Unions of hyperplanes, unions of spheres, and some related estimates

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By a hyperplane in $\mathbb{R}^d$ we mean any translate of a $(d-1)$-plane. The collection $\mathcal{H}$ of all hyperplanes P in $\mathbb{R}^d$ can be parametrized by $\Sigma^{(d-1)} \times [0,\infty)$ if one identifies P with (σ, t) whenever $P = \sigma^\perp + t\sigma$. Following the capacitarian definition of Hausdorff dimension, we say that a compact set $\mathcal{K}$ of hyperplanes has dimension $\alpha > 0$ if, for each small ϵ, $\mathcal{K}$ carries a Borel probability measure μ such that

$$(1_H) \qquad \int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(P_1)\ d\mu(P_2)}{(|\sigma_1 - \sigma_2| + |t_1 - t_2|)^{\alpha-\epsilon}} < \infty.$$

Similarly, let $S(x,r)$ stand for the sphere in $\mathbb{R}^d$ with center x and radius r. Identifying the collection of all such spheres with $\mathcal{S} \doteq \mathbb{R}^d \times (0,\infty) \subseteq \mathbb{R}^{d+1}$, we will say that a compact set $\mathcal{K}$ of spheres has dimension $\alpha > 0$ if, for each small ϵ, $\mathcal{K}$ carries a Borel probability measure μ such that

$$(1_S) \qquad \int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(S_1) d\mu(S_2)}{(|x_1 - x_2| + |r_1 - r_2|)^{\alpha-\epsilon}} < \infty.$$

In both cases we are interested in what can be said about the size of

$$(2) \qquad \cup_{T \in \mathcal{K}} T$$

in terms of the Hausdorff dimension of $\mathcal{K}$. Since the dimension of a hyperplane or sphere is $d-1$, intuition suggests the conjectures that

(a) the union (2) should have positive d-dimensional Lebesgue measure whenever $\dim(\mathcal{K}) > 1$, and

(b) if $0 < \alpha < 1$ and $\dim(\mathcal{K}) = \alpha$, then (2) should have dimension at least $d - 1 + \alpha$.

In these situations (though not always in similar ones), such intuition appears to be correct. For example, considering hyperplanes and the case $\dim(\mathcal{K}) > 1$, one may define a truncated Radon transform R_0 by

$$R_0 f(\sigma, t) = \int_{\sigma^\perp \cap B(0,1)} f(p + t\sigma)\ d\mathcal{L}^{d-1}(p).$$

The following theorem is from [1].

Theorem 1. *Suppose μ is a nonnegative Borel measure on a compact set $\mathcal{K} \subseteq \mathcal{H}$ and suppose that μ satisfies (1) for $\alpha - \epsilon > 1$. Then*

$$\|R_0 \chi_E\|_{L_\mu^{\alpha-\epsilon,\infty}} \lesssim \mathcal{L}^d(E)^{1/2}$$

for Borel $E \subseteq \mathbb{R}^d$.

Now suppose that $\mathcal{K} \subseteq \mathcal{H}$ and $\dim(\mathcal{K}) = \alpha > 1$. Let μ be a Borel probability measure satisfying (1_H). If E is the set (2) then $R_0\chi_E(\sigma, t) \geq c > 0$ for each $\sigma^\perp + t\sigma \in \mathcal{K}$, and so it follows from Theorem 1 that $\mathcal{L}^d(E) \geq c^2 > 0$. Thus (a) is true for hyperplanes. For $d \geq 3$ the paper [2] contains an analogue of Theorem 1 for the spherical average operator $Tf(x,r) = \int_{\Sigma(d-1)} f(x - r\sigma) d\sigma$. It therefore follows that, when $d \geq 3$, (a) is also true for spheres. (When $d = 2$ the circle version of (a) is a significantly more difficult question, answered in the affirmative in Wolff's paper [3].) The papers [1] and [2] also contain results which imply the following theorem.

Theorem 2. *Suppose that $\mathcal{K}$ is either a compact set of hyperplanes or, if $d \geq 3$, a compact set of spheres. Suppose that $\dim(\mathcal{K}) = \alpha \in (0,1)$ and that $\mathcal{K}$ either lies on a smooth curve or has a certain Cantor set structure. Then if $E = \cup_{T\in\mathcal{K}} T$ we have $\dim(E) \geq d - 1 + \alpha$.*

Theorem 2 verifies (b) for hyperplanes in case $d = 2$ but applies only in special cases if $d > 2$. Another approach to results like (b) begins by recalling that $E \subseteq \mathbb{R}^d$ has Hausdorff dimension $\beta \in (0, d)$ if and only if, for each $\epsilon > 0$, E carries a Borel probability measure $\widetilde{\mu}$ satisfying

$$\int_{\mathbb{R}^d} \frac{|\widehat{\widetilde{\mu}}(\xi)|^2}{|\xi|^{d-\beta+\epsilon}} d\xi < \infty.$$

That is, $\dim(E) = \beta$ if, for $\epsilon > 0$, E supports a nontrivial nonnegative distribution in the Sobolev space $W^{2,-(d-\beta+\epsilon)/2}$. Thus, for example, (b) is equivalent to the conjecture that, if $0 < \alpha < 1$, $\dim(K) = \alpha$, and $\epsilon > 0$, then $\cup_{T\in\mathcal{K}} T$ should support a nonnegative distribution in $W^{2,(\alpha-1)/2-\epsilon}$. On the other hand, the dimension of $\mathcal{H} = \Sigma^{(d-1)} \times [0, \infty)$ is $d \geq 2$ and the dimension of $\mathcal{S} = \mathbb{R}^d \times (0, \infty)$ is $d + 1$ but if $\mathcal{K}$ has dimension as small as $1 + \epsilon$ then we know already that $\cup_{T\in\mathcal{K}}T$ has positive measure. It is therefore natural to wonder if more than this (i.e., more than that $\cup_{T\in\mathcal{K}}T$ has positive measure) can be said when $\dim(\mathcal{K}) > 1$. In particular, in view of the just-mentioned reformulation of (b), one might conjecture that, no matter the $\alpha \in (0, d)$, if $\dim(\mathcal{K}) = \alpha$, then, for any $\epsilon > 0$, $\cup_{T\in\mathcal{K}}T$ should support a nonnegative and nontrivial measure in $W^{2,(\alpha-1)/2-\epsilon}$. Our main result is that this is true in certain cases.

Theorem 3_H. *If $\mathcal{K} \subseteq \mathcal{H}$ and $\dim(\mathcal{K}) = \alpha \in (0, d]$ then, for $\epsilon > 0$, $\cup_{P\in\mathcal{K}}P$ supports a nonnegative measure (function if $\alpha > 1$) in $W^{2,(\alpha-1)/2-\epsilon}$.*

We note that, for hyperplanes, Theorem 3_H implies (a) as well as (b). For spheres our result is less satisfactory.

Theorem 3_S. *If $\mathcal{K} \subseteq \mathcal{S}$ and $\dim(\mathcal{K}) = \alpha \in (0, (d-1)/2)$ then, for $\epsilon > 0$, $\cup_{S\in\mathcal{K}}S$ supports a nonnegative measure in $W^{2,(\alpha-1)/2-\epsilon}$.*

Theorem 3_S implies (a) only when $d \geq 4$ and (b) only when $d \geq 3$ (though, in its range of validity, the partial result for (b) in dimension 2 is a little more general than Wolff's observation in [3] that, for $0 < \alpha < 1$, the union of a set of circles in the plane has dimension at least $1+\alpha$ if the set of centers of those circles has dimension α).

Results like Theorems 3_H and 3_S are often connected with estimates for operators like R and T. That is the case here, and we begin with the Radon transform estimate which goes with Theorem 3_H. Suppose $\psi \in \mathcal{S}(\mathbb{R}^{d-1})$ is a nonnegative radial function with Fourier transform $\widehat{\psi}$ equal to 1 on $B(0,1)$ and supported in $B(0,2)$. For $\sigma \in S^{(d-1)}$ fix an orthogonal linear map O_σ from $\sigma^\perp \subseteq \mathbb{R}^d$ to $\mathbb{R}^{d-1}$. Define a Radon transform $\widetilde{R}$ by

$$\widetilde{R}f(\sigma,t) = \int_{\sigma^\perp} f(p+t\sigma)\psi(O_\sigma(p))\ d\mathcal{L}^{d-1}(p).$$

The estimate we have in mind is the following.

Theorem 4_H. *Suppose μ is a nonnegative Borel measure on a compact set $\mathcal{K} \subseteq \mathcal{H}$ and suppose that μ satisfies the condition (slightly stronger than (1_H))*

$$\mu(\{(\sigma,t) : |\sigma-\sigma_0| + |t-t_0| < \tau\}) \lesssim \tau^\alpha$$

for some $\alpha \in (0, d]$ and for all $(\sigma_0, t_0) \in \mathcal{H}$ and $\tau > 0$. Then, for $\epsilon > 0$,

$$\|\widetilde{R}f\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}.$$

If also $\alpha > 1$, then, for small $\epsilon > 0$ and

$$\frac{1}{p} = \frac{1}{2} + \frac{\alpha-1}{2d} - \epsilon$$

there is the estimate

$$\|\widetilde{R}f\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Here is the corresponding result for spheres.

Theorem 4_S **.** *Suppose μ is a nonnegative Borel measure on a compact set $\mathcal{K} \subseteq \mathcal{S}$ and suppose that, for $\alpha \in (0,(d-1)/2)$, μ satisfies the condition*

$$\mu(\{(x,r) : |x-x_0| + |r-r_0| < \tau\}) \lesssim \tau^\alpha$$

for all $(x_0,r_0) \in \mathcal{S}$ and $\tau > 0$. Then, for $\epsilon > 0$,

$$\|Tf\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}.$$

If also $\alpha > 1$, then, for small $\epsilon > 0$ and

$$\frac{1}{p} = \frac{1}{2} + \frac{\alpha-1}{2d} - \epsilon$$

there is the estimate

$$\|Tf\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof of Theorem 3_H: Suppose that μ is a measure on $\mathcal{K}$ satisfying

$$\int_{\mathcal{K}}\int_{\mathcal{K}} \frac{d\mu(P_1)\ d\mu(P_2)}{(|\sigma_1-\sigma_2| + |t_1-t_2|)^\alpha} < \infty.$$

With ψ as above, define a measure $\widetilde{\mu}$ on $\mathbb{R}^d$ by

$$\langle f, \widetilde{\mu}\rangle = \int_{\mathcal{K}}\int_{\sigma^\perp} f(p+t\sigma)\psi(O_\sigma(p))\ d\mathcal{L}_{d-1}(p)\ d\mu(\sigma,t) = \langle \widetilde{R}f, \mu\rangle.$$

We will show that, for $\epsilon > 0$,

$$\int_{\mathbb{R}^d} |\widehat{\widetilde{\mu}}(\xi)|^2 |\xi|^{\alpha-1-2\epsilon} d\mathcal{L}_d(\xi) < \infty. \tag{3}$$

Replacing α by $\alpha-\epsilon$ then shows that Theorem 3_H is true. Suppose ρ is a nonnegative C^∞ function supported in $[1/2,4]$ and equal to one on $[1,2]$. We will establish (3) by showing that

$$\int_{\mathbb{R}^d} |\widehat{\widetilde{\mu}}(\xi)|^2 \rho^2(2^{-j}|\xi|) d\mathcal{L}_d(\xi) \tag{4}$$

is $\lesssim 2^{-j(\alpha-1)}$. Thus we begin by fixing j. If, for $\sigma \in S^{(d-1)}$, π_σ denotes the projection of $\mathbb{R}^d$ into $\sigma^\perp$ and $\Pi_\sigma = O_\sigma \circ \pi_\sigma$, then (4) is equal to

$$\int_{\mathbb{R}^d}\int_{\mathcal{K}}\int_{\mathcal{K}} e^{-i\xi\cdot(t_1\sigma_1 - t_2\sigma_2)}\widehat{\psi}\big(\Pi_{\sigma_1}(\xi)\big)\widehat{\psi}\big(\Pi_{\sigma_2}(\xi)\big)d\mu(\sigma_1,t_1)\,d\mu(\sigma_2,t_2)\rho^2(2^{-j}|\xi|)d\mathcal{L}_d(\xi) =$$

$$\int_{\mathcal{K}}\int_{\mathcal{K}} b(\sigma_1,\sigma_2,t_1\sigma_1 - t_2\sigma_2)d\mu(\sigma_1,t_1)\,d\mu(\sigma_2,t_2) \tag{5}$$

where

$$b(\sigma_1,\sigma_2,x) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\widehat{\psi}(\Pi_{\sigma_1}(\xi))\widehat{\psi}(\Pi_{\sigma_2}(\xi))\rho^2(2^{-j}|\xi|)d\mathcal{L}_d(\xi).$$

If $b(\sigma_1,\sigma_2,\cdot)$ is not identically 0, then the tubes of radius 2 through the origin in the directions of σ_1 and σ_2 must intersect at some ξ satisfying $|\xi| \sim 2^j$. This implies that $|\sigma_1 \pm \sigma_2| \lesssim 2^{-j}$. There is no loss of generality in assuming that if (σ_1,t_1) and (σ_2,t_2) are both in support of μ, then $|\sigma_1+\sigma_2| \geq 1$ (for this can be achieved by decomposing μ into a finite sum of measures with small supports). Thus we may assume that, unless $b(\sigma_1,\sigma_2,\cdot) \equiv 0$, $|\sigma_1 - \sigma_2| \lesssim 2^{-j}$. Now, with

$$a(\sigma,x) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\widehat{\psi}(\Pi_\sigma(\xi))\rho(2^{-j}|\xi|)d\mathcal{L}_d(\xi),$$

we have $b(\sigma_1,\sigma_2,\cdot) = a(\sigma_1,\cdot) * a(\sigma_2,\cdot)$. Let P_σ be the plate

$$B(0,1) \cap \{x \in \mathbb{R}^d : |x\cdot\sigma| \leq 2^{-j}\}.$$

Assume for the moment the following standard result (which will be proved later):

Lemma 1. *For $N \in \mathbb{N}$ we have*

$$|a(\sigma,\cdot)| \leq C_N 2^j \sum_{n=1}^{\infty} 2^{-nN}\chi_{2^nP_\sigma}. \tag{6}$$

Then it follows that

$$|b(\sigma_1,\sigma_2,\cdot)| \lesssim 2^{2j}\sum_{m,n=1}^{\infty} 2^{-(m+n)N}\chi_{2^nP_{\sigma_1}} * \chi_{2^mP_{\sigma_2}}. \tag{7}$$

If $|\sigma_1 - \sigma_2| \lesssim 2^{-j}$ and $m \leq n$, we have

$$\chi_{2^nP_{\sigma_1}} * \chi_{2^mP_{\sigma_2}} \lesssim 2^{dm-j}\chi_{2^{n+2}P_{\sigma_1}}$$

and so, if $N > d$,

$$2^{2j}\sum_{n=1}^{\infty}\sum_{m=1}^{n}2^{-(m+n)N}\chi_{2^nP_{\sigma_1}}*\chi_{2^mP_{\sigma_2}}\lesssim 2^{2j}\sum_{n=1}^{\infty}\sum_{m=1}^{n}2^{-(n+m)N}2^{dm-j}\chi_{2^{n+2}P_{\sigma_1}}\lesssim$$

$$2^j\sum_{n=1}^{\infty}2^{-nN}\chi_{2^{n+2}P_{\sigma_1}}.$$

It therefore follows from (7) that (5), and so (4), is controlled by

$$(8)\qquad 2^j\sum_{n=1}^{\infty}2^{-nN}\int\int_{\{|\sigma_1-\sigma_2|\lesssim 2^{-j}\}}\chi_{2^{n+2}P_{\sigma_1}}(t_1\sigma_1-t_2\sigma_2)d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2).$$

Now if $t_1\sigma_1-t_2\sigma_2\in 2^{n+2}P_{\sigma_1}$, then

$$|t_1-t_2+t_2(\sigma_1-\sigma_2)\cdot\sigma_1|=|(t_1\sigma_1-t_2\sigma_1)\cdot\sigma_1+t_2(\sigma_1-\sigma_2)\cdot\sigma_1|=|(t_1\sigma_1-t_2\sigma_2)\cdot\sigma_1|\lesssim 2^{n-j}.$$

If also $|\sigma_1-\sigma_2|\lesssim 2^{-j}$, then $|t_2|\lesssim 1$ gives $|t_1-t_2|\lesssim 2^{n-j}$ and so

$$|\sigma_1-\sigma_2|+|t_1-t_2|\lesssim 2^{n-j}.$$

Thus (8) is bounded by

$$(9)\qquad \sum_{n=1}^{\infty}2^{-nN}2^j\int\int_{\{|\sigma_1-\sigma_2|+|t_1-t_2|\lesssim 2^{n-j}\}}d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2).$$

Since

$$\int\int_{\{|\sigma_1-\sigma_2|+|t_1-t_2|\leq\tau\}}d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2)\leq\tau^\alpha\int_{\mathcal{K}}\int_{\mathcal{K}}\frac{d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2)}{(|\sigma_1-\sigma_2|+|t_1-t_2|)^\alpha}\lesssim\tau^\alpha,$$

we may bound (9), and so (4), by

$$\sum_{n=1}^{\infty}2^{-nN}2^j2^{(n-j)\alpha}\lesssim 2^{-j(\alpha-1)}.$$

This completes the proof of Theorem 3_H.

Proof of Lemma 1: Without loss of generality let $\sigma=(1,0,\dots,0)$. Writing $\xi=(\xi_1,\xi')$ and identifying $\sigma^\perp$ with $\mathbb{R}^{d-1}$, we have

$$(10)\qquad a(\sigma,x)=\int\int e^{-i\xi\cdot x}\widehat{\psi}(\xi')\rho(2^{-j}|\xi|)d\mathcal{L}_{d-1}(\xi')d\mathcal{L}_1(\xi_1).$$

Suppose $x \in 2^{n+1}P_\sigma \sim 2^n P_\sigma$. Writing $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$, assume first that $|x| \geq 2^n$ so that, if $j > 1$, $|x'| \geq 2^{n-1}$. Then, considering the support of $\widehat{\psi}$,

$$\Big| \int e^{-i\xi' \cdot x'} \widehat{\psi}(\xi') \rho(2^{-j}|\xi|) d\mathcal{L}_{d-1}(\xi') \Big| = \Big| \int_{B(0,2)} e^{-i\xi' \cdot x'} \widehat{\psi}(\xi') \rho(2^{-j}|\xi|) d\mathcal{L}_{d-1}(\xi') \Big|.$$

Integrating by parts N times, this is bounded by $C_N 2^{-nN}$. Thus (10) is bounded by $C_N 2^j 2^{-nN}$ since $|\xi_1| \lesssim 2^j$. Suppose now that $x \in 2^{n+1}P_\sigma \backslash 2^n P_\sigma$ and $|x| < 2^n$. Then $|x_1| > 2^{n-j}$. Now

$$\int e^{-i\xi_1 x_1} \rho(2^{-j}|\xi|) d\xi_1 = 2^j \int e^{-i\widetilde{\xi}_1 2^j x_1} \rho\Big(\sqrt{\widetilde{\xi}_1^2 + |2^{-j}\xi'|^2}\Big) d\widetilde{\xi}_1. \tag{11}$$

Since $|2^j x_1| \sim 2^n$, integrating by parts N times bounds (11) by $C_N 2^{j-nN}$. Since $\widehat{\psi}$ is supported in $B(0,2)$, the same bound applies to (10).

Proof of Theorem 4_H*:* Theorem 4_H will follow from the estimate

$$\|\widetilde{R}^* \chi_{\mathcal{E}}\|_{W^{2,(\alpha-1)/2-\epsilon}} \lesssim (\mu(\mathcal{E}))^{1/2}, \quad \mathcal{E} \subseteq \mathcal{H},$$

dual to

$$\|\widetilde{R} f\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}$$

and, if $\alpha > 1$, the Sobolev embedding theorem. Thus, for Borel $\mathcal{E} \subseteq \mathcal{H}$ and for suitable f, we note that

$$\langle f, \widetilde{R}^* \chi_{\mathcal{E}} \rangle = \langle \widetilde{R} f, \chi_{\mathcal{E}} \mu \rangle = \int_{\mathcal{E}} \int_{\sigma^\perp} f(p + t\sigma) \psi(O_\sigma(p)) \, d\mathcal{L}_{d-1}(p) \, d\mu(\sigma, t).$$

Following the proof of Theorem 3 with μ replaced by $\chi_{\mathcal{E}}\mu$ (see (9)) shows that

$$\|\widetilde{R}^* \chi_{\mathcal{E}}\|^2_{W^{2,(\alpha-1)/2-\epsilon}}$$

is controlled by the sum on j of the terms

$$2^{j(\alpha-1-2\epsilon)} \sum_{n=1}^{\infty} 2^{-nN} 2^j \int_{\mathcal{E}} \int_{\{|\sigma_1 - \sigma_2| + |t_1 - t_2| \lesssim 2^{n-j}\}} d\mu(\sigma_1, t_1) d\mu(\sigma_2, t_2) \lesssim$$

$$2^{j(\alpha-1-2\epsilon)} \sum_{n=1}^{\infty} 2^{-nN} 2^j \mu(\mathcal{E}) 2^{\alpha(n-j)} \lesssim 2^{-2j\epsilon} \mu(\mathcal{E}).$$

This yields the desired result.

Proof of Theorem 3_S: Here we write σ for Lebesgue measure on $S^{(d-1)}$. The proof is generally parallel to that of Theorem 3_H. Thus suppose that μ is a measure on $\mathcal{K}$ satisfying

$$\int_{\mathcal{K}}\int_{\mathcal{K}} \frac{d\mu(S_1)\ d\mu(S_2)}{(|x_1-x_2|+|r_1-r_2|)^{\alpha}} < \infty$$

and define $\widetilde{\mu}$ on $\mathbb{R}^d$ by

$$\langle f,\widetilde{\mu}\rangle = \int_{\mathcal{K}}\int_{S^{(d-1)}} f(x+r\zeta)\ d\sigma(\zeta)\ d\mu(x,r) = \langle \widetilde{T}f,\mu\rangle.$$

With ρ as in the proof of Theorem 3, we would like to show that

$$\int_{\mathbb{R}^d} |\widehat{\widetilde{\mu}}(\xi)|^2\rho(2^{-j}|\xi|)d\mathcal{L}_d(\xi) \lesssim 2^{-j(\alpha-1)}. \tag{12}$$

We begin by rewriting (12) as

$$\int_{\mathbb{R}^d}\int_{\mathcal{K}}\int_{\mathcal{K}} \widehat{\sigma}(r_1\xi)\ \widehat{\sigma}(r_2\xi)\ e^{-i(x_1-x_2)\cdot\xi}d\mu(x_1,r_1)\ d\mu(x_2,r_2)\rho(2^{-j}|\xi|)d\mathcal{L}_d(\xi)$$

Changing to polar coordinates on $\mathbb{R}^d$ and abusing notation by writing $\widehat{\sigma}(|\xi|)$ to stand for $\widehat{\sigma}(\xi)$, this is

$$\int_{\mathcal{K}}\int_{\mathcal{K}}\int_0^{\infty} \widehat{\sigma}(r_1 r)\ \widehat{\sigma}(r_2 r)\ \widehat{\sigma}(|x_1-x_2|r)\rho(2^{-j}r)r^{d-1}dr\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)\ =$$

$$\int_{\mathcal{K}}\int_{\mathcal{K}} b(r_1,r_2,|x_1-x_2|)\ d\mu(x_1,r_1)\ d\mu(x_2,r_2) \tag{13}$$

if

$$b(r_1,r_2,s) = \int_0^{\infty} \widehat{\sigma}(r_1 r)\ \widehat{\sigma}(r_2 r)\ \widehat{\sigma}(sr)\rho(2^{-j}r)r^{d-1}dr.$$

We will use the following notation: if $S_1 = S(x_1,r_1)$ and $S_2 = S(x_2,r_2)$ are spheres, then $\delta = \delta(S_1,S_2)$ will stand for the distance $|x_1-x_2|+|r_1-r_2|$ between S_1 and S_2 while $\Delta = \Delta(S_1,S_2)$ while stand for $||x_1-x_2|-|r_1-r_2||$. We also observe that on the compact subset $\mathcal{K}$ of $\mathcal{S}$, r is bounded away from 0. We will estimate (13), and therefore establish (12), by considering the different cases which result from splitting the integral in a certain way.

Case I: $\int\int_{\{\Delta<\delta/2\}} b(r_1,r_2,|x_1-x_2|)\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)$

If $\Delta < \delta/2$ then $\delta \sim |x_1-x_2|$. Now $|b(r_1,r_2,|x_1-x_2|)| \lesssim 2^j$ follows from

$$|\widehat{\sigma}(s)| \lesssim s^{(1-d)/2} \tag{14}$$

(recall that the r_j are bounded away from 0 and that $|\widehat{\sigma}|$ is bounded). Thus the portion of the Case I integral where $|x_1 - x_2| \leq 2^{-j}$ is controlled by

$$2^j \int\int_{\{\delta \lesssim 2^{-j}\}} d\mu(x_1, r_1)\ d\mu(x_2, r_2) \lesssim 2^{-j(\alpha-1)},$$

where the last inequality follows (as in the proof of Theorem 3_H) from the capacitarian assumption on μ. If $|x_1 - x_2| \gtrsim 2^{-j}$ then (14) and $\delta \sim |x_1 - x_2|$ imply that the relevant integral is controlled by

$$\frac{2^j}{(2^j|x_1 - x_2|)^{(d-1)/2}} \lesssim \frac{1}{\delta^{(d-1)/2} 2^{j(d-3)/2}} \lesssim$$

$$\frac{1}{\delta^{\alpha} 2^{-j[(d-1)/2-\alpha]}} \frac{1}{2^{j(d-3)/2}} = \frac{1}{\delta^{\alpha} 2^{j(-1+\alpha)}}.$$

Here the second inequality follows from $\delta \gtrsim 2^{-j}$ and $\alpha \leq (d-1)/2$. Thus the Case I integral is controlled by $2^{-j(\alpha-1)}$.

Case II: $\int\int_{\{\delta < 4 \cdot 2^{-j}\}} b(r_1, r_2, |x_1 - x_2|)\ d\mu(x_1, r_1)\ d\mu(x_2, r_2)$

Since

$$\int\int_{\{\delta < 4 \cdot 2^{-j}\}} d\mu(x_1, r_1)\ d\mu(x_2, r_2) \lesssim 2^{-j\alpha}$$

and $|b(r_1, r_2, |x_1 - x_2|)| \lesssim 2^j$, the desired bound of $2^{-j(1-\alpha)}$ is immediate.

Case III: $\int\int_{\{4 \cdot 2^{-j} \leq \delta \leq 2\Delta\}} b(r_1, r_2, |x_1 - x_2|)\ d\mu(x_1, r_1)\ d\mu(x_2, r_2)$

Recall that

$$b(r_1, r_2, |x_1 - x_2|) = \int_a^b \widehat{\sigma}(r_1 r)\ \widehat{\sigma}(r_2 r)\ \widehat{\sigma}(|x_1 - x_2| r) \rho(2^{-j} r) r^{d-1} dr$$

where $a \gtrsim 2^j$. Utilizing the asymptotic expansion of $\widehat{\sigma}$ and recalling that r_1 and r_2 are bounded away from 0, the principal term in this integral is controlled by the largest of

$$\Big| \int_a^b \frac{e^{i(\pm r_1 \pm r_2 \pm |x_1 - x_2|) r}}{(r|x_1 - x_2|)^{(d-1)/2}} dr \Big|. \tag{15}$$

After rescaling and then multiplying μ by a cutoff function of x, we may assume that $r_1, r_2 \geq 1/2$ and $|x_1 - x_2| \leq 1/2$. One can check that then $\Delta = ||r_1 - r_2| - |x_1 - x_2||$ minimizes $| \pm r_1 \pm r_2 \pm |x_1 - x_2||$. An integration by parts bounds (15) by some multiple of

$$|x_1 - x_2|^{-(d-1)/2} \Big(\Big| \int_a^b \int_a^r e^{i\Delta s} ds\ r^{-(d+1)/2} dr \Big| + 2^{-j(d-1)/2} \Big| \int_a^b e^{i\Delta s} ds \Big| \Big).$$

Since $a \geq 2^j$, it follows that

$$|(15)| \lesssim \frac{2^{-j(d-1)/2}}{\Delta \cdot |x_1 - x_2|^{(d-1)/2}} \lesssim \frac{2^{-j(d-1)/2}}{\Delta^{(d+1)/2}} \lesssim \frac{2^{-j(d-1)/2}}{\Delta^{\alpha}\ 2^{-j[(d+1)/2-\alpha]}},$$

where the last inequality follows from $\Delta \gtrsim 2^{-j}$ and $\alpha \leq (d-1)/2 < (d+1)/2$. Thus

$$\int \int_{\{4 \cdot 2^{-j} \leq \delta \leq 2\Delta\}} |(15)|\ d\mu(x_1, r_1)\ d\mu(x_2, r_2) \lesssim 2^{-j(1-\alpha)}$$

by the capacitarian assumption on μ. The nonprincipal terms are controlled similarly. For example, the term coming from the principal terms of $\widehat{\sigma}(r_i r)$ and the second order term from $\widehat{\sigma}(|x_1 - x_2|r)$ is controlled by

$$\int_1^b \frac{dr}{(r|x_1 - x_2|)^{(d+1)/2}} \lesssim \frac{1}{\Delta^{(d+1)/2} 2^{j(d-1)/2}}$$

and so may be treated as was $|(15)|$. This completes the proof of Theorem 3_S.

The changes to the proof of Theorem 3_S which are required in order to prove Theorem 4_S are analogous to the changes in the proof of Theorem 3_H which yield the proof of Theorem 4_H.

References

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