Error Estimation of goal oriented functional arising from iterative solution of Euler Equation

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Error estimation of goal oriented functional arising from an iteration stopping criterion is considered for solution of the steady Euler problem. The functional error is calculated using an iteration residual and adjoint parameters. Numerical tests demonstrate the applicability of this approach for the steady 2-D Euler equations.

Keywords: steady Euler equations, iteration error, adjoint equations.

1. INTRODUCTION

The quantitative evaluation of errors caused by different components of a numerical algorithm including approximation error and iteration error are of significant current interest [1-3]. Iterative methods are commonly used for solving steady CFD problems. The simplest technique involves the temporal evolution from an initial guess to obtain the steady solution. This approach implies iterations along the time or a certain pseudo-temporal variable [4-5]. Different variants of preconditioning [6-10] are used to improve the relaxation rate. Commonly used a priori estimates of iteration convergence [5] linking the error with iteration residual in some norms, contain constants that are in general case (for nonlinear non self-adjoint operators) unknown. Usually, iterations are terminated when some convergence criterion (for example, $\max_i \left|\rho_i^{n+1} - \rho_i^n\right| \le 10^m$, [4]) is satisfied. This provides for a decrease the of iteration error, however the magnitude of this error remains unknown. When the problem under the consideration is characterized by an important functional, it is natural to observe the error of this functional. This error may be calculated using adjoint equations [11-12]. In the present paper we estimate the error in the goal functional via iteration residual and adjoint parameters. The flow density at a certain reference point is chosen here as the goal functional. From another viewpoint, the approach used here may be considered as 'a posteriori' estimation of error caused by variation of the physical model [13-19] using adjoint equations. The results provided by coarse and fine mesh physical models are compared in [13, 14] for several problems including the viscous incompressible fluid governed by Navier-Stokes or Stokes models. The influence of a coefficient's oscillations and nonlinearity for the Poisson and convection-diffusion-reaction equations is estimated in [17]. The deviation of solutions governed by Helmholtz and Poisson models is considered in [18]. Ref. [19] discusses the impact of viscosity on the flow parameters comparing the Euler and parabolized Navier-Stokes equations. In present paper we address the issue of comparing steady and unsteady inviscid gas flows. This approach was employed for the heat transfer equation in [20].

2. Algorithm outline

Let us consider the formal scheme of the considered algorithm. We solve a steady nonlinear equation by time iterations.

$$N\tilde{f} = w \quad in\ \Omega \subset R^n,\ \tilde{f}(\partial\Omega) = \tilde{f}_B(x) \in L_2(\partial\Omega); \qquad (1)$$

Herein N - is the nonlinear differential operator ($H^k(\Omega) \to L_2(\Omega)$).

The time iterations mean solving the equation

$$\partial f / \partial t - Nf = w \quad \text{in } Q = \Omega \times (0, t_f),\quad f(\partial\Omega) = f_B(x) \in L_2(\partial\Omega); \qquad (2)$$

with an initial guess $f(\Omega,0) = f_0(x) \in L_2(\Omega)$;

The iteration residual is equal to $q = w - Nf = \dfrac{\partial f}{\partial t}$.

Consider $N\tilde{f} = w$ as an exact equation while $\partial f / \partial t - Nf = w$ is considered as a disturbed one. The exact and disturbed solutions are related by
$f(t,x) = \tilde{f}(x) + \Delta f(t,x)$.
The operator N is considered to be Frechet differentiable, the corresponding derivative is denoted as N_f. Then the following relation holds

$$N(\tilde{f} + \Delta f) = N(\tilde{f}) + N_f \Delta f \text{ with the error of } O(\|\Delta f\|^2), \tag{3}$$

To the first order of accuracy the disturbance for time instant t is governed by the steady equation

$$\frac{\partial f}{\partial t} - N_f \Delta f = q - N_f \Delta f = 0 \text{ in } \Omega \subset R^n, \Delta f(\partial\Omega) = 0; \tag{4}$$

where $\frac{\partial f}{\partial t} = q(t,x)$ is considered as source-like term.

Let us use a Frechet differentiable goal functional $\varepsilon(\cdot): H^k(\Omega) \to R^1$. The variation of this functional is linear continuous functional and may be represented according to Riesz theorem using inner product in $L_2(\Omega)$ as

$$\Delta\varepsilon = (\varepsilon_f, \Delta f)_{L_2}. \tag{5}$$

Using (4) this expression may be rewritten as

$$\Delta\varepsilon = (\Delta f, \varepsilon_f)_{L_2} = (N_f^{-1} q, \varepsilon_f)_{L_2} = (q, N_f^{-1*}\varepsilon_f)_{L_2} = (q, \Psi)_{L_2} \tag{6}$$

where $\Psi = N_f^{-1*}\varepsilon_f$ is a formal solution of adjoint problem:

$$N_f^* \Psi = \varepsilon_f \tag{7}$$

The detailed form of adjoint problem may be obtained from the bilinear identity

$(N_f^* \Psi, f)_{L_2} = (N_f f, \Psi)_{L_2}$ using integration by parts [12].
Thus, the goal functional variation caused by the iteration residual may be expressed as

$$\Delta\varepsilon = \int_\Omega q(t,x)\Psi \, d\Omega \tag{8}$$

where Ψ is the solution of adjoint problem

$$N_f^* \Psi - \varepsilon_f = 0 \text{ in } \Omega \subset R^n \text{ with boundary conditions } \Psi(\partial\Omega) = 0. \tag{9}$$

The adjoint problem can be solved using some iterative method, herein the time relaxation of the following form was used.

$$\partial\Psi / \partial\tau - N_f^* \Psi + \varepsilon_f = 0 \text{ in } Q = \Omega \times (0, \tau_f), \; \Psi(\partial\Omega) = 0, \; \Psi(\Omega, 0) = 0; \tag{10}$$

It should be noted that problem (10) is not connected with (2) neither by a single temporal interval nor by the form of iterations, which is unusual for adjoint problems.

3. TEST PROBLEM

Let us consider the approach described above for a steady two-dimensional compressible inviscid flow. The iterations are based on temporal relaxation using the unsteady form of Euler equations.

$$\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho U^k\right)}{\partial X^k}=0\,; \tag{11}$$

$$\frac{\partial\left(\rho U^i\right)}{\partial t}+\frac{\partial\left(\rho U^k U^i+P\delta_{ik}\right)}{\partial X^k}=0\,; \tag{12}$$

$$\frac{\partial\left(\rho(e+0.5(U^2+V^2)\right)}{\partial t}+\frac{\partial\left(\rho U^k(\gamma e+0.5(U^2+V^2)\right)}{\partial X^k}=0\,; \tag{13}$$

$$P=(\gamma-1)\rho e\,;h_0=(U^2+V^2)/2+h\,,h(\rho,P)=\gamma e\,,\theta=\frac{1}{2}(U^2+V^2)$$

$(X,Y)\in\Omega=(0<X<X_{max};\,0<Y<Y_{max}),\,0<t<t_{max});$

The boundary conditions were considered as steady ones. The calculation of the steady flow-field was performed using time evolution starting from a spatially uniform initial guess.

The pointwise density estimation was used as the goal functional $\rho(X^{est},Y^{est})$.

$$\rho^{est}=\varepsilon=\int_\Omega\tilde\rho(X,Y)\delta(Y-Y^{est})\delta(X-X^{est})dXdY \tag{14}$$

The corresponding adjoint problem may be obtained by standard means [12- 22] and assumes the form

$$\frac{\partial\Psi_\rho}{\partial\tau}+(-U^2+\theta(\gamma-1))\frac{\partial\Psi_U}{\partial x}-UV\frac{\partial\Psi_V}{\partial x}+(-Uh_0+U\theta(\gamma-1))\frac{\partial\Psi_E}{\partial x}+$$
$$-UV\frac{\partial\Psi_U}{\partial y}+(-V^2+\theta(\gamma-1))\frac{\partial\Psi_V}{\partial y}+(-Vh_0+V\theta(\gamma-1))\frac{\partial\Psi_E}{\partial y}-\delta(X-X^{est})\delta(Y-Y^{est})=0 \tag{15}$$

The δ- form source in equation for Ψ_ρ corresponds to the location of the reference point.

$$\frac{\partial\Psi_U}{\partial\tau}+\frac{\partial\Psi_\rho}{\partial x}+(2U-U(\gamma-1))\frac{\partial\Psi_U}{\partial x}+V\frac{\partial\Psi_V}{\partial x}+(h_0-U^2(\gamma-1))\frac{\partial\Psi_E}{\partial x}+$$
$$V\frac{\partial\Psi_U}{\partial y}-U(\gamma-1)\frac{\partial\Psi_V}{\partial y}-UV(\gamma-1)\frac{\partial\Psi_E}{\partial y}=0 \tag{16}$$

$$\frac{\partial\Psi_V}{\partial\tau}-V(\gamma-1))\frac{\partial\Psi_U}{\partial x}+U\frac{\partial\Psi_V}{\partial x}-UV(\gamma-1))\frac{\partial\Psi_E}{\partial x}+$$
$$\frac{\partial\Psi_\rho}{\partial y}+U\frac{\partial\Psi_U}{\partial y}+(2V-V(\gamma-1))\frac{\partial\Psi_V}{\partial y}+(h_0-V^2(\gamma-1))\frac{\partial\Psi_E}{\partial y}=0 \tag{17}$$

$$\frac{\partial\Psi_E}{\partial\tau}+(\gamma-1))\frac{\partial\Psi_U}{\partial x}+\gamma U\frac{\partial\Psi_E}{\partial x}+(\gamma-1)\frac{\partial\Psi_V}{\partial y}+\gamma V\frac{\partial\Psi_E}{\partial y}=0 \tag{18}$$

The parameters $(\Psi_\rho,\Psi_U,\Psi_V,\Psi_E)$ are the adjoint analogues of density, velocity components and energy, respectively.

Initial conditions: $\Psi_{\rho,U,V,e}\Big|_{\tau=0}=0;$ (19)

Boundary conditions (Y=0*;* Y=Y_{max}): $\Psi_{f,\partial\Omega}=0$ (20)

The mollification of the δ- form source term was used according to [23-24].
The convergence error estimate (analogue of Eq. (8)) has a form:

$$\delta\varepsilon = \iint\limits_{\Omega} \left(\Psi_{\rho}(t, X, Y) \frac{\partial \rho}{\partial t} + \Psi_U \frac{\partial U}{\partial t} + \Psi_V \frac{\partial V}{\partial t} + \Psi_e \frac{\partial e}{\partial t} \right) dXdY \qquad (21)$$

This expression quantitatively determines the deviation of the calculation from the exact steady value due to interruption of iterations (evolution at moment *t*).

4. Numerical tests

A first order finite-difference scheme (donor cells [4, 19]) was used in the numerical tests. The flowfield engendered by two crossing shock waves ($\alpha = \pm 22.23^{\circ}$, M=4) was calculated. Fig. 1 illustrates the isolines of the density in flowfield and Fig. 2 illustrates the isolines of the adjoint density (a concentration of isolines corresponds to a reference point).

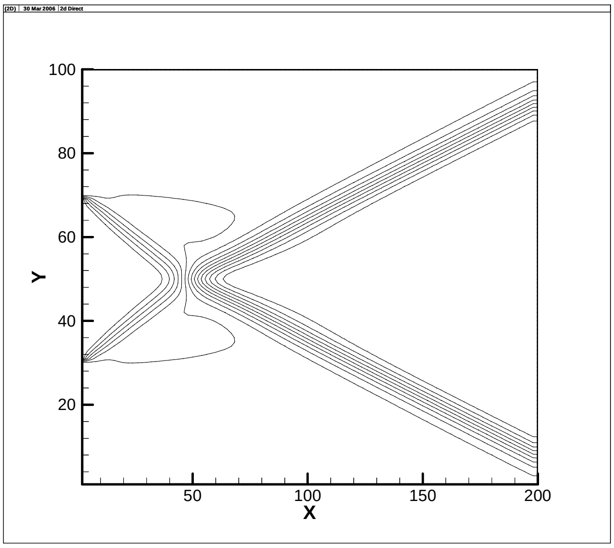

Fig. 1 Isolines of density

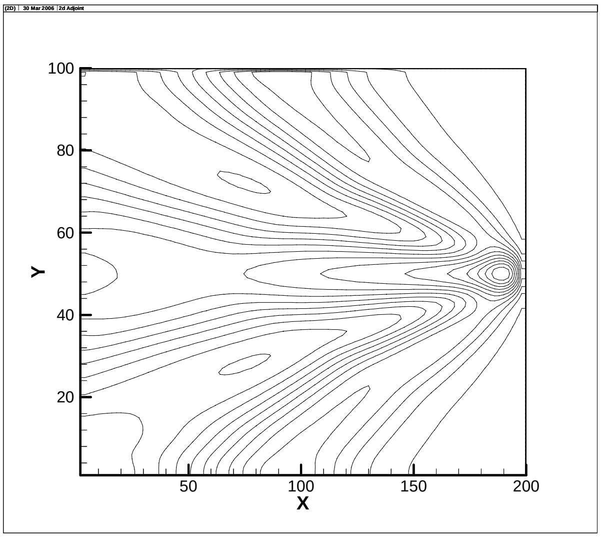

Fig. 2 Isolines of adjoint density

The time evolution starts from a spatially uniform initial guess. During the time relaxation (past every 50 steps) the adjoint problem was solved and the value of (21) was estimated. Fig. 3 presents the history of the density past crossing shocks at reference point as a function of time. The difference between calculation and analytic solution, the convergence error estimated via adjoint parameters and the convergence indicator $\max\limits_{i,j} \left| \rho_{ij}^{n+1} - \rho_{ij}^{n} \right|$ (multiplied by a coefficient 200 for better display visibility) may be compared. The convergence indicator $\max\limits_{i,j} \left| \rho_{ij}^{n+1} - \rho_{ij}^{n} \right|$ provides qualitatively the correct pattern of time evolution but it does not involve quantitative information on the distance from the exact solution. The adjoint estimation of the convergence error at the initial stage of relaxation deviates significantly from the exact value due to the nonlinearity of the problem (terms marked in Eq. (2) as $O(\|\Delta f\|^2)$ are rather large).

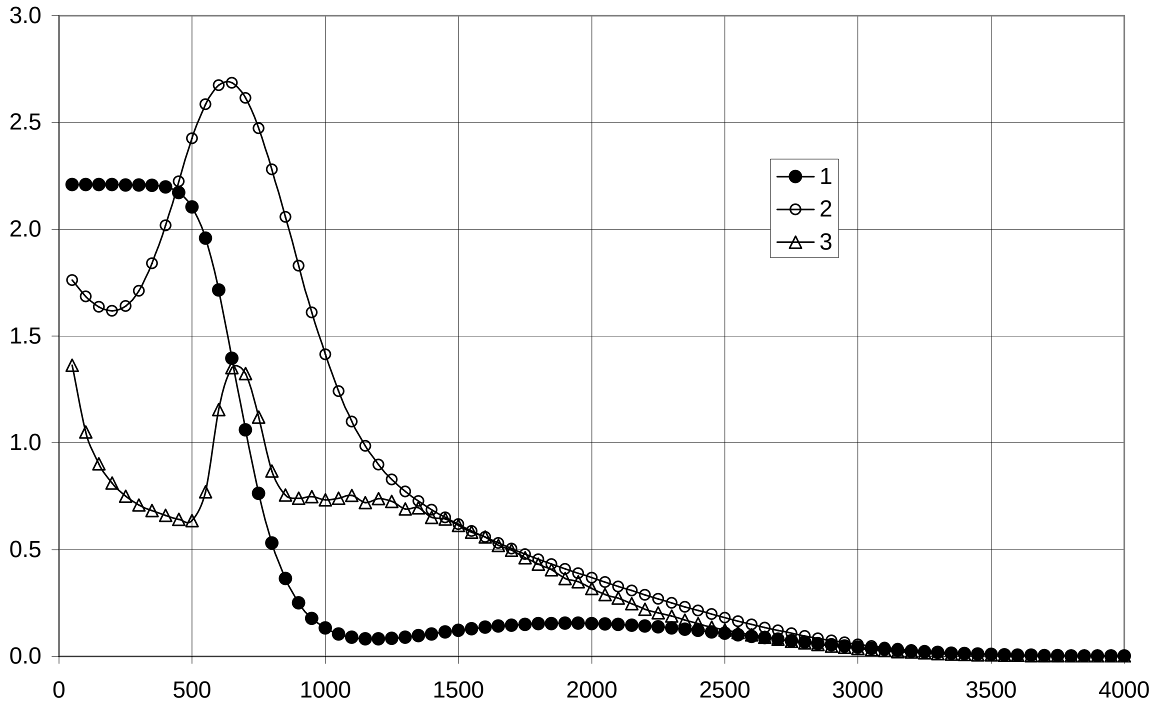

Fig. 3. The history of density relaxation as a function of number of time steps.
1- deviation of numerical solution from analytic one, 2- adjoint estimation of convergence error, 3-convergence indicator $\max_{i,j}\left|\rho_{ij}^{n+1}-\rho_{ij}^{n}\right| * 200$

Fig. 4 provides the final part of the temporal evolution. Curve 1 presents the deviation of the numerical solution from analytic one (shifted by the value of error caused by the spatial approximation). Line 2 presents the error estimation using iteration residual and adjoint parameters. The error of the spatial approximation at the final iterative stage is marked by 3.

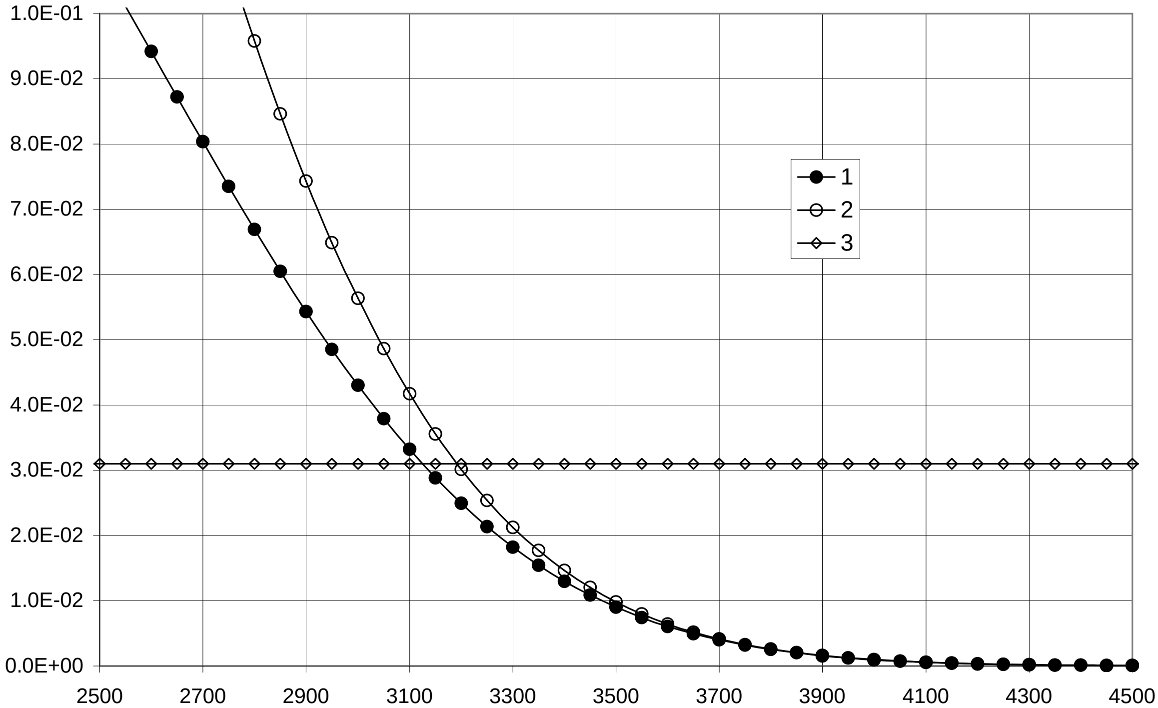

Fig. 4. The history of density relaxation (continuation of Fig. 3).
1- deviation of numerical solution from analytic one, 2- adjoint estimation of convergence error, 3-the error of spatial approximation

For the final stage of time relaxation the adjoint estimation (21) is close to the convergence error.

5. DISCUSSION

The present paper considers the convergence error estimate for the simplest form of iterations (time relaxation). The analysis presented is also valid for cases when iterations occur along some pseudo-time with use of preconditioning. This approach may also be applied for discrete iterations if a discrete statement of the adjoint problem is used.

At the beginning stage of the time relaxation the adjoint error estimation significantly deviates from exact error due to nonlinearity (this effect is considered also in [20]). At the final stage the adjoint error estimation is quite close to the exact error (if the spatial approximation error is taken into account).

In general, the considered estimate may serve for checking the convergence stopping criterion, if one considers the necessary tolerance of the goal functional as a stopping criterion. However, this involves a rather large computational effort due to elliptic nature of the corresponding adjoint problem. If we need to track the iterations by estimating the convergence quality according to (21) we should solve the adjoint problem at every check point which implies a large computational burden. Thus, the number of time points where estimation is performed should be limited.

The widely used convergence indicator $\max_{i,j}\left|\rho_{ij}^{n+1}-\rho_{ij}^{n}\right|$ [4] qualitatively correctly reflects the convergence but does not provide a quantitative estimation of the deviation from the steady state. Fig. 3 presents a comparison between the adjoint estimation and the value $\max_{i,j}\left|\rho_{ij}^{n+1}-\rho_{ij}^{n}\right|$, multiplied by coefficient 200, chosen for the purpose of visual presentation. The adjoint error estimate at initial stage of iterations exhibits a large deviation from the exact values due to nonlinearity. At the final stage the adjoint estimate is precise enough. Unfortunately, if the exact solution is unknown, it is difficult to determine when the adjoint estimate approaches the true error. An estimation of the spatial approximation error [15,16,19 and 21] may serve as an auxiliary criterion aimed at determining applicability of the adjoint estimation of convergence error.

6. CONCLUSIONS

The error in the goal functional caused by truncating the number of iterations required for solving the steady problem may be calculated using adjoint variables and the value of iteration residual.

Numerical tests demonstrate that for 2-D Euler equations use of this approach enables us to calculate the error of the goal functional (density at a checkpoint) caused by the time iterations.

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