A DISCRETE KATO TYPE THEOREM ON INVISCID LIMIT OF NAVIER-STOKES FLOWS *

WENFANG (WENDY) CHENG AND XIAOMING WANG †

Abstract. The inviscid limit of wall bounded viscous flows is one of the unanswered central questions in theoretical fluid dynamics. Here we present a result indicating the difficulty in numerical study of the problem. More precisely, we show that numerical solutions of the incompressible Navier-Stokes equations converge to the exact solution of the Euler equations at vanishing viscosity provided that small scales of the order of ν/U in the direction tangential to the boundary in an appropriate boundary layer is not resolved in the scheme. Here ν is the kinematic viscosity of the fluid and U is the typical velocity taken to be the maximum of the shear velocity at the boundary for the inviscid flow. Such a result is somewhat surprising since such a small scale is smaller than any of the known small scales predicted by conventional theory of turbulence and boundary layer theory. On the other hand, such a result can be viewed as a discrete version of our early result (Wang 2001) which generalized earlier result of Kato (1984) where the relevance of a scale proportional to the kinematic viscosity to the problem of vanishing viscosity is first discovered.

Key words. Zero viscosity limit, Navier-Stokes equations, Euler equations, energy dissipation rate, boundary layer, small scale, kinematic viscosity

AMS subject classifications. 65M99, 76D09, 35Q30

1. Introduction.

One of the central and most useful system in fluid dynamics is the **Navier-Stokes system** for incompressible homogeneous Newtonian fluids which governs the motion of fluids like air and water under normal conditions:

$$\frac{\partial \mathbf{u}^\nu}{\partial t} + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu - \nu\Delta\mathbf{u}^\nu + \nabla p^\nu = \mathbf{f}, \text{ in } \Omega, \tag{1.1}$$

$$\text{div } \mathbf{u}^\nu = 0 \text{ in } \Omega, \tag{1.2}$$

$$\mathbf{u}^\nu = \mathbf{b} \text{ on } \Gamma, \tag{1.3}$$

$$\mathbf{u}^\nu = \mathbf{u}_0 \text{ at } t = 0, \tag{1.4}$$

where $\mathbf{u}^\nu = (u_1^\nu, u_2^\nu, u_3^\nu)$ is the velocity field in the Eulerian coordinates, p^ν is the kinematic pressure, and $\mathbf{f} = (f_1, f_2, f_3)$ is the external body force, the positive constant ν is the kinematic viscosity. The velocity $\mathbf{b}$ at the boundary satisfies the no-penetration condition

$$\mathbf{b} \cdot \mathbf{n} = 0, \tag{1.5}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\Gamma = \partial\Omega$. This includes the case of Taylor-Couette type flows among others. The boundary condition sometimes is referred to as characteristic boundary condition since the boundary consists of stream lines all the time.

There is an abundant literature on the Navier-Stokes systems. The interested reader may consult the books by Constantin and Foias (1988), Doering and Gibbon (1995), Ladyzhenskaya (1969), Majda and Bertozzi (2001) or Temam (2001) for the mathematical theories of the Navier-Stokes equations.

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† Florida State University, Tallahassee, FL 32306

For realistic fluids like air and water, the kinematic viscosity is very small and hence we may formally set it to zero and arrive at the **Euler system** for incompressible inviscid (dry) fluids:

$$\frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 + \nabla p^0 = \mathbf{f}, \text{ in } \Omega, \tag{1.6}$$

$$\text{div } \mathbf{u}^0 = 0 \text{ in } \Omega, \tag{1.7}$$

$$\mathbf{u}^0 \cdot \mathbf{n} = 0 \text{ on } \Gamma, \tag{1.8}$$

$$\mathbf{u}^0 = \mathbf{u}_0 \text{ at } t = 0. \tag{1.9}$$

More importantly, if the characteristic fluid velocity U is large or the characteristic length scale L of the motion is large, the Reynolds number defined as

$$Re = \frac{LU}{\nu} \tag{1.10}$$

is large and the non-dimensionalized Navier-Stokes system takes the same form except the kinematic viscosity is replaced by the reciprocal of the Reynolds' number which is very small. This provides another scenario where inviscid approximation is needed.

Such an approximation has been utilized in many applications. A physically important question is then whether such an approximation can be justified via the zero viscosity limit of the Navier-Stokes equations.

The mathematical investigation of such a problem is extremely difficult due to the nonlinear nonlocal nature of the systems involved and to the singular nature of the problem which involves a boundary layer. There have been extensive efforts on resolving this inviscid limit problem which lead to many partial results (see for instance Prandtl (1905), von Karman (1930), Schlichting (1979) etc from the physical perspective, and Bona and Wu (2002), E and Engquist (1997), Kato (1984), Ladyzhenskaya (1969), Oleinik (1963), Oleinik and Samokhin (1999), Matsui (1984), Sammartino and Caflisch (1995, 1996), Temam and Wang (1996, 1998), Wang (2001), Xin and Zhang (2004) for some of the mathematical results).

Confronted with such a difficult problem, we naturally resort to numerical methods, especially with today's powerful computer and efficient and accurate numerical schemes. A natural question to ask is if we can trust the numerical results. More precisely, let

$$\mathbf{u}^k = \mathbf{u}_{h_k}^{\nu_k}$$

be a sequence of numerical solutions of an appropriate numerical scheme with kinematic viscosity ν_k and mesh size h_k satisfying the vanishing viscosity and mesh size assumption

$$\nu_k \to 0, \quad h_k \to 0 \quad \text{as } k \to \infty$$

our questions are:

$$\text{Does } \lim_{k\to\infty} \mathbf{u}_{h_k}^{\nu_k} = \mathbf{u}^0 \text{implies} \lim_{k\to\infty} \mathbf{u}^{\nu_k} = \mathbf{u}^0? \tag{1.11}$$

$$\text{Does } \lim_{k\to\infty} \mathbf{u}_{h_k}^{\nu_k} \neq \mathbf{u}^0 \text{implies} \lim_{k\to\infty} \mathbf{u}^{\nu_k} \neq \mathbf{u}^0? \tag{1.12}$$

What we will demonstrate below is that the convergence of the numerical solutions may have nothing to do with the convergence of the continuous solutions (solutions of the Navier-Stokes system (1.1))at vanishing viscosity to the solution of the Euler system (1.6) if small scales of the order of $\frac{\nu}{U}$ is not resolved in the scheme in an appropriate boundary layer in the direction tangential to the boundary. This indicates the difficulty in studying such an inviscid limit problem. Such a result is eluded in Wang (2001) and is somewhat surprising since this scale is smaller than any small scale predicted by conventional theory of turbulence or boundary layer theory.

The rest of the manuscript is organized as follows. In the next section we introduce the notion of appropriate truncation of the Navier-Stokes system and formulate our main result. We then compare the small scale in our theorem with other small scales predicted by conventional theory of turbulence and boundary layer theory. We then give a sketch of the proof of the main result in the third section, and we offer our concluding remarks in the last section.

2. The Main Result and Remarks. It is apparent that the convergence of numerical solutions of the Navier-Stokes system to that of the Euler system should not be expected for arbitrary truncation, but for suitable approximations of the Navier-Stokes system. Thus we need to introduce the notion of appropriate truncation. Also the problem involves several limits: time step, spatial scale and viscosity. The essential ingredients of an appropriate truncation are the consistency (as required by all convergent numerical schemes) and a bound on the truncated time averaged energy dissipation rate that is independent of the kinematic viscosity (as is consistent with the Kolmogorov theory, see for instance Doering and Gibbon 1995, Foias, Manley, Rosa and Temam 2001).

In order to focus on the main issue and for the sake of exposition, we consider flow in a 2D channel. Moreover, we consider discretization in the direction tangential to the boundary only (no time discretization or spatial discretization in the direction normal to the wall). This allows us to concentrate on the phenomena related to tangential (to the wall) spatial discretization only as it is the focus of our main result. The result stated here remains valid for 3D general domain with discretization in the directions tangential to the wall in a boundary layer done using local curvilinear coordinates, and the additional assumption that the Euler system possesses a smooth enough solution on that fixed time interval under consideration.

For the channel geometry with periodicity in the horizontal direction, it is natural to use Fourier spectral truncation in the horizontal direction and thus a natural (suitable) truncation would be the following Galerkin truncation

$$\frac{\partial \mathbf{u}^k}{\partial t} + P_k((\mathbf{u}^k \cdot \nabla)\mathbf{u}^k) - \nu_k \Delta \mathbf{u}^k + \nabla p^k = P_k \mathbf{f}, \tag{2.1}$$

$$\text{div } \mathbf{u}^k = 0, \tag{2.2}$$

$$\mathbf{u}^k|_{z=0,h} = P_k \mathbf{b}, \tag{2.3}$$

$$\mathbf{u}^k|_{t=0} = P_k \mathbf{u}_0 \tag{2.4}$$

where P_k is the projection onto the first K_k modes in x, i.e.

$$P_k \mathbf{u} = \sum_{|j| \leq K_k} e^{2\pi i j x/L} \mathbf{u}^j, \quad (\mathbf{u} = \sum_j e^{2\pi i j x/L} \mathbf{u}^j). \tag{2.5}$$

The consistency of such a truncation is obvious. An appropriate bound on the energy dissipation rate will be derived later in the next section.

Our main result is

Theorem 1. *Suppose that we have a smooth solution* $\mathbf{u}^0$ *of the Euler system (1.6) on the time interval* $[0,T]$[1]. *Let* $\mathbf{u}^k$ *be the solution of the truncated Navier-Stokes system (2.1) with kinematic viscosity* ν_k. *Assume that the following conditions are satisfied*

$$K_k \to \infty \quad (consistency) \tag{2.6}$$

$$\nu_k \to 0 \quad (vanishing\ viscosity) \tag{2.7}$$

$$K_k \frac{\nu_k}{LU} \to 0 \quad (under-resolved\ condition) \tag{2.8}$$

Then

$$\mathbf{u}^k \to \mathbf{u}^0. \tag{2.9}$$

More precisely, there exists a generic constant κ *independent of* k *such that*

$$\|\mathbf{u}^k - \mathbf{u}^0\|_{L^\infty(0,T;L^2)} \tag{2.10}$$

$$\leq \kappa((K_k\nu_k)^{\frac{1}{5}} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^2(0,T;H^1)} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)}). \tag{2.11}$$

The under-resolved condition (2.8) can be written in terms of the smallest scale, denoted l_s, resolved by the numerical method in the direction tangential to the boundary. Indeed, since $K_k\ l_s = L$, the under-resolved condition is equivalent to

$$\frac{\nu_k/U}{l_s} \to 0. \tag{2.12}$$

This means that scales of the order ν/U are not resolved in the scheme and this is what we mean by under-resolved situation.

The appearance of this small scale is a little bit surprising since it is smaller than any of the known scales predicted by conventional theory of turbulence and boundary layer theory. Here we recall a few well-known small scales (Foias, Manley, Rosa and Temam 2001, Doering and Gibbon 1995, Prandtl 1905, Frisch 1995 among others)

- Prandtl boundary layer thickness

$$\sqrt{\nu T} \tag{2.13}$$

- Kolmogorov dissipation length (3D)

$$(\frac{\nu^3}{\varepsilon})^{\frac{1}{4}} \sim \nu^{\frac{3}{4}} \tag{2.14}$$

  where ε is the energy dissipation rate per unit volume and is presumably independent of the kinematic viscosity.
- Kraichnan dissipation length (2D)

$$(\frac{\nu^3}{\eta})^{\frac{1}{6}} \sim \nu^{\frac{1}{2}} \tag{2.15}$$

  where η is the enstrophy dissipation rate per unit volume which is presumably independent of the kinematic viscosity.

[1]This is guaranteed in the 2D case with smooth enough data satisfying certain compatibility condition, see Temam 1975

- Taylor micro length

$$(\frac{\nu U^2}{\varepsilon})^{\frac{1}{2}} \tag{2.16}$$

Notice that these small length scales are all much bigger than ν/U. Even the thickness of a viscous sublayer ($\frac{\nu}{U}\log Re$) predicted by some boundary layer theory is bigger than $\frac{\nu}{U}$ at large Reynolds number. Thus, if one follows the conventional wisdom, one would just resolve the small scales predicted by conventional theory and thus the numerical results would indicate convergence of numerical solutions to that of the Euler system (see for instance Johnston, Liu and E (2000)).

Of course, ν/U appear as the natural small scale in certain circumstances such as the boundary layer thickness in the presence of suction at the boundary. The appearance of the thickness ν/U is directly related to the suction which makes the boundary layer thinner and stable (see Temam and Wang 2000, 2002). Even in that case, the scale of ν/U appears *only* in the direction *normal* to the boundary in the boundary layer.

The relevance of small scales of the order of ν/U to the inviscid limit problem was first discovered by Kato (1984) and was improved to the case of small scale of the order of ν/U in the directions tangential to the boundary in an appropriate boundary layer by Temam and Wang (1998) and Wang (2001). The main result here is basically a discrete version of the main result stated in Wang (2001) and thus a discrete Kato type result.

3. Sketch of the Proof. Throughout this section, κ will denote a generic constant independent of the kinematic viscosity ν or truncation wave number K_k.

Our proof is along the line of Kato (1984) and Temam and Wang (1998) with some modification. The basic idea is to construct a so called *background* flow (Hopf type technique, 1955) with a free parameter α which interpolates between the viscous sublayer (Kato type result) and laminar boundary layer (Prandtl theory).

For simplicity we will consider channel flow (flat boundary) and two dimensional case only. The case with curved boundary can be treated in the same way as in our previous work Temam and Wang (1998) using curvilinear coordinates. The three dimensional case is very similar to our work on energy dissipation rate Wang (2000).

Our approach is close to the idea of Vishik and Lyusternik (1957) (see also Lions 1973) in the sense that we seek a corrector which approximates the difference between the viscous and inviscid solution. Hence it is slightly different from Kato's (1984) approach.

Since we are interested in the asymptotic behavior of the solution $\mathbf{u}^k$ to the Galerkin truncated Navier-Stokes system (2.1), we naturally compare $\mathbf{u}^k$ to the spectral truncation of the solution to the Euler equation, namely $P_k\mathbf{u}^0$. Notice that $P_k\mathbf{u}^0$ satisfies the system

$$\frac{\partial}{\partial t}P_k\mathbf{u}^0 + P_k((P_k\mathbf{u}^0\cdot\nabla)P_k\mathbf{u}^0) + \nabla P_k p^0 = P_k\mathbf{f} + \mathbf{g}_k \tag{3.1}$$

$$\operatorname{div} P_k\mathbf{u}^0 = 0 \tag{3.2}$$

$$P_k\mathbf{u}^0\cdot\mathbf{n}|_{z=0,h} = 0 \tag{3.3}$$

$$P_k\mathbf{u}^0|_{t=0} = P_k\mathbf{u}_0 \tag{3.4}$$

where

$$\mathbf{g}_k = -P_k(((I-P_k)\mathbf{u}^0\cdot\nabla)\mathbf{u}^0) - P_k((P_k\mathbf{u}^0\cdot\nabla)(I-P_k)\mathbf{u}^0). \tag{3.5}$$

It is easy to see that $\mathbf{g}_k$ is small for large k due to the consistency assumption and the smoothness assumption on the inviscid solution $\mathbf{u}^0$. Indeed

$$\begin{aligned}\|\mathbf{g}_k\|_{L^2} &\le \kappa(\|\nabla\mathbf{u}^0\|_{L^\infty}\|(I-P_k)\mathbf{u}^0\|_{L^2}+\|P_k\mathbf{u}^0\|_{L^\infty}\|\nabla(I-P_k)\mathbf{u}^0\|_{L^2}\\ &\le \kappa\|\nabla\mathbf{u}^0\|_{L^\infty}\|(I-P_k)\mathbf{u}^0\|_{H^1}.\end{aligned} \tag{3.6}$$

We now follow the strategy of the continuous case and compare $\mathbf{u}^k$ to $P_k\mathbf{u}^0$ with the aid of a corrector (background flow). For this purpose we need to first establish upper bound on the energy dissipation rate independent of the kinematic viscosity for the truncated Navier-Stokes system (2.1) just as in the continuous case.

Let ϕ be a fixed (smooth) incompressible flow that matches $\mathbf{b}$ on the boundary of the domain. The existence of such flows is classical (see for instance the text book by Temam 2001, Wang 2001).

Consider

$$\mathbf{v}^k = \mathbf{u}^k - P_k\phi.$$

We then deduce that $\mathbf{v}^k$ satisfies the following system

$$\begin{aligned}\frac{\partial\mathbf{v}^k}{\partial t}+P_k((\mathbf{v}^k\cdot\nabla)\mathbf{v}^k)+P_k((\mathbf{v}^k\cdot\nabla)P_k\phi)+P_k((P_k\phi\cdot\nabla)\mathbf{v}^k)-\nu_k\Delta\mathbf{v}^k+\nabla p^k&\\ = P_k\mathbf{f}-\frac{\partial}{\partial t}P_k\phi-P_k((P_k\phi\cdot\nabla)P_k\phi)+\nu_k\Delta P_k\phi&,\\ \text{div } \mathbf{v}^k = 0&,\\ \mathbf{v}^k|_{z=0,h} = 0&,\\ \mathbf{v}^k|_{t=0} = P_k(\mathbf{u}_0-\phi(0))&.\end{aligned}$$

Multiplying both sides by $\mathbf{v}^k$, integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\mathbf{v}^k|^2_{L^2}+\nu_k|\nabla\mathbf{v}^k|^2_{L^2} &\le |\nabla P_k\phi|_{L^\infty}|\mathbf{v}^k|^2_{L^2}\\ +\kappa(|\mathbf{f}|_{L^2}+|\frac{\partial\phi}{\partial t}|_{L^2}&+|\phi|_{H^2}|\nabla\phi|_{L^2}+\nu_k|\Delta\phi|_{L^2})|\mathbf{v}^k|_{L^2}\end{aligned}$$

which implies

$$\|\mathbf{v}^k\|_{L^\infty(0,T;L^2)} \le \kappa$$

which further implies

$$\nu_k\int_0^T|\nabla\mathbf{v}^k|^2_{L^2}\,dt \le \kappa$$

where κ is a constant independent of k (or ν_k). Since $\mathbf{u}^k$ and $\mathbf{v}^k$ differ by $P_k\phi$, we also have

$$\nu_k\int_0^T|\nabla\mathbf{u}^k|^2_{L^2}dt \le \kappa. \tag{3.7}$$

Next we move onto the issue of convergence of $\mathbf{u}^k$ to $\mathbf{u}^0$ under the under-resolved condition. We first introduce a corrector (background flow) just as in the continuous case. The key idea, in addition to the ones that we had for the continuous case, is a reverse Poincaré inequality which implies smallness of energy dissipation rate due to the tangential derivative of the flow.

Define the stream function

$$\psi^k(x,z,t) = P_k(b_1(x,0,t) - u_1^0(x,0,t)) \int_0^z \rho(\frac{\alpha U s}{\nu_k})\, ds, \tag{3.8}$$

where the cut-off function ρ satisfies the following properties

$$\begin{aligned} \rho &\in C^\infty[0,\infty), \\ \rho(0) &= 1, \\ \rho'(0) &= 0, \\ \operatorname{supp} \rho &\subset [0,1), \\ \int_0^1 \rho &= 0, \\ |\rho|_{L^\infty} &\le 1, \\ |\rho'|_{L^\infty} &\le 2, \end{aligned}$$

and the typical velocity U is defined as

$$U = \sup_k \max_{[0,T]\times\Gamma} \{|P_k(b_1 - u_1^0)|\}. \tag{3.9}$$

The corresponding velocity field is

$$\theta^k(x,z,t) = \operatorname{curl} \psi^k(x,z,t) = (\frac{\partial \psi^k}{\partial z}, -\frac{\partial \psi^k}{\partial x}). \tag{3.10}$$

The typical velocity defined here is a natural generalization of the continuous one ($\max_{[0,T]\times\Gamma} |b_1 - u_1^0|$) to this truncated case. This new typical velocity dominates the continuous version since we have, for smooth enough $b_1 - u_1^0$,

$$\lim_{k\to\infty} P_k(b_1 - u_1^0) = b_1 - u_1^0.$$

Next, we consider the adjusted differences

$$\mathbf{w}^k = \mathbf{u}^k - P_k \mathbf{u}^0 - \theta^k. \tag{3.11}$$

Our goal is to prove $\mathbf{w}^k \to 0$ which implies our final result since $\theta^k \to 0$ in $L^\infty(0,T;L^2)$ and $P_k\mathbf{u}^0 \to \mathbf{u}^0$ in $L^\infty(0,T;L^2)$ as k approaches infinity.

It is easy to verify that $\mathbf{w}^k$ satisfies

$$\begin{aligned} \frac{\partial \mathbf{w}^k}{\partial t} + P_k((\mathbf{u}^k\cdot\nabla)\mathbf{w}^k) - \nu_k\Delta\mathbf{w}^k + \nabla q^k &= -\frac{\partial\theta^k}{\partial t} + \nu_k\Delta\mathbf{u}^0 + \nu_k\Delta\theta^k \\ &\quad -P_k((\theta^k\cdot\nabla)\theta^k) - P_k((\mathbf{w}^k\cdot\nabla)\theta^k) - P_k((\mathbf{u}^0\cdot\nabla)\theta^k) \\ &\quad -P_k((\mathbf{w}^k\cdot\nabla)P_k\mathbf{u}^0) - P_k((\theta^k\cdot\nabla)P_k\mathbf{u}^0) + \mathbf{g}_k \end{aligned} \tag{3.12}$$

$$\operatorname{div} \mathbf{w}^k = 0, \tag{3.13}$$

$$\mathbf{w}^k\big|_{z=0,h} = 0, \tag{3.14}$$

$$\mathbf{w}^k\big|_{t=0} = 0. \tag{3.15}$$

Thanks to the explicit construction of our θ^k, we have

$$
\begin{aligned}
|\frac{\partial\theta^k}{\partial t}|_{L^2}^2 &\le U_t^2\frac{L\nu_k}{\alpha U}+U_{tx}^2\frac{L\nu_k^3}{\alpha^3U^3},\\
|\nabla\theta^k|_{L^2}^2 &\le 2U_x^2\frac{L\nu_k}{\alpha U}+U^2\frac{L\alpha U}{\nu_k}+U_{xx}^2\frac{L\nu_k^3}{\alpha^3U^3},\\
|P_k(\theta^k\cdot\nabla)\theta^k|_{L^2}^2 &\le 5U^2U_x^2\frac{L\nu_k}{\alpha U}+U^2U_{xx}^2\frac{L\nu_k^3}{\alpha^3U^3}+U_x^4\frac{L\nu_k^3}{\alpha^3U^3},\\
|P_k(P_k\mathbf{u}^0\cdot\nabla)\theta^k|_{L^2}^2 &\le 2(|P_ku_1^0|_{L^\infty}^2|\frac{\partial\theta^k}{\partial x}|_{L^2}^2+|\frac{P_ku_2^0}{z(h-z)}|_{L^\infty}^2|z(h-z)\frac{\partial\theta^k}{\partial z}|_{L^2}^2)\\
&\le \kappa(\frac{L\nu_k}{\alpha U}+\frac{L\nu_k^3}{\alpha^3U^3}),\\
|P_k(\mathbf{w}^k\cdot\nabla)P_k\mathbf{u}^0|_{L^2} &\le |\nabla P_k\mathbf{u}^0|_{L^\infty}|\mathbf{w}^k|_{L^2},\\
|P_k(\theta^k\cdot\nabla)P_k\mathbf{u}^0|_{L^2}^2 &\le |\nabla P_k\mathbf{u}^0|_{L^\infty}^2(U^2\frac{L\nu_k}{\alpha U}+U_x^2\frac{L\nu_k^3}{\alpha^3U^3})\\
&\le \kappa|\mathbf{u}^0|_{H^3}^2(U^2\frac{L\nu_k}{\alpha U}+U_x^2\frac{L\nu_k^3}{\alpha^3U^3})
\end{aligned}
$$

where we have used the impermeable wall boundary condition (3.3), and

$$
\begin{aligned}
U_t &= \sup_k \max_{[0,T]\times\Gamma}\{|P_k(\frac{\partial b_1}{\partial t}-\frac{\partial u_1^0}{\partial t})|\},\\
U_x &= \sup_k \max_{[0,T]\times\Gamma}\{|P_k(\frac{\partial b_1}{\partial x}-\frac{\partial u_1^0}{\partial x})|\},\\
U_{tx} &= \sup_k \max_{[0,T]\times\Gamma}\{|P_k(\frac{\partial^2 b_1}{\partial x\partial t}-\frac{\partial^2 u_1^0}{\partial x\partial t})|\},\\
U_{xx} &= \sup_k \max_{[0,T]\times\Gamma}\{|P_k(\frac{\partial^2 b_1}{\partial x^2}-\frac{\partial^2 u_1^0}{\partial x^2})|\}.
\end{aligned}
$$

We then deduce, via standard energy method,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\mathbf{w}^k|_{L^2}^2+\nu_k|\nabla\mathbf{w}^k|_{L^2}^2 &\le \nu_k\sqrt{2U_x^2\frac{L\nu_k}{\alpha U}+U^2\frac{L\alpha U}{\nu_k}+U_{xx}^2\frac{L\nu_k^3}{\alpha^3U^3}}|\nabla\mathbf{w}^k|_{L^2}\\
&\quad+\nu_k^2|\Delta\mathbf{u}^0|_{L^2}^2+\kappa|\mathbf{w}^k|_{L^2}^2+\kappa|\mathbf{u}^0-P_k\mathbf{u}^0|_{H^1}^2\\
&\quad+\kappa(\frac{\nu_k}{\alpha}+\frac{\nu_k^3}{\alpha^3})+\int_\Omega(\mathbf{w}^k\cdot\nabla)\mathbf{w}^k\cdot\theta^k.
\end{aligned}\tag{3.16}
$$

Notice the last (nonlinear) term can be rewritten as

$$
\int_\Omega(\mathbf{w}^k\cdot\nabla)\mathbf{w}^k\cdot\theta^k=\int_\Omega w_1^k\frac{\partial w_1^k}{\partial x}\theta_1^k+\int_\Omega w_3^k\frac{\partial w_1^k}{\partial z}\theta_1^k+\int_\Omega w_1^k\frac{\partial w_3^k}{\partial x}\theta_3^k+\int_\Omega w_3^k\frac{\partial w_3^k}{\partial z}\theta_3^k,\tag{3.17}
$$

and hence we have the following estimates on the nonlinear term, thanks to the explicit construction of the corrector (see Wang 2001),

$$
\begin{aligned}
2\int_\Omega w_1^k\frac{\partial w_1^k}{\partial x}\theta_1^k &= \int_\Omega\frac{\partial}{\partial x}(w_1^k)^2\theta_1^k\\
&= -\int_\Omega(w_1^k)^2\frac{\partial\theta_1^k}{\partial x}\\
&\le U_x|w_1^k|_{L^2}^2,
\end{aligned}
$$

$$2\int_\Omega w_3^k \frac{\partial w_1^k}{\partial z}\theta_1^k \le 2U|w_3^k|_{L^2(\Gamma_\delta)}|\frac{\partial w_1^k}{\partial z}|_{L^2(\Gamma_\delta)}$$
$$\le 2U\delta|\frac{\partial w_3^k}{\partial z}|_{L^2(\Gamma_\delta)}|\frac{\partial w_1^k}{\partial z}|_{L^2(\Gamma_\delta)}$$
$$= \frac{2\nu}{\alpha}|\frac{\partial w_1^k}{\partial x}|_{L^2(\Gamma_\delta)}|\frac{\partial w_1^k}{\partial z}|_{L^2(\Gamma_\delta)}$$
$$\le \frac{\nu}{4}|\frac{\partial w_1^k}{\partial z}|^2_{L^2(\Gamma_\delta)} + \frac{4\nu}{\alpha^2}|\frac{\partial w_1^k}{\partial x}|^2_{L^2(\Gamma_\delta)},$$

$$2\int_\Omega w_1^k \frac{\partial w_3^k}{\partial x}\theta_3^k \le \kappa\frac{\nu}{\alpha}|w_1^k|_{L^2(\Gamma_\delta)}|\frac{\partial w_3^k}{\partial x}|_{L^2(\Gamma_\delta)}$$
$$\le \frac{\nu}{4}|\frac{\partial w_3^k}{\partial x}|^2_{L^2(\Gamma_\delta)} + \kappa\frac{\nu}{\alpha^2}|w_1^k|^2_{L^2(\Gamma_\delta)}.$$

Similarly

$$2\int_\Omega w_3^k \frac{\partial w_3^k}{\partial z}\theta_3^k \le \frac{\nu}{4}|\frac{\partial w_3^k}{\partial z}|^2_{L^2(\Gamma_\delta)} + \kappa\frac{\nu}{\alpha^2}|w_3^k|^2_{L^2(\Gamma_\delta)}.$$

Thus we have

$$\int_\Omega (\mathbf{w}^k\cdot\nabla)\mathbf{w}^k\cdot\theta^k \le \frac{\nu_k}{4}|\nabla \mathbf{w}^k|^2_{L^2} + \frac{4\nu_k}{\alpha^2}|\frac{\partial w_1^k}{\partial x}|_{L^2(\Gamma_\delta)} + \kappa\frac{\nu_k}{\alpha^2}|\mathbf{w}^k|^2_{L^2} + U_x|\mathbf{w}^k|^2_{L^2} \tag{3.18}$$

where

$$\delta = \frac{\nu_k}{\alpha U} \tag{3.19}$$

is the thickness of the boundary layer.

We now make the following assumption on the free parameter α (and thus δ)

$$\alpha = \alpha_k \to 0, \text{ as } k\to\infty; \quad \text{and } \frac{\nu_k}{\alpha_k^2} \le 1. \tag{3.20}$$

The first part of the condition is equivalent to saying that the chosen boundary layer must be thicker than ν_k/U since $\delta_k = \frac{\nu_k}{\alpha_k U}$, and the second part of the assumption is equivalent to saying that the thickness of the chosen boundary layer is at most that of the laminar boundary layer $\sqrt{\nu T}$ since

$$\frac{\delta_k^2}{\nu_k} = \frac{\nu_k}{\alpha_k^2 U^2}.$$

The condition also implies that

$$\frac{\nu_k}{\alpha_k} = U\delta_k \to 0, \text{ as } k\to\infty.$$

It is then easy to see, that under the assumption on the parameter (3.20), together with the vanishing viscosity condition (2.7), and the key estimate on trilinear term (3.18), the energy inequality on $\mathbf{w}^k$ becomes,

$$\frac{d}{dt}|\mathbf{w}^k|^2_{L^2} + \nu_k|\nabla\mathbf{w}^k|^2_{L^2} \le \kappa(|\mathbf{w}^k|^2_{L^2} + |\mathbf{u}_0 - P_k\mathbf{u}_0|^2_{H^1} + \frac{\nu_k}{\alpha}) + \alpha L U^3 + \frac{8\nu_k}{\alpha^2}|\frac{\partial w_1^k}{\partial x}|^2_{L^2(\Gamma_\delta)}$$

which implies, after utilizing Gronwall inequality,

$$\|\mathbf{w}^k\|_{L^\infty(0,T;L^2)} \le \kappa(\sqrt{\frac{\nu_k}{\alpha}} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^2(0,T;H^1)} + (\alpha LU^3 + \frac{8\nu_k}{\alpha^2}\frac{1}{T}\int_0^T\int_{\Gamma_{\delta_k}} |\frac{\partial w_1^k}{\partial x}|^2)^{\frac{1}{2}}).$$

Therefore

$$\begin{aligned}\|\mathbf{u}^k - \mathbf{u}^0\|_{L^\infty(0,T;L^2)} &\le \|\mathbf{w}^k\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)} + \|\theta^k\|_{L^\infty(0,T;L^2)} \\ &\le \kappa(\sqrt{\frac{\nu_k}{\alpha}} + \|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^2(0,T;H^1)} + \|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^\infty(0,T;L^2)}) \\ &\quad +\kappa(\alpha LU^3 + \frac{8\nu_k}{\alpha^2}\frac{1}{T}\int_0^T\int_{\Gamma_\delta} |\frac{\partial w_1^k}{\partial x}|^2)^{\frac{1}{2}} \qquad (3.21)\end{aligned}$$

Here α is a free parameter that we may adjust provided the constraints specified in (3.20) are met.

Next, we estimate the integral on the right hand side of (3.21) as follows. Notice that

$$\begin{aligned}\nu_k\int_0^T\int_{\Gamma_\delta}|\frac{\partial}{\partial x}u_1^k|^2 &\le 2\nu_k\int_0^T\int_{\Gamma_\delta}|\frac{\partial}{\partial x}(u_1^k - P_k\phi_1)|^2 + 2\nu_k\int_0^T\int_{\Gamma_\delta}|\frac{\partial}{\partial x}P_k\phi_1|^2 \\ &\le 2\nu_k\delta^2\int_0^T\int_{\Gamma_\delta}|\frac{\partial^2}{\partial x\partial z}(u_1^k - P_k\phi_1)|^2 + \kappa\nu_k\delta \\ &\le \kappa\nu_k\delta^2K_k^2\int_0^T\int_{\Gamma_\delta}|\frac{\partial}{\partial z}(u_1^k - P_k\phi_1)|^2 + \kappa\nu_k\delta \\ &\le \kappa\nu_k\delta^2K_k^2\int_0^T\int_{\Gamma_\delta}|\frac{\partial u_1^k}{\partial z}|^2 + \kappa\nu_k\delta^2K_k^2\int_0^T\int_{\Gamma_\delta}|\frac{\partial}{\partial z}P_k\phi_1|^2 + \kappa\nu_k\delta \\ &\le \kappa(\delta^2K_k^2 + \nu_k\delta) \\ &\le \kappa(\frac{\nu_k^2K_k^2}{\alpha^2} + \frac{\nu_k^2}{\alpha})\end{aligned}$$

where we have applied the direct and inverse Poincaré inequality, and utilized the bound on energy dissipation rate (3.7). This further implies,

$$\begin{aligned}\nu_k\int_0^T\int_{\Gamma_\delta}|\frac{\partial w_1^k}{\partial x}|^2 &\le 2\nu_k\int_0^T\int_{\Gamma_\delta}|\frac{\partial u_1^k}{\partial x}|^2 + 2\nu_k\int_0^T\int_{\Gamma_\delta}|\frac{\partial(P_k\mathbf{u}^0 - \theta^k)}{\partial x}|^2 \\ &\le \kappa(\delta^2K_k^2 + \nu_k\delta) \\ &\le \kappa(\frac{\nu_k^2K_k^2}{\alpha^2} + \frac{\nu_k^2}{\alpha}).\end{aligned}$$

We may then rewrite the estimates on $\mathbf{u}^k - \mathbf{u}^0$ as

$$\begin{aligned}\|\mathbf{u}^k - \mathbf{u}^0\|_{L^\infty(0,T;L^2)} &\le \kappa(\|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^2(0,T;H^1)} + \|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^\infty(0,T;L^2)}) \\ &\quad +\kappa(\frac{\nu_k}{\alpha} + \alpha + \frac{\nu_k^2K_k^2}{\alpha^4} + \frac{\nu_k^2}{\alpha^3})^{\frac{1}{2}} \\ &\le \kappa(\|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^2(0,T;H^1)} + \|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^\infty(0,T;L^2)}) \\ &\quad +\kappa(\alpha + \frac{\nu_k^2K_k^2}{\alpha^4})^{\frac{1}{2}} \qquad (3.22)\end{aligned}$$

since ν_k/α is dominated by α as $\frac{\nu_k/\alpha}{\alpha} = \frac{\nu_k}{\alpha^2} \le 1$, while $\frac{\nu_k^2}{\alpha^3}$ is dominated by $\frac{\nu_k}{\alpha}$ as $\frac{\nu_k^2/\alpha^3}{\nu_k/\alpha} = \frac{\nu_k}{\alpha^2} \le 1$.

The last piece of work is to choose an appropriate α which minimizes the expression $\alpha + \frac{\nu_k^2 K_k^2}{\alpha^4}$. This is roughly accomplished if we set

$$\alpha = \alpha_k = \left(\frac{\nu_k K_k}{LU}\right)^{\frac{2}{5}}. \tag{3.23}$$

Obviously α_k approaches zero as k approaches infinity thanks to the under-resolved condition (2.8). Moreover,

$$\frac{\nu_k}{\alpha_k^2} = \nu_k^{\frac{1}{5}} K_k^{-\frac{4}{5}} (LU)^{\frac{4}{5}} \to 0, \text{ as } k \to \infty$$

thanks to the consistency condition (2.6) and the vanishing viscosity condition (2.7). Thus the α determined by (3.23) satisfies the constraint (3.20) and hence is allowed.

In the last step, we plug (3.23) into (3.22) and we deduce

$$\|\mathbf{u}^k - \mathbf{u}^0\|_{L^\infty(0,T;L^2)} \le \kappa(\|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^2(0,T;H^1)} + \|\mathbf{u}_0 - P_k\mathbf{u}_0\|_{L^\infty(0,T;L^2)} + (\nu_k K_k)^{\frac{1}{5}}) \tag{3.24}$$

which is exactly what we desired. This ends the proof.

4. Conclusion and Remarks. We have shown that if small scales of the order $\frac{\nu}{U}$ were not resolved in the direction tangential to the boundary in numerical scheme for NSE (1.1), the numerical solutions will always converge to the solution of the Euler system (1.6) at vanishing viscosity and mesh size for any suitable (reasonable) numerical scheme. Numerical results performed by Johnston, Liu and E (2000) as well as ours confirm this fact. Of course the numerics can be interpreted in two different ways.

1. No small scales of the order $\frac{\nu}{U}$ or smaller are detected in the numerical experiment, and thus numerics provide further evidence that the inviscid limit of viscous flows is the inviscid Euler flow.
2. Small scales of the order $\frac{\nu}{U}$ are not resolved in the numerics and thus the numerical solutions must converge to the solution of the inviscid Euler system (1.6) regardless whether the solutions of the Navier-Stokes system (1.1) converge to the solution of the Euler system at vanishing viscosity. In another word, the numerical results may have nothing to do with the continuous problem.

This indicates that in order to guarantee that the convergence of the numerical solutions implies the convergence of the continuous solutions, i.e., providing an affirmative answer to (1.11), small scales of the order of $\frac{\nu}{U}$ must be resolved in the numerical scheme. This gives us a flavor on the difficulty of the problem of numerical investigation of the vanishing viscosity problem.

A natural question to ask then is what is the smallest scale that has to be resolved in the numerics in order to ensure that convergence of numerical solutions imply convergence of continuous solutions , i.e., we have an affirmative answer to (1.11). It is natural to speculate that the smallest scale needs to be resolved is of the order of $\frac{\nu}{U}$. This small scale can be inferred from several results including the result we proved here, and in terms of determining modes, nodes and dimension of global attractors (see for instance Foias, Manley, Rosa and Temam 2001, Doering and Gibbon 1995). Unfortunately we can not establish such a small scale in a rigorous fashion. What we can prove is that if we resolve an exponentially small scale ($L\, exp(-c_0 \frac{\nu_k}{LU})$), then $\mathbf{u^k} \to \mathbf{u^0}$ does imply $\mathbf{u}^{\nu_\mathbf{k}} \to \mathbf{u^0}$. Of course such a small scale is physically irrelevant. The appearance of such a small scale is due to the very presence of boundary layer and is typical in rigorous analysis of wall bounded flows (see for instance Temam 1997, Foias, Manley, Rosa and Temam 2001). It still remains a great challenge to establish that the effective smallest scale is an algebraic function of the Reynolds number.

We also remark that a similar result involving small scales in the direction normal to the boundary in the boundary layer can be derived as well.

THEOREM 2. *If the smallest scales resolved in the direction normal to the boundary in a thick enough boundary layer is at most of the order of ν/U, then we always observe numerical convergence of the solutions to the suitably truncated Navier-Stokes system to that of the Euler system at vanishing viscosity and mesh size.*

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REFERENCES

[1] J. BONA AND J. WU, *The zero viscosity limit of the two dimensional Navier-Stokes equations*, Stud. Appl. Math. 109, no.4 (2002), pp265-278.

[2] P. CONSTANTIN AND C. FOIAS, *Navier-Stokes Equations,* Chicago University Press, 1988.

[3] CH. DOERING AND J.D. GIBBON, *Applied Analysis of the Navier-Stokes Equations,* Cambridge University Press, 1995.

[4] W. E, *Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equations,* Acta Math Sinica, vol. 16, No. 2 (2000), pp. 207-218.

[5] W. E AND B. ENGQUIST, *Blow-up of solutions of the unsteady Prandtl's equation,* Comm. Pure Appl. Math. 50, no. 12 (1997), 1287–1293.

[6] C. FOIAS, O. MANLEY, R. ROSA AND R. TEMAM, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, United Kingdom, 2001.

[7] U. FRISCH, *Turbulence*, Cambridge University Press, Cambridge, UK, 1995.

[8] E. HOPF, *On nonlinear partial differential equations*, Lecture Series of the Symposium on Partial Differential Equations, Berkeley, 1955 (The Univ. of Kansas, ed.), 1957, pp.1-29.

[9] H. JOHNSTON, J.-G. LIU AND W. E A Numerical Study of the Zero-Viscosity Limit of the Flow Around a Cylinder, in preparation, 2000.(*www.math.temple.edu/ ∼ hej/*)

[10] TH. VON KÁRMÁN, 1930, *Mechanische Ähnlichkeit und Turbulenz*, in *Collected Works of Theodore von Kármán*, vol. 2, 337-346, London, Butterworths Scientific Publications, 1956.

[11] T. KATO, *Remarks on the zero viscosity limit for nonstationary Navier-Stokes flows with boundary,* in Seminar on PDE, edited by S.S. Chern, Springer, NY, 1984.

[12] O.A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flows,2ed* Gordon and Breach, New York, 1969.

[13] J.L. LIONS , *Perturbations singulières dans les problèmes aux limites et en controle optimal,* Lecture Notes in Math., vol 323, Springer-Verlag, New York, 1973.

[14] S. MATSUI, *Example of zero viscosity limit for two dimensional nonstationary Navier-Stokes flows with boundary,* Japan J. Industrial and Appl. Math., 11 (1994), pp. 155 - 173.

[15] O. OLEINIK, *The Prandtl system of equations in boundary layer theory,* Soviet Math. Dokl., 4 (1963), pp. 583 - 586.

[16] O. Oleinik and V. Samokhin, *Mathematical models in boundary layer theory*, Chapman and Hall/CRC, Boca Raton, FL, 1999.

[17] L. Prandtl, *Verber Flüssigkeiten bei sehr kleiner Reibung*, Verk. III Intem. Math. Kongr. Heidelberg (1905), pp. 484-491, Teuber, Leibzig.

[18] M. Sammartino and R. E. Calflisch, *Zero viscosity limit for analytic solutions of the Navier-Stokes equations*, Proceedings of the VIII International Conference on Waves and Stability in Continuous Media, Part II (Palermo, 1995). Rend. Circ. Mat. Palermo (2) Suppl. No. 45, part II (1996), pp. 595-605.

[19] H. Schlichting, *Boundary Layer Theory, 7ed*, McGraw-Hill, New York, 1979.

[20] R. Temam, *On the Euler equations of incompressible perfect fluids*, J. Funct. Anal., 20 (1975), pp. 32-43.

[21] R. Temam, *Navier-Stokes Equations*, American Mathematical Society, Rhode Island, 2001.

[22] R. Temam and X. Wang, *Asymptotic Analysis of Oseen Type Equations in a Channel at Small Viscosity*, Indiana University Mathematics Journal, **45** (1996), pp. 863-916.

[23] R. Temam and X. Wang, *On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity*, Annali della Scuola Normale Superiore de Pisa, volume dedicated to the memory of E. De Giorgi, Serie IV, Vol. **XXV** (1998), 807-828.

[24] R. Temam and X. Wang, *Remarks on the Prandtl type equations with permeable wall*, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM). 80 (2000), 11-12, 835-843.

[25] R. Temam and X. Wang, *Boundary Layer Associated with the Incompressible Navier-Stokes Equations: the non-characteristic boundary case*, J. Diff. Eqs. , Vol.179 (2002), 647-686.

[26] M. Vishik and Lyusternik, *Regular degeneration and boundary layer for linear differential equations with small parameter*, Uspekki Mat. Nauk, vol 12 (1957), pp. 3-122.

[27] X. Wang, *Effect of tangential derivatives in the boundary layer on time averaged energy dissipation rate,* Physica D 144 (2000) 142-153.

[28] X. Wang, *A Kato Type Theorem on Zero Viscosity Limit of Navier-Stokes Flows*, Indiana University Mathematics Journal, Vol.50, No.1 (2001), 223-241.

[29] Z. Xin and L. Zhang, *On the global existence of solutions to the Prandtl's system*. Adv. Math. 181 (2004), no. 1, 88–133.