HILBERT COEFFICIENTS AND THE DEPTHS OF ASSOCIATED GRADED RINGS

Sam Huckaba and Thomas Marley[1]

§1. Introduction

This work was motivated in part by the following general question: Given an ideal I in a Cohen-Macaulay (abbr. CM) local ring R such that $\dim R/I = 0$, what information about I and its associated graded ring can be obtained from the Hilbert function and Hilbert polynomial of I? By the Hilbert (or Hilbert-Samuel) function of I, we mean the function $H_I(n) = \lambda(R/I^n)$ for all $n \geq 1$, where λ denotes length. Samuel ([24]) showed that for large values of n, $H_I(n)$ coincides with a polynomial $P_I(n)$ of degree $d = \dim R$. This polynomial is referred to as the Hilbert, or Hilbert-Samuel, polynomial of I. The Hilbert polynomial is often written in the form

$$P_I(n) = \sum_{i=0}^{d}(-1)^i e_i(I)\binom{n+d-i-1}{d-i}$$

where $e_0(I), \ldots, e_d(I)$ are integers uniquely determined by I. These integers are known as the Hilbert coefficients of I.

The first coefficient, $e_0(I)$, is called the multiplicity of I, and, owing to its geometric significance, has been studied extensively (eg. [23]). For instance, a classical result due to Nagata ([13]) says that $e_0(I) = 1$ if and only if R is regular and I is the maximal ideal of R. The other coefficients are not as well understood, either geometrically or in terms of how they are related to algebraic properties of the ideal or ring. One motivation for the work in this article is to pursue a better understanding of the interplay between between the Hilbert coefficients and the depth of $G(I)$, where $G(I) = R/I \oplus I/I^2 \oplus I^2/I^3 \cdots$ is the associated graded ring of I. On the one hand, it is known that if $\operatorname{depth} G(I)$ is sufficiently large (e.g., $\operatorname{depth} G(I) \geq d-1$), then the Hilbert coefficients of I possess some nice properties ([11]). A more difficult task is to find conditions on the Hilbert coefficients which force $G(I)$ to have large depth. There are several results of this kind, all of which involve $e_1(I)$. The first was given by Northcott, who proved that $e_1(I) = 0$ if and only if I is generated by a system of parameters. Consequently, if $e_1(I) = 0$ then $G(I)$ is CM. Later, Huneke ([8]) and Ooishi ([17]) proved that $\lambda(R/I) = e_0(I) - e_1(I)$ if and only if $I^2 = JI$, where J is some (any) minimal reduction of I. (Here we are assuming the residue field of R is infinite.

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See §2 for background on reductions.) Using the fact that $e_0(I) = \lambda(R/J)$, we can restate the Huneke-Ooishi theorem as $e_1(I) = \lambda(I/J)$ if and only if $I^2 = JI$. By a well-known result of Valla and Valabrega [25, Proposition 3.1], $I^2 = JI$ implies $G(I)$ is CM. In [21], Sally proved that if $e_1(I) = \lambda(I/J) + 1$ and $e_2(I) \neq 0$ then $I^3 = JI^2$, $\lambda(I^2/JI) = 1$ and $\operatorname{depth} G(I) \geq d-1$. Recently, the first author showed that $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1})$ for some minimal reduction J of I if and only if $\operatorname{depth} G(I) \geq d-1$ ([7, Theorem 3.1]).

One of the main results of this paper (Corollary 4.8) is the following: for any minimal reduction J of I and any integer $r \geq 0$,

$$e_1(I) \geq \sum_{n=1}^{r} \lambda((I^n, J)/J) \text{ with equality if and only if } G(I) \text{ is CM and } JI^r = I^{r+1}.$$

If $r = 0$ we interpret the sum on the right-hand side of the inequality to be zero and set $I^0 = R$. This result extends the Northcott, Huneke and Ooishi theorems to any reduction number. It also characterizes the CM property of $G(I)$ in terms of $e_1(I)$. Combining Corollary 4.8 with a theorem of Goto and Shimoda ([4]), we obtain the following criterion for when the Rees algebra of I, $R(I)$, is CM (Corollary 4.10):

$$R(I) \text{ is CM if and only if } e_1(I) = \sum_{n=1}^{d-1} \lambda((I^n, J)/J).$$

These results hold not only for Hilbert functions defined by powers of an ideal, but for Hilbert functions of what we call "Hilbert filtrations" (see §2). These include the function $\lambda(R/\overline{I^n})$ (provided R is analytically unramified), where $\overline{I^n}$ denotes the integral closure of I^n, and $\lambda(R/\widetilde{I^n})$, where $\widetilde{I^n}$ denotes the Ratliff-Rush closure of I^n (see §4). Our proofs allow this sort of generality with very little extra effort, but it is important to note that Hilbert filtrations are worth considering even if one is only interested in the I-adic case. For instance, Corollary 4.13 states that if $\dim R = 2$ then

$$e_1(I) = \sum_{n \geq 1} \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}}) \qquad \text{and} \qquad e_2(I) = \sum_{n \geq 2} (n-1)\lambda(\widetilde{I^n}/J\widetilde{I^{n-1}}).$$

These formulas can be used in much the same way as Proposition 2.4 of [22] and Lemma 3.1 of [5], which relate $P_I(n)$ to the local cohomology of $R(I)$ and $G(I)$, respectively. As an illustration, we use Corollary 4.13 to give a short proof of the result of Sally ([21]) mentioned above.

Our work on this paper began with the idea of finding a homological proof for the result of the first author ([7, Theorem 3.1]) mentioned above. This led us naturally to work of the second author ([10]), where machinery had already been developed for an earlier study of Hilbert coefficients. In particular, we make use of the following complex:

$$(1.1) \qquad 0 \to R/I^{n-d} \to (R/I^{n-d+1})^d \to (R/I^{n-d+2})^{\binom{d}{2}} \to \cdots \to R/I^n \to 0,$$

where n is an arbitrary integer and the maps are induced by the maps from the Koszul complex $K_.(x_1, \ldots, x_d; R)$ for some minimal reduction $(x_1, \ldots, x_d)$ of I. A version of this complex was subsequently studied in the Ph.D. thesis [3] of A. Guerrieri. In §3 of this paper, we give another formulation of this complex, different from those given in either [10] or [3], which is based on mapping cylinders. We then make several observations on its homology leading up to Theorem 3.7. (We note that Theorem 3.7 also follows from the work of Guerrieri [3], at least in the I-adic case.) This theorem is the foundation upon which our results in §4 on Hilbert functions rest. In §2, we define the terminology used in §3 and §4 and prove a couple of preliminary lemmas on filtrations.

§2. Definitions and notation

Throughout this paper, R denotes a commutative Noetherian local ring with maximal ideal m. The Krull dimension of R ($\dim R$) is denoted by d and is always assumed to be positive. In general, we adopt the notation and terminology of [12].

For any function $f : \mathbf{Z} \to \mathbf{Z}$, we define the function $\Delta(f) \colon \mathbf{Z} \to \mathbf{Z}$ by $\Delta(f)(n) = f(n) - f(n-1)$ for all integers n. For $i > 1$, we define $\Delta^i(f) = \Delta(\Delta^{i-1}(f))$. Also, we set $\Delta^0(f) = f$. By abuse of notation, we usually write $\Delta(f(n))$ for $\Delta(f)(n)$

A set of ideals $\mathcal{F} = \{I_n\}_{n\in\mathbf{Z}}$ of R is called a *filtration* if $I_0 = R$, $I_1 \neq R$, and for all n, m, $I_{n+1} \subseteq I_n$ and $I_n \cdot I_m \subseteq I_{n+m}$. Given any filtration $\mathcal{F}$ we can construct the following two graded rings:

$$R(\mathcal{F}) = R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots$$

$$G(\mathcal{F}) = R/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots$$

We call $R(\mathcal{F})$ the Rees algebra of F and $G(F)$ the associated graded ring of $\mathcal{F}$.

If $\mathcal{F}$ is an I-adic filtration (i.e., $\mathcal{F} = \{I^n\}$ for some ideal I) we denote $R(\mathcal{F})$ and $G(\mathcal{F})$ by $R(I)$ and $G(I)$, respectively. A filtration $\mathcal{F}$ is called *Noetherian* if $R(\mathcal{F})$ is a Noetherian ring. By adapting the proof of [12, Theorem 15.7], one can prove that if $\mathcal{F}$ is Noetherian then $\dim G(F) = d$. We let $G(\mathcal{F})_+$ denote the ideal of $G(\mathcal{F})$ generated by the homogeneous elements of positive degree. If M is the unique homogeneous maximal ideal of $G(\mathcal{F})$, we often write $\operatorname{depth} G(\mathcal{F})$ for $\operatorname{depth}_M G(\mathcal{F})$.

Recall that Bourbaki [1] defined a filtration $\mathcal{F}$ to be an I_1-good filtration if $R(\mathcal{F})$ is a finite $R(I_1)$-module. Adding to this, we will call a filtration $\mathcal{F} = \{I_n\}$ a *Hilbert* filtration if $\mathcal{F}$ is I_1-good and I_1 is an m-primary ideal. Note that by Theorem III.3.1.1 and Corollary III.3.1.4 of [1], $R(\mathcal{F})$ is a finite $R(I_1)$-module if and only if there exists an integer k such that $I_n \subseteq (I_1)^{n-k}$ for all n. Examples of non-I-adic Hilbert filtrations are abundant: for an m-primary ideal I, $\{\widetilde{I^n}\}$ is a Hilbert filtration, and $\{\overline{I^n}\}$ is a Hilbert filtration if and only if R is analytically unramified ([19]). (Here, $\overline{I^n}$ denotes the integral closure of I^n.) Moreover, if R is an analytically unramified ring containing a field and $(I^n)^*$ denotes the tight closure of I^n, then $\{(I^n)^*\}$ is a Hilbert filtration.

If $\mathcal{F}$ is a Hilbert filtration then $G(\mathcal{F})$ is a finite $G(I_1)$-module and so for large n, $H_{\mathcal{F}}(n) = \lambda(R/I_n)$ coincides with a polynomial $P_{\mathcal{F}}(n)$ of degree d ([12, Theorem 13.2]). We call $H_{\mathcal{F}}$ and $P_{\mathcal{F}}$ the *Hilbert function* and *Hilbert polynomial* of $\mathcal{F}$, respectively. We set $n(\mathcal{F}) = \sup\{n \in \mathbf{Z} \mid H_{\mathcal{F}}(n) \neq P_{\mathcal{F}}(n)\}$. As in the case with adic filtrations, there exist unique integers $e_0(\mathcal{F}), \ldots, e_d(F)$ (called the *Hilbert coefficients* of $\mathcal{F}$) such that for all $n \in \mathbf{Z}$

$$P_{\mathcal{F}}(n) = \sum_{i=0}^{d} (-1)^i e_i(\mathcal{F}) \binom{n+d-1-i}{d-i}.$$

Here we adopt the convention that for $m, k \in \mathbf{Z}$ with $k \geq 1$,

$$\binom{m}{k} = \frac{(m)(m-1)\cdots(m-k+1)}{k!} \qquad \text{and} \qquad \binom{m}{0} = 1.$$

For any filtration $\mathcal{F} = \{I_n\}$ and any ideal J of R, we let $\mathcal{F}/J$ denote the filtration $\{(I_n, J)/J\}$ in the ring R/J. It is easy to see that if $\mathcal{F}$ is Noetherian (Hilbert) then $\mathcal{F}/J$ is Noetherian (Hilbert).

A *reduction* of a filtration $\mathcal{F}$ is an ideal $J \subseteq I_1$ such that $JI_n = I_{n+1}$ for all large n. Equivalently, $J \subseteq I_1$ is a reduction of $\mathcal{F}$ if and only if $R(\mathcal{F})$ is a finite $R(J)$-module ([1, Theorem III.3.1.1]). A *minimal reduction* of $\mathcal{F}$ is a reduction of $\mathcal{F}$ minimal with respect to containment. If J is a reduction of the filtration $\{I^n\}$, we say simply that J is a reduction of I. By [16], minimal reductions of ideals always exist; moreover, if R/m is infinite and I is an m-primary ideal, then a reduction J of I is a minimal if and only if J is generated by d elements. If $R(F)$ is a finite $R(I_1)$-module, then J is a reduction of $\mathcal{F}$ if and only if J is a reduction of I_1. Therefore, minimal reductions of Hilbert filtrations always exist and are generated by d elements if R/m is infinite. For a minimal reduction J of $\mathcal{F}$ we set $r_J(\mathcal{F}) = \sup\{n \in \mathbf{Z} \mid I_n \neq JI_{n-1}\}$. The *reduction number* of $\mathcal{F}$, denoted $r(\mathcal{F})$, is defined to be the least $r_J(\mathcal{F})$ over all possible minimal reductions J of $\mathcal{F}$.

Let $\mathcal{F}$ be a Noetherian filtration. For any element $x \in I_1$ we let x^* denote the image of x in $G(\mathcal{F})_1 = I_1/I_2$. We note that if x^* is a regular element of $G(F)$ then x is a regular element of R and $G(\mathcal{F}/(x)) \cong G(\mathcal{F})/(x^*)$.

An element $x \in I_1$ is called *superficial* for $\mathcal{F}$ if there exists an integer c such that $(I_{n+1} : x) \cap I_c = I_n$ for all $n \geq c$. In terms of the associated graded ring of $\mathcal{F}$, x is superficial for $\mathcal{F}$ if and only if $(0 :_{G(\mathcal{F})} x^*)_n = 0$ for all n sufficiently large. If $\operatorname{grade} I_1 \geq 1$ and x is superficial for $\mathcal{F}$ then x is a regular element of R and (by the Artin-Rees Theorem) $(I_{n+1} : x) = I_n$ for all n sufficiently large. (To see that x is not a zero-divisor, note that if $ux = 0$ then $(I_1)^c u \subseteq \bigcap_n ((I_{n+1} : x) \cap I_c) = \bigcap_n I_n = 0$. Hence $u = 0$.)

A sequence $x_1, \ldots, x_k$ is called a *superficial sequence* for $\mathcal{F}$ if x_1 is superficial for $\mathcal{F}$ and x_i is superficial for $\mathcal{F}/(x_1, \ldots, x_{i-1})$ for $2 \leq i \leq k$. If $\mathcal{F}$ is a Hilbert filtration, $x_1, \ldots, x_k$ is a superficial sequence for $\mathcal{F}$, and $\operatorname{depth} R \geq k$ then

$$\begin{aligned} P_{\overline{\mathcal{F}}}(n) &= \Delta^k(P_{\mathcal{F}}(n)) \\ &= \sum_{i=0}^{d-k} (-1)^i e_i(\mathcal{F}) \binom{n+d-1-i}{d-i} \end{aligned}$$

where $\overline{\mathcal{F}} = \mathcal{F}/(x_1, \ldots, x_k)$ (see for instance [9, Lemma 4]). In particular, $e_i(\overline{\mathcal{F}}) = e_i(\mathcal{F})$ for $i = 0, \ldots, d-k$.

As a final bit of notation, if $\mathcal{F} = \{I^n\}$ where I is an m-primary ideal, we use $r(I)$, $n(I)$, P_I, H_I and $e_i(I)$ (for $0 \leq i \leq d$) to denote $r(\mathcal{F})$, $n(\mathcal{F})$, $P_{\mathcal{F}}$, $H_{\mathcal{F}}$ and $e_i(\mathcal{F})$, respectively.

Lemma 2.1. *Let $\mathcal{F}$ be a Noetherian filtration and $x_1, \ldots, x_k$ a superficial sequence for $\mathcal{F}$. If* $\operatorname{grade} G(\mathcal{F})_+ \geq k$ *then $x_1^*, \ldots, x_k^*$ is a regular sequence.*

proof. By induction it is enough to prove the case $k = 1$. But

$$(0 : x_1^*) \subseteq H^0_{G(\mathcal{F})_+}(G(\mathcal{F})) = 0.$$

Lemma 2.2. *Let $\mathcal{F} = \{I_n\}$ be a Noetherian filtration and $x_1, \ldots, x_k$ a superficial sequence for $\mathcal{F}$. If* $\operatorname{grade} G(\mathcal{F}/(x_1, \ldots, x_k))_+ \geq 1$ *then* $\operatorname{grade} G(\mathcal{F})_+ \geq k + 1$.

proof. We first consider the case $k = 1$. Let x denote x_1. Using that $\operatorname{grade} G(\mathcal{F}/(x))_+ \geq 1$, let y be an element in I_t for some $t > 0$ such that the image of y in $G(\mathcal{F}/(x))_t$ is not a zero-divisor in $G(\mathcal{F}/(x))$. Then $(I_{n+tj} : y^j) \subset (I_n, x)$ for all n, j. Since x is superficial for $\mathcal{F}$ there exists an integer c such that $(I_{n+j} : x^j) \cap I_c = I_n$ for all $j \geq 1$ and $n \geq c$. Let n and j be arbitrary and p any integer greater than c/t. Then

$$\begin{aligned} y^p(I_{n+j} : x^j) &\subseteq (I_{n+pt+j} : x^j) \cap I_c \\ &\subseteq I_{n+pt}. \end{aligned}$$

Therefore

$$\begin{aligned} (I_{n+j} : x^j) &\subseteq (I_{n+pt} : y^p) \\ &\subseteq (I_n, x). \end{aligned}$$

Thus $(I_{n+j} : x^j) = I_n + x(I_{n+j} : x^{j+1})$ for all n and j. Iterating this formula n times, we get that

$$\begin{aligned} (I_{n+j} : x^j) &= I_n + xI_{n-1} + x^2 I_{n-2} + \cdots + x^n(I_{n+j} : x^{j+n}) \\ &= I_n \end{aligned}$$

Hence x^* is a regular element of $G(\mathcal{F})$. As $G(\mathcal{F}/(x)) \cong G(\mathcal{F})/(x^*)$, $\operatorname{grade} G(\mathcal{F})_+ \geq 2$.

Now suppose $k > 1$. By the $k = 1$ case we have that $\operatorname{grade} G(\mathcal{F}/(x_1, \ldots, x_{k-1}))_+ \geq 2$. By induction on k, $\operatorname{grade} G(\mathcal{F})_+ \geq k$ and $x_1^*, \ldots, x_k^*$ is a regular sequence on $G(\mathcal{F})$ (Lemma 2.1). Since $G(\mathcal{F}/(x_1, \ldots, x_k) \cong G(\mathcal{F})/(x_1^*, \ldots, x_k^*)$, we obtain that $\operatorname{grade} G(\mathcal{F})_+ \geq k + 1$.

§3. A modified Koszul complex

In [8], Huneke proves several interesting results on the Hilbert function and Hilbert polynomial of an m-primary ideal I in a two-dimensional CM local ring R. Most of these results rely upon his "Fundemental Lemma" ([8, Lemma 2.4]). (See [7] for a generalization of this lemma.) The proof of Huneke's lemma essentially comes down to showing the exactness of the the sequence

$$0 \longrightarrow R/(I^n : (x, y)) \xrightarrow{\alpha} (R/I^n)^2 \xrightarrow{\beta} (x, y)/I^n(x, y) \longrightarrow 0,$$

where n is an arbitrary integer, (x, y) is a reduction of I, $\alpha(\overline{r}) = (-\overline{ry}, \overline{rx})$ and $\beta(\overline{s}, \overline{t}) = \overline{sx + ty}$. Inspired by Huneke's success in exploiting this sequence, the second author defined in [10] the complex (1.1) mentioned in the introduction. This complex yields the same information as Huneke's sequence in dimension two but is defined in all (positive) dimensions. By studying the homology of this complex, he showed that if depth $G(I) \geq d-1$ then H_I has many nice properties (see also [11]). In this section, we give a different formulation of this complex and make a closer examination of its homology.

Let (R, m) be a d-dimensional local ring, $\mathcal{F} = \{I_n\}$ a filtration and $x_1, \ldots, x_k \in I_1$. For any integer n, we construct the complex $C_.(x_1, \ldots, x_k, \mathcal{F}, n)$ as follows: for $k = 1$ we define $C_.(x_1, \mathcal{F}, n)$ to be the complex

$$0 \longrightarrow R/I_{n-1} \xrightarrow{x_1} R/I_n \longrightarrow 0.$$

For $k > 1$, assume the $C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n)$ has been defined and consider the chain map $f\colon C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n-1) \to C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n)$ given by multiplication by x_k. Define $C_.(x_1, \ldots, x_k, \mathcal{F}, n)$ to be the mapping cylinder of f (see [20, p. 175]). One can show that $C_.(x_1, \ldots, x_k, \mathcal{F}, n)$ has the form

$$(3.0) \qquad 0 \longrightarrow R/I_{n-k} \longrightarrow (R/I_{n-k+1})^k \longrightarrow (R/I_{n-k+2})^{\binom{k}{2}} \longrightarrow \cdots \longrightarrow R/I_n \longrightarrow 0$$

and that there is a natural surjective chain map $K_.(x_1, \ldots, x_k; R) \to C_.(x_1, \ldots, x_k, \mathcal{F}, n)$. Let $C_.(n) = C_.(x_1, \ldots, x_k, \mathcal{F}, n)$ and $C'_.(n) = C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n)$. For any n there is an exact sequence of complexes

$$0 \longrightarrow C'_.(n) \longrightarrow C_.(n) \longrightarrow C'_.(n-1)[-1] \longrightarrow 0,$$

where $C'_.(n-1)[-1]$ is the complex $C'_.(n-1)$ shifted to the left by one degree. Consequently, we have the corresponding long exact sequence on homology:

$$(3.1) \qquad \cdots \longrightarrow H_i(C'_.(n)) \longrightarrow H_i(C_.(n)) \longrightarrow H_{i-1}(C'_.(n-1)) \xrightarrow{\pm x_k} H_{i-1}(C'_.(n)) \longrightarrow \cdots$$

(These facts follow from exercices 6.13–6.15 of [20].)

Some of the homology of $C_.(n)$ can be calculated explicitly:

Lemma 3.2. *(cf. [10, Lemma 3.5]) Let $\mathcal{F} = \{I_n\}$ be a Hilbert filtration, $\underline{x} = x_1, \ldots, x_k$ elements of I_1 and $C_.(n) = C_.(\underline{x}, \mathcal{F}, n)$. Then*

(a) $H_0(C_.(n)) = R/(I_n, \underline{x})$.
(b) $H_k(C_.(n)) = (I_{n-k+1} : \underline{x})/I_{n-k}$.
(c) *If $x_1, \ldots, x_k$ is a regular sequence then* $H_1(C_.(n)) \cong ((\underline{x}) \cap I_n)/(\underline{x})I_{n-1}$.

proof. Parts (a) and (b) can be proved straightforwardly using (3.1) and induction. For part (c) we first note that since $x_1, \ldots, x_k$ is a regular sequence,

$$R^{\binom{k}{2}} \xrightarrow{f} R^k \xrightarrow{g} (\underline{x}) \longrightarrow 0$$

is an exact sequence (where f and g are the appropriate maps from the Koszul complex $K_.(\underline{x}, R)$). Tensoring this sequence with R/I_{n-1}, we see that the sequence

$$(R/I_{n-1})^{\binom{k}{2}} \xrightarrow{\alpha} (R/I_{n-1})^k \to (\underline{x})/(\underline{x})I_{n-1} \to 0$$

is also exact. Note that if $\phi_i\colon C_i(n) \to C_{i-1}(n)$ is the ith differential of $C_.(n)$ then $\operatorname{im}(\phi_2) = \operatorname{im}(\alpha)$. Thus, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\phi_2) \arrow[r] \arrow[d] & (R/I_{n-1})^k \arrow[r] \arrow[d, "\psi"] & (\underline{x})/(\underline{x})I_{n-1} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \ker(\phi_1) \arrow[r] & (R/I_{n-1})^k \arrow[r, "\phi_1"'] & R/I_n \arrow[r] & 0
\end{tikzcd}$$

Since ψ is the identity map, we obtain that

$$\begin{aligned}
H_1(C_.(n)) &= \ker(\phi_1)/\operatorname{im}(\phi_2) \\
&\cong \ker(\beta) \\
&\cong ((\underline{x}) \cap I_n)/(\underline{x})I_{n-1}.
\end{aligned}$$

The next result shows that the grade of $(\underline{x}^*)$ is determined by position the of the last non-zero homology module of $C_.(\underline{x}, \mathcal{F}, n)$ for some n:

Proposition 3.3. *Let $\mathcal{F} = \{I_n\}$ be a Noetherian filtration, $\underline{x} = x_1, \ldots, x_k$ elements of I_1, $\underline{x}^* = x_1^*, \ldots, x_k^*$, and $C_.(n) = C_.(x_1, \ldots, x_k, \mathcal{F}, n)$. Then*

$$\text{grade } (\underline{x}^*) = \min\{j \mid H_{k-j}(C_.(n)) \neq 0 \text{ for some } n\}.$$

proof. Let $K_. = K_.(\underline{x}^*; G(\mathcal{F}))$. By [12, Thm. 16.8], grade $(\underline{x}^*) = \min\{j \mid H_{k-j}(K_.) \neq 0\}$. Thus, it is enough to show that $H_i(K_.) = 0$ for all $i \geq j$ if and only if $H_i(C_.(n)) = 0$ for all n and $i \geq j$. Since each x_i^* is homogeneous of degree 1, the complex $K_.$ is the direct sum of complexes of the form

$$0 \to I_{n-k-1}/I_{n-k} \to (I_{n-k}/I_{n-k+1})^k \to \cdots \to I_{n-1}/I_n \to 0.$$

Let $K_.(n)$ denote the above complex. Then for each n we have the exact sequence of complexes:

$$0 \to K_.(n) \to C_.(n) \to C_.(n-1) \to 0.$$

¿From the corresponding long exact sequence on homology, it is clear that if $H_i(C_.(n)) = 0$ for all n and $i \geq j$ then $H_i(K_.) = \oplus_n H_i(K_.(n)) = 0$ for $i \geq j$. Conversely, if $H_i(K_.) = 0$ for $i \geq j$ then $0 \to H_i(C_.(n)) \to H_i(C_.(n-1))$ is exact for all n and $i \geq j$. Since $H_i(C_.(n)) = 0$ for $n \leq 0$, we see that $H_i(C_.(n)) = 0$ for all n and $i \geq j$.

The complex $C_.(n)$ also satisfies a certain kind of rigidity:

Lemma 3.4. *Let $\mathcal{F}$, $\underline{x}$, and $C_.(n)$ be as in Proposition 3.3. Suppose that for some $j \geq 1$, $H_j(C_.(n)) = 0$ for all n. Then $H_i(C_.(n)) = 0$ for all n and $i \geq j$.*

proof. For $k = 1$ there is nothing to prove, so assume $k > 1$. Assume the result holds for the complex $C'_.(n) = C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n)$. Then by (3.1) we have the exact sequence

$$H_{j+1}(C_.(n)) \longrightarrow H_j(C'_.(n-1)) \longrightarrow H_j(C'_.(n)) \longrightarrow 0.$$

Since $H_j(C'_.(n)) = 0$ for $n \leq 0$, we see that $H_j(C'_.(n)) = 0$ for all n. Thus, $H_i(C'_.(n)) = 0$ for all n and $i \geq j$. By (3.1), this implies that $H_i(C_.(n)) = 0$ for all n and $i \geq j$.

We will occasionally make use of the following proposition, which generalizes to filtrations a result of Valabrega and Valla ([25, Corollary 2.7]):

Proposition 3.5. *Let $\mathcal{F}$ and $\underline{x}$ be as in Proposition 3.3. Then $\underline{x}^*$ is a regular sequence if and only if $\underline{x}$ is a regular sequence and $(\underline{x}) \cap I_n = (\underline{x})I_{n-1}$ for all $n \geq 1$.*

proof. This follows from Lemma 3.2, Proposition 3.3 and and Lemma 3.4.

We now wish to establish some special properties of $C_.(\underline{x}, \mathcal{F}, n)$ in the case that $\underline{x}$ is a superficial sequence for $\mathcal{F}$.

Lemma 3.6. *Let $\underline{x}$, $\mathcal{F}$ and $C_.(n)$ be as in Proposition 3.3. Suppose $\underline{x}$ is a regular sequence on R and a superficial sequence for $\mathcal{F}$. Then $H_i(C_.(n)) = 0$ for $i \geq 1$ and n sufficiently large.*

proof. For $k = 1$ we have that $H_1(C_.(n)) = (I_n : x_1)/I_{n-1}$. Since x_1 is a non-zero-divisor and superficial for $\mathcal{F}$, $(I_n : x_1) = I_{n-1}$ for n sufficiently large. Suppose $k > 1$ and assume the lemma is true for $C'_.(n) = C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n)$. By (3.1), we see that $H_i(C_.(n)) = 0$ for $i \geq 2$ and n sufficiently large. For $i = 1$, we have that the sequence

$$0 \longrightarrow H_1(C_.(n)) \longrightarrow R/(I_{n-1}, x_1, \ldots, x_{k-1}) \xrightarrow{\alpha_n} R/(I_n, x_1, \ldots, x_{k-1}) \longrightarrow 0$$

is exact for n sufficiently large (where α_n is multiplication by x_k). Let $\overline{\mathcal{F}}$ denote the filtration $\mathcal{F}/(x_1, \ldots, x_{k-1})$ and $\overline{x}_k$ the image of x_k in the ring $R/(x_1, \ldots, x_{k-1})$. By Lemma 3.2(b), $H_1(C_.(\overline{x}_k, \overline{\mathcal{F}}, n)) = \ker(\alpha_n)$ for all n. Thus, by the $k = 1$ case, $\ker(\alpha_n) = 0$ for n sufficiently large and so $H_1(C_.(n)) = 0$ for n sufficiently large.

Let $\mathcal{F}$ be a Hilbert filtration and $\underline{x} = x_1, \ldots, x_k$ a regular sequence on R and a superficial sequence for $\mathcal{F}$. Since $\mathcal{F}$ is a Hilbert filtration, $H_i(C_.(\underline{x}, \mathcal{F}, n)$ has finite length for all i and n. For $i \geq 1$, we define

$$h_i(\underline{x}, \mathcal{F}) := \sum_{n \geq 1} \lambda(H_i(C_.(\underline{x}, \mathcal{F}, n))).$$

These integers are well-defined by Lemma 3.6. Of course, $h_i(\underline{x}, \mathcal{F}) = 0$ for $i > k$. The following theorem will be crucial for the results on Hilbert functions in the next section. In the case that $\mathcal{F}$ is an I-adic filtration, this result follows from some results in the Ph.D. thesis of A. Guerrieri (see Theorem 3.4 and the proof of Theorem 3.10 of [3]). The proof we present here uses different methods from those in [3] and is valid for any Hilbert filtration.

Theorem 3.7. *Let $\mathcal{F}$ be a Hilbert filtration and and $\underline{x} = x_1, \ldots, x_k$ a regular sequence on R and a superficial sequence for $\mathcal{F}$. Then for each $i \geq 1$*

$$\sum_{j\geq i}(-1)^{j-i}h_j(\underline{x}, \mathcal{F}) \geq 0.$$

Moreover, equality occurs if and only if grade $(\underline{x}^*) \geq k - i + 1$.

proof. We use induction on k. For $k = 1$ the result follows directly from Proposition 3.3, so assume that $k > 1$. Let $C_.(n) = C_.(\underline{x}, \mathcal{F}, n)$ and $C'_.(n) = C_.(x_1, \ldots, x_{k-1}, \mathcal{F}, n)$. Fix $i \geq 1$ and for each n let B_n be the kernel of the map $H_i(C_.(n)) \to H_{i-1}(C'_.(n-1))$ given in (3.1). Then for each n we have the exact sequence

$$0 \longrightarrow H_k(C_.(n)) \longrightarrow H_{k-1}(C'_.(n-1)) \longrightarrow \cdots \longrightarrow H_i(C'_.(n)) \longrightarrow B_n \longrightarrow 0.$$

Therefore, for each n we have

$$\lambda(B_n) = \sum_{j\geq i+1}(-1)^{j-i-1}\lambda(H_j(C_.(n)) + \sum_{j\geq i}(-1)^{j-i}\Delta(\lambda(H_j(C'_.(n)))).$$

Summing these equations over all $n \geq 1$ and using the fact that $\sum_{n\geq 1} \Delta(\lambda(H_j(C'_.(n)))) = 0$ for $j \geq 1$ (by Lemma 3.6), we see that

$$h_i(\underline{x}, \mathcal{F}) \geq \sum_{n\geq 1}\lambda(B_n) = \sum_{j\geq i+1}(-1)^{j-i-1}h_j(\underline{x}, \mathcal{F}). \tag{3.7}$$

This proves the first part of the theorem.

By Proposition 3.3, If grade $(\underline{x}^*) \geq k - i + 1$ then $H_j(C_.(n)) = 0$ for all n and $j \geq i$. Thus, $h_j(\underline{x}, \mathcal{F}) = 0$ for $j \geq i$. Conversely, suppose

$$\sum_{j\geq i}(-1)^{j-i}h_j(\underline{x}, \mathcal{F}) = 0.$$

Then by (3.7) we must have that $h_i(\underline{x}, \mathcal{F}) = \sum_{n\geq 1}\lambda(B_n)$. Consequently, $B_n = H_i(C_.(n))$ for all $n \geq 1$. Therefore, the map

$$H_{i-1}(C'_.(n-1)) \xrightarrow{x_k} H_{i-1}(C'_.(n))$$

is injective for all $n \geq 1$. If $i > 1$, then by Lemma 3.6 $H_{i-1}(C'_.(n)) = 0$ for all n. Lemma 3.4 then implies that $H_j(C'_.(n)) = 0$ for all n and $j \geq i-1$. It then follows from (3.1) that $H_j(C_.(n)) = 0$ for all n and $j \geq i$. By Proposition 3.3, grade $(\underline{x}^*) \geq k - i + 1$.

If $i = 1$ then the map

$$R/(I_{n-1}, x_1, \ldots, x_{k-1}) \xrightarrow{x_k} R/(I_n, x_1, \ldots, x_{k-1})$$

is injective for all n. Thus, grade $G(\mathcal{F}/(x_1, \ldots, x_{k-1}))_+ \geq 1$. By Lemma 2.1 and 2.2, we see that $x_1^*, \ldots, x_k^*$ is a regular sequence.

§4. Hilbert functions in a Cohen-Macaulay ring

Throughout this section R will always represent a d-dimensional ($d > 0$) CM local ring with maximal ideal m. We will also assume that the residue field R/m is infinite.

Let $\mathcal{F}$ be a Hilbert filtration and J a minimal reduction of $\mathcal{F}$. Since R/m is infinite we can find a superficial sequence $\underline{x} = x_1, \dots, x_d$ for $\mathcal{F}$ such that $J = (\underline{x})$ ([6, Lemma 2.11]). Let $C_.(n) = C_.(\underline{x}, \mathcal{F}, n)$. By (3.0), it follows that for each integer n, $\Delta^d(H_{\mathcal{F}}(n)) = \sum_i (-1)^i \lambda(C_i(n)) = \sum_i (-1)^i \lambda(H_i(C_.(n)))$. Using Lemma 3.2 and the fact that $\Delta^d(P_{\mathcal{F}}(n)) = \lambda(R/J)$, we get that for each n

$$\Delta^d(P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n)) = \lambda(R/J) - \sum_{i=0}^{d} (-1)^i \lambda(H_i(C_.(n))) \tag{4.1}$$

$$= \lambda((I_n, J)/J) - \sum_{i=1}^{d} (-1)^i \lambda(H_i(C_.(n))) \tag{4.2}$$

$$= \lambda(I_n/JI_{n-1}) - \sum_{i=2}^{d} (-1)^i \lambda(H_i(C_.(n))). \tag{4.3}$$

We remark that (4.2) gives a homological interpretation of the integers $w_n(J, I)$ used in [7, Theorem 2.4] to extend Huneke's Fundemental Lemma. Namely,

$$w_n(J, I) = -\sum_{i \geq 2} (-1)^i \lambda(H_i(C_.(n-1))).$$

Equation (4.2) also shows that if $\operatorname{depth} G(\mathcal{F}) \geq d - 1$ then $\Delta^d(P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n)) = \lambda(I_n/JI_{n-1})$ for all n. As an easy consequence of this formula, we recover the following result found in [10]: if $\operatorname{depth} G(\mathcal{F}) \geq d - 1$ and J is any reduction of $\mathcal{F}$ then

$$r_J(\mathcal{F}) = n(\mathcal{F}) + d. \tag{4.4}$$

We now wish to use the results from the previous section to obtain information on the Hilbert coefficients of $\mathcal{F}$. To do this, we first recall Proposition 2.9 of [7]: for $1 \leq i \leq d$

$$e_i(\mathcal{F}) = \sum_{n \geq i} \binom{n-1}{i-1} \Delta^d(P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n)). \tag{4.5}$$

This fact enables us to derive nice formulas for the Hilbert coefficients of $\mathcal{F}$ in the case that $\operatorname{depth} G(\mathcal{F}) \geq d - 1$:

Proposition 4.6. *Let $\mathcal{F}$ be a Hilbert filtration and suppose* depth $G(\mathcal{F}) \geq d - 1$. *Then for $1 \leq i \leq d$*

$$e_i(\mathcal{F}) = \sum_{n \geq i} \binom{n-1}{i-1} \lambda(I_n/JI_{n-1}).$$

proof. By Proposition 3.3 we have that $H_i(C_.(n)) = 0$ for $i \geq 2$ and all n. The result now follows from (4.3) and (4.5).

The above result was first proved using different methods by the first author for I-adic filtrations ([7, Corollary 2.11]). In that paper he showed that the converse is also true ([7, Theorem 3.1]). This rather surprising result led us to prove the following:

Theorem 4.7. *Let $\mathcal{F}$ be a Hilbert filtration and J a minimal reduction of $\mathcal{F}$. Then*

(a) $\quad e_1(\mathcal{F}) \geq \sum_{n\geq 1} \lambda((I_n, J)/J)$ *with equality if and only if $G(\mathcal{F})$ is CM, and*

(b) $\quad e_1(\mathcal{F}) \leq \sum_{n\geq 1} \lambda(I_n/JI_{n-1})$ *with equality if and only if* $\operatorname{depth} G(\mathcal{F}) \geq d-1$.

proof. Combining (4.5) with (4.2) and (4.3) we get that

$$\begin{aligned} e_1(\mathcal{F}) &= \sum_{n\geq 1} \lambda((I_n, J)/J) + \sum_{i\geq 1}(-1)^{i-1} h_i(\underline{x}, \mathcal{F}) \\ &= \sum_{n\geq 1} \lambda(I_n/JI_{n-1}) - \sum_{i\geq 2}(-1)^{i-2} h_i(\underline{x}, \mathcal{F}) \end{aligned}$$

where $\underline{x}$ is chosen to be a superficial sequence for $\mathcal{F}$ which generates J. Both (a) and (b) now follow from Theorem 3.7.

Two fundemental facts about the Hilbert coefficients of an m-primary ideal I are that $e_1(I) \geq 0$ with equality if and only if $r(I) = 0$ ([13]) and $\lambda(R/I) \geq e_0(I) - e_1(I)$ with equality if and only if $r(I) \leq 1$ ([15], [8] and [17]). If J is a minimal reduction of I we can rewrite the second statement as $e_1(I) \geq \lambda(I/J)$ with equality if and only if $r_J(I) \leq 1$. The following corollary provides a natural generalization of these two results:

Corollary 4.8. *Let $\mathcal{F}$ be a Hilbert filtration and J a minimal reduction of $\mathcal{F}$. Then for any $r \geq 0$*

$$e_1(\mathcal{F}) \geq \sum_{n=1}^{r} \lambda((I_n, J)/J) \quad \text{with equality if and only if } G(\mathcal{F}) \text{ is CM and } r_J(\mathcal{F}) \leq r.$$

proof. The inequality and the "if" part of the statement follow easily from part (a) of Theorem 4.7. If equality holds then by Theorem 4.7(a), $G(\mathcal{F})$ is CM and $I_n \subseteq J$ for $n > r$. By Proposition 3.5, $J \cap I_n = JI_{n-1}$ for $n \geq 1$. Hence $I_{n+1} = JI_n$ for $n \geq r$.

As special cases of Corollary 4.8 we obtain filtration versions for the two facts about $e_1(I)$ mentioned above:

Corollary 4.9. *Let $\mathcal{F}$ be a Hilbert filtration. Then*

(a) $e_1(\mathcal{F}) \geq 0$ *with equality if and only if* $r(\mathcal{F}) = 0$.

(b) $\lambda(R/I_1) \geq e_0(\mathcal{F}) - e_1(\mathcal{F})$ *with equality if and only if* $r(\mathcal{F}) \leq 1$.

proof. Note that if $r_J(\mathcal{F}) \leq 1$ then $G(\mathcal{F})$ is CM (by Proposition 3.5, for instance). The result now follows from Corollary 4.8.

As another application of Theorem 4.7, we get the following criterion for when $R(\mathcal{F})$ is CM in terms of $e_1(I)$:

Corollary 4.10. *Let $\mathcal{F}$ be a Hilbert filtration and J a minimal reduction of $\mathcal{F}$. Then*

$$R(\mathcal{F}) \text{ is CM if and only if } \quad e_1(\mathcal{F}) = \sum_{n=1}^{d-1} \lambda((I_n, J)/J).$$

proof. By the filtration version of the Goto-Shimoda Theorem ([4]) proved by Viet ([26]), $R(\mathcal{F})$ is CM if and only if $G(\mathcal{F})$ is CM and $r(\mathcal{F}) < d$. Now apply Corollary 4.8.

Remark (4.11). We remark here that we can recover a filtration version of a result due to Guerrieri ([3, Theorem 2.13]) using Theorem 4.7: namely, if $\sum_{n\geq 2} \lambda((J \cap I_n)/JI_{n-1}) = 1$ then $\operatorname{depth} G(\mathcal{F}) = d - 1$. To see this, we restate Theorem 4.7 as follows:

$$\sum_{n\geq 1} \lambda(I_n/(J \cap I_n)) \leq e_1(\mathcal{F}) \leq \sum_{n\geq 1} \lambda(I_n/JI_{n-1}),$$

where equality holds on the left if and only if $\operatorname{depth} G(\mathcal{F}) = d$, and equality holds on the right if and only if $\operatorname{depth} G(\mathcal{F}) \geq d - 1$. Since $G(F)$ can not be CM (by Proposition 3.5), we have

$$0 < e_1(\mathcal{F}) - \sum_{n\geq 1} \lambda(I_n/(J \cap I_n)) \leq \sum_{n\geq 2} \lambda((J \cap I_n)/JI_{n-1}) = 1.$$

Thus, the right equality holds, but not the left, so $\operatorname{depth} G(\mathcal{F}) = d - 1$.

For any ideal I the set of ideals $\{(I^{n+1} : I^n)\}$ forms an ascending chain. Let $\widetilde{I}$ denote the union of these ideals. Ratliff and Rush ([18]) showed that $\widetilde{I}$ is the largest ideal containing I which has the same Hilbert polynomial as I. It is easily seen that grade $G(I)_+ \geq 1$ if and only if $\widetilde{I^n} = I^n$ for all $n \geq 1$.

Lemma 4.12. *Let I be an m-primary ideal of R and $\mathcal{F} = \{\widetilde{I^n}\}$. Then $\mathcal{F}$ is a Hilbert filtration, $P_F = P_I$ and* $\operatorname{depth} G(\mathcal{F}) \geq 1$.

proof. Since $\widetilde{I^n} = I^n$ for n sufficiently large, $R(\mathcal{F})$ is a finite $R(I)$-module and $H_{\mathcal{F}}(n) = H_I(n)$ for n sufficiently large. Thus $\mathcal{F}$ is a Hilbert filtration and $P_{\mathcal{F}} = P_I$. To prove the last statement, it is enough to check that $(\widetilde{I^{k+1}} : I) = \widetilde{I^k}$ for all $k \geq 1$. To see this, first note that $\widetilde{I^k}$ is the union of the ideals in the chain $\{(I^{n+k} : I^n)\}$. If $u \in (\widetilde{I^{k+1}} : I)$ then $uI \subseteq (I^{n+k+1} : I^n)$ for some n. Thus $uI^{n+1} \subseteq I^{k+n+1}$ and hence $u \in \widetilde{I^k}$.

In particular, the Hilbert coefficients of I are the same as the Hilbert coefficients of the filtration $\{\widetilde{I^n}\}$. Applying these facts in dimension two, we get the following formulas for $e_1(I)$ and $e_2(I)$:

Corollary 4.13. *Suppose* $\dim R = 2$ *and let I be an m-primary ideal of R. Then*

(1) $e_1(I) = \sum_{n\geq 1} \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})$
(2) $e_2(I) = \sum_{n\geq 2} (n-1)\lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})$
(3) $e_2(I) = 0$ *if and only if* $r(\widetilde{I}) \leq 1$

proof. Both (a) and (b) follow from Lemma 4.12 and Proposition 4.6. We prove (c). For convenience we set $\widetilde{I} = K$. If $e_2(I) = 0$ then $\widetilde{I^n} = \widetilde{JI^{n-1}}$ for all $n \geq 2$ by (b). It holds generally that $K^2 \subseteq \widetilde{I^2}$, hence $K^2 \subseteq JK$ showing that $r(K) \leq 1$. Conversely suppose $r(K) \leq 1$. Then $\operatorname{depth}(G(K)) \geq 1$ (in fact, $G(K)$ is Cohen-Macaulay), hence $\widetilde{K^n} = K^n$ for all $n \geq 0$. But $\widetilde{K^n} = \widetilde{I^n}$ for all $n \geq 0$, thus $\widetilde{I^n} = K^n$ for all $n \geq 0$. Therefore $\widetilde{JI^{n-1}} = \widetilde{I^n}$ for all $n \geq 2$ which implies $e_2(I) = 0$ by (b).

Remark (4.14). Part (b) gives another proof of Narita's result that $e_2(I) \geq 0$ ([14]). In fact, $e_2(\mathcal{F}) \geq 0$ for any Hilbert filtration $\mathcal{F}$. To see this, one can reduce to the dimension two case by using superficial elements. Since $R(\mathcal{F})$ is Noetherian, there exists an integer $k \geq 1$ such that $I_{kn} = (I_k)^n$ for all $n \geq 1$ (see [1], for instance). Let $\mathcal{F}' = \{I_{kn}\}_n$. Then $e_2(\mathcal{F}) = e_2(\mathcal{F}') = e_2(I_k) \geq 0$.

Corollary 4.13 can be used to give new proofs of several known results regarding Hilbert functions of ideals in two-dimensional CM rings (for instance, [8, Theorem 2.1 and Theorem 4.6] and [5, Theorem 3.3]). Part of the inspiration for this work grew out of an effort to understand some results of J. Sally concerning the relationship between the Hilbert coefficients of I and the depth of $G(I)$ (see [21], [22]). We end by giving another proof of one of her results using the techniques developed in this paper.

Corollary 4.15 ([21, Theorem 1.4]). *Suppose* $\dim R = d \geq 2$ *and let* I *be an* m*-primary ideal of* R. *Suppose* $e_2(I) \neq 0$. *Then* $\lambda(R/I) = e_0(I) - e_1(I) + 1$ *if and only if for some (every) minimal reduction* J *of* I, $\lambda(I^2/JI) = 1$ *and* $r_J(I) = 2$. *If such conditions hold then* $e_2(I) = 1$ *and* depth $G(I) \geq d-1$.

proof. $\Rightarrow$: We have that $e_1(I) = \lambda(I/J)+1$ for any reduction J of I. If $\operatorname{depth} G(I) \geq d-1$ then by Proposition 4.6, $\lambda(I^2/JI) = 1$, $r_J(I) = 2$ and $e_2(I) = 1$. Thus, it suffices to show $\operatorname{depth} G(I) \geq d-1$. If $d > 2$ let $\underline{x} = x_1, \ldots, x_{d-2}$ be a superficial sequence for I. Then $e_i(I/(\underline{x})) = e_i(I)$ for $i = 0, 1, 2$. If $\operatorname{depth} G(I/(\underline{x})) \geq 1$ then by Lemma 2.2, $\operatorname{depth} G(I) \geq d-1$. Hence, it is enough to prove that if $\dim R = 2$ and $e_1(I) = \lambda(I/J)+1$ then $\operatorname{depth} G(I) \geq 1$. By Corollary 4.13(a),

$$\lambda(\widetilde{I}/I) + \sum_{n\geq 2} \lambda(\widetilde{I^n}/\widetilde{JI^{n-1}}) = 1.$$

If $I \neq \widetilde{I}$ then $e_2(\mathcal{F}) = 0$ by Corollary 4.13(b). Therefore $I = \widetilde{I}$, $\lambda(\widetilde{I^2}/JI) = 1$ and $\widetilde{I^n} = \widetilde{JI^{n-1}}$ for $n \geq 3$. (If $\widetilde{I^2} = JI$ then $r_J(I) \leq 1$, again contradicting that $e_2(I) \neq 0$.) Since $\lambda(\widetilde{I^2}/JI) = \lambda(\widetilde{I^2}/I^2) + \lambda(I^2/JI)$, we see that $\widetilde{I^2} = I^2$ and consequently $\widetilde{I^n} = I^n$ for all $n \geq 1$. Therefore depth $G(I) \geq 1$.

$\Leftarrow$: Since $\lambda(I^2/JI) = 1$ and $r_J(I) = 2$, then by Theorem 4.7(b) $e_1(I) \leq \lambda(I/J) + 1$. But $e_1(I) > \lambda(I/J)$ by Northcott's theorem ([15]) and the Huneke-Ooishi theorem ([8], [17]). Hence, $\lambda(R/I) = e_0(I) - e_1(I) + 1$.

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Department of Mathematics, Florida State University, Tallahassee, FL, 32306-3027
E-mail address: huckaba@gauss.math.fsu.edu

Department of Mathematics and Statistics, University of Nebraska, Lincoln, NE 68588
E-mail address: tmarley@unlinfo.unl.edu