Degrees of Generators of Ideals Defining Subschemes of Projective Space

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In this paper, we are interested in giving an upper bound for the degrees of the generators for an ideal defining a curve in projective space, and in investigating the properties of curves for which this bound is realized. While our bound works for subschemes of any dimension in any projective space, our characterizations for when the bound is realized only work for curves in $\mathbb{P}^3$. Using the known structure theory for liaison classes of curves in $\mathbb{P}^3$ allows us to give a reasonably complete picture of the liaison classes containing the extremal curves, mainly in terms of the dimensions of the components of the deficiency module for the liaison class. We are also able to give some similar conditions for when a liaison class contains an integral curve satisfying our upper bound for degrees of generators.

In contrast to the minimal degree of a generator, in most cases, the maximal degree is not readily computed in terms of other invariants, for example, the Hilbert function. There is, of course, Castelnuovo–Mumford regularity, [Mu], bounding the degrees of generators in terms of the cohomology modules (and more generally, bounding the degrees of generators of the syzygies). Another result which gives information about a certain class of curves is in [DGM], where they show that the defining ideals of arithmetically Cohen–Macaulay curves have generators whose degree is bounded above by one plus the last integer for which the second difference of the Hilbert function is non-zero. Furthermore, in [CGO], most arithmetically Cohen–Macaulay curves achieving this bound are classified; generally speaking, with some exceptions, the arithmetically Cohen–Macaulay curves whose ideals have a generator of maximal degree are linked in the minimal number of steps to plane curves. In that paper, however, they note that little is known about the non-arithmetically Cohen–Macaulay case, or the non-codimension two case. This is what sparked our interest in the problem, and we offer a relatively complete solution in the present paper, at least for the case of curves.

Now, we describe the contents of this paper more precisely. In the first section, we set up our notation and state and prove the upper bound for curves in $\mathbb{P}^n$ (Proposition 1.4), and more generally for subschemes of arbitrary codimension in $\mathbb{P}^n$ (Proposition 1.10). We also interpret the bound as a condition on the cohomology of the curve, which we designate *equal cohomology*. In the second section, we specialize to $\mathbb{P}^3$, and study the even liaison classes of curves in an effort to determine those classes having a curve with the equal cohomology property necessary for having a high degree generator. This classification depends heavily on the structure theory for even liaison classes. Our main result of this section is Theorem 2.5, which we use to characterize the liaison classes having curves with equal cohomology, in terms of the Hilbert function of the minimal curve in the class. In the third section, we show that the curves with equal cohomology enjoy a strong Lazarsfeld–Rao structure, in the sense that liaison classes of such curves have minimal curves, and the other curves are obtained by basic double links of low degree and deformations. The fourth section is devoted to describing which of the curves with equal cohomology can possibly be integral, by using the machinery developed by Nollet of examining the postulation characters of curves. Finally, in the last section, we reconsider the problem of generators of high degree, and interpret our previous results in this context. As a result, we are able to give a mostly complete picture in terms of cohomology of which curves have high degree generators. This picture is particularly clear for the Buchsbaum curves, since their cohomology is quite well-known. We are also able to give some necessary conditions for integral curves to have generators of high degree, and again, the Buchsbaum case provides the best statement.

1 Degrees of Generators

Throughout this paper, we will work over an algebraically closed field k, of arbitrary characteristic. Let $S = k[x_0, \ldots, x_n]$ be the polynomial ring over k, and $\mathbb{P}^n$ the n-dimensional projective space over k. We will furthermore consider only subschemes of $\mathbb{P}^n$ which are locally Cohen–Macaulay and equidimensional. It is well-known that this is equivalent to requiring that all the intermediate cohomology modules $H^i_*(\mathbb{P}^n, V)$, $1 \leq i \leq \dim V$, have finite length.

Given a subscheme V of $\mathbb{P}^n$, with $\dim V = d$, let I_V denote its homogeneous, saturated defining ideal in S. Thus S/I_V is a standard graded k-algebra, and so we can define the Hilbert function of

S/I_V by

$$H(S/I_V,t) = \dim_k [S/I_V]_t.$$

Alternatively, we sometimes write $H(V,t)$ for $H(S/I_V,t)$. It is a standard fact that there is a polynomial $P(S/I_V,t)$, having degree d, such that $H(S/I_V,t) = P(S/I_V,t)$ for all $t \gg 0$. We furthermore define the nth difference of $H(S/I_V,t)$ inductively as follows:

$$\begin{aligned} \Delta^1 H(S/I_V,t) &= H(S/I_V,t) - H(S/I_V,t-1) \\ \Delta^n H(S/I_V,t) &= \Delta^1(\Delta^{n-1} H(S/I_V,t)). \end{aligned}$$

Now, since $H(S/I_V,t)$ is eventually a polynomial of degree d, the function $\Delta^{d+1}H(S/I_V,t)$ is eventually zero, and we define

$$\sigma(S/I_V) = \min\{\, k : \Delta^{d+1} H(S/I_V,t) = 0 \text{ for all } t \geq k \,\}.$$

Again, we will sometimes write $\sigma(V)$ for $\sigma(S/I_V)$. It is worth noting that if $\dim V = d$, then the Hilbert function of V and the Hilbert polynomial of V are equal in all degrees $\geq t+d+1$ if and only if $\sigma(V) = t$. The degree at which the Hilbert function and the Hilbert polynomial agree from then on is sometimes called, at least in the context of local algebra, the postulation number, and has played an important role in questions about Cohen–Macaulayness and related invariants in local rings.

Given an ideal I, we will write $\alpha(I)$ for the minimal degree of a minimal generator, and $\omega(I)$ for the maximal degree of a minimal generator of I.

Next, let $H^i_*(\mathbb{P}^n, \mathcal{I}_V) = \oplus_{s\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{I}_V(s))$ be the cohomology modules of V. We will put $h^i(\mathbb{P}^n, \mathcal{I}_V(t)) = \dim_k H^i(\mathbb{P}^n, \mathcal{I}_V(t))$. We let $e(V) = \max\{\, t : h^{d+1}(\mathbb{P}^n, \mathcal{I}_V(t)) \neq 0 \,\}$ denote the index of speciality of V. Also, note that by our assumption that subschemes be locally Cohen–Macaulay and equidimensional, $h^i(\mathbb{P}^n, \mathcal{I}_V(t))$ is non-zero for only finitely many t, when $1 \leq i \leq d$. Hence for a non-arithmetically Cohen–Macaulay curve C in $\mathbb{P}^n$, with notation following Martin-Deschamps and Perrin, we can define

$$r_a(C) = \min\{\, n \in \mathbb{Z} : h^1(\mathbb{P}^n, \mathcal{I}_C(n)) \neq 0 \,\} \qquad r_o(C) = \max\{\, n \in \mathbb{Z} : h^1(\mathbb{P}^n, \mathcal{I}_C(n)) \neq 0 \,\},$$

and $\operatorname{diam} H^1_*(\mathbb{P}^n, C) = r_o(C) - r_a(C) + 1$, the number of components between the first and the last non-zero components, inclusive. Note that some of the intermediate components may have dimension zero, but we also count them. If C is arithmetically Cohen–Macaulay, then $H^1_*(\mathbb{P}^3, C) = 0$, and we will put $\operatorname{diam} H^1_*(\mathbb{P}^3, C) = 0$.

In this section, we prove a statement about the maximal degree of a generator for the defining ideal of a curve, in terms of the Hilbert function and the cohomology of the curve. The relationship between these two objects is well-known, and we spell it out explicitly in the first lemma.

Lemma 1.1 *Let C be a curve in $\mathbb{P}^n$. Then*

$$\begin{aligned}\Delta^2 H(C,t) &= h^2(\mathbb{P}^n,\mathcal{I}_C(t)) - 2h^2(\mathbb{P}^n,\mathcal{I}_C(t-1)) + h^2(\mathbb{P}^n,\mathcal{I}_C(t-2)) \\ &\qquad - h^1(\mathbb{P}^n,\mathcal{I}_C(t)) + 2h^1(\mathbb{P}^n,\mathcal{I}_C(t-1)) - h^1(\mathbb{P}^n,\mathcal{I}_C(t-2))\end{aligned}$$

Proof: First, recall that the Hilbert polynomial is given by $P(C,t) = h^0(\mathbb{P}^n,\mathcal{O}_C(t)) - h^1(\mathbb{P}^n,\mathcal{O}_C(t))$ (see [H, Exercise III.5.2]). Thus, from the short exact sequence

$$0 \to I_t \to S_t \to H^0(\mathbb{P}^n,\mathcal{O}_C(t)) \to H^1(\mathbb{P}^n,\mathcal{I}_C(t)) \to 0,$$

we obtain

$$\begin{aligned}H(C,t) &= h^0(\mathbb{P}^n,\mathcal{O}_C(t)) - h^1(\mathbb{P}^n,\mathcal{I}_C(t)) \\ &= P(C,t) + h^2(\mathbb{P}^n,\mathcal{I}_C(t)) - h^1(\mathbb{P}^n,\mathcal{I}_C(t)).\end{aligned}$$

Now, since $P(C,t)$ is a polynomial of degree 1, on taking second differences we obtain

$$\begin{aligned}\Delta^2 H(C,t) &= h^2(\mathbb{P}^n,\mathcal{I}_C(t)) - 2h^2(\mathbb{P}^n,\mathcal{I}_C(t-1)) + h^2(\mathbb{P}^n,\mathcal{I}_C(t-2)) \\ &\qquad - h^1(\mathbb{P}^n,\mathcal{I}_C(t)) + 2h^1(\mathbb{P}^n,\mathcal{I}_C(t-1)) - h^1(\mathbb{P}^n,\mathcal{I}_C(t-2))\end{aligned}$$

which is what we wanted to show. $\square$

The following corollary is an immediate consequence.

Corollary 1.2 *$\Delta^2 H(C,t) = 0$ for all $t \geq k+2$ if and only if $h^2(\mathbb{P}^n,\mathcal{I}_C(t)) = h^1(\mathbb{P}^n,\mathcal{I}_C(t))$ for all $t \geq k$.* $\square$

Proposition 1.3 *Suppose C is a curve in $\mathbb{P}^n$ defined by an ideal $I = I_C$. If $\omega(I) = \sigma(C) + k$ for some $k \geq 1$, then $e(C) = r_o(C)$ and $h^2(\mathbb{P}^n,\mathcal{I}_C(t)) = h^1(\mathbb{P}^n,\mathcal{I}_C(t))$ for $t \geq e(C) - k + 1$.*

Proof: We first show that $e(C) = r_o(C)$. If not, let $m = \max\{\, e(C), r_o(C)\,\}$. Then by Castelnuovo–Mumford regularity [Mu], we have

$$\sigma(C) < \sigma(C) + k = \omega(I) \leq \operatorname{reg}(I) \leq m+3.$$

But $\Delta^2 H(C, m+2) \neq 0$, because of Lemma 1.1 and so $\sigma(C) = m+3$, which is a contradiction. Thus, we must have $e(C) = r_o(C) = m$. In particular, $\text{reg}\,(I) = m+3$, and so again by Castelnuovo–Mumford regularity, we have $\sigma(C) = \omega(I) - k \leq \text{reg}\,(I) - k = m+3-k$. Thus, we have $\Delta^2 H(C,t) = 0$ for $t \geq m+3-k$, and so by Lemma 1.1, $h^1(\mathbb{P}^n, \mathcal{I}_C(t)) = h^2(\mathbb{P}^n, \mathcal{I}_C(t))$ for $t \geq m-k+1$. $\square$

Proposition 1.4 *Suppose $I = I_C$ defines a curve C in $\mathbb{P}^n$. Then $\omega(I) \leq \sigma(S/I) + \text{diam } H^1_*(\mathbb{P}^n, \mathcal{I}_C)$.*

Proof: This follows immediately from the previous proposition, since $h^1(\mathbb{P}^n, \mathcal{I}_C(t))$ and $h^2(\mathbb{P}^n, \mathcal{I}_C(t))$ can only possibly be non-zero and equal for $k = \text{diam } H^1_*(\mathbb{P}^n, \mathcal{I}_C)$ degrees. $\square$

Remark 1.5 Also note that Proposition 1.4 includes the case that C is arithmetically Cohen–Macaulay, and says that $\omega(I_C) \leq \sigma(S/I_C)$. This is a result in [DGM]; see also [CGO, Proposition 1.2].

Remark 1.6 Some comments about the proof of this result are in order. First of all, if either $e(C) \neq r_o(C)$ or if $e(C) = r_o(C)$ and $h^1(\mathbb{P}^n, \mathcal{I}_C(r_o)) \neq h^2(\mathbb{P}^n, \mathcal{I}_C(r_o))$, we get $\omega(I) \leq \sigma(S/I)$. This is the same bound as when C is assumed to be arithmetically Cohen–Macaulay. Thus, the cases of most interest occur when $H^1_*(\mathbb{P}^n, \mathcal{I}_C)$ and $H^2_*(\mathbb{P}^n, \mathcal{I}_C)$ both become zero at the same degree, and moreover have equal dimensions for some number of preceding degrees. Essentially, Proposition 1.3 says that having a generator of high degree forces $h^1(\mathbb{P}^n, \mathcal{I}_C(t))$ and $h^2(\mathbb{P}^n, \mathcal{I}_C(t))$ to be equal in a large number of degrees.

Curves which are not arithmetically Cohen–Macaulay and have a generator of maximum degree in the sense of the above proposition, must be "almost Buchsbaum." This means that a general linear form L induces a multiplication on $H^1_*(\mathbb{P}^n, \mathcal{I}_C)$ which has non-trivial kernel in each degree. As notation, if L is a linear form, let K_L be the kernel of the multiplication on $H^1_*(\mathbb{P}^n, \mathcal{I}_C)$ induced by L.

Proposition 1.7 *Let C be a non-arithmetically Cohen–Macaulay curve in $\mathbb{P}^3$, and suppose that $h^1(\mathbb{P}^n, \mathcal{I}_C(t)) = h^2(\mathbb{P}^n, \mathcal{I}_C(t))$ in the last r degrees. Let L be a general linear form and let $K = K_L$. Then $\dim_k K_t > \dim_k K_{t+1} > 0$ for all $t = r_o(C) - r + 1, \ldots, r_o(C) - 1$.*

Proof: Let L be a general general linear form defining a hyperplane H, and let $Z = C \cap H$ be the hyperplane section of C considered as a subscheme of $\mathbb{P}^{n-1}$. Then it is easy to see that

$$\Delta^2 H(C,t) = \Delta^1 H(Z,t) + \Delta^1 \dim_k K_{t-1}.$$

Let $p = r_o(C)$, and note that by the condition on cohomology and by Lemma 1.1, we have

$$0 = \Delta^2 H(C, p+2) = \Delta^1 H(Z, p+2) + \Delta^1 K_{p+1}.$$

But $\dim_k K_{p+1} = 0$ and $\dim_k K_p > 0$, so $\Delta^1 K_{p+1} < 0$. This implies that $\Delta^1 H(Z, p+2) > 0$, and since $\Delta^1 H(Z,t)$ is non-increasing in the range in which we are interested (see [DGM]), then $\Delta^1 H(Z,t) > 0$ for all $t = r_o(C) - r + 3, \ldots, r_o(C) + 2$. But the assumptions on the cohomology of C then imply

$$0 = \Delta^2 H(C,t) = \Delta^1 H(Z,t) + \Delta^1 \dim_k K_{t-1}$$

for $t = r_o(C) - r + 3, \ldots, r_o(C) + 2$, and so $\Delta^1 \dim_k K_{t-1} < 0$. That is, $\dim_k K_t > \dim_k K_{t+1} > 0$ for $t = r_o(C) - r + 1, \ldots, r_o(C) - 1$. $\square$

As an immediate corollary, we obtain the following statement:

Corollary 1.8 *Suppose C is a curve in $\mathbb{P}^n$ having a generator of degree $\sigma(C) + \operatorname{diam} H^1_*(\mathbb{P}^n, \mathcal{I}_C)$. Then for each general linear form L, $\dim K_t > \dim K_{t+1} > 0$, for $t = r_a(C), \ldots, r_o(C) - 1$.* $\square$

We can also use Proposition 1.7 to refine the bound on the maximum degree of a generator.

Corollary 1.9 *If C is a curve in $\mathbb{P}^n$ defined by an ideal $I = I_C$, then $\omega(I) \leq \sigma(C) + \operatorname{diam} K$.*

Proof: Again, this follows from Proposition 1.3 and Proposition 1.7. $\square$

More generally, we have the following result for subschemes of dimension d in $\mathbb{P}^n$:

Proposition 1.10 *Let V be a subscheme of $\mathbb{P}^n$ having dimension d, defined by an ideal $I = I_V$. Then $\omega(I) \leq \sigma(V) + \max\{\operatorname{diam} H^i_*(\mathbb{P}^n, \mathcal{I}_V) : i = 1, \ldots, d\}$.*

Proof: Since the proof of this result is quite similar to the case of curves, we will only give an outline. The Hilbert polynomial of V is given by

$$P(V,t) = \sum_{i=0}^{d} (-1)^i h^i(\mathbb{P}^n, \mathcal{O}_V(t)),$$

and so from the exact sequence

$$0 \to I_t \to S_t \to H^0(\mathbb{P}^n, \mathcal{O}_V(t)) \to H^1(\mathbb{P}^n, \mathcal{I}_V(t)) \to 0,$$

we see that the Hilbert function of V is

$$H(V,t) = P(V,t) + \sum_{i=1}^{d} (-1)^i h^i(\mathbb{P}^n, \mathcal{I}_V(t)).$$

Since $P(V,t)$ is a polynomial of degree d, when we take $(d+1)$th differences, we get

$$\Delta^{d+1} H(V,t) = \sum_{i=1}^{d} (-1)^i \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} h^i(\mathbb{P}^n, \mathcal{I}_V(t-j)).$$

Now we argue by cases. If none of the cohomology modules end in the same place, it is easy to see by Castelnuovo–Mumford regularity that $\omega(I) \leq \sigma(V)$. Suppose, on the other hand, that some of the cohomology modules end in the same degree t, say, and the others end in degrees $< t$, and let m be the maximum of the diameters of the intermediate cohomologies. Then $\mathrm{reg}\,(V) = t + d + 1$, and by the formula above, $\sigma(V) \geq t - m + d + 1$, and since $\omega(I) \leq \mathrm{reg}\,(I)$, the required inequality follows. $\square$

We can be a bit more precise in a few cases. For instance, suppose V is a surface in $\mathbb{P}^n$. Then there are only three non-zero cohomology modules, and as in the proof above the Hilbert function of V is given by

$$H(V,t) = P(V,t) - h^3(\mathbb{P}^n, \mathcal{I}_V(t)) + h^2(\mathbb{P}^n, \mathcal{I}_V(t)) - h^1(\mathbb{P}^n, \mathcal{I}_V(t)).$$

Because the h^3 term and the h^1 term have the same sign, the only cancellation that can occur comes from the h^2 term, and using the same argument as above, we get the inequality $\omega(I) \leq \sigma(I) + \mathrm{diam}\ H^2_*(\mathbb{P}^n, \mathcal{I}_V)$. In particular, if V were a non-arithmetically Cohen–Macaulay surface, with $H^2_*(\mathbb{P}^n, \mathcal{I}_V) = 0$, then $\omega(I) \leq \sigma(V)$, which is the same bound as in the arithmetically Cohen–Macaulay case.

2 Curves with equal cohomology

The previous section showed that the property of having a generator of high degree is very closely related to having equal (non-zero) cohomology dimensionally in a large number of degrees. This section is devoted to characterizing when an even liaison class of curves in $\mathbb{P}^3$ has this property. By a slight abuse of terminology, we make the following definition:

Definition 2.1 *A curve C in $\mathbb{P}^3$ is said to have* equal cohomology *if $e(C) = r_o(C)$ and $h^1(\mathbb{P}^3, \mathcal{I}_C(t)) = h^2(\mathbb{P}^3, \mathcal{I}_C(t))$ for $t = r_a(C), \dots, r_o(C)$.*

We first recall the structure theory for curves in $\mathbb{P}^3$ (which holds more generally for codimension 2 subschemes of $\mathbb{P}^n$) initiated by Lazarsfeld and Rao, and developed in a series of papers, of which [BBM] contains the most general statement and proof. See also the book [M] for a comprehensive overview of liaison theory and the Lazarsfeld–Rao structure theory. First, let C be a curve, and choose a form $F \in I_C$ of degree f and a form $G \in S$ of degree g so that F and G have no common components. Then $I_Z = G \cdot I_C + (F)$ defines a curve Z in $\mathbb{P}^3$, called a basic double link of C, and denoted

$$C : (g, f) \to Z.$$

It is easy to see that Z is evenly linked to C, and that there is a short exact sequence

$$0 \to S(-g-f) \xrightarrow{[F\ G]} I_C(-g) \oplus S(-f) \xrightarrow{\phi} I_Z \to 0,$$

where $\phi(r, s) = rG + sF$.

The Lazarsfeld–Rao property says essentially that even liaison classes of curves are built up by this process of basic double linkage. More precisely, let $\mathcal{L}$ be an even liaison class of curves in $\mathbb{P}^3$. Then the cohomology module $M = H^1_*(C)$ is invariant up to shifts in grading as C varies in $\mathcal{L}$, and there is a leftmost shift of M which actually occurs as the deficiency module of a curve in $\mathcal{L}$, and every rightward shift is realized. Thus $\mathcal{L}$ is parameterized by shifts of M, and a curve C_0 which has $H^1_*(C_0)$ in the leftmost shift is called a *minimal curve*. Every other curve C in $\mathcal{L}$ is obtained from C_0 as follows: there is a curve C_m which is a deformation of C through curves having constant cohomology, and a series of basic double links

$$C_0 : (1, d_0) \to C_1 : (1, d_1) \to \cdots \to C_{m-1} : (1, d_{m-1}) \to C_m. \tag{1}$$

We can moreover choose the degrees to satisfy $d_0 = \cdots = d_s < d_{s+1} < \cdots < d_{m-1}$. Note furthermore that C_i is in the ith shift of $\mathcal{L}$; that is, $H^1_*(C_i)$ is a rightward shift by i degrees of $H^1_*(C_0)$. Also, for curves in $\mathbb{P}^3$, the book [MDP2] gives much more information about the behavior of invariants along liaison classes, and moreover gives an algorithm for computing the minimal curve in the liaison class from the deficiency module associated to the class.

We will need to have some information about how Hilbert functions change as we move along an even liaison class by basic double linkage. The following result is quite elementary.

Lemma 2.2 *Suppose $I = I_C$ defines a curve C in $\mathbb{P}^3$. Let L be a general hyperplane and let F define a surface of degree d containing C. Form the basic double link Z of C by L and F. Then*

$$\Delta^2 H(Z,t) = \begin{cases} \Delta^2 H(C,t-1)+1 & \text{if } 1 \le t \le d-1 \\ \Delta^2 H(C,t-1) & \text{if } t \ge d. \end{cases}$$

Proof: Since Z is a basic double link of C, we have a short exact sequence

$$0 \to S(-d-1) \to I_C(-1) \oplus S(-d) \to I_Z \to 0.$$

Using the additive properties of Hilbert functions, it is easy to see that $\Delta^2 H(Z,t) = \Delta^2 H(C,t-1) + \Delta^3 H(F,t)$. Then the statement follows, since $\Delta^3 H(F,t) = 1$ for $0 \le t \le d-1$, and is zero otherwise.

Note that this is also in [N, Corollary 2.3.5], in terms of postulation characters. $\square$

Definition 2.3 Suppose there is a chain

$$\mathcal{C} : C_0 : (1,d_0) \to C_1 : (1,d_1) \to \cdots \to C_{m-1} : (1,d_{m-1}) \to C_m$$

of basic double links by surfaces F_i having degrees d_i. Then define $\delta(\mathcal{C},t,s)$ to be the number of F_i such that $d_i \ge t-s+i+1$, for $i = 1,\ldots,s-1$.

It is easy to see that $\delta(\mathcal{C},t,s) \le \delta(\mathcal{C},t-1,s)$.

The following result is a straightforward calculation using Lemma 2.2.

Corollary 2.4 *Let $\mathcal{C}$ as above be a chain of basic double links. Then for each $s = 0,\ldots,m$,*

$$\Delta^2 H(C_s,t) = \Delta^2 H(C_0,t-s) + \delta(\mathcal{C},t,s). \qquad \square$$

The main result of this section concerns when an even liaison class has a curve with h^1 and h^2 equal for some number of places, and gives a characterization in terms of the Hilbert function of the minimal curve in the liaison class. We first state a more general version, from which we can trivially make a statement about minimal curves.

Theorem 2.5 *Let $\mathcal{L}$ be an even liaison and let C_0 be a curve in $\mathcal{L}$. Then a sequence*

$$C_0 \to \ldots \to C_m = C$$

of basic double links can be constructed with C having $h^1(\mathbb{P}^3,\mathcal{I}_C(t)) = h^2(\mathbb{P}^3,\mathcal{I}_C(t))$ in the last r places if and only if C_0 has $e(C_0) \le r_o(C_0)$ and has Hilbert function satisfying

$$\Delta^2 H(C_0, r_o(C_0)-r+3) \le \Delta^2 H(C_0, r_o(C_0)-r+4) \le \cdots \le \Delta^2 H(C_0, r_o(C_0)+2) \le 0.$$

Proof: Suppose that C_0 has $e(C_0) \le r_o(C_0)$ and Hilbert function

$$\Delta^2 H(C_0, t) = \quad \cdots \quad t_1 \quad t_2 \quad \cdots \quad t_r \quad 0 \quad \cdots$$

where $t_1 \le t_2 \le \cdots \le t_r \le 0$, and the term t_r occurs in degree $r_o(C_0) + 2$. If not all the t_i are equal, let s, $1 \le s \le r$, be the first integer for which $t_{s-1} < t_s$. Otherwise, let $s = r + 1$. Note that I_{C_0} must have elements in degree $\le r_o(C_0) - r + 3$, so we can form the basic double link $C_0 : (1, r_o(C_0) - r + s + 1) \to C_1$. Then by Lemma 2.2, C_1 has Hilbert function

$$\Delta^2 H(C_1, t) = \quad \cdots \quad t_1 + 1 \quad \cdots \quad t_{s-1} + 1 \quad t_s \quad \cdots \quad t_r \quad 0 \quad \cdots,$$

where now the term t_r occurs in degree $r_o(C_0)+3$. Continuing by induction, we construct a sequence of basic double links to a curve C_m having Hilbert function

$$\Delta^2 H(C_m, t) = \quad \cdots \quad u_1 \quad \cdots \quad u_r \quad 0 \quad \cdots,$$

where $u_1 = \cdots = u_r = 0$, and where the term u_r occurs in degree $r_o(C_0) + 2 + m$.

We claim that C_m has the cohomology property. To see this, note that $r_o(C_m) = r_o(C_0) + m$, and that $\sigma(C_m) \le r_o(C_0) + m - r + 3$, so that $\Delta^2 H(C_m, t) = 0$ for all $t \ge r_o(C_0) + m - r + 3$. But by Corollary 1.2, we see that $h^2(\mathbb{P}^3, \mathcal{I}_{C_m}(t)) = h^1(\mathbb{P}^3, \mathcal{I}_{C_m}(t))$ for all $t \ge r_0(C_0) + m - r - 1 = r_o(C_m) - r + 1$. That is, $h^2(\mathbb{P}^3, \mathcal{I}_{C_m}(t)) = h^1(\mathbb{P}^3, \mathcal{I}_{C_m}(t))$ in the last r places.

For the converse, suppose $C \in \mathcal{L}$ has the cohomology property and that there is a sequence of basic double links

$$\mathcal{C} : C_0 : (1, d_0) \to C_1 : (1, d_1) \to \ldots \to C_m = C.$$

First, note that by [BM2, Lemma 1.14], $e(Z)$ increases by at least one each time we move up in the liaison class. But $r_o(Z)$ increases by exactly one each time. Since the cohomology property for C_m in particular means that $e(C_m) = r_o(C_m)$, then we must have $e(C_0) \le r_o(C_0)$.

Next we show that the Hilbert function of C_0 has the given form. First,

$$\Delta^2 H(C_0, r_o(C_0) + 2) = \Delta^2 H(C_m, r_o(C_0) + 2 + m) - \delta(\mathcal{C}, r_o(C_0) + m, m),$$

and since $r_o(C_0)+2+m = r_o(C_m)+2$, we have $\Delta^2 H(C_m, r_o(C_0)+2+m) = 0$, because of Corollary 1.2 and the fact that $h^1(\mathbb{P}^3, \mathcal{I}_C(t)) = h^2(\mathbb{P}^3, \mathcal{I}_C(t))$ for $t \ge r_0(C_m)$. Thus, $\Delta^2 H(C_0, r_o(C_0) + 2) \le 0$. Moreover, for each $i = 1, \ldots, r-1$, we have

$$\begin{aligned} \Delta^2 H(C_0, r_o(C_0) + 2 - i) &= \Delta^2 H(C_m, r_o(C_0) + 2 - i + m) - \delta(\mathcal{C}, r_o(C_0) + 2 - i + m, m) \\ &= -\delta(\mathcal{C}, r_o(C_0) + 2 - i + m, m). \end{aligned}$$

As above, the second equality follows from the fact that $r_o(C_0)+2-i+m = r_o(C_m)+2-i$ and because the cohomology property for C_m implies $\Delta^2 H(C_m, r_o(C_m)+2-i) = 0$ via Corollary 1.2. But by our observation following Definition 2.3, we have

$$\begin{aligned}
\Delta^2 H(C_0, r_o(C_0)+2-i) &= -\delta(\mathcal{C}, r_o(C_0)+2-i+m, m) \\
&\leq -\delta(\mathcal{C}, r_o(C_0)+2-(i-1)+m, m) \\
&= \Delta^2 H(C_0, r_o(C_0)+2-(i-1)),
\end{aligned}$$

and so the proof is finished. $\square$

Since for any curve C in an even liaison class $\mathcal{L}$ there is a sequence of basic double links from a minimal curve in $\mathcal{L}$ to a curve C_m, followed by a deformation with constant cohomology to C, the Theorem has the immediate consequence:

Corollary 2.6 *An even liaison class of curves in $\mathbb{P}^3$ contains a curve C having $h^1(\mathbb{P}^3, \mathcal{I}_C(t)) = h^2(\mathbb{P}^3, \mathcal{I}_C(t))$ in the last r places if and only if the minimal curve C_0 in $\mathcal{L}$ has $e(C_0) \leq r_o(C_0)$ and has Hilbert function satisfying*

$$\Delta^2 H(C_0, r_o(C_0)-r+3) \leq \Delta^2 H(C_0, r_o(C_0)-r+4) \leq \ldots \leq \Delta^2 H(C_0, r_o(C_0)+2) \leq 0.$$

3 Liaison Classes of Curves with Equal Cohomology

As we saw in the previous section, the presence or non-presence of a curve in an even liaison class $\mathcal{L}$ having equal cohomology can be detected by looking at the Hilbert function of the minimal curve in $\mathcal{L}$. This raises some immediate questions: if an even liaison class $\mathcal{L}$ contains curves with equal cohomology, in what shifts of $\mathcal{L}$ do they occur, and "how many" such curves are there? To answer these questions, we first make some remarks concerning how the equal cohomology property behaves with respect to basic double linkage.

Remark 3.1 (a.) Let $\mathcal{L}$ be an even liaison class with associated deficiency module of diameter r. Suppose $\mathcal{L}$ contains curves with equal cohomology. Then by the previous corollary, the minimal curve in the liaison class has Hilbert function ending in a non-decreasing sequence of r non-positive terms, beginning in degree $r_a(C_0)+2$. Say $\Delta^2 H(C_0, t) = \cdots \quad t_1 \quad \cdots \quad t_r$ is this sequence. Let

$$C_0 : (1, b_0) \to C_1 : (1, b_1) \to C_m : (1, b_{m-1}) \to C_m$$

be a sequence of basic double links. By Lemma 2.2, if $\Delta^2 H(C_i, t)$ ends in negative terms, and if the basic double linkage $C_i : (1, b_i) \to C_{i+1}$ changes one of these negatives, then it also changes every term preceding. In particular, it must change the left-most negative term. Moreover, each basic double link which changes negatives increases the left-most negative term by exactly one. Note that a link $C_i : (1, b_i) \to C_{i+1}$ changes negative terms if and only if $b_i \geq r_a(C_i) + 2 = r_a(C_0) + i + 2$.

(b.) Related to this is the observation that if there are more than $-t_1$ basic double links which change negative terms, then C_m cannot have equal cohomology. More precisely, if we have $b_i \geq r_a(C_0) + i + 2$ for more than $-t_1$ indices i, then the t_1 term eventually becomes positive, and this forces $h^1(\mathbb{P}^3, \mathcal{I}_C(t))$ and $h^2(\mathbb{P}^3, \mathcal{I}_C(t))$ to be non-equal in at least the leftmost degree.

(c.) Continuing with that theme, it follows rather trivially that if we have more than $-t_1$ links which change negative terms, then no further basic double link can possibly produce a curve with equal cohomology.

(d.) As a final remark, note that if $C : (1, d) \to D$ is a basic double link, and if C has equal cohomology, then D has equal cohomology if and only if $d \leq r_a(C) + 3$. This follows directly from Corollary 1.2 and Lemma 2.2.

Now, we begin our description of which curves in the liaison class have equal cohomology by showing that there is a unique minimal such curve.

Proposition 3.2 *Suppose $\mathcal{L}$ is an even liaison class which contains curves having equal cohomology. Then up to deformation through curves with constant cohomology, there is a unique minimal curve with equal cohomology.*

Proof: This is quite easy, given our remarks above. If $\Delta^2 H(C_0, t) = \cdots \quad t_1 \quad \cdots \quad t_r$ are the final non-decreasing non-positive terms in the Hilbert function of C_0, as guaranteed by Corollary 2.6, then it requires at least $-t_1$ basic double links to reach a curve D for which $\Delta^2 H(D, t) = 0$ for $t \geq r_a(D) + 2$, and by Corollary 1.2, D then has equal cohomology. On the other hand, the construction in Theorem 2.5 produces a curve with equal cohomology in exactly $-t_1$ steps. Also, if we reach a curve in equal cohomology in $-t_1$ steps, then every basic double link of degree b_i satisfies $b_i \geq r_a(C_0) + i + 2$. Thus, by Lemma 2.2, no matter what sequence of basic double links we take, the Hilbert function of the resulting curve is invariant. Thus any two curves in this shift with equal cohomology are deformations through curves with constant cohomology; see [BM3, Proposition 3.1]. □

Now, we need to recall a result from [BM3] on "equivalence" of basic double links. In the proof of [BM3, Lemma 5.2], they show the following: suppose

$$C_1 : (1, b_1) \to C_2 : (1, b_2) \to C_3$$

are basic double links with $b_1 < b_2$. Then the sequence b_1, b_2 is equivalent to the sequence b_2-1, b_1+1 in the sense that if we make the basic double links

$$C_1 : (1, b_2 - 1) \to C_2' : (1, b_1 + 1) \to C_3',$$

then C_3 and C_3' are deformations of each other, through curves with constant cohomology. Note that implicit in this is the fact that the basic double linkage of degree $b_2 - 1$ can actually be made; this is also noted in their proof.

We will use this idea of "flipping" adjacent degrees in the next result.

Proposition 3.3 *Suppose $\mathcal{L}$ is an even liaison class containing curves with equal cohomology, and that C is a curve in $\mathcal{L}$ having equal cohomology. Then C is obtained from the minimal curve having equal cohomology by a sequence of basic double linkages followed by a deformation through curves with constant cohomology, if necessary. Each curve in the sequence also has equal cohomology.*

Proof: Assume that the deficiency module associated to $\mathcal{L}$ has diameter r. First, since C is in $\mathcal{L}$, then by the Lazarsfeld–Rao property, after deforming C through curves with constant cohomology if necessary, we may assume that there is a sequence of basic double links

$$C_0 : (1, b_0) \to C_1 : (1, b_1) \to \cdots \to C_m = C \tag{2}$$

where C_0 is the (absolute) minimal curve in $\mathcal{L}$, and we can assume that $b_0 \leq \ldots \leq b_{m-1}$. Since $\mathcal{L}$ possesses curves with equal cohomology, C_0 has Hilbert function given by

$$\Delta^2 H(C_0, t) = \quad \cdots \quad t_1 \quad \cdots \quad t_r$$

where $t_1 \leq \ldots \leq t_r \leq 0$, and t_1 is in degree $r_a(C_0) + 2$. Now, since C has equal cohomology, then exactly $-t_1$ of the basic double links in the sequence (2) change negatives. That is, exactly $-t_1$ of the b_i satisfy $b_i \geq r_a(C_0) + i + 2$. This follows from Remark 3.1(a).

Choose the first index $s \geq 0$ for which $b_i \geq r_a(C_0) + i + 2$, and using the equivalence outlined above, flip this degree down to the first position. Note that this is possible since our original sequence of b_i's is non-decreasing and $b_{s-j} < r_a(C_0) + s + 2 - j \leq b_s - j$ for $0 \leq j \leq s$. This creates an equivalent sequence of basic double links of degrees $b'_0, \ldots, b'_{m-1}$, where $b'_0 = b_s - s$, $b'_i = b_{i-1} + 1$ for $0 < i \leq s$, and $b'_i = b_i$ for $i \geq s+1$. In particular, exactly $-t_1$ of the b'_i satisfy $b'_i \geq r_a(C_0)+i+2$, and moreover $b'_0 \geq r_a(C_0)+2$. Continue in the same manner: find the second time that a b'_i changes negatives, and flip it down to the second position, and so forth. Then we end up with a sequence $c_0, \ldots, c_{m-1}$ of basic double links which is equivalent to the one we started with, and which moreover has $c_i \geq r_a(C_0) + i + 2$ for $i = 0, \ldots, -t_1 - 1$. Hence, by Remark 3.1, since we change exactly $-t_1$ negative terms, all in the first $-t_1$ links, and since we eventually end up with a curve having equal cohomology, then the curve C_{-t_1} and each curve from C_{-t_1} on, must also have equal cohomology. This follows from Remark 3.1. $\square$

We recapitulate what we have proven in the next statement:

Theorem 3.4 *Suppose $\mathcal{L}$ is an even liaison class containing curves with equal cohomology. Then there is a minimal shift $\mathcal{L}^t$ which contains a curve with equal cohomology; the curves with equal cohomology in the minimal shift are unique up to deformation through curves with constant cohomology; every curve in $\mathcal{L}$ with equal cohomology is obtained from the minimal one by basic double linkage and deformation through curves with constant cohomology; and finally every rightward shift of $\mathcal{L}^t$ contains, up to deformation, a finite, non-zero number of curves with equal cohomology.*

Proof: The only part which remains to be proven is the final statement. If C is a curve with equal cohomology in some shift $\mathcal{L}^s$ of the liaison class, then $\Delta^2 H(C,t) = 0$ for all $t \geq r_a(C) + 2$. This implies in particular that I_C contains non-zero elements of degree $\geq r_a(C) + 2$. Thus, we can make a basic double link $C : (1, r_a(C) + 2) \to D$, and $D \in \mathcal{L}^{s+1}$ also has equal cohomology. On the other hand, if we make a basic double link of degree $> r_a(C) + 3$, then the resulting curve does not have equal cohomology, so there is only a finite number of allowable degrees. $\square$

Theorem 3.4 shows that the curves with equal cohomology have a strong Lazarsfeld-Rao property, in the sense that there are unique minimal curves, every other curve is obtained by basic double linkage, and in each allowable shift, there are only a finite number of curves, up to deformation.

In the case of Buchsbaum liaison classes, we can actually count the number of curves in each shift which have equal cohomology.

Proposition 3.5 *Suppose $\mathcal{L}$ is a Buchsbaum even liaison class having curves with equal cohomology, and let $\mathcal{L}^s$ be the minimal shift in which such a curve occurs. Then for each $t \geq s$, there are exactly 2^{t-s} curves, up to flat deformations, having equal cohomology.*

Proof: This follows from the fact that if $D \in \mathcal{L}^h$, then $\alpha(I_D) = 2N + h$, where $N = \sum \dim H^1_*(\mathbb{P}^3, \mathcal{I}_D(i))$, and the description of the Hilbert function of minimal Buchsbaum curves in [BM2, Proposition 2.1]. In particular, the minimal curve C having equal cohomology has Hilbert function

$$\Delta^2 H(C,t) = 1 \quad 2 \quad \cdots \quad 2\alpha + s.$$

In order to move from $\mathcal{L}^s$ to $\mathcal{L}^{s+1}$ and preserve the cohomology property, we can only make basic double links of degree $2\alpha + s$ or $2\alpha + s + 1$. Similarly, we can only move from $\mathcal{L}^{s+1}$ to $\mathcal{L}^{s+2}$ by basic double links of degree $2\alpha + s + 1$ or $2\alpha + s + 2$. Continuing inductively, the statement is proven. $\square$

We can also give a proof more along the lines of the original proof that the liaison classes of curves in $\mathbb{P}^3$ have the Lazarsfeld–Rao property. It is much less constructive in nature, but, in some sense, points out the naturality of our cohomological criterion of equal cohomology. Since it is so different in spirit from our previous argument, we felt it necessary to include it here.

Lemma 3.6 *Let $\mathcal{F}$ be a rank $(r+1)$ vector bundle on $\mathbb{P}^3$ with $H^2_*(\mathbb{P}^3, \mathcal{F}) = 0$. Let*

$$\phi_1 : \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{F}, \qquad a_1 \leq \ldots \leq a_r$$

$$\phi_1 : \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-b_i) \to \mathcal{F}, \qquad b_1 \leq \ldots \leq b_r$$

be morphisms whose degeneracy loci are curves C_1 and C_2 with equal cohomology. Then there exists a morphism

$$\phi : \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-c_i) \to \mathcal{F}, \qquad c_i = \min\{a_i, b_i\}$$

whose degeneracy locus is also a curve with equal cohomology.

Proof: Notice that C_1 and C_2 are evenly linked, in the even liaison class determined by the stable equivalence class of $\mathcal{F}$, according to Rao's classification [R]. By [BBM, Lemma 2.1], there exists such a ϕ whose degeneracy locus is a curve C. We just have to prove that C has equal cohomology.

Twisting and relabeling if necessary, we may assume that we have locally free resolutions

$$0 \to \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{F} \to \mathcal{I}_{C_1} \to 0$$

$$0 \to \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-b_i) \to \mathcal{F} \to \mathcal{I}_{C_2}(h) \to 0.$$

Notice that the deficiency module of C_2 is shifted h places to the right of that of C_1. We then get

$$0 \to H^2(\mathbb{P}^3, \mathcal{I}_{C_1}(t)) \to H^3(\mathbb{P}^3, \bigoplus \mathcal{O}_{\mathbb{P}^3}(t-a_i)) \to H^3(\mathbb{P}^3, \mathcal{F}(t)) \to 0$$

$$0 \to H^2(\mathbb{P}^3, \mathcal{I}_{C_2}(t+h)) \to H^3(\mathbb{P}^3, \bigoplus \mathcal{O}_{\mathbb{P}^3}(t-b_i)) \to H^3(\mathbb{P}^3, \mathcal{F}(t)) \to 0.$$

(The first 0 comes from the assumption on the vanishing of the cohomology of $\mathcal{F}$ and the second from the fact that $h \geq 0$.) By the assumption of equal cohomology, for $t \geq r_a(C_1)$ the first term in the first sequence has the same dimension as the first term in the second sequence. Hence for $t \geq r_a(C_1)$, also the second terms are equal. Therefore

$$\{a_i \mid r_a(C_1) - a_i \leq -4\} = \{b_i \mid r_a(C_1) - b_i \leq -4\}$$

(since these are the terms which contribute to the middle cohomology space in the degrees $t \geq r_a$). That is,

$$\{a_i \mid a_i \geq r_a + 4\} = \{b_i \mid b_i \geq r_a + 4\}$$

Call this set A.

Now, for any curve Y with locally free resolution

$$0 \to \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-d_i) \to \mathcal{F} \to \mathcal{I}_Y(\delta) \to 0,$$

Y has equal cohomology if and only if $\{d_i \mid d_i \geq r_a(Y) + 4\} = A$ (since this set determines $h^3(\mathbb{P}^3, \bigoplus \mathcal{O}(-d_i + t))$ and hence $h^2(\mathcal{I}_Y(t+\delta)$ in the desired range). The proof of the lemma follows immediately from this fact. $\square$

Corollary 3.7 *The set of curves in a given even liaison class which have equal cohomology satisfy the Lazarsfeld–Rao property.*

Proof: The proof is identical to that in [BBM]. The lemma above replaces [BBM, Lemma 2.1]. Then [BBM, Proposition 2.2] goes through to prove the uniqueness of the minimal element. Similarly, [BBM, Proposition 2.3] goes through to show the relation between the minimal element and any other curve in the even liaison class with equal cohomology. Finally, [BBM, Theorem 2.4] still works to show how to produce a curve with equal cohomology as a sequence of basic double links followed by a deformation, starting with a minimal curve with equal cohomology. The proof in [BBM] shows that such a sequence exists. The fact that we start with equal cohomology and end with equal cohomology shows that every step in between has equal cohomology too. □

Remark 3.8 We do not yet know of any examples of even liaison classes for which the absolute minimal curve C_0 is also the minimal curve with equal cohomology.

4 Integral Curves with Equal Cohomology

There has been much recent progress on further clarifying the structure of even liaison classes by giving conditions for the presence within the liaison classes of nice curves. In particular, there is information on where in a given class one can find integral curves [N], or smooth and connected curves in Buchsbaum classes [MDP1], and on how these curves are related to each other and to the minimal curve in the class. The paper [PR] shows that at least in Buchsbaum classes, smooth and connected curves share the same Lazarsfeld–Rao properties as irreducible curves, and imply that one can obtain the integral curves within a given shift of a liaison class by deforming irreducible curves. Their calculations are based also on the work of Nollet, as well as on [MDP1].

In this section, we are interested in using some of the results of [N] to obtain some information on when an even liaison class contains curves which are integral and have equal cohomology. We first recall the relevant definitions and some results from [N].

Let C be a curve in $\mathbb{P}^3$, defined by an ideal $I = I_C$. The postulation character of C is given by $\gamma_C(n) = -\Delta^3 H(C, n)$. There are three natural invariants to attach to C:

$$\begin{aligned} s(C) &= \min\{\, n : \gamma_C(n) \geq 0 \,\} \\ t(C) &= \min\{\, n : \gamma_C(n) > 0 \,\} \\ t_1(C) &= \text{smallest degree of a surface containing } C \text{ which meets} \\ &\qquad\qquad \text{a surface of degree } s(C) \text{ containing } C \text{ properly.} \end{aligned}$$

We note for clarity that $s(C) = \alpha(C)$, the minimal degree of a generator of I_C, and $t_1(C) = \beta(C)$, the minimal degree for which $I_{\leq t} = \oplus_{i \leq t}[I]_i$ generates an ideal of codimension 2. Next, say that C dominates a curve D at height h if C can be obtained from D by a sequence of h basic double links, followed by a deformation. The central definition for this section is the following: suppose C dominates the minimal curve C_0 in $\mathcal{L}$ at height h. Then

$$\theta_C(n) = \begin{cases} \gamma_C(n), & \text{if } s(C) \leq n < s(C_0) + h \\ \gamma_C(n) - \gamma_{C_0}(n-h), & \text{if } n \geq s(C_0) + h \\ 0, & \text{otherwise.} \end{cases}$$

(This definition appears in [PR] and is clearly equivalent to the one in [N].) We say θ_C is connected in degrees $\geq a$ if $\theta_C(b) > 0$ for $b \geq a$ implies $\theta_C(n) > 0$ for all $a \leq n \leq b$, and similarly θ_C is connected in degrees $\leq b$ if $\theta_C(a) > 0$ for some $a \leq b$ implies $\theta_C(n) > 0$ for all $a \leq n \leq b$. Finally, θ_C is connected about an interval $[a, b]$ if it is connected in degrees $\geq a$ and in degrees $\leq b$, and if $\theta_C(n) > 0$ for all $a \leq n \leq b$.

Now, Nollet proves the following theorem in [N]:

Theorem 4.1 *Let $\mathcal{L}$ be an even liaison class of curves in $\mathbb{P}^3$ with minimal curve C_0.*

- (a.) ([N, Theorem 5.2.1]) *If $C \in \mathcal{L}$ is an integral curve of height h, then θ_C is connected about $[t(C_0) + h, t_1(C_0) + h - 1]$.*
- (b.) ([N, Theorem 5.2.5]) *Conversely, suppose C dominates at height h an integral curve D in $\mathcal{L}$, which is generically Cartier on a surface of minimal degree and has either $\theta_D \neq 0$ or $t(D) \leq e(D) + 4$. If θ_C is connected about $[t(C_0) + h, t_1(C_0) + h - 1]$, then C can be deformed to an integral curve.*

Thus, having θ_C connected about the interval $[t(C_0)+h, t_1(C_0)+h-1]$ is very close to having C integral. As it turns out, this condition is relatively easy to check for curves with equal cohomology. We begin with some elementary calculations. Throughout, let $\mathcal{L}$ be an even liaison class of curves, which contains curves having equal cohomology, and let C_0 be the (absolute) minimal curve in $\mathcal{L}$ and C the minimal curve in $\mathcal{L}$ with equal cohomology.

Lemma 4.2 *Suppose the minimal curve C with equal cohomology has height h over C_0. Then:*

$$\begin{aligned} s(C) &= s(C_0) + h \\ t(C) &= t(C_0) + h \\ \theta_C(n) &= \begin{cases} -\gamma_{C_0}(n-h) & \text{if } r_a(C_0) + h + 2 < n \leq \sigma(C_0) + h \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

Proof: Note that in order to move up the liaison class from the minimal curve C_0 to the curve C, in order to get C with equal cohomology, we must take basic double links $C_i \to C_{i+1}$ of degree large enough to change the negative signs in $\Delta^2 H(C_i, t)$. Clearly, this degree d, say, is strictly larger than $t(C_i)$. By [N, Corollary 2.3.5], then, $\gamma_{C_{i+1}}(n) = \gamma_{C_i}(n-1)$ for $n \leq d$. In particular, $s(C_{i+1}) = s(C_i)+1$ and $t(C_{i+1}) = t(C_i)+1$, and so the first two statements are done by induction. The assertion about θ_C then follows from the definition of θ_C and the fact that $\gamma_C(n) = 0$ for $n \geq r_a(C_0)+h+2$, since C has equal cohomology, and using that $s(C_0) \leq r_a(C_0)+2$ and $s(C) = s(C_0)+h$. $\square$

Now we are ready to determine which curves have both equal cohomology and connected θ. Our first result takes care of a rather trivial case.

Proposition 4.3 *Suppose the minimal curve C with equal cohomology has height h over C_0 and has θ_C connected about the interval $[t(C_0)+h, t_1(C_0)+h-1]$. Then $C = C_0$, up to deformation, and $t(C_0) = t_1(C_0)$.*

Conversely, if C_0 has equal cohomology and $t(C_0) = t_1(C_0)$, then θ_{C_0} is connected about the interval $[t(C_0), t_1(C_0)-1]$.

Proof: First note that since C has equal cohomology, we must have $t(C) \leq r_a(C)+2$, and this then implies by Lemma 4.2 that $t(C_0) \leq r_a(C_0)+2$. But again by Lemma 4.2, this means that $\theta_C(t(C_0)+h) = 0$, so $\theta_C > 0$ on the interval $[t(C_0)+h, t_1(C_0)+h-1]$ if and only if this interval is empty. Clearly, this is equivalent to $t(C_0) = t_1(C_0)$. Now, θ_C is connected in degrees $\geq t(C_0)+h$ if and only $\theta_C = 0$ in degrees $\geq t(C_0)+h$, and again by Lemma 4.2, this occurs if and only if $\sigma(C_0) = r_a(C_0)+2$. By Corollary 1.2, this implies that C_0 has equal cohomology, and so $C = C_0$, up to deformation.

The other direction is quite trivial, since $\theta_{C_0} = 0$ and $[t(C_0), t_1(C_0)-1]$ is empty. $\square$

Our next proposition is the main result of this section, and tells us when an even liaison class contains curves with equal cohomology and connected θ. Note that it is identical in spirit to Theorem 2.6.

Proposition 4.4 *Suppose $\mathcal{L}$ is an even liaison class of curves with minimal curves satisfying $t(C_0) < t_1(C_0)$. Then $\mathcal{L}$ contains a curve D of height $\overline{h}$, say, having equal cohomology and θ_D*

connected about $[t(C_0)+\overline{h}, t_1(C_0)+\overline{h}-1]$ *if and only if* $t_1(C_0) \le \sigma(C_0)+1$ *and the Hilbert function of* C_0 *satisfies*

$$\Delta^2 H(C_0, r_a(C_0)+2) < \Delta^2 H(C_0, r_a(C_0)+3) < \cdots < \Delta^2 H(C_0, \sigma(C_0)-1) < 0.$$

Proof: First, suppose C_0 has $t_1(C_0) \le \sigma(C_0)+1$ and satisfies the condition on the Hilbert function. Note that the condition on the Hilbert function implies that $t(C_0) \le r_a(C_0)+2$. Then by Lemma 4.2, the minimal curve C with equal cohomology has $\theta_C(t) = 0$ for $t(C_0)+h \le t \le r_a(C_0)+h+2$ and $\theta_C(t) > 0$ for $r_a(C_0)+h+2 < t \le \sigma(C_0)+h$, where C has height h over C_0. Now perform a sequence of $r_a(C_0)+2-t(C_0)$ basic double links all of degree $r_a(C_0)+h+3$ to reach a curve D of height $\overline{h} = h + r_a(C_0)+2-t(C_0)$. Then by a repeated application of [N, Corollary 2.3.5], it is easy to check that

$$\theta_D(t) = \begin{cases} 1 & \text{for } t(C_0)+\overline{h} \le t \le r_a(C_0)+\overline{h}+2 \\ \theta_C(t+h-\overline{h}) & \text{for } r_a(C_0)+\overline{h}+2 < t \le \sigma(C_0)+\overline{h} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, θ_D is connected about $[t(C_0)+\overline{h}, t_1(C_0)+\overline{h}-1]$. Also, since D was obtained from C by basic double links of low degree, D still has equal cohomology; see Remark 3.1(d.).

Conversely, suppose D is a height $\overline{h}$ curve with equal cohomology, and with θ_D connected about the interval $[t(C_0)+\overline{h}, t_1(C_0)+\overline{h}-1]$. Then there is a sequence of basic double links

$$C : (1, b_0) \to C_1 : (1, b_1) \to \cdots \to D$$

where C is the minimal curve in $\mathcal{L}$ with equal cohomology (of height h, say) and where each b_i satisfies $b_i \le r_a(C_0)+h+i+3$ (see Remark 3.1(d.) or the proof of Corollary 3.7). If $C = C_0$, then $r_a(C_0)+2 = \sigma(C_0)$, so there is nothing to show. Hence we may assume that C_0 does not have equal cohomology, and this means that $\Delta^2 H(C_0, r_a(C_0)+2) < 0$, which in turn implies $\gamma_{C_0}(r_a(C_0)+3) < 0$ and $t(C_0) \le r_a(C_0)+2$.

By a repeated use of [N, Corollary 2.3.5], $\gamma_D(n) = \gamma_C(n-\overline{h}+h)$ for $r_a(C_0)+\overline{h}+3 \le n$. But by Lemma 4.2, $\gamma_C(n-\overline{h}+h) = -\gamma_{C_0}(n-\overline{h})$ for $r_a(C_0)+\overline{h}+3 \le n \le \sigma(C_0)+\overline{h}$.

Now, θ_D is connected in degree $\ge t(C_0)+\overline{h}$, and our assumption that $t(C_0) < t_1(C_0)$ implies in particular that $\theta_D(t(C_0)+\overline{h}) > 0$. Also, $\theta_D(\sigma(C_0)+\overline{h}) = -\gamma_{C_0}(\sigma(C_0)) > 0$, so $\theta_D > 0$ on the interval $[t(C_0)+\overline{h}, \sigma(C_0)+\overline{h}]$.

Next, note that θ_D is positive on the interval $[r_a(C_0)+\overline{h}+3, \sigma(C_0)+\overline{h}]$, since this interval is contained in the interval $[t(C_0)+\overline{h}, \sigma(C_0)+\overline{h}]$. Thus, we have $0 < \theta_D(t) = -\gamma_{C_0}(t-\overline{h})$ for

$r_a(C_0) + \overline{h} + 3 \leq t \leq \sigma(C_0) + \overline{h}$. This clearly implies that $\gamma_{C_0}(t) < 0$ for $r_a(C_0) + 3 \leq t \leq \sigma(C_0)$, and this means that $\Delta^2 H(C_0, t)$ is strictly increasing in the given range.

Finally, to see that $t_1(C_0) \leq \sigma(C_0) + 1$, note that the connectedness property of θ_D implies that $\theta_D > 0$ on $[r_a(C_0) + \overline{h} + 3, t_1(C_0) + \overline{h}]$, but clearly $\theta_D(t) = 0$ for $t > \sigma(C_0) + \overline{h}$, by the argument above. $\square$

5 Degrees of Generators and Liaison Classes of Curves

In this section, we go back to studying degrees of generators of the ideals defining space curves, by using the results on cohomology given in the previous sections. We are able to give some nice conditions on the degrees of the components of the deficiency module associated to a liaison class in order for the class to contain a curve whose ideal has a generator of maximal degree. Since knowledge about curves with equal cohomology in a given liaison class depends so crucially on knowing the Hilbert function of the minimal curve in the liaison class, our characterizations for when a liaison class contains curves with generators of high degree work best when we already know the Hilbert function of the minimal curve. To do this, we have to make some extra assumptions on the liaison class. We have concentrated on cohomological criteria, and two very clean statements are given in Proposition 5.4 and in Proposition 5.5. Similarly, using our results on integral curves, we give some results on existence within a liaison class of integral curves with generators of maximal degree.

As we showed in the previous sections, if C is defined by an ideal $I = I_C$, then the property of I_C having a generator of high degree is very closely related to having $h^1(\mathbb{P}^3, \mathcal{I}_C(t)) = h^2(\mathbb{P}^3, \mathcal{I}_C(t))$ in a large number of places, and furthermore, curves with this cohomology property are easily constructed by basic double linkage, as long as we know the minimal curve in a liaison class. However, we should remark that in order to construct a curve whose ideal has a high degree generator, we need to choose the degrees of the basic double links carefully, since it is possible to make $h^1 = h^2$ in the maximum number of places, without introducing a high degree generator. The following example should clarify this somewhat.

Example 5.1 Start with the Buchsbaum liaison class $\mathcal{L} = \mathcal{L}_{4,1}$, whose deficiency module has two consecutive components of dimensions 4 and 1, respectively. Then the minimal curve C_0 in $\mathcal{L}$ has Hilbert function (see [BM2, Corollary 2.18])

$$\Delta^2 H(C_0, t) = 1 \quad 2 \quad \cdots \quad 10 \quad -2 \quad -1.$$

Taking the two basic double links

$$C_0 : (1, 13) \to C_1 : (1, 13) \to C_2$$

produces the curve C_2, for which $h^1 = h^2$ in the last two places, and which has $\sigma(C_2) = 12$, but $\omega(I_{C_2}) = 13$. Thus the bound in Proposition 1.4 is not obtained, even though C_2 does have the cohomology property.

On the other hand, note that if we take the basic double links

$$C_0 : (1, 12) \to C_1' : (1, 14) \to C_2',$$

then C_2' has $h^1 = h^2$ in the last two places, and also has $\sigma(C_2') = 12 = \omega(I_{C_2'}) - 2$. So the bound is achieved in this case.

Generally speaking, this second sequence of basic double links produces curves whose defining ideals have a high degree generator. Note that it is essentially the procedure given by Theorem 2.5. However, we need to require that there are no "trailing zeroes" on the end of the second difference of the Hilbert function. We formalize this in the next statement.

Proposition 5.2 *Suppose $\mathcal{L}$ is an even liaison class of curves in $\mathbb{P}^3$, with minimal curve C_0, and let $r = \operatorname{diam} H^1_*(\mathbb{P}^3, \mathcal{I}_C)$. If the Hilbert function of C_0 satisfies*

$$\Delta^2 H(C_0, r_a(C_0) + 2) \leq \ldots \leq \Delta^2 H(C_0, r_o(C_0) + 2) < 0,$$

then there exists a curve C in $\mathcal{L}$ whose defining ideal I_C satisfies $\omega(I_C) = \sigma(S/I_C) + r$.

Proof: We follow the construction of basic double links in the first part of Theorem 2.5. Perform the given sequence of basic double links, up to the next to the last stage. Since $\Delta^2 H(C_0, r_o(C_0)+2) < 0$, then at this stage we see that

$$\Delta^2 H(C_{m-1}, t) = \cdots \quad -1 \quad -1 \quad \cdots \quad -1 \quad 0 \quad \cdots,$$

where there are r terms of -1, and the rightmost term occurs in degree $r_o(C_0) + 2 + m$. Thus our final basic double link $C_{m-1} : (1, r_o(C_0) + m + 4) \to C_m$ produces the curve C_m, whose ideal I_{C_m} has a generator of degree $r_o(C_0) + m + 4$, and such that $\sigma(C_m) = r_o(C_0) + m - r + 4$. Hence $\omega(I_{C_m}) = \sigma(C_m) + r$, which is what we wanted to show. $\square$

In fact, using the structure theory for curves with equal cohomology developed in the previous section, we can say more about the curves in an even liaison class having maximal degree generators. Namely, the minimal such curve occurs in the shift $\mathcal{L}^s$ of $\mathcal{L}$, where $s = -\Delta^2 H(C_0, r_a+2)$, and every other such curve is obtained by a sequence of basic double links from this minimal one, followed if necessary by a deformation. Moreover, up to a flat deformation, there are only a finite number of curves with a maximal degree generator in each allowable shift.

Remark 5.3 It is interesting to note that curves in a non-arithmetically Cohen–Macaulay liaison class which have a generator of maximal degree in fact have all of their generators of relatively high degree. Indeed, since the minimal such curve C lies in the shift $\mathcal{L}^s$ as above, $\alpha(I_C) = \alpha(I_{C_0})+s$. So, at least if the minimal curve does not already have a maximal degree generator (see Remark 3.8), we are forced to increase all the degrees of the generators.

This is in some contrast with the arithmetically Cohen–Macaulay case, where there are curves D with quadric generators and for which $\omega(D) = \sigma(D)$, the maximum possible. For example, let D be the union of a plane curve of degree m and a line, which meet at one point. Then it is easy to see that I_D has quadric generators and $\omega(D) = \sigma(D) = m$.

On the other hand, even within a non-arithmetically Cohen–Macaulay liaison class, we can make the difference $\omega(I_C) - \alpha(I_C)$ arbitrarily large when $\omega(I_C)$ is maximal. For this, simply take the minimal curve C with a maximal degree generator, and form m basic double links all of degree $\alpha(I_C)$. Then the resulting curve C_m still has $\omega(I_{C_m})$ maximal (see Remark 3.1(d.)), and has $\omega(I_{C_m})-\alpha(I_{C_m}) = \omega(I_C) - \alpha(I_C) + m$.

Next, we want to interpret the conditions on Hilbert functions only in terms of cohomology. As we noted above, to do this we need to make some extra assumptions to allow us to calculate the Hilbert function of the minimal curve. Our first result is for maximal corank curves, and the second for Buchsbaum curves.

Corollary 5.4 *Suppose $\mathcal{L}$ is an even liaison class of curves in $\mathbb{P}^3$, with minimal curve C_0. Assume that $e(C_0) < r_a(C_0)$ (i.e., C_0 has maximal corank). Then $\mathcal{L}$ contains a curve C such that $\omega(I_C) = \sigma(C) + \mathrm{diam}\ H^1_*(\mathbb{P}^3, \mathcal{I}_C)$ if and only if*

$$h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t)) \geq 3h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t+1)) - 3h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t+2)) + h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t+3)),$$

for $t = r_a(C_0), \ldots, r_o(C_0)$.

Proof: The conditions on the cohomology modules guarantee via Lemma 1.1 that

$$\Delta^2 H(C_0, r_o - r + 2) \leq \ldots \leq \Delta^2(r_o + 2, C_0) < 0,$$

and so sufficiency follows from Proposition 5.2.

To see necessity, note that if $\mathcal{L}$ contains a curve with a generator of maximal degree, then by Proposition 1.3, that curve has equal cohomology, and so by Corollary 2.6, the minimal curve in $\mathcal{L}$ has Hilbert function whose second difference ends in a sequence of non-decreasing negative terms, and then Lemma 1.1 translates this back to the required statement about cohomology. □

Recall that a Buchsbaum liaison class is completely determined by the dimensions of the graded components of the associated deficiency module. We will write $\mathcal{L}_{n_1 \ldots n_r}$ for the Buchsbaum class associated to the graded module $M = \oplus [M]_i$, where $\dim_k [M]_i = n_i$ and is zero otherwise, and where we assume that $n_1, n_r > 0$ and $n_i \geq 0$ for $1 < i < r$.

Proposition 5.5 *Suppose $\mathcal{L} = \mathcal{L}_{n_1 \ldots n_r}$ is a Buchsbaum even liaison class. Then $\mathcal{L}$ contains a curve C such that $\omega(I_C) = \sigma(I_C) + r$ if and only if $n_i \geq 3n_{i+1}$ for $i = 1, \ldots r - 1$.*

Proof: It follows from [BM2, Corollary 2.18] that the conditions on the deficiency module guarantee that the Hilbert function of C_0 satisfies the conditions given in Proposition 5.2, and we can therefore use that result to prove sufficiency.

On the other hand, if C is a curve in $\mathcal{L}$ whose ideal has a generator of maximal degree, then by Proposition 1.3, C has equal cohomology. Thus by Corollary 2.6, the minimal curve $C_0 \in \mathcal{L}$ has Hilbert function satisfying

$$\Delta^2 H(C_0, r_o(C_0) - r + 2) \leq \ldots \leq \Delta^2 H(C_0, r_o(C_0) + 2) \leq 0.$$

But now it follows from the description of the Hilbert function for minimal Buchsbaum curves given in [BM2, Corollary 2.18] that $n_i \geq 3n_{i+1}$, for each $i = 1, \ldots, r - 1$. □

Remark 5.6 Proposition 5.2 is, in a sense, a complete answer to the problem of determining which even liaison classes $\mathcal{L}$ contain a curve C whose defining ideal I_C satisfies $\omega(I_C) = \sigma(S/I_C) + \operatorname{diam} H^1_*(\mathbb{P}^3, \mathcal{I}_C)$. However, it is generally not easy to tell, for a given even liaison class, what the Hilbert function of the corresponding minimal curve is. So it is worth noting that there are also times when one can tell directly from the associated deficiency module M (defined up to shift) that such a curve does not exist.

If M is annihilated by the maximal ideal (i.e. if C is Buchsbaum), then the necessary and sufficient condition for the existence of such a curve is given in terms of the dimensions of the components of M (Proposition 5.5). Similarly, if $\mathcal{L}$ contains any curve with maximal corank then the minimal curve has maximal corank as well, since any basic double link increases r_a by exactly 1 and e by at least 1 ([BM2, Lemma 1.14]). Hence again it reduces to a question of the dimensions of the module components.

It is also true that $\mathcal{L}$ contains curves of maximal rank (i.e. $r_0 < \alpha$) if and only if the minimal curve in $\mathcal{L}$ has maximal rank, since a basic double link increases r_o by exactly 1 and increases α by at most 1. (This was first observed in [BM1, Theorem 2.1].) Then it follows from Theorem 2.5 that if the deficiency module M associated to $\mathcal{L}$ has diameter 3 or more, and if $\mathcal{L}$ contains any curve of maximal rank, then $\mathcal{L}$ does not contain any curve achieving our bound on ω.

Finally, we observe that it follows from Proposition 1.3 and Corollary 1.8 that if diam $K <$ diam M then $\mathcal{L}$ contains no curve achieving our bound. This immediately rules out a huge number of even liaison classes, since "most" classes will have a module containing at least one pair of consecutive components, for which multiplication by a general linear form is injective.

By using the results in Section 4, we can prove similar results for the existence of integral curves with generators of maximal degree. However, because the theorems in that section only go one direction, and because deformations do not in general preserve integrality or degrees of generators, we can only get necessity.

Theorem 5.7 *Suppose $\mathcal{L}$ is a liaison class whose minimal curve C_0 satisfies $t(C_0) < t_1(C_0)$. If $\mathcal{L}$ contains an integral curve with a generator of maximal degree, then the minimal curve C_0 has Hilbert function which satisfies*

$$\Delta^2 H(C_0, r_a(C_0)+2) < \cdots < \Delta^2 H(C_0, \sigma(C_0)-1) < 0.$$

Proof: This follows immediately from Proposition 4.4. $\square$

As before, with extra assumptions on the liaison class, we can give a cohomological criterion.

Proposition 5.8 *Suppose $\mathcal{L}$ is a liaison class whose minimal curve C_0 satisfies $t(C_0) < t_1(C_0)$.*

(a.) *If C_0 has maximal corank, and if $\mathcal{L}$ contains integral curves with generators of maximal degree, then*

$$h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t)) > 3h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t+1)) - 3h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t+2)) + h^1(\mathbb{P}^3, \mathcal{I}_{C_0}(t+3)),$$

for $t = r_a(C_0), \ldots, r_o(C_0)$.

(b.) *If $\mathcal{L}_{n_1 \ldots n_r}$ is a Buchsbaum liaison class containing integral curves with generators of maximal degree, then $n_i > 3n_{i+1}$ for $i = 1, \ldots, n_r$.*

Proof: This follows exactly as before, using Proposition 4.4. □

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