The Apollonian Metric in Iwasawa Groups

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Abstract

We introduce the apollonian metric in Carnot groups using capacity. Extending Beardon's result for euclidean space, we give an equivalent definition using the cross ratio in Iwasawa groups. We also show that the apollonian metric is bounded above by twice the quasihyperbolic metric in domains in Iwasawa groups.

Key words: Iwasawa groups; cross ratio; conformal capacity; apollonian metric; quasihyperbolic metric

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1 Introduction

We define an apollonian metric in Carnot groups in terms of the modulus of ring domains. This uses recent results from potential theory in Carnot groups,[4],[5],[6],[7] and [20]. Also we define a cross ratio metric on Carnot groups. This cross ratio metric is invariant with respect to the conformal Möbius group on the Iwasawa groups. Using the stereographic projection from the boundaries of the corresponding rank one symmetric spaces we show that these metrics agree on the compactifications of the Iwasawa groups (Theorem 4). The equivalence of these two definitions of the apollonian metric on $\bar{\mathbb{R}}^n$ was given in [8]. See also [21] and [22]. We also show that the apollonian metric is dominated by twice the quasihyperbolic metric in domains in Iwasawa groups (Theorem 5). This also appears in [8] in the euclidean case. The similarity of the proofs here to those in [8] indicates that these extensions are natural.

2 Carnot Groups

A Carnot group is a connected, simply connected, nilpotent Lie group G of topological dim$G = N \geq 2$ equipped with a graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$

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so that $[V_1, V_i] = V_{i+1}$ for i=1,2,...,r-1 and $[V_1, V_r] = 0$. As usual, elements of $\mathcal{G}$ will be identified with left-invariant vectors fields on G. We fix a left-invariant Riemannian metric g on G with $g(X_i, X_j) = \delta_{ij}$. We denote the inner product with respect to this metric, as well as all other inner products, by $\langle , \rangle$. We assume that $\dim V_1 = m \geq 2$ and fix a basis of $V_1 : X_1, X_2, ..., X_m$. The horizontal tangent bundle of G, HT , is the subbundle determined by V_1 with horizontal tangent space HT_x the fiber $span[X_1(x), ..., X_m(x)]$. We use a fixed global coordinate system as $\exp : \mathcal{G} \to G$ is a diffeomorphism. We extend $X_1, ..., X_m$ to a basis $X_1, ..., X_m, T_1, ..., T_{N-m}$ of $\mathcal{G}$. We denote by Q the homogeneous dimension of the Carnot group G defined by $Q = \sum_{i=1}^{r} i \dim V_i$.

The family of dilations on G, $\{\delta_\lambda : \lambda > 0\}$, is the lift to G of the automorphism δ_λ of $\mathcal{G}$ which acts on each V_i by multiplication by λ^i. A path in G is called horizontal if its tangents lie in V_1. The (left-invariant) Carnot-Carathéodory distance , $d_c(x, y)$, between x and y is the infimum of the lengths, measured in the Riemannian metric g, of all horizontal paths which join x to y. A homogeneous norm is given by $|x| = d_c(0, x)$. All homogeneous norms on G are equivalent as such $|\cdot|$ is equivalent to the homogeneous norms below. We write $B_r(x) = \{y \in G : |x^{-1}y| < r\}$ for the ball centered at x of radius r. Since the Jacobian determinant of the dilation δ_λ is λ^Q and we have normalized the measure, $|B_\lambda| = \lambda^Q$. For information about Carnot groups we refer to [19],[29] and [20].

For $g = \exp \xi$, with $\xi \in \mathcal{G}$, $\xi = \xi_1 + \xi_2 + \cdots + \xi_r$, $\xi_i \in V_i$,, we define a norm on G by

$$N(g) = (\Sigma_{i=1}^{r} |\xi_i|^{2r!/i})^{\frac{1}{2r!}}. \tag{1}$$

This is called the non-isotropic gauge. See [18] and [23]. We also define the gauge distance,

$$d(g, g') = N(g^{-1}g'). \tag{2}$$

Being homogeneous this distance is equivalent to the Carnot-Carathéodory distance.

3 The Cross Ratio

Definition 1 *For g_1, g_2, g_3, g_4 in a Carnot group G we define the cross ratio using the norm N from (1).*

$$|g_1, g_2, g_3, g_4| = \frac{N(g_3^{-1}g_1)N(g_4^{-1}g_2)}{N(g_4^{-1}g_1)N(g_3^{-1}g_2)} = \frac{d(g_1, g_3)d(g_2, g_4)}{d(g_1, g_4)d(g_2, g_3)}. \tag{3}$$

Clearly the cross ratio is invariant under isometric transformations with respect to d and under dilations by the homogeneity of N. See [26] and [24].

We repeat a general construction of pseudo-metrics as explained in [8]. See also [25],[27]. Let $\mathcal{F}$ be a non-empty class of positive real-valued functions on a non-empty set Ω. The class satisfies the *Harnack condition* if for each $x, y \in \Omega$,

$$sup\{\frac{f(x)}{f(y)} : f \in \mathcal{F}\} < \infty.$$

Under this condition the function

$$\mu_\Omega(x, y) = sup\{|log\frac{f(x)}{f(y)}| : f \in \mathcal{F}\},$$

is finite and defines a pseudo-metric on Ω. If also $\mathcal{F}$ separates points, (for $x \neq y$ there exists an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$), then μ is a metric on Ω.

Notice that we can define a cross ratio metric of the form μ. Given a domain Ω in G we define $\mathcal{F}$ as the class of functions

$$f_{a,b}(g) = \frac{d(g, a)}{d(g, b)}$$

indexed by pairs $a, b \in \partial\Omega \times \partial\Omega$. If $\partial\Omega \times \partial\Omega$ is compact, then

$$\beta_\Omega(g, g') = sup\{|log\frac{f_{a,b}(g)}{f_{a,b}(g')}| : a, b \in \partial\Omega\}$$

is a pseudo-metric on Ω.

We give some examples of Carnot groups. We write $H(n, m)$ for a two step group with $\dim V_1 = n$ and $\dim V_2 = m$ as well as for its one-point compactification. In the following examples, s is any positive integer.

Euclidean space $\mathbb{R}^n$ with its usual abelian group structure is a Carnot group. Here $Q = n$, $X_i = \partial/\partial x_i$ and $\dim V_2 = 0$.

The usual Heisenberg groups occur when $\dim V_2$=1. Each Heisenberg group $H(2s, 1)$, is homeomorphic to $\mathbb{R}^{2s+1}$ for each $s \geq 1$. Denoting points by (z, t) with $z = (z_1, ..., z_s) \in \mathbb{C}^s$ and $t \in \mathbb{R}$ we have the group law given as

$$(z, t) \circ (z', t') = (z + z', t + t' + 2\sum_{j=1}^{n} \Im(z_j \bar{z}'_j)). \tag{4}$$

With the notation $z_j = x_j + iy_j$, the horizontal space V_1 is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t} \tag{5}$$

$$Y_j = \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t}. \tag{6}$$

where $j = 1, ..., s$. The one dimensional center V_2 is spanned by the vector field $T = \partial/\partial t$ with commutator relations $[X_j, Y_j] = 4T$. All other brackets of $\{X_1, Y_1, ..., X_s, Y_s\}$ are zero. The homogeneous dimension of $H(2s, 1)$ is $Q = 2s + 2$.

An example with $\dim V_2 = 3$ are the quaternionic Heisenberg groups. See [31] for the vector fields and bracket structure in this case. Here $Q = 4s + 6$.

These examples arise as boundary groups which form the nilpotent part of the Iwasawa decomposition of the isometries of hyperbolic space. We refer to [11],[12] and [13].

We use here the unit ball of $\mathbb{K}^r, r = 2, 3, ...$ as the model of the hyperbolic spaces $H^r(\mathbb{K})$. The Iwasawa groups G occur as the four cases, $s = r - 1$:

a) $\mathbb{K} = \mathbb{R}$, the real numbers, $G = H(s, 0)$, Euclidean spaces,
b) $\mathbb{K} = \mathbb{C}$, the complex numbers, $G = H(2s, 1)$, the Heisenberg groups,
c) $\mathbb{K} = \mathbb{H}$, the quaternions, $G = H(4s, 3)$, the quaternionic Heisenberg groups,
d) $\mathbb{K} = \mathbb{O}$, the octonions, $G = H(8, 7)$, the boundary of the Cayley plane.

See [28].

We assume that G is one of these groups throughout the rest of this section.

The product law in G, with $s, t \in \Im\mathbb{K}$ and $z, w \in \mathbb{K}^s$ is given by

$$(z, t) \circ (w, s) = (z + w, t + s + 2\Im\langle z, w\rangle)$$

where $\langle z, w\rangle = \Sigma z_i \bar{w}_i$.

With $X = H^r(\mathbb{K})$, the inverse stereographic projection $\pi^{-1} : \bar{G} \to \partial X$ is given by

$$\pi^{-1}(z,t) = (\frac{1+|z|^2+t}{|1+|z|^2+t|^2}(1-|z|^2+t), 2\frac{1+|z|^2+t}{|1+|z|^2+t|^2}z).$$

It is important here that this Cayley map

$$\pi : \partial X \to \bar{G}$$

is conformal in the case of the Iwasawa groups. See [3] and [14].

We define a chordal metric on ∂X:

$$\chi(x,y)^2 = \sqrt{|1-\langle x,y\rangle|^2 + 2R(x,y)}.$$

Here $R(v,w) = 0$ in all cases except the Cayley hyperbolic case, $H^2(\mathbb{O})$, where $R(v,w) = \Re(v_1\bar{v_2})(w_2\bar{w_1}) - \Re(\bar{v_2}w_2)(\bar{w_1}v_1)$ with $v = (v_1, v_2)$ and $w = (w_1, w_2)$. See [24],[28] and [14].

Theorem 2 *For $x, y, u, v \in \partial X$,*

$$|\pi(x), \pi(y), \pi(u), \pi(v)| = \frac{\chi(u,x)\chi(v,y)}{\chi(v,x)\chi(u,y)} \tag{7}$$

This appears in [24]. See [8] in the real case and [14] in the complex case. Hence the cross ratio on G is the same as the cross ratio in the χ-metric under projection. The isometries of X extend continuously to ∂X and the cross ratio in the χ-metric in invariant under these isometries [24]. As such the cross ratio on G is invariant under the Möbius group of G which arises from projection. Notice that the group of Möbius transformations in particular includes left translations, dilations and inversions.

We define the chordal metric on $\bar{G}$, $\chi(g,g') = \chi(\pi^{-1}(g), \pi^{-1}(g'))$. We note a case of (7):

$$\frac{\chi(g,\infty)\chi(g',0)}{\chi(g',\infty)\chi(g,0)} = |\infty, 0, g, g'| = N(g')/N(g). \tag{8}$$

Here we define a β-metric for Ω in $\bar{G}$ using the function class $\mathcal{F}$,

$$\{f_{a,b}(g) = \frac{\chi(g,a)}{\chi(g,b)} : a, b \in \partial\Omega\}$$

as

$$\beta_\Omega(g,g') = sup\{|log\frac{\chi(g,a)\chi(g',b)}{\chi(g,b)\chi(g',a)}| : a,b \in \partial\Omega\}. \tag{9}$$

Notice here that for Ω contained in $\bar{G}$, $\partial\Omega \times \partial\Omega$ is compact so that β defines a pseudo-metric in Ω.

We use global coordinates in G, $(x_1, x_2, ..., x_n, t_1, t_2, ..., t_m)$, given by the exponential map. The corresponding homogeneous norms are then a special case of (1):

$$N(g) = (|x|^4 + |t|^2)^{1/4}. \tag{10}$$

Appropriate functions of these norms are fundamental solutions of the sub-p-Laplacian given by the corresponding vector fields. See [16],[17],[4],[5],[6] and [7].

For information about these groups in the context of H-type groups and Damek-Ricci spaces see [15],[9] and [30].

4 Capacity of Condensers

All integrals will be with respect to the bi-invariant Harr measure on a Carnot group G which arises as the push-forward of the Lebesgue measure in $\mathbb{R}^N$ under the exponential map. We write $|v|^2 = \langle v, v\rangle$. We use the following spaces where U is an open set in G :

$C^\infty(U)$: infinitely differentiable functions in U,

$C_0^\infty(U)$: compactly supported functions in $C^\infty(U)$,

$HW^{1,p}(U)$: horizontal Sobolev space of functions $u \in L^p(U)$ such that the distributional derivatives $X_i u \in L^p(U)$ for $i = 1$,...,m. When u is in the local horizontal Sobolev space $HW^{1,p}_{loc}(U)$ we write the horizontal differential as $d_0 u = X_1 u dx_1 + ... + X_m u dx_m$. (The horizontal gradient $\nabla_0 u = X_1 u X_1 + ... + X_m u X_m$ appears also in the literature.)

Let U be a domain in a Carnot group G of homogeneous dimension Q. We write 0 for the group identity. A weak solution to the sub-Q-Laplacian is a function $u \in HW^{1,Q}_{loc}(U)$ which satisfies

$$\int_U |d_0 u|^{Q-2} \langle d_0 u, d_0 w \rangle = 0 \tag{11}$$

for all test functions $w \in C_0^\infty(U)$. Notice that this is the Euler-Lagrange equation for the variational integral

$$\int_U |d_0 u|^Q. \tag{12}$$

Let E and F be closed sets contained in the closure of U. The Q-capacity of the condenser $(E$,F;$U)$ is given by

$$Cap_Q(E, F; U) = \inf_u \int_U |d_0 u|^Q \tag{13}$$

where the infimum is over all functions $u \in C^\infty(U)$ with $lim_{g\to\xi} u(g) = 0$ for all $\xi \in E$ and $lim_{g\to\xi} u(g) = 1$ for all $\xi \in F$. This is the conformal capacity. Also when U is a ring R (i.e a domain whose complement is exactly two connected components) with boundary components E and F, then we write $CapR = Cap_Q(E, F; U)$. We call u a Green's function for the Q-Laplacian with pole at 0 when u is any continuous Q-harmonic function in $G\backslash\{0\}$(i.e. satisfies the sub-Q-Laplacian) which satisfies $lim_{g\to 0} u(g) = \infty$ and $lim_{|g|\to\infty} u(g) = -\infty$ and $Cap\{g : \beta < u(g) < \alpha\} = (\alpha - \beta)^{1-Q}$.

In general Carnot groups, there exists a constant $\gamma = \gamma(G)$ such that $N_u(g) = exp(-\gamma u(g))$ is a homogeneous norm ([4],[20],[5],[6],[7]) so that $N_u \circ \delta_t = tN_u$ and $N_u \circ i = N_u$, where $i(g) = g^{-1}$ is the group inversion. In the case of the Iwasawa groups the norms above agree : $N_u = N, d_u = d$. See [4],[6],[16] and [17].

For $0 < a < b$ we define

$$R_{ab} = \{g \in G : a < N_u(g) < b\}. \tag{14}$$

If R is a ring in a Carnot group G, then we define the modulus of R as

$$\text{mod } R = \gamma(G)(CapR)^{1/(1-Q)}. \tag{15}$$

With this, $\text{mod } R_{ab} = log(b/a)$.

We remark that the conformal capacity is invariant under conformal maps and quasiinvariant under quasiconformal maps. See [20],[4] and in the case of the

Iwasawa groups [13].

5 The Apollonian Metric

Using the norm $N_u(g)$ from the previous section we define a left-invariant distance $d_u(g', g) = N_u(g^{-1}g')$. Given a domain Ω in a Carnot group G and distinct points $g, g' \in \Omega$ we define the apollonian balls centered at g and g' in Ω, $\mathcal{B}_g$ and $\mathcal{B}_{g'}$ as the closure of the maximal balls in Ω of the form

$$\{h \in \bar{G} : \frac{d_u(h,g)}{d_u(h,g')} < k\}, \{h \in \bar{G} : \frac{d_u(h,g')}{d_u(h,g)} < k\} \tag{16}$$

respectively. The complement in G of $\mathcal{B}_g \cup \mathcal{B}_{g'}$ is an open ring domain $A_{g,g'}$.

Definition 3 *The apollonian distance between g and g' with respect to Ω is given by*

$$\alpha_\Omega(g, g') = mod(A_{g,g'})$$

if $A_{g,g'} \neq \varnothing$ and = 0 otherwise, or if $g = g'$.

This definition was given for $\mathbb{R}^n$ by Beardon [8].

Theorem 4 *For domains Ω in an Iwasawa group $\bar{G}$*

$$\beta_\Omega = \alpha_\Omega.$$

Here β_Ω is given by (9).

Proof: Our proof is similar to the proof in [8]. The metrics are both invariant under Möbius transformations. This is true for the apollonian metric α since the Möbius transformations are conformal and it is true for the β-metric since the cross ratio is Möbius-invariant. Given g and g', there exists a Möbius transformation γ such that $\gamma(g) = 0$ and $\gamma(g') = \infty$. As such we may assume that $\{g : N(g) < r\}$ and $\{g : N(g) > R\}$ are the apollonian balls centered at 0 and ∞, respectively, which lie in $\gamma(\Omega)$. Hence with $a, b \in \partial\gamma(\Omega)$ and using (8) and (15),

$$\begin{aligned} \beta_\Omega(g, g') &= \beta_{\gamma(\Omega)}(0, \infty) \\ &= sup_{a,b}|log\frac{\chi(0,a)\chi(\infty,b)}{\chi(0,b)\chi(\infty,a)}| \end{aligned}$$

$$= sup_{a,b}|log\frac{N(a)}{N(b)}|$$
$$= log\frac{R}{r}$$
$$= \alpha_{\gamma(\Omega)}(0,\infty)$$
$$= \alpha_{\Omega}(g,g').$$

6 The Quasihyperbolic Metric

Given $g_1, g_2 \in \Omega$ we define the quasihyperbolic distance

$$k_{\Omega}(g_1,g_2) = \inf \int_{\gamma} \frac{|dg|}{d_{\Omega}(g)}. \tag{17}$$

Here $|dg|$ is arclength , $d_{\Omega}(g)$ is the distance between g and $\partial\Omega$ in the metric $d(\cdot,\cdot)$ above and the infimum is over all rectifiable curves γ in this metric which join g_1 to g_2. We also define the j-metric

$$j_{\Omega}(g_1,g_2) = log(1 + \frac{d(g_1,g_2)}{\min(d_{\Omega}(g_1), d_{\Omega}(g_2))}). \tag{18}$$

Theorem 5 *Let Ω be a domain in an Iwasawa group G. We have*

$$\alpha_{\Omega} \leq 2j_{\Omega} \leq 2k_{\Omega}.$$

Proof : Let $g, g' \in \Omega$. Define $B_g = \{h \in \Omega : d(g,h) < d_{\Omega}(g)\}$. Let k_1 and k_2 be the largest numbers such that

$$\mathcal{B}'_g = \{h : \frac{d(h,g)}{d(h,g')} < k_1\} \subset B_g$$

and

$$\mathcal{B}'_{g'} = \{h : \frac{d(h,g')}{d(h,g)} < k_2\} \subset B_{g'}.$$

If $\mathcal{B}_g$ and $\mathcal{B}_{g'}$ are the apollonian balls centered at g and g', then $\mathcal{B}'_g \subset \mathcal{B}_g$ and $\mathcal{B}'_{g'} \subset \mathcal{B}_{g'}$. Hence with $A = (\mathcal{B}'_g \cup \mathcal{B}'_{g'})^c$ we have (recall $d = d_u$ here)

$$\alpha_{\Omega}(g,g') \leq \mod (A) = \log \frac{1}{k_1 k_2}.$$

With $h \in \bar{\mathcal{B}}'_g \cap \partial B_g$,

$$k_1 = \frac{d(h,g)}{d(h,g')} \geq \frac{d_{\Omega}(g)}{d_{\Omega}(g) + d(g,g')}.$$

Hence

$$\log\frac{1}{k_1} \leq \log(1+\frac{d(g,g')}{d_\Omega(g)}) \leq j_\Omega(g,g').$$

A similar argument gives this for $\log\frac{1}{k_2}$ and so $\alpha_\Omega \leq 2j_\Omega$. Since $j_\Omega \leq k_\Omega$ in a locally compact, rectifiably connected noncomplete metric space [10], Theorem 5 follows.

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