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Triangulations and moduli spaces of Riemann surfaces with group actions *

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Summary. We study that subset of the moduli space $\overline{M^g}$ of stable genus g, $g > 1$, Riemann surfaces which consists of such stable Riemann surfaces on which a given finite group F acts. We show first that this subset is compact. It turns out that, for general finite groups F, the above subset is not connected. We show, however, that for $\mathbb{Z}_2$ actions this subset *is* connected. Finally, we show that even in the moduli space of *smooth* genus g Riemann surfaces, the subset of those Riemann surfaces on which $\mathbb{Z}_2$ acts *is* connected. In view of deliberations of Klein ([8]), this was somewhat surprising.

These results are based on new coordinates for moduli spaces. These coordinates are obtained by certain *regular* triangulations of Riemann surfaces. These triangulations play an important role also elsewhere, for instance in approximating eigenfunctions of the Laplace operator numerically.

1. Introduction

The moduli space $\overline{M^g}$, $g > 1$, of stable compact Riemann surfaces is a projective algebraic variety. It is a compactification of the moduli space M^g of the compact smooth Riemann surfaces of genus g.

It was shown in [12] that essentially the same compactification can be constructed for moduli spaces of compact Riemann surfaces which admit an antiholomorphic involution. Such Riemann surfaces correspond to real algebraic

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curves. Let us for that reason refer to them as *real Riemann surfaces.* The moduli space of real stable genus g Riemann surfaces is a compactification of the moduli space of smooth real genus g Riemann surfaces.

The construction of [12] was based on an extension of a lemma concerning pants decompositions of Riemann surfaces. It is well known that a compact Riemann surface X of genus g admits a decomposition into pairs of pants, also called Y-pieces, in such a way that the lengths of the decomposing curves are bounded by a constant depending only on g.

In [4] it was shown that this constant can be taken to be $21g$ and that if X admits an antiholomorphic involution, then the decomposition into Y-pieces can be chosen in such a way that the involution maps the set of decomposing curves onto itself.

It was shown further in [4] that the above result is *not* true for more general group actions.

As a consequence we are lead to the following question. Let F be a finite group and let

$$\overline{M^g_F} = \{[X] \in \overline{M^g} \mid F \subset \mathrm{Isom}(X)\}, \tag{1}$$

where $\mathrm{Isom}(X)$ is the group of isometries of X onto itself. For which groups F is $\overline{M^g_F}$ compact? The considerations of [12] imply that $\overline{M^g_F}$ is compact for $\mathbb{Z}_2$ acting via antiholomorphic involutions (see also [11]).

We prove in this note that for any finite group action F the space $\overline{M^g_F}$ is a compact subspace of the moduli space $\overline{M^g}$.

The proof is based on size-controlled triangulations which may be taken to be invariant with respect to finite group actions. We point out that such triangulations are also important for numerical considerations. Part of our emphasis will therefore be to construct these triangulations rather explicitly.

In the special case of the $\mathbb{Z}_2$-actions we consider the question whether the space $\overline{M^g_{\mathbb{Z}_2}}$ is connected or not. This is related to a conjecture of Klein. In 1892 Klein conjectured that the subspace of $\overline{M^g}$ consisting of all real stable Riemann surfaces of genus g, $g > 1$, is connected ([8]). This was shown to be the case in [12].

In the same paper Klein seems to have thought that the smooth real Riemann surfaces of genus g do not form a connected subspace of $\overline{M^g}$. We show in this paper that, on the contrary, the set of all real smooth Riemann surfaces is a connected subset of the complex moduli space M^g.

We assume that the general properties of the moduli spaces $\overline{M^g}$ are known. For more details we refer to [3] or to [13].

It is a pleasure to express here our thanks to Gareth Jones who communicated to us a result about $\mathbb{Z}_3$-actions. We shall reproduce this result at the end of the paper.

2. Triangulations and the Mahler Compactness Theorem

In order to describe our approach in a simple case we begin with a new proof of the so-called Mahler compactness theorem (see [10] or [9]).

Let $\epsilon > 0$ and consider the set

$$M^g_\epsilon = \{[X] \in M^g \mid \text{ the injectivity radius of } X \text{ is } \geq \epsilon\}. \tag{2}$$

Here $[X]$ is the the conformal equivalence class of a compact Riemann surface X of genus g, $g \geq 2$. We always assume that X is equipped with the unique riemannian metric of constant curvature -1 which is compatible with the conformal structure (the so-called Poincaré metric). All geometric constructions shall be defined in terms of this metric. The *injectivity radius* of X is defined as the radius of the largest circular open disc which can be isometrically embedded in X. Likewise it is one half of the length of the shortest simple closed geodesic on X.

The theorem is the following.

Theorem 1. *The space M^g_ϵ is compact.*

Proof. Let $\mathcal{P}$ be a finite set of points in X such that

1. Any two points of $\mathcal{P}$ lie at distance $\geq \epsilon/2$ from each other.
2. For all $x \in X$ there exists $p \in \mathcal{P}$ such that the distance from x to p is $< \epsilon/2$.

Such a set $\mathcal{P}$ is easily obtained by successively marking points in X at pairwise distances $\geq \epsilon/2$ until there is no more room for such points.

The following argument is due to Fejes Tóth, [6] and [3, pages 116 – 121]. Consider circles of radius $< \frac{\epsilon}{2}$ going through three (or accidentally more) points of $\mathcal{P}$ but such that no point of $\mathcal{P}$ lies in the interior of any of these circles. A continuity argument involving families of circles with growing radii shows that any point of $\mathcal{P}$ lies on such a circle. Moreover, if for each such circle we draw the inscribed polygon whose vertices are the points of $\mathcal{P}$ on that circle (usually this yields a triangle), then we get a tessellation of X as shown in Figure 1. We refer to [3, pages 116 – 121] for more details about this part. On those polygons which

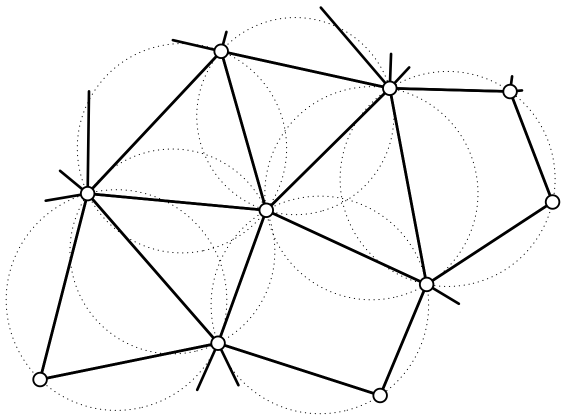

Fig. 1. Tessellation of X.

are not triangles we add diagonals coming out of some fixed vertex as shown in Figure 2. In this way we get a triangulation of X. This triangulation has the property that for some universal positive constant θ, all the interior angles of the occurring triangles lie in the interval $[\theta, \pi - \theta]$:

$$\theta \leq \alpha,\, \beta,\, \gamma \leq \pi - \theta. \tag{3}$$

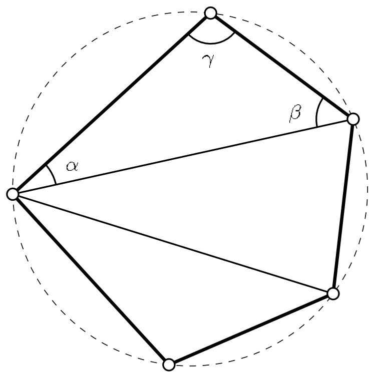

Fig. 2. Triangulation of X.

Also, any side length s satisfies

$$\frac{\epsilon}{2} \leq s < \epsilon. \tag{4}$$

The constant θ may be computed explicitly but we do not need this here.

We prove the compactness of M_ϵ^g by showing that any infinite sequence has a converging subsequence.

Let $[X_n]$, $n \in \mathbb{N}$, be a sequence in M_ϵ^g. Triangulate each X_n in the above way. By (3) and (4) there is a lower bound on the area of any of the occurring triangles. Hence, there is a uniform upper bound on the number of triangles on each X_n. Therefore, there exists an infinite subsequence of $[X_n]$ for which the triangulation is combinatorially always the same. Without loss of generality we assume that the subsequence is $[X_n]$ itself. We then also label the triangles and their sides on any X_n in the same combinatorial way denoting by $T_{n,1}, \dots, T_{n,r}$ the triangles and by $a_{n,j}$, $b_{n,j}$ and $c_{n,j}$ their sides, $j = 1, \dots, r$. As usual these letters also denote the lengths of the sides.

Passing to subsequences several times, if necessary, we may assume that the lengths of all sides $a_{n,j}$, $b_{n,j}$ and $c_{n,j}$ converge as $n \to \infty$. Let a_j, b_j and c_j be the respective limits.

Since the angles and the side lengths of $T_{n,j}$ are nicely bounded, the limit triangles T_j with sides of lengths a_j, b_j and c_j exist and have their interior angles in the interval $[\theta, \pi - \theta]$. This implies that no T_j is degenerate.

Now let us paste the triangles T_j together following the common pattern of the triangulations of the Riemann surfaces X_n. This yields a Riemann surface X, since, by continuity, the sum of the angles at each vertex is 2π. It is clear that we have $[X_n] \to [X]$ in M^g with respect to the usual topology of M^g.

We add a remark about the "usual topology of M^g" just mentioned. The above triangulations of X_n and X are prompting some kind of intuitive topology: the sides converge, "hence", the triangles converge, "hence", the surfaces converge. This can be made precise as follows. Using triangular coordinates on the triangles $T_{n,j}$ and T_j as in [3, page 68] we get a quasiconformal mapping

$\phi_{n,j} : T_{n,j} \to T_j$ sending points with given coordinates on $T_{n,j}$ to points with the same coordinates on T_j. These mappings fit together on the boundaries of the triangles and define a quasiconformal mapping $\phi_n : X_n \to X$. As $n \to \infty$ the maximal dilatation of ϕ_n converges to 1. This is exactly what is required for the convergence $[X_n] \to [X]$ in M^g (see e.g. [13, 5.6]).

3. Triangulations and the Baily–Mumford Compactification

Let us now modify the above construction to give a compactification of M^g. The compactification that we proceed to describe is the usual Baily–Mumford compactification of the moduli space by adding stable Riemann surfaces. We are presenting a new approach to this compactification which allows a convenient generalization to the case of Riemann surfaces with group actions. For the ease of exposition we postpone the generalization to the next section.

We begin with the construction and check afterwards what it actually yields.

Let $[X_n]$ be again an infinite sequence, but now in M^g, and assume that the injectivity radius becomes arbitrarily small. By the collar theorem ([3, Theorem 4.1.6, page 97], there exists a constant ϵ_∞ (in fact, $\epsilon_\infty = 2\ \mathrm{arcsinh}(1)$) such that on any X_n at most $3g-3$ simple closed geodesics are shorter than ϵ_∞. Hence, by passing to a subsequence, if necessary, we have an integer $m \leq 3g-3$ and some positive constant ρ with the following property. There are closed geodesics $\gamma_{n,1}, \ldots, \gamma_{n,m}$ on X_n whose lengths go to 0 as $n \to \infty$ and all other closed geodesics on X_n are longer than 2ρ. For convenience we assume that $\rho \leq \frac{1}{16}$.

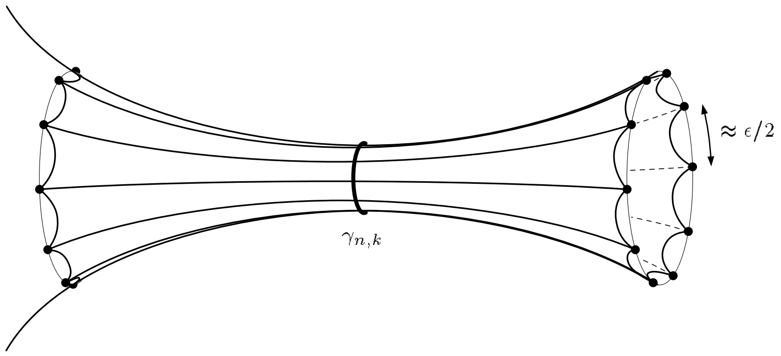

Fig. 3. Collar at geodesic $\gamma_{n,k}$. The boundary is marked with points of $\mathcal{P}$.

Again by the collar theorem ([3, Theorems 4.1.1, 4.1.6, pages 94–97]), there are distance neighborhoods $C(\gamma_{n,k})$, the *collars*, with the following properties. Each $C(\gamma_{n,k})$ is a topological cylinder as shown in Figure 3. The boundary components (dotted curves) of $C(\gamma_{n,k})$ are of length 1, the injectivity radius at points near the boundary is $\geq \frac{1}{4}$, and the collars have pairwise distances $\geq \frac{1}{2}$. (For this last property the widths of the collars here have been chosen slightly smaller than in [3].)

This gives us a "thin and thick decomposition" of each surface: the *thin part* is defined to be the union of all collars, the *thick part* is the complement. Note that for each X_n and for each point p in the thick part of X_n the injectivity radius at p is $\geq \rho$.

Now let ϵ be a positive constant, $\epsilon < \rho$, which will be kept fixed. (We may take $\epsilon = \frac{\rho}{2}$, but in view of future applications we want to allow smaller ϵ.) We define a finite set $\mathcal{P}$ on X_n as follows. On the boundary components of the collars we put points at equal consecutive distances $\geq \frac{\epsilon}{2}$ and as close to $\frac{\epsilon}{2}$ as possible. This is shown in Figure 3. In the thick part we then proceed as in the proof of Theorem 1. Namely, starting with the points already given on the boundary we successively plant points at pariwise distances $\geq \frac{\epsilon}{2}$ until there is no more room for such points in the thick part.

In the thin part, that is, in the interior of each $C(\gamma_{n,k})$, we proceed in a different way. For any point of $\mathcal{P}$ on the boundary of $C(\gamma_{n,k})$ we take the perpendicular geodesic segment to the "core" geodesic $\gamma_{n,k}$. This yields a decomposition of $C(\gamma_{n,k})$ into geodesic quadrilaterals as shown in Figures 3 and 4. For better reference we call them *regular quadrilaterals*. We assume that the points of $\mathcal{P}$ on the boundary are placed symmetrically with respect to $\gamma_{n,k}$ so that each side of a quadrilateral on $\gamma_{n,k}$ coincides with the side of a second such quadrilateral. The surface X_n is thus tessellated with triangles in the thick part and with regular quadrilaterals in the thin part.

The angles of each triangle are between θ and $\pi - \theta$ as before. The acute angles of the quadrilaterals satisfy

$$\theta \leq \alpha_{n,k} = \beta_{n,k} \leq \pi - \theta. \tag{5}$$

The notation is as in Figure 4. We recall that ϵ is small so that the angles $\alpha_{n,k}$ and $\beta_{n,k}$ are almost equal $\frac{\pi}{2}$.

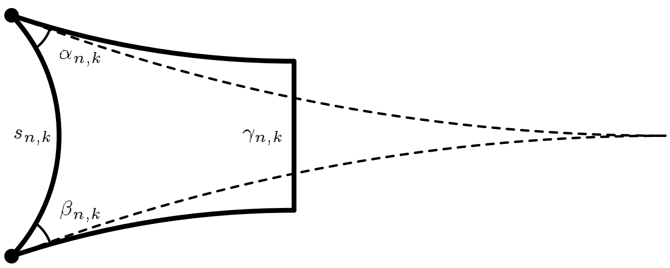

Fig. 4. Regular quadrilateral and limit quadrilateral

Now let $n \to \infty$. Take a subsequence as in Section 2 such that the combinatorial pattern of the tessellation and the naming of the sides and angles is always the same and such that the lengths of the sides and the angles converge. The limit triangles exist as before. The limit quadrilaterals exist also, but in a degenerate form: they are triangles with a point at infinity. We set $\alpha_k = \lim \alpha_{n,k}$, $\beta_k = \lim \beta_{n,k}$, $s_k = \lim s_{n,k}$. By pasting the triangles and the infinite triangles together according to the common pattern of all tessellations we get a limit surface X_∞. The convergence $[X_n] \to [X_\infty]$ follows from the following observations:

Thick part. The thick part of X_n is tessellated by the ordinary triangles and converges to the thick part of X_∞ as in the case of M^g_ϵ.

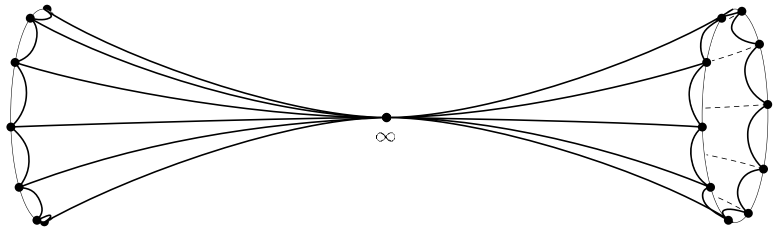

Fig. 5. Cusps with node. The boundary is marked with points of $\mathcal{P}$.

Thin part. The thin part of X_n is tessellated by the quadrilaterals and converges to the thin part of X_∞ which is tessalated by the limit triangles. The convergence follows from the fact that, by construction, $(\alpha_{n,k}, \beta_{n,k}, s_{n,k}) \to (\alpha_k, \beta_k, s_k)$.

Our construction is completed and it remains to check what we have obtained with it.

Since all X_n have combinatorially the same tessellation they also have combinatorially the same pants decomposition. Moreover, some of the geodesics have lengths converging to 0. This implies that the limit surface X_∞ is a *stable Riemann surface with nodes* in the sense of Bers [2]. Inversely, any such surface may be obtained as a limit in the above way. The limit surfaces are therefore exactly the ones on the boundary of $\overline{M^g}$ where $\overline{M^g}$ is the Baily–Mumford compactification. Thus, we have constructed $\overline{M^g}$.

As for the topology, normally the topology of $\overline{M^g}$ is defined by using the Fenchel–Nielsen coordinates (or equivalent ones). Here we used as coordinates side lengths of the triangles and quadrilaterals of our tessellation. It is an elementary excercise to prove that these both ways define the same topology. In fact, since all tessellations are combinatorially the same, there exists a fixed set of analytic formulae which compute the Fenchel–Nielsen parameters in terms of the side lengths and vice versa. Together with Theorem 1 we have thus shown that, with respect to the usual topology, any infinite sequence in M^g has a converging subsequence in $\overline{M^g}$. This is sufficient to show that $\overline{M^g}$ is compact.

Remarks. Another way to prove the convergence is to proceed as in the case of M^g_ε by defining (based on the tessellations) for each n a quasiconformal mapping ϕ_n from X_n to a suitable compact subset $X_\infty(n)$ of X_∞ where the following hold: $X_\infty(n)$ contains the thick part of X_∞, the union of all $X_\infty(n)$ is X_∞, the maximal dilatation of ϕ_n converges to 1.

With some extra effort it is possible to smooth the mappings ϕ_n so that we get convergence even in the C^∞-topology.

We are ready to consider group actions.

4. Group actions on degenerating Riemann surfaces

Let F be a finite group. We say that F *acts* on a Riemann surface X if F is isomorphic to a subgroup of the group of isometries of X.

For the ease of exposition we restrict ourselves to orientation preserving isometries. The following ideas may, however, be extended to group actions involving orientation reversing isometries.

Any orientation preserving isometry of a compact Riemann surface which has fixed points lifts to an elliptic Möbius transformation of the universal cover (the upper half-plane or the unit disk). Our first observation is that there are uniform lower bounds for the distances of such fixed points.

Lemma 1. *Let g and h be orientation preserving isometries of the hyperbolic plane with respective fixed points p and q. If g and h generate a discrete group and not both are of order two, then* $\operatorname{dist}(p,q) \geq 0.283\ldots$.

Proof. The proof is similar to the proof of [1, Theorem 11.6.7 (page 313)], and we only give a brief outline. Consider the geodesic rays at p and q forming angles $\frac{\pi}{P}$ and $\frac{\pi}{Q}$ with the segment pq and lying on the same side of pq. Here P and Q are the orders of g and h. If the rays do not intersect each other then the distance is greater or equal the distance in the model situation where $P=2$, $Q=3$ and where the two rays meet at infinity. If the rays intersect each other, then the intersection point is a fixed point of an element generated by g and h, and the latter must have some finite order R. An elementary argument shows that in this case the smallest possible distance is obtained in the model situation where $P=2$, $Q=3$ and $R=7$. Here we have a triangle with the angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{7}$, and trigonometry yields $\operatorname{dist}(p,q) = 0.283\ldots$ for the latter.

We now assume that F acts on the Riemann surface X. Our aim is to construct a tessellation with triangles and regular quadrilaterals as in the preceding section but such that it is invariant under the action of F.

We let ρ be a positive constant,

$$\rho \leq \frac{1}{16}\,, \tag{6}$$

and denote by $\gamma_1,\ldots,\gamma_m$ ($m \leq 3g-3$) the set of *all* simple closed geodesics on X with lengths $\leq 2\rho$. Again we have the collars $C(\gamma_k)$, $k=1,\ldots,m$, with boundary components of length 1. These form the thin part of X (with respect to our choice of ρ), and the complement is the thick part. We recall that the injectivity radius at any point in the thick part is $> \rho$.

We first look how $g \in F$ acts on the collars $C(\gamma_1),\ldots,C(\gamma_m)$. Since they are pairwise disjoint, any of these collars is either mapped onto itself or onto another such collar. Now assume $g(C(\gamma_k)) = C(\gamma_k)$. Since γ_k is the only simple closed geodesic in $C(\gamma_k)$ we get $g(\gamma_k)=\gamma_k$. This is only possible if either g has no fixed points in γ_k or if g is of order two with two opposite fixed points on γ_k. From what is known about discrete groups of isometries of $\mathbf{S^1}$ we conclude that the subgroup F_k of F which fixes $C(\gamma_k)$ is either the identity or a cyclic group with no fixed points in $C(\gamma_k)$ or a dihedral group. In the latter case, the fixed points of F in $C(\gamma_k)$ are of order two, their number is equal to the order of the dihedral group and they form a set of equidistant points on γ_k. Therefore, there are at most $\#F$ such fixed points on γ_k, where $\#F$ denotes the order of the group. We now tessellate the collars accordingly using quadrilaterals. For this we let ϵ be a small positive constant which will be kept fixed. We must assume that

$$\epsilon \leq \frac{\rho}{\#F}. \tag{7}$$

On γ_1 we plant N_1 equidistant points (N_1 shall be determined below) which contain among them all the fixed points of F_1. Since the points are equidistant they are permuted by F_1. From any of these points we draw perpendicular geodesic arcs on either side of γ_1 up to the boundary of $C(\gamma_1)$. This is again as shown in Figure 3. The endpoints of the arcs on the boundary of $C(\gamma_1)$ will be part of the point set $\mathcal{P}$ which we shall define in a similar way as in the preceding sections. N_1 is taken such that these points have consecutive distances $\geq \frac{\epsilon}{2}$ and as close to $\frac{\epsilon}{2}$ as possible. Since the boundary components have length 1 and since $\epsilon \leq \frac{1}{16\#F}$ this distance is, in fact very close to $\frac{\epsilon}{2}$. Connecting consecutive points by geodesic arcs we get a tessellation of the collar with regular quadrilaterals. This tessellation is invariant under the action of F_1

Whenever $C(\gamma_k) = g(C(\gamma_1))$ for some $g \in F$ and some k, $k = 1, \ldots, m$, then we take the image tessellation on $C(\gamma_k)$ with respect to g. This gives a tessellation of all images under F of $C(\gamma_1)$. Repeating the procedure on the remaining collars we eventually get an F-invariant tessellation of the thin part. Its vertices form an F-invariant set $\mathcal{P}'$.

In order to proceed in the thick part we need the following.

Lemma 2. *The fixed points in the thick part have pairwise distances* $\geq \frac{2\rho}{\#F}$.

Proof. Let φ_p, φ_q be the rotations having p and q as fixed points. If p, q are not both of order two, then, by Lemma 1, they have distance $\geq 0.283 > 2\rho$. Now let p and q be fixed points of order two and assume that their distance is less than $\frac{2\rho}{\#F}$. Consider the geodesic through them as in Figure 6. $\varphi_p(q)$ is on the

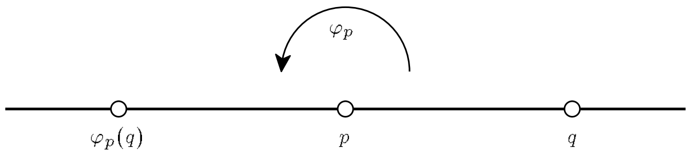

Fig. 6. Geodesic through fixed points of order two.

same geodesic and so are $\varphi_{\varphi_p(q)}$ etc. So there is a sequence of equidistant points $q_1 := q, q_2 := p, q_3 = \varphi_p(q_1), \ldots$ on this geodesic, each the center of a rotation of order 2. There are at most $\#F$ such points. Hence, the geodesic is closed and has length $< \frac{2\rho}{\#F}\#F = 2\rho$. This is not possible in the thick part.

Now let $\mathcal{P}''$ be the union of $\mathcal{P}'$ and the set of all fixed points of F in the thick part. This set is again F-invariant. Moreover, those points of $\mathcal{P}''$ which lie in the thick part have pairwise distances $\geq \frac{2\rho}{\#F} > 2\epsilon$. This follows from Lemma 2 and from the fact that the collars have pairwise distances $\geq \frac{1}{2}$.

If $p_0 \in \mathcal{P}''$ is a fixed point in the thick part corresponding to a rotation $g \in F$ of order $k \leq 6$ (in the following g is meant to be a *generator* of the cyclic group fixing p_0 so that k is the order of p_0) and if p is any point at distance $\frac{\epsilon}{2}$ from p_0 then its images under $g, g^2, \ldots, g^k$, etc., have pairwise distances $\geq \frac{\epsilon}{2}$. If p_0 has order $k \geq 7$ this no longer holds. For each $p_0 \in \mathcal{P}''$ of order $k \geq 7$

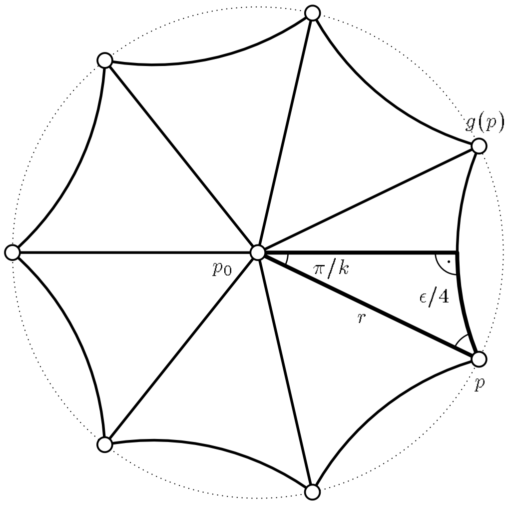

Fig. 7. Crown at fixed point.

with corresponding generator g we construct therefore a g-invariant "crown" as shown in Figure 7 in the following way. We let p be a point so close to p_0 that its images $g(p), \ldots, g^k(p) = p$ have consecutive distances $\frac{\epsilon}{2}$. The crown is the union of the triangles $p_0 p g(p)$, $p_0 g(p) g^2(p)$, ..., $p_0 g^{k-1}(p) p$ as shown in the figure. As we did with the tessellations of the collars we proceed in such a way that the group F permutes these crowns. The set $\mathcal{P}'''$ formed by $\mathcal{P}''$ and by the vertices of the crowns is then invariant under F.

We show that the points of $\mathcal{P}'''$ in the thick part have pairwise distances $\geq \frac{\epsilon}{2}$. For this we use the notation as in Figure 7. The triangles in the crown consist of two right angled ones for which

$$\sinh(\frac{\epsilon}{4}) = \sinh(r)\ \sin(\frac{\pi}{k})$$

where r is the distance from p_0 to p. As $k \geq 7$ and since ϵ and r are very small we get from this

$$\frac{\epsilon}{2} < r < \frac{k}{8}\epsilon \leq \frac{\epsilon \# F}{8} \leq \frac{\rho}{8}\ . \tag{8}$$

The center p_0 has distance $\geq \frac{1}{4}$ to any collar $C(\gamma_k)$ (for otherwise $C(\gamma_k)$ and $g(C(\gamma_k))$ are different collars with distance less than $\frac{1}{2}$). The vertices of the crown have therefore distances $\geq \frac{\epsilon}{2}$ to $\mathcal{P}'$. They also have distances $\geq \frac{\epsilon}{2}$ to the fixed points of F and distances $\geq \frac{\epsilon}{2}$ among each other. Finally, if p_0' is the center of another crown, then the distances of the vertices of the two crowns are $\geq \frac{\epsilon}{2}$ because the centers p_0 and p_0' lie further apart than 0.283 (Lemma 2 and inequality 8). This proves our claim about $\mathcal{P}'''$.

Once $\mathcal{P}'''$ is defined we proceed as earlier by adding points at pairwise distances $\geq \epsilon/2$ in the thick part (beginning with $\mathcal{P}'''$) until there is no more room for such points. This defines a finite set of points $\mathcal{P}$ on X as earlier. To make this set F-invariant we let F act at each step. That is, assume the F-invariant set $\mathcal{P}_k$ has been obtained at step k and we want to add p in the thick part whose distance to $\mathcal{P}_k$ is $\geq \frac{\epsilon}{2}$. Then the entire orbit $F(p)$ has distance $\geq \frac{\epsilon}{2}$ to $\mathcal{P}_k$, the points inside the orbit have pariwise distances $\geq \frac{\epsilon}{2}$ and we define $\mathcal{P}_{k+1} = \mathcal{P}_k \cup F(p)$.

We have thus shown that each compact Riemann surface X admits a size-controlled F-invariant tessellation with regular quadrangles in the thin part and triangles in the thick part. Here *size-controlled* means that the side lengths of the triangles are between $\frac{\epsilon}{2}$ and ϵ and that the angles of the triangles and the quadrangles lie between θ and $\pi - \theta$ where now θ is a positive constant depending on F (but not on X).

If X with $[X] \in \overline{M^g}$ is a stable Riemann surface then we have a similar tessellation except that in addition to the collars around the small geodesics (tessellated by regular quadrangles) we also have cusps around the nodes tessellated by infinite triangles which are limits of regular quadrangles as shown in Figure 5.

Now let $[X_n]$ be a sequence in $\overline{M^g}$, where each X_n comes with an action of F. Tessellate each X_n as explained. By passing to a subsequence we may assume, as earlier, that the combinatorial pattern of the tessellation is always the same and that as $n \to \infty$ the angles and sides of all triangles converge. The X_n then converge again to a limit surface X. By passing to a subsequence once more, if necessary, we may further assume that F operates on the tessellation of each X_n in the same combinatorial way. This implies that F also operates on the tessellation of X in this way and thus F operates on X.

This proves the compactness theorem below. As our construction may be generalized to group actions which contain orientation reversing isometries, we state the theorem for arbitrary isometric actions of F.

Theorem 2. *For any finite group F the set*

$$\overline{M^g_F} = \{[X] \in \overline{M^g} | F \textit{ is isomorphic to a subgroup of } \mathrm{Isom}(X)\} \tag{9}$$

is a compact subspace of the Baily–Mumford compactification of the moduli space of stable genus g Riemann surfaces. □

Actually we have proved a stronger statement. A finite group F can act on an oriented genus g surface Σ by homeomorphisms in several topologically different ways. We say that two such actions F on Σ and F' on Σ' are *topologically the same* or *of the same topological type* if there is a homeomorphism $f : \Sigma \to \Sigma'$ such that $F' = fFf^{-1}$. More generally, if Σ and/or Σ' has nodes then F and F' act topologically in the same way if there are invariant dense open subsets $S \subset \Sigma$, $S' \subset \Sigma'$ and a homeomorphism $f : S \to S'$ such that $F'|S' = f(F|S)f^{-1}$.

We use the notation $M^g_{F,T}$ (respectively, $\overline{M^g_{F,T}}$) to denote the subset of the moduli space M^g (respectively, $\overline{M^g}$) formed by the Riemann surfaces admitting an F-action of fixed topological type T.

In the proof of the above compactness theorem we passed to a subsequence of $[X_n]$ where all tessellations are combinatorially equivalent and where F always operates in the same combinatorial way. Accordingly, all F-actions are topologically the same and thus we have shown that $\overline{M^g_{F,T}}$ is compact.

5. Topological classification of $\mathbb{Z}_2$-actions

For the rest of the paper we are concerned with the connectedness of subspaces of M^g defined by group actions. We shall look in particular at $\mathbb{Z}_2$ which is best understood. For convenience we review in this section the possible $\mathbb{Z}_2$-actions.

An orientation reversing involution of a compact oriented topological genus g surface Σ, $\sigma : \Sigma \to \Sigma$ is determined topologically by the following invariants:

- $n(\sigma)$, the number of components of the fixed point set R of σ,
- $k(\sigma) = 2 -$ the number of components of $\Sigma \setminus R$.

Theorem 3 (Klein, Weichhold). *Let $\sigma : \Sigma \to \Sigma$ be an orientation reversing involution of a compact oriented genus g surface Σ. The invariants $n(\sigma)$ and $k(\sigma)$ satisfy*

1. $0 \leq n(\sigma) \leq g + 1$.
2. *If* $k(\sigma) = 0$, *then* $n(\sigma) \equiv g + 1 \, mod \; (2)$.
3. *If* $k(\sigma) = 1$, *then* $n(\sigma) \neq g + 1$.

All possible combinations within these restrictions may occur. The topological type of σ is uniquely determined by $n(\sigma)$ and $k(\sigma)$.

By Riemann–Hurwitz we have an analogous statement in the orientation preserving case:

Theorem 4. *Let $s : \Sigma \to \Sigma$ be an orientation preserving involution of a compact oriented genus g surface Σ. The number $n(s)$ of fixed points of the mapping s satisfies:*

1. $0 \leq n(s) \leq 2g + 2$,
2. $n(s) \equiv 2g + 2 \, mod \; (4)$.

All possible cases within these restrictions may occur. The topological type of s is uniquely determined by $n(s)$.

These two well-known theorems classify all $\mathbb{Z}_2$-actions on compact oriented surfaces. In view of this classification we use the following notation.

- $M^g_{[n,k]}$ denotes the subspace of M^g formed by the surfaces which admit an orientation reversing involution σ with the invariants $n(\sigma) = n$ and $k(\sigma) = k$.
- $M^g_{(n)}$ denotes the subspace of M^g formed by the surfaces which admit an orientation preserving involution with n fixed points.

$M^g_{[n,k]}$ may also be described as the moduli space of the real Riemann surfaces (or, equivalently, the real algebraic curves) whose real structure σ is of the topological type determined by n and k. If $k = 0$ the fixed point set R (the "real part") separates the surface into two connected components and we call R *dividing*. If $k = 1$ the complement of R is connected and we call R *non-dividing*.

6. Connectedness theorems

We now let $g > 1$ and assume that F is a finite group admitting a topological action on genus g surfaces. For any topological type T of these actions the following holds.

Theorem 5. *The subset $M^g_{F,T} \subset M^g$ is connected.*

Proof. This is a well-known result and follows from the considerations of Earle ([5]). For the benefit of the reader we sketch a proof here.

Let $[X]$ and $[Y]$ be two points of $M^g_{F,T} \subset M^g$. It suffices to construct a continuous path from $[X]$ to $[Y]$ which lies in $M^g_{F,T}$.

We view the points of M^g as being equivalence classes of complex structures on a given fixed topological surface Σ. The assumption $[X] \in M^g_{F,T}$ and $[Y] \in M^g_{F,T}$ means that the groups of isometric selfmappings of X and of Y contain subgroups F_X and F_Y isomorphic to F. Furthermore, since both actions are of the same topological type, there exists a homeomorphism $f : Y \to X$ such that $F_X = fF_Yf^{-1}$.

Let $(f^{-1})^*(X)$ denote that complex structure of Σ for which the mapping

$$f^{-1} : (\Sigma, (f^{-1})^*(X)) \to (\Sigma, X)$$

is a holomorphic mapping between Riemann surfaces. As points of the moduli space M^g, $[X] = [(f^{-1})^*(X)]$. We may, therefore, replace $[X]$ by $[(f^{-1})^*(X)]$. This implies that, without loss of generality, we may assume that the mapping f conjugating F_X to F_Y is the identity mapping of the topological surface Σ.

Let now $f_e : (\Sigma, Y) \to (\Sigma, X)$ be the extremal quasiconformal mapping homotopic to the identity. The uniqueness part of the Teichmüller extremal mapping theorem implies that the mapping f_e also conjugates F_X to F_Y. Let μ_f be the complex dilatation of f_e. Then $[0, 1] \ni t \mapsto t\mu_f$ is a continuous path joining $[Y]$ to $[X]$ in $M^g_{F,T} \subset M^g$. This proves the theorem.

Now we are going to apply this to $\mathbb{Z}_2$, first in the orientation preserving case.

Theorem 6. *Let $M^g_{\mathbb{Z}_2,+}$ be the subspace of M^g formed by the surfaces on which $\mathbb{Z}_2$ has an orientation preserving action. Then $M^g_{\mathbb{Z}_2,+}$ is connected.*

Proof. We prove this by showing that for each possible n, $M^g_{(n)}$ has a non empty intersection with $M^g_{(2)}$ respectively $M^g_{(0)}$ depending upon the parity of g. In view of Theorems 4 and 5 this will suffice.

Thus, we have to construct an example of a Riemann surface $X^g_{(n)}$ which has two involutions, the first with 2, respectively 0 fixed points, the second with n fixed points. We shall obtain this example as the Schottky double of a circle domain $D^g_{(n)} \subset \mathbb{C}$ as shown in Figure 8. The details are as follows. Let $n \equiv 2g + 2 \mathrm{mod}\ (4), 2 < n \leq 2g + 2$. (For $n = 2$, respectively $n = 0$ there is nothing to prove.)

If g is even we set $r = (n - 2)/4$, $m = 0$. If g is odd we set $r = (n - 4)/4$, $m = 1$.

The domain $D^g_{(n)}$ is obtained by cutting away, from the closed unit disk in the complex plane, the interiors of g disjoint closed smaller disks of equal size placed

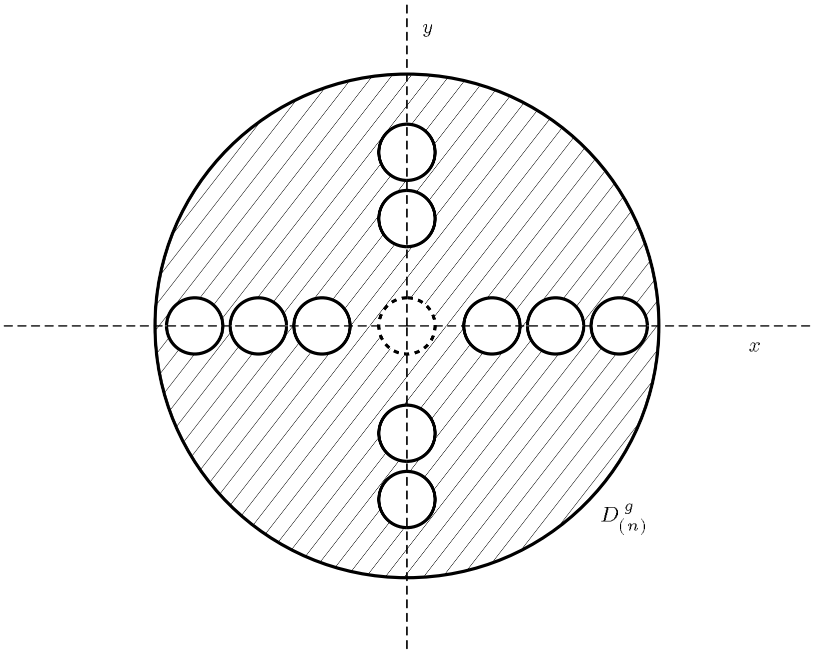

Fig. 8. Circle domain in the complex plane.

symmetrically with respect to the origin as shown in Figure 8 where m disks have their center at the origin, 2r disks have their centers on the x-axis and the remaining disks have their centers on the y-axis. $D^g_{(n)}$ is invariant under a 180^0 rotation about the origin and the x-axis is an axis of symmetry. The Schottky double $X^g_{(n)}$ of $D^g_{(n)}$ (two copies of $D^g_{(n)}$ pasted together by identifying corresponding boundary points) has an orientation reversing involution (exchanging the copies of $D^g_{(n)}$); it also has an orientation preserving involution with $2(1-m)$ fixed points and another such involution with $2 + 2m + 4r = n$ fixed points.

Let us next look at the Riemann surfaces with orientation reversing $\mathbb{Z}_2$-actions, that is, the real Riemann surfaces, or, equivalently, the real algebraic curves.

We first recall the following result ([11, Theorem 4.2]).

Theorem 7. *The real hyperelliptic Riemann surfaces form a connected subset of M^g.*

With this we prove

Theorem 8. *The subspace of all real Riemann surfaces in M^g is connected.*

Proof. If a real Riemann surface is *hyperelliptic* then the fixed point set R of the orientation reserving involution σ has the following properties (see [7]). If R is non dividing, (i.e. $k = k(\sigma) = 1$), then R may have any number n of connected components for $0 \leq n < g+1$. If R is dividing, (i.e. $k=0$), then there are only two possibilities, either $n = g+1$ or $n = 1$ if g is even, respectively $n = 2$ if g is odd.

This allows us to reduce much of the proof to Theorem 7 but not all of it.

First we remark that within the above restrictions, any topological type of the action of σ exists on hyperelliptic surfaces. An easy way to build examples is to consider algebraic curves defined by $y^2 = P(x)$, where P is a real polynomial of degree $2g+2$. Then, if the number $2n$ of real roots of P is positive, R has n connected components and is non dividing unless $n = g+1$. If $n = 0$, then R has either 0 components or else R is dividing and has 1 or 2 components depending on the parity of g.

By Theorems 5 and 7 it remains show that $M^g_{[n,0]}$ and $M^g_{[g+1,0]}$ have a non-empty intersection for all n such that $0 < n < g+1$ and $n \equiv g+1 \bmod (2)$.

For this last point we proceed as in the proof of Theorem 6. If g is even we set $r = (n-1)/2$, $m = 0$. If g is odd we set $r = (n-2)/2$, $m = 0$. Then we take the circle domain $D^g_{(2n)}$ again as shown in Figure 8 with m circles at the origin, $2r$ circles placed symmetrically along the x-axis and the remaining circles placed symmetrically along the y-axis. On the Schottky double $X^g_{(2n)}$ we have an orientation reversing involution with $g+1$ fixed components (the one which exchanges the copies of $D^g_{(2n)}$) and another such involution with $2r+1+m = n$ fixed components (the reflection along the x-axis and its copy).

Since the real hyperelliptic surfaces also form a subset of $M^g_{\mathbb{Z}_2,+}$, we obtain altogether the following.

Theorem 9. *$M^g_{\mathbb{Z}_2}$ is connected.*

Example ($\mathbb{Z}_3$–actions). We finish this paper by giving an example of a group action for which the subspace M^g_F is not connected. Explicitly we prove that $M^g_{\mathbb{Z}_3}$ is not connected. The argument is due to Gareth Jones whom we would like to thank again at this point.

Recall that on a genus 3 Riemann surface X two types of $\mathbb{Z}_3$-actions are possible, and only two. One of them has 2 fixed points (the quotient is then of genus 1), the other has 5 fixed points (the quotient is then of genus 0). We are going to show that not both actions can occur simultaneously on X.

Assume that this were the case and let φ_1 and φ_2 be two order 3 automorphisms, one of each type, on X. Since φ_1 and φ_2 are not conjugate we may assume, replacing one of them by a conjugate if necessary, that they both belong to the same 3-Sylow sub-group, S_3, of $\mathrm{Aut}(X)$. Since 3 is prime, S_3 has a non trivial center and this center contains an element of order 3, φ_3. Since there are only two topological types of actions, we may assume that φ_3 is of the same type as φ_1. But now φ_2 and φ_3 commute, in particular the fixed points of φ_2 are globally fixed by φ_3. Since the number of these is 2 or 5 and φ_3 is of order 3, we find that the two automorphisms must actually have a common fixed point x. Now in a neighborhood of x either mapping is a rotation of angle $2\pi/3$ around x. Hence, in this neighborhood $\varphi_2 = \varphi_3$ or $\varphi_2 = \varphi_3^{-1}$. As the mappings are conformal, the equality holds globally, a contradiction.

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