat{P}^k$, and since $f \in \widehat{T}$ and $g_1,\dots,g_s \in \widehat{T}$, the relation $f(k) \in (g_1(k),\dots,g_s(k))R_k$ means

$$f \in (g_1,\dots,g_s)\widehat{T} + \widehat{P}^k = \widehat{P}^n + \widehat{P}^k = \widehat{P}^n.$$

Hence $M^nS \cap \widehat{T} \subseteq \widehat{P}^n$, and equality holds. Therefore

$$\widehat{T} \cap Q = \widehat{T} \cap (\cap_{n=1}^{\infty} M^nS) = \cap_{n=1}^{\infty} \widehat{P}^n = (0).$$

(3): As observed in (2), $J(R) = \prod_{n=1}^{\infty}(P/P^n)$, so $R/J(R) \cong \prod_{n=1}^{\infty}(T/P)_n$ is a product of countably many copies of K. It is clear that R/J(R) and R admit the same residue fields, and if K is finite, Theorem 5.3 of [GH4] shows that, up to isomorphism, K is the only residue field of $\prod_{n=1}^{\infty}(K)_n$. Hence each maximal ideal of R has residue field K if K is finite.

To prove the assertions in the second sentence of (3), it suffices, in view of (2) and the fact that $\phi : R \to S$ is a surjection [GH6, Prop. 2.4], to show that $S \subseteq \widehat{T} + Q$. Since $S/MS \cong \widehat{T}/\widehat{P} \cong K$ and since $S/M^nS = \widehat{T}/\widehat{P}^n$ for each positive integer n, it follows that $S = \widehat{T} + M^nS$ for each n. Let $s \in S$. For each $n \in \mathbb{Z}^+$, we have $s = a_n + c_n$, where $a_n \in \widehat{T}$ and $c_n \in M^nS$. Since $a_n - a_{n+1} = c_{n+1} - c_n \in M^nS \cap \widehat{R} = \widehat{P}^n$ for each n, $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\widehat{T}$. Let $a = \lim a_n$. We claim that $s - a \in M^nS$ for each $n \in \mathbb{Z}^+$. To see this, observe that $a - a_k \in \widehat{P}^n$ for some $k \geq n$, and

$$s - a = s - a_k + a_k - a = c_k + a_k - a \in M^kS + \widehat{P}^n \subseteq M^kS + M^nS = M^nS.$$

Therefore $s - a \in Q$ and $s \in \widehat{T} + Q$. This establishes the desired inclusion $S \subseteq \widehat{T} + Q$. □

(3.3) Under the notation as in Theorem 3.2, let $\mathcal{U} = \mathcal{Y}(M)$. Since $\mathcal{U}$ is free, for each $i \in \mathbb{Z}^+$ there exists $U_i \in \mathcal{U}$ such that $i \notin U_i$. Let $Y_m = \bigcap_{i=1}^m U_i$ for $m \in \mathbb{Z}^+$. Then each $Y_m \in \mathcal{U}$, and $\{Y_m\}_{m=1}^{\infty}$ is a decreasing sequence such that $\bigcap_{m=1}^{\infty} Y_m = \emptyset$. Omitting duplication among the sets Y_i, we obtain a strictly decreasing sequence $X_1 > X_2 > \dots$ of elements of $\mathcal{U}$ such that $\bigcap_{i=1}^{\infty} X_i = \emptyset$.

Theorem 3.4. Suppose (T, P) is a local ring, and for each positive integer n, let $R_n = T/P^n$. Let $R = \prod_{n=1}^{\infty} R_n$ and identify T with its image under the canonical injection of

T into R. Let M be a free maximal ideal of R, let $S = R_M$, and let $Q = \bigcap_{n=1}^{\infty} M^n S$. Assume that $\{x_i\}_{i=1}^{k}$ is a generating set for P. If $f \in R$ is such that $fS \subseteq Q$, then there exists an ideal $J = (g_1, \ldots, g_k)R$ of R such that $JS \subseteq Q$ and $fS \subseteq \bigcap_{n=1}^{\infty} J^n S$.

Proof. Let $Y = \mathcal{Y}(f)$. Since $f \in M$, $Y \in \mathcal{U}$. Let $X_1 > X_2 > \ldots$ be a decreasing sequence of elements of $\mathcal{U}$ with empty intersection as in (3.3). For a finitely generated ideal $L = (h_1, \ldots, h_r)R$ of R and for $i \in \mathbb{Z}^+$ we denote by $L(i)R_i$ the ideal $(h_1(i), \ldots, h_r(i))R_i$. For $n \in \mathbb{Z}^+$, let $X_n^* = \{i \in \mathbb{Z}^+ : f(i) \in P^{kn^2}(i)R_i\}$. Since, by assumption, $fS \subseteq M^{kn^2}S = P^{kn^2}S$, each $X_n^* \in \mathcal{U}$. Let $Y_n = X_n^* \cap X_n$. Then $\{Y_n\}_{n=1}^{\infty}$ is a decreasing sequence of elements of $\mathcal{U}$ with empty intersection. For $1 \le i \le k$, define $g_i \in R$ as follows:

$$g_i(a) = \begin{cases} 1, & \text{if } a \in \mathbb{Z}^+ - (Y_1 \cap Y) \\ x_i^n, & \text{if } a \in (Y_n \cap Y) - (Y_{n+1} \cap Y). \end{cases}$$

Since $\bigcap_{n=1}^{\infty} Y_n = \emptyset$, g_i is defined on all of $\mathbb{Z}^+$.

To show that $JS \subseteq Q$, we show that $JS \subseteq P^t S$ for each positive integer t. Since $x_i^t \in P^t$, we have $\{j \in \mathbb{Z}^+ : g_i(j) \in P^t(j)R_j\} \supseteq Y_t \cap Y$, an element of $\mathcal{U}$. Hence $g_i S \subseteq P^t S$ by [GH6, Prop. 2.4]. Therefore $JS \subseteq P^t S$ for all t and $JS \subseteq Q$.

To show that $fS \subseteq J^n S$ for each $n \in \mathbb{Z}^+$, it suffices to show that $f(j) \in J^n(j)R_j$ for each $j \in Y_n \cap Y$. To do this, we prove that $f(j) \in J^n(j)R_j$ for $j \in (Y_m \cap Y) - (Y_{m+1} \cap Y)$ for each $m \ge n$. Since $Y_m \subseteq X_m^*$, we have $f(j) \in P^{km^2}R_j$. The ideal P^{km^2} is generated by all products $\prod_{i=1}^{k} x_i^{e_i}$, where $\sum_{i=1}^{k} e_i = km^2$. In each such sum, some e_i is at least m^2, so

$$P^{km^2} \subseteq (x_1^{m^2}, \ldots, x_k^{m^2}) \subseteq (x_1^m, \ldots, x_k^m)^m \subseteq (x_1^m, \ldots, x_k^m)^n.$$

Therefore $P^{km^2}(j)R_j \subseteq (x_1^m, \ldots, x_k^m)^n(j)R_j \subseteq (g_1(j), \ldots, g_k(j))^n R_j = J^n(j)R_j$. This completes the proof. $\square$

We use in Theorem 3.6 the following lemma.

Lemma 3.5. Let S be a ring and let $P \in \mathrm{Spec}(S)$. Suppose that $J \subseteq P$ is a finitely generated ideal of S and $f \in \bigcap_{n=1}^{\infty} J^n$. If $Q \subseteq P$ is a prime ideal such that $\dim(R_P/QR_P) = 1$, then $f \in Q$.

Proof. By passage to R_P/QR_P, it suffices to show that if $J \subseteq M$ is a finitely generated ideal of a one-dimensional quasilocal domain (D, M), then $\bigcap_{n=1}^{\infty} J^n = (0)$. If $x \in M - (0)$, then $M = \mathrm{rad}(xD)$ so $J^k \subseteq xD$ for some $k \in \mathbb{Z}^+$. Since $J^k > J^{k+1}$, it follows that $x \notin \bigcap_{n=1}^{\infty} J^n$. $\square$

Theorem 3.6. As in Theorem 3.4, let (T, P) be a local ring, let $R_n = T/P^n$, and let $R = \prod_{n=1}^{\infty} R_n$. Suppose M is a free maximal ideal of R, let $S = R_M$, and let $Q = \bigcap_{n=1}^{\infty} M^n S$. If $P_0 \in \mathrm{Spec}(S)$ is such that S/P_0 is finite-dimensional, then $Q \subseteq P_0$.

Proof. We use induction on $d = \dim(S/P_0)$. If $d = 0$, then $P_0 = MS$ and $Q \subseteq P_0$. Assume the result is true for prime ideals P_1 such that $\dim(S/P_1) < d$, where $d > 0$, and consider the case where $\dim(S/P_0) = d$. Let $P_1 \supseteq P_0$ be a prime such that $\dim(S/P_1) = d - 1$. Suppose $f \in Q$ and choose, by Theorem 3.4, a finitely generated ideal $J \subseteq Q$ such that $f \in \bigcap_{n=1}^{\infty} J^n$. The induction hypothesis implies that $f \in P_1$ and $J \subseteq P_1$, and it then follows from Lemma 3.5 that $f \in P_0$. Therefore $Q \subseteq P_0$ and the proof is complete. □

Corollary 3.7. With notation as in Theorem 3.6, if L is an ideal of S such that S/L is finite-dimensional, then $Q \subseteq L$.

Proof. Theorem 3.6 implies that $Q \subseteq \mathrm{rad}(L)$. Suppose $f \in Q$ and choose, by Theorem 3.4, a finitely generated ideal $J \subseteq Q$ such that $f \in \bigcap_{n=1}^{\infty} J^n$. Then $J \subseteq \mathrm{rad}(L)$, so $J^k \subseteq L$ for some $k \in \mathbb{Z}^+$. Since $f \in J^k$, it follows that $Q \subseteq L$. □

Theorem 3.8. Suppose (T, P) is a local ring with finite residue field, and for each positive integer n, let $R_n = T/P^n$. Let $R = \prod_{n=1}^{\infty} R_n$ and let $\widehat{T}$ denote the completion of T in the P-adic topology. If $d = \dim(T) = \dim(\widehat{T}) = 0$, then R is zero-dimensional. If $d > 0$, a nonzero homomorphic image of R has dimension $0, 1, \ldots, d$, or ∞, and each of these values is realized as the dimension of a homomorphic image of R.

Proof. By Theorem 3.4 of [GH2], $\dim(T) = 0$ implies that $\dim(R) = 0$, and $\dim(T) = d > 0$ implies that $\dim(R) = \infty$. For $\dim(T) = d > 0$, the theorem is equivalent to the statement that $\{0, 1, \ldots, d\}$ is the set of integers that occur as the dimension of a finite-dimensional nonzero homomorphic image of R. By part(3) of Theorem 3.2, $\widehat{T}$ is a homomorphic image of R. Hence each of the integers $0, \ldots d$ is realized as the dimension of a homomorphic image of R.

Suppose R admits a finite-dimensional homomorphic image R_0 of dimension $k > d$. Without loss of generality we may assume that R_0 is an integral domain. Hence $R_0 \cong R/P_0$ for some $P_0 \in \mathrm{Spec}(R)$. Since P_0 is contained in a unique maximal ideal M of R [GH6, Prop. 2.3], $R/P_0 \cong S/P_0S$, where $S = R_M$. If $Q = \bigcap_{n=1}^{\infty} M^n S$, Theorem 3.6 shows that S/P_0S is a homomorphic image of S/Q, and part(3) of Theorem 3.2 shows that $S/Q \cong \widehat{T}$. Therefore $R_0 \cong S/P_0S$ has dimension at most d, a contradiction. Thus each finite-dimensional nonzero homomorphic image of R has dimension at most d. This completes the proof of Theorem 3.8. □

Remark 3.9. As in Theorem 3.8, let (T, P) be a local ring with finite residue field, let $R_n = T/P^n$, and let $R = \prod_{n=1}^{\infty} R_n$. Suppose M_1 and M_2 are distinct free maximal ideals of R. Let $S_i = R_{M_i}$ and $Q_i = (\bigcap_{n=1}^{\infty} M_i^n S_i) \cap R$, for $i = 1, 2$. By Theorem 3.2, we have $R/Q_1 \cong \widehat{T} \cong R/Q_2$. We observe that the ideals Q_1 and Q_2 are comaximal. For a proof, note that if P_1 is a minimal prime of Q_1, then P_1 is contracted from S_1 since Q_1 is obviously contracted from S_1; hence P_1 is contained in M_1. By part(iv) of Proposition 2.3 of [GH6], M_1 is the unique maximal ideal of R containing P_1. Hence M_1 is the unique maximal ideal of R containing Q_1. Similarly, M_2 is the unique maximal ideal of R containing Q_2, so $Q_1 + Q_2 = R$.

The following example shows that it is possible to have an infinite product of zero-dimensional local rings for which there exists a homomorphic image of dimension d for each positive integer d.

Example 3.10. Let $\{x_i\}_{i=1}^{\infty}$ be a set of indeterminates over a finite field k and for each positive integer n, let $R_n = k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^n$. Let $R = \prod_{n=1}^{\infty} R_n$. For d a positive integer and $n > d$, let $S_n = R_n/(x_{d+1}, \ldots, x_n)R_n$. Then $\prod_{n=d+1}^{\infty} S_n$ is a homomorphic image of R. An argument similar to that for part(3) of Theorem 3.2 shows that the d-dimensional ring $k[[x_1, \ldots, x_d]]$ is a homomorphic image of $\prod_{n=d+1}^{\infty} S_n$, and hence a homomorphic image of R. Therefore R admits a homomorphic image of dimension d for every positive integer d.

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