A Note on the Emergence of Large Scale Coherent Structure under Small Scale Random Bombardments: the Discrete Case

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Abstract

We continue our study on mathematical justification of the emergence of large scale coherent structure in a two dimensional fluid system under small scale random bombardments. We treat the case of small scale random bombardments at discrete times which is different from our earlier work [16] where we approximated the small scale random kicks by a continuous in time random process. In the absence of geophysical effects, the large scale structure emerging out of the small scale random forcing is the same as the case of continuous in time forcing that we studied before.

Keywords: *One layer barotropic quasi-geostrophic equations, large coherent structure, random small scale forcing, generalized Grashof number*

1 Introduction

We continue our investigation on the emergence and persistence of large scale coherent structures in geophysical flows under small scale random bombardments. In our previous work [16], we have demonstrated that large scale coherent structures could emerge and persist under small scale forcing . Our analysis shows that the large scale structure emerging out of the small scale random forcing is not the one predicted by equilibrium statistical mechanics. But the error is very small which explains earlier successful prediction of the large scale structure based on equilibrium statistical mechanics. However, the discrete in time random kicks was replaced by a continuous in time random process in that work. The purpose of this note is to treat the case of the original discrete in time random kick forcing as was used in the original numerics [10, 11]. We will show that in the absence of geophysical effects, the same kind of large scale coherent structure will emerge out of our analysis as suggested by robust numerical experiments. The main result here is somewhat weaker than the corresponding part for the continuous case [16] since stochastic calculus tools are not applicable here.

Here we consider an extremely simplified (idealized) situation of a one-layer barotropic system without geophysical effects except Ekman damping in a square under the influence of random small scale vortices in the presence of a (small eddy) viscosity. The interested reader is referred to [9, 14, 17, 20] for justifications of one-layer model in geophysical fluid dynamics. More precisely, we consider the following one layer barotropic quasi-geostrophic model with Ekman damping and eddy viscosity in a square with free-slip boundary condition and impulse forcing of small scale

$$\begin{aligned} \frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= -d\,q + \nu\Delta q + \mathcal{F}, \qquad (1)\\ q &= \Delta\psi \qquad (2) \end{aligned}$$

equipped with initial condition

$$q|_{t=0} = q_0 \qquad (3)$$

and no-penetration, free-slip boundary condition

$$\psi = q = 0, \text{on } \partial Q \qquad (4)$$

where the fluid occupies the square

$$Q = [0,\pi] \times [0,\pi] \qquad (5)$$

and $d > 0$ is the Ekman damping coefficient, $\nu > 0$ is the eddy viscosity.

The random small scale kick forcing is given by

$$\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - jd_t)\, A\omega_r(\vec{x} - \vec{x}_j)\xi_j \tag{6}$$

where A is the amplitude of the small scale bombardment, $\vec{x}_j$ is the (random) center, the small scale vortex ω_r takes the form,

$$\omega_r(\vec{x}) = \begin{cases} \left(1 - |\vec{x}|^2/r^2\right)^2, & |\vec{x}|^2 \le r^2 \quad , \\ 0, & |\vec{x}|^2 > r^2 \quad , \end{cases} \tag{7}$$

and the center of the small vortices, $\vec{x}_j$, satisfies a distribution μ on $Q_{r_0} = [r_0, \pi - r_0] \times [r_0, \pi - r_0]$ where $r_0(\ge r)$ is a fixed constant (μ was taken as the uniform distribution on Q_{r_0} in our previous study [16]), while ξ_j are i.i.d. (independent of $\vec{x}_j$) binomial random variable with

$$\mathrm{Prob}(\xi_j = 1) = p > \frac{1}{2},\ \mathrm{Prob}(\xi_j = -1) = 1 - p < \frac{1}{2} \tag{8}$$

so that the flow is bombarded by predominantly positive vortices ($p = 1$ in our previous work [16]).

The interested reader is referred to [16, 17] for the relevance and motivation of such small scale random bombardments.

Since ω_r is piecewise smooth with compact support and C^1, we see that

$$\omega_r(\vec{x} - \vec{x}_j) \in H_0^2(Q) \tag{9}$$

with norm independent of j. In fact, $\omega_r(\vec{x} - \vec{x}_j) \in W^{2,\infty}(Q)$.

Here we have followed our earlier work [16] and used q for vorticity instead of the standard ω. This is because ω is a standard notation for point in probability space, and q is the standard notation for potential vorticity in Geophysical Fluid Dynamics which reduces to the usual vorticity in our specific setting.

It is then easy to see that there are two different stages in the dynamics, a stage of pure decay from $(jd_t)^+$ to $((j+1)d_t)^-$, governed by the decaying barotropic flow

$$\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q = -d\, q + \nu \Delta q, \tag{10}$$

$$q = \Delta \psi, \tag{11}$$

and a stage of instantaneous forcing

$$q((jd_t)^+) = q((jd_t)^-) + A\omega_r(\vec{x} - \vec{x}_j)\xi_j. \tag{12}$$

Numerical simulation in the regime of weak forcing and weak damping [10, 11, 16] indicates the emergence and persistence of a large coherent structure when μ is the uniform distribution, $d = 0$ and $p = 1$ (see also the contour plot in figure 1). More precisely, numerical experiments demonstrate that the flow field reaches a quasi-equilibrium state in terms of energy, enstrophy, circulation etc, and the contour plot of the vorticity field looks like a large vortex plus small (random) perturbation. For the special case of zero initial data and uniform distribution μ as well as $p = 1$ and $d = 0$, such a phenomenon is termed spin-up from rest [10, 17]. This large coherent structure resembles very much the ground state mode of the Laplace operator, i.e., $sin(x)sin(y)$, with a correlation between the vorticity field q and $sin(x)sin(y)$ above 0.97. The interested reader is referred to [17, 15] for more on equilibrium statistical theories for basic geophysical flows and large scale coherent structures.

The ground state mode is in fact the predicted most probable mean field of equilibrium statistical mechanics theory utilizing energy and enstrophy as conserved quantities (see for instance [17]). Thus the numerical evidence can be viewed as evidence toward applicability of equilibrium statistical mechanics in this damped driven case. If one applies a more sophisticated equilibrium statistical theory such as the point vortex energy-circulation theory which leads to a sinh-Poisson type mean field equation (see for instance [10, 17] among others), one gets better prediction.

The purpose of this paper is to provide a rigorous theoretical underpinning of such success. More precisely, we will show, under appropriate assumptions on the smallness of parameters (generalized Grashof number defined in the next section and time step d_t), that the long time dynamics is that of a large coherent vortex q^0. This large scale coherent structure is close to (but not equal to) the ground state mode $sin(x)sin(y)$, plus small deterministic and random perturbations if μ is the uniform distribution and $d = 0$. Such a result indicates that neither the energy-enstrophy statistical theory (which predicts the ground state mode) nor the point vortex energy-circulation statistical theory (which predicts a sinh-Poisson type mean field equation not satisfied by q^0) predicts the exact statistical equilibrium. However the error is so small (less than 2%) which establishes the practical applicability of these equilibrium statistical mechanics theories to this damped driven situation.

The rest of the paper is organized as follows. In section 2 we state and prove the main result of this note, i.e., the emergence and persistence of suitable large scale coherent structure under appropriate assumptions on the parameters. We provide concluding remarks at the end.

2 Main Result

Recall that the numerically observed emergence of large scale coherent structure is under the bombardment of random small scale forcing at discrete time. Our earlier analysis [16] treated the case when the random bombardment was replaced by the sum of a deterministic forcing and a small white in time noise. Here we would like to study the original setting of random bombardments of small scale vortices at discrete time.

Again, the observed large scale coherent structure is best explained by the existence of a unique invariant measure whose support is concentrated around the observed large scale coherent structure. However, it is easy to see that there is no invariant measure for the whole system since the only measure that is invariant in the freely decaying period is the delta measure centered at the origin, and this measure is obviously not the right one for the instantaneous bombardment. The remedy is to consider one of the following two discrete time Markov processes based on the behavior of the system right before or right after the bombardments:

$$\eta_-^{j+1} = S(d_t)(\eta_-^j + A\omega_r(\vec{x} - \vec{x}_j)\xi_j), \tag{13}$$

$$\eta_+^{j+1} = S(d_t)(\eta_+^j) + A\omega_r(\vec{x} - \vec{x}_j)\xi_j \tag{14}$$

where $S(t)$ denotes the solution semigroup of the freely decaying process [18, 17, 13].

Apparently the process after the instantaneous bombardment (the $\eta_+ s$) is less regular than the process before the instantaneous bombardment. Therefore, we will focus on the $\eta_+ s$ and drop the $+$ from now on. Of course, we may derive the same kind of result for the $\eta_- s$ as well.

Similar to the continuous case [16], it is useful to introduce the following new parameter

$$c_A = \frac{A}{d_t} \tag{15}$$

which is the ratio of the amplitude of the random vortices and the time step between two consecutive bombardments.

It is easy to check that the Markov process possesses invariant measures. Indeed, it is easy to verify, after utilizing the optimal decay rate of $(d+\lambda_1\nu)d_t$ for the freely decaying period and the instantaneous forcing, that the vorticity field q has an absorbing ball of radius R in L^2 (the same as absorbing ball for the velocity field in H^1, or absorbing ball for the stream function in H^2) given by

$$R = \frac{A|\omega_r|_2}{1-e^{-(d+\lambda_1\nu)d_t}} \leq \frac{A|\omega_r|_2}{(d+\lambda_1\nu)d_t(1-(d+\lambda_1\nu)d_t/2)} = \frac{c_A}{(d+\lambda_1\nu)(1-(d+\lambda_1\nu)d_t/2)}|\omega_r|_2 \tag{16}$$

where

$$|\omega_r|_2 = \max_{\vec{x}_j} |\omega_r(\vec{x}-\vec{x}_j)|_{L^2(Q)} = |\omega_r(\vec{x}-\vec{x}_j)|_{L^2(Q)} \simeq r. \tag{17}$$

which is clear, thanks to the explicit form of ω_r given in (7), and $\lambda_1 = 2$ is the first eigenvalue of $-\Delta$ with the given boundary condition.

The existence of absorbing ball ensures the existence of invariant measure for the velocity field $\vec{v} = \nabla^{\perp}\psi$ in $L^2(Q)$ and the stream function ψ in $H^1(Q)$ by the usual tightness argument [18, 1].

It is also easy to check that the vorticity field q has an absorbing ball of radius R_4 in $L^4(Q)$ (the same as absorbing ball for the velocity field in $W^{1,4}$, or absorbing ball for the stream function in $W^{2,4}$) given by

$$\begin{aligned} R_4 &= \frac{A|\omega_r|_4}{1-e^{-(d+3\lambda_1\nu/4)d_t}} \\ &\leq \frac{A|\omega_r|_4}{(d+3\lambda_1\nu/4)d_t(1-(d+3\lambda_1\nu/4)d_t/2)} \\ &= \frac{c_A}{(d+3\lambda_1\nu/4)(1-(d+3\lambda_1\nu/4)d_t/2)}|\omega_r|_4 \end{aligned} \tag{18}$$

where

$$|\omega_r|_4 = \max_{\vec{x}_j} |\omega_r(\vec{x}-\vec{x}_j)|_{L^4(Q)} = |\omega_r(\vec{x}-\vec{x}_j)|_{L^4(Q)} \sim r^{\frac{1}{2}}. \tag{19}$$

The set of invariant measures will be unique if we have small enough data (amplitude). Indeed, the freely decaying stage will be contractive with a contraction constant $\kappa = e^{-(d+\lambda_1\nu)d_t/2}$ for the velocity field on the absorbing ball provided that the generalized Grashof number (defined below) is small

$$Gr \overset{def}{=} \frac{A|\omega_r|_4}{d_t\lambda_1^{1/4}\nu^{1/2}(d+\lambda_1\nu)^{3/2}} \leq c \tag{20}$$

where c is a generic constant originated in various Sobolev inequalities. This is very much similar to the case of contraction of incompressible Navier-Stokes flow with small Grashoff number [3, 4, 8, 23]. Of course, the same estimates hold for the linearized flow bombarded with the same random forcing.

This small generalized Grashof number assumption is guaranteed if we have

$$\frac{c_A r^{1/2}}{\lambda_1^{1/4}\nu^{1/2}(d+\lambda_1\nu)^{3/2}} \leq c. \tag{21}$$

This is a smallness assumption on c_A and/or the radii r of the small scale random vortices relative to the eddy viscosity ν and overall dissipation effect $d+\lambda_1\nu$.

Our main goal is to show that the unique invariant measure is supported around a large scale coherent structure.

In order to get info on the (potentially) large scale coherent structure, we consider the linear regime of the parameters where we could approximate the freely decaying flow by the linearized flow on the absorbing ball. Indeed, let $q(t)$ be the solution of the freely decaying flows and let $q^s(t)$ be the solution of the linearized flow. Then the difference $\delta q = q - q^s$ satisfies the following equation

$$\frac{\partial \delta q}{\partial t} + d\delta q - \nu\Delta\delta q = -\nabla^{\perp}\psi\cdot\nabla q. \tag{22}$$

Multiplying this equation by $-\delta\psi$ and integration over the domain Q we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\nabla\delta\psi|_{L^2}^2 + d|q|_{L^2}^2 + \nu|\delta q|_{L^2}^2 &= \int \nabla^{\perp}\psi\cdot\nabla q\,\delta\psi \\
&= -\int \nabla^{\perp}\psi\cdot\nabla\delta\psi\, q \\
&\leq |\nabla\psi|_{L^4}|\nabla\delta\psi|_{L^2}|q|_{L^4} \\
&\leq cR\,R_4|\nabla\delta\psi|_{L^2}
\end{aligned}$$

This implies

$$|\nabla\delta\psi(d_t)|_{L^2} \leq cRR_4\frac{1-e^{-(d+\lambda_1\nu)d_t}}{d+\lambda_1\nu} \leq cR\,R_4\,d_t \tag{23}$$

assuming q and q^s satisfy the same initial data.

Now we define $\tilde{\eta}^j$ as the process corresponding to the linearized dynamics, i.e.,

$$\tilde{\eta}^{j+1} = e^{-(d-\nu\Delta)d_t}\tilde{\eta}^j + A\omega_r(\vec{x}-\vec{x}_j)\xi_j. \tag{24}$$

We then have

$$\begin{aligned}\eta^{j+1}-\tilde{\eta}^{j+1} &= S(d_t)\eta^j - e^{-(d-\nu\Delta)d_t}\tilde{\eta}^j \\ &= S(d_t)\eta^j - S(d_t)\tilde{\eta}^j + S(d_t)\tilde{\eta}^j - e^{-(d+\nu\Delta)d_t}\tilde{\eta}^j.\end{aligned}$$

Therefore, in terms of the $H^{-1}(Q)$ norm of the vorticity which is the same as the $L^2(Q)$ norm for the velocity, we have

$$\begin{aligned}|\eta^{j+1}-\tilde{\eta}^{j+1}|_{H^{-1}} &\leq |S(d_t)\eta^j - S(d_t)\tilde{\eta}^j|_{H^{-1}} + |S(d_t)\tilde{\eta}^j - e^{-(d-\nu\Delta)d_t}\tilde{\eta}^j|_{H^{-1}} \\ &\leq \kappa|\eta^j-\tilde{\eta}^j|_{H^{-1}} + cR\,R_4\,d_t \end{aligned}\tag{25}$$

where we have used the contraction of $S(d_t)$ with Lipschitz constant κ and the estimate on the difference between the freely decaying barotropic flow and the linearized flow.

Iterating in j and noting that the two processes share identical initial value, we have

$$|\eta^{j+1}-\tilde{\eta}^{j+1}|_{H^{-1}} \leq cR\,R_4\,d_t\frac{1-\kappa^{j+1}}{1-\kappa} \leq \frac{c}{1-\kappa}(\frac{c_A}{d+\lambda_1\nu})^2 d_t|\omega_r|_2|\omega_r|_4. \tag{26}$$

This tells us that the process would be close to the linearized process provided that the freely decaying process is contractive on the absorbing ball with Lipschitz constant $\kappa = e^{-\frac{(d+\lambda_1\nu)d_t}{2}} < 1$ and the time interval d_t between two consecutive bombardments is small.

Next we consider the linearized flow under small scale bombardments, i.e., $\tilde{\eta}^j$.

For this purpose we decompose the random kick into a mean part and a random fluctuation part as in our previous work [16]

$$\omega_r(\vec{x}-\vec{x}_j)\xi_j = \bar{\omega}_r + \omega_r'(j), \tag{27}$$

where the mean part is defined as

$$\bar{\omega}_r = \mathbf{E}(\omega_r(\vec{x}-\vec{x}_j)\xi_j) = (2p-1)\mathbf{E}(\omega_r(\vec{x}-\vec{x}_j)) \tag{28}$$

with $\mathbf{E}$ being the mathematical expectation operator. It is easy to see that, thanks to (9),

$$\begin{aligned}
\bar{\omega}_r &\in H_0^2(Q), \quad \omega_r'(j) \in H_0^2(Q). && (29)\\
\mathbf{E}(\omega_r'(j)) &= 0, && (30)\\
\mathbf{E}(\|\Delta\omega_r'(j)\|_{L^2}^2) &= \mathbf{E}(\|\Delta\omega_r(\vec{x}-\vec{x}_j)\|_{L^2}^2) + \mathbf{E}(\|\Delta\bar{\omega}_r\|_{L^2}^2)\\
&\quad -2\int_Q \Delta\bar{\omega}_r \mathbf{E}(\Delta\omega_r(\vec{x}-\vec{x}_j)) < \infty && (31)
\end{aligned}$$

This means that the $\{\omega_r'(j)\}s$ are $H_0^2(Q)$ valued *i.i.d.* random variables.

We now have

$$\begin{aligned}
\tilde{\eta}^{j+1} &= e^{-(d-\nu\Delta)d_t}\tilde{\eta}^j + A\omega_r(\vec{x}-\vec{x}_j)\xi_j\\
&= e^{-(d-\nu\Delta)d_t}\tilde{\eta}^j + A\bar{\omega}_r + A\omega_r'(j)\\
&= e^{-(j+1)(d-\nu\Delta)d_t}\eta_0 + A\sum_{l=0}^{j} e^{-l(d-\nu\Delta)d_t}\bar{\omega}_r + A\sum_{l=0}^{j} e^{-l(d-\nu\Delta)d_t}\omega_r'(l) \qquad (32)
\end{aligned}$$

It is easy to check that

$$|e^{-(j+1)(d-\nu\Delta)d_t}\eta_0|_{H^{-1}} \le e^{-(j+1)(d+\lambda_1\nu)d_t}|\eta_0|_{H^{-1}} \to 0, \quad \text{as} \quad j \to \infty. \qquad (33)$$

As for the deterministic part, we have

$$\begin{aligned}
A\sum_{l=0}^{j} e^{-l(d-\nu\Delta)d_t}\bar{\omega}_r &= \frac{A}{d_t}(d-\nu\Delta)^{-1}\bar{\omega}_r - \frac{A}{d_t}e^{-(j+1)(d-\nu\Delta)d_t}(d-\nu\Delta)^{-1}\bar{\omega}_r\\
&\to \frac{A}{d_t}(d-\nu\Delta)^{-1}\bar{\omega}_r \quad \text{as} \quad j \to \infty\\
&= c_A(d-\nu\Delta)^{-1}\bar{\omega}_r\\
&= q^0. && (34)
\end{aligned}$$

We then speculate that the large scale coherent structure may resemble $(d-\nu\Delta)^{-1}\bar{\omega}_r$ which is very close to $(-\nu\Delta)^{-1}(1)$ just as in the continuous case [16] if μ is the uniform distribution on Q_{r_0} and $d = 0$. Although this predicted flow field is not the one predicted by equilibrium statistical mechanics theory, the error is very small (less than 10% [16]) and thus this can be viewed as supporting evidence on the applicability of equilibrium statistical theory in this weakly damped driven environment.

We still need to show that the pure fluctuation part is small enough. For this purpose we decompose the pure fluctuation in terms of the Fourier modes as

$$\omega_r'(j) = \sum_{\vec{k}} \hat{\omega}_{\vec{k}} \zeta_{\vec{k},j} \tag{35}$$

where $\hat{\omega}_{\vec{k}} = \frac{1}{\pi} e^{i\vec{k}\cdot\vec{x}}$ form an orthonormal basis for $L^2(Q)$, and the random coefficients $\{\zeta_{\vec{k},j}\}$ are i.i.d. random variables with mean zero for each fixed $\vec{k}$.

We now look at the variance of the purely fluctuation part (the mean is zero by our decomposition of mean and fluctuation).

$$\begin{aligned}
\mathbf{E}(|A\sum_{l=0}^{j} e^{-l(d-\nu\Delta)d_t}\omega_r'(l)|^2_{L^2(Q)}) &= A^2 \sum_{\vec{k}} \sum_{l_1=0,l_2=0}^{j} e^{-(l_1+l_2)(d+\nu|\vec{k}|^2)d_t} \mathbf{E}(\zeta_{\vec{k},l_1}\zeta^*_{\vec{k},l_2}) \\
&= A^2 \sum_{\vec{k}} \sum_{l=0}^{j} e^{-2l(d+\nu|\vec{k}|^2)d_t} \mathrm{Var}(\zeta_{\vec{k},l}) \\
&= A^2 \sum_{\vec{k}} \frac{1-e^{-2(j+1)(d+\nu|\vec{k}|^2)d_t}}{1-e^{-2(d+\nu|\vec{k}|^2)d_t}} \mathrm{Var}(\zeta_{\vec{k}}) \\
&\leq A^2 \frac{1}{1-e^{-2(d+\lambda_1\nu)d_t}} \sum_{\vec{k}} \mathrm{Var}(\zeta_{\vec{k}}) \\
&= \frac{A^2}{1-e^{-2(d+\nu\lambda_1)d_t}} \mathbf{E}(|\omega_r'|^2_{L^2}) \\
&\leq \frac{A^2}{2(d+\lambda_1\nu)d_t(1-(d+\lambda_1\nu)d_t)} |\omega_r'|_2^2 \\
&= \frac{c_A^2}{2(d+\lambda_1\nu)(1-(d+\lambda_1\nu)d_t)} d_t |\omega_r'|_2^2 \\
&\sim \frac{c_A^2}{2(d+\lambda_1\nu)} d_t r^2
\end{aligned} \tag{36}$$

where

$$|\omega_r'|_2 = \max_{\vec{x}_j} |\omega_r'(j)|_{L^2} \sim r. \tag{37}$$

Combining all the results we have the following theorem

Theorem 1 *The Markov process η^j enjoys a unique invariant measure provided the small generalized Grashof number assumption (20) or (21) is satisfied. Moreover, it has the following asymptotic expansion*

$$\eta^{j+1} = c_A(d-\nu\Delta)^{-1}\bar{\omega}_r + err_d(j) + err_r(j) \tag{38}$$

where the deterministic error $err_d(j)$ and the random fluctuation error $err_r(j)$ are bounded by

$$\begin{aligned} |err_d(j)|_{H^{-1}(Q)} &\leq e^{-(j+1)(d+\lambda_1\nu)d_t}|\eta_0|_{H^{-1}} + \frac{c_A}{d+\lambda_1\nu}e^{-(j+1)(d+\lambda_1\nu)d_t}|\bar{\omega}_r|_{L^2} \\ &\quad + \frac{c}{1-\kappa}(\frac{c_A}{d+\lambda_1\nu})^2 d_t|\omega_r|_2|\omega_r|_4, & (39) \\ \mathbf{E}(|err_r(j)|^2_{H^{-1}}) &\leq \frac{c_A^2}{2(d+\lambda_1\nu)}d_t|\omega'_r|_2^2 & (40) \end{aligned}$$

Hence the unique invariant measure is supported around the large scale coherent structure $c_A(d-\nu\Delta)^{-1}(\bar{\omega}_r)$.

We would like to remark here that the asymptotics above shows that the asymptotic behavior of the Markov process is close to a large scale coherent structure $c_A(d-\nu\Delta)^{-1}\bar{\omega}_r$ ($\sim (2p-1)c_Ar^2(d-\nu\Delta)^{-1}(1)$ in the case of uniform distribution μ). Indeed, the first two terms in the deterministic error are exponentially small for large j (time), while the last term in the deterministic error is of the order of $\frac{r^{3/2}}{d+\lambda_1\nu}(\frac{c_A}{d+\lambda_1\nu})^2$ which is smaller than the order of the large scale structure ($\frac{c_Ar^2}{d+\lambda_1\nu}$) for small c_A. The random error is at most of the order of $\frac{c_Ar\sqrt{d_t}}{\sqrt{d+\lambda_1\nu}}$ which is also of lower order to the large scale structure for small d_t.

3 Concluding remarks

We have demonstrated that small scale random bombardments at discrete time may induce large scale coherent structure in two dimension flow problems. Moreover, the large scale coherent structure is well predicted by equilibrium statistical theory utilizing energy-enstrophy as conserved quantities or energy-circulation as conserved quantities [17] although the mean field predicted by the rigorous theory is different from the mean field predicted by the equilibrium statistical theory.

There have been many works on the existence and uniqueness of invariant measures for systems with random kick forcing (see for instance [5, 13, 18, 19] and the references therein). However, we are not aware of any work that demonstrate the support of the unique invariant measure is around a large scale coherent structure except our earlier work [16].

As for many rigorous results, our analytical results are not as sharp as the numerics have suggested. Namely, the rigorous results here imposes quite severe restrictions on the generalized Grashof number Gr and d_t although our numerical experiments suggest that such restrictions are not necessary. It would be interesting to see rigorous results without stringent restriction as those imposed here. Even the continuous case where tools of stochastic calculus are applicable is very difficult [5, 6, 7, 12, 19] in the case of degenerate noise as we have here. The discrete case also suffers another technical setback since stochastic calculus is not available and hence our result here in the discrete case is somewhat weaker than our earlier result in the continuous case [16].

The main result can be generalized in some straightforward fashion to include more geophysical effects such as β-plane, F-plane, topography, hyper-viscosity etc. The large scale coherent structure predicted would depend on the geophysical effects and the probability distributions μ for the center $\vec{x_j}$ of the random small scale forcing (see figure 1 for the influence of μ).

Lastly, we treated both the continuous and discrete kicking problems. We may then naturally ponder the relationship between the two. It is easy to see that the discrete one converges to the continuous one in the limit so long as various smallness assumptions here for the discrete case and those for the continuous case imposed in [16] are satisfied. Notice that the discrete random kicking only approximate the continuous random forcing process in the weak sense as in the invariance principle [16, 2]. Therefore we may only anticipate that the statistics of the continuous process be close to those of the discrete in time Markov process. Indeed, this is a somewhat universal issue related to models of stochastic differential equations with white in time noise if we interpret white noise as cumulative effect of i.i.d. random kicks via the invariance principle. This is similar to the issue of strong versus weak schemes for stochastic differential equations [21, 22]. We will provide detailed analysis to this and some other related problems in the near future in a separate manuscript.

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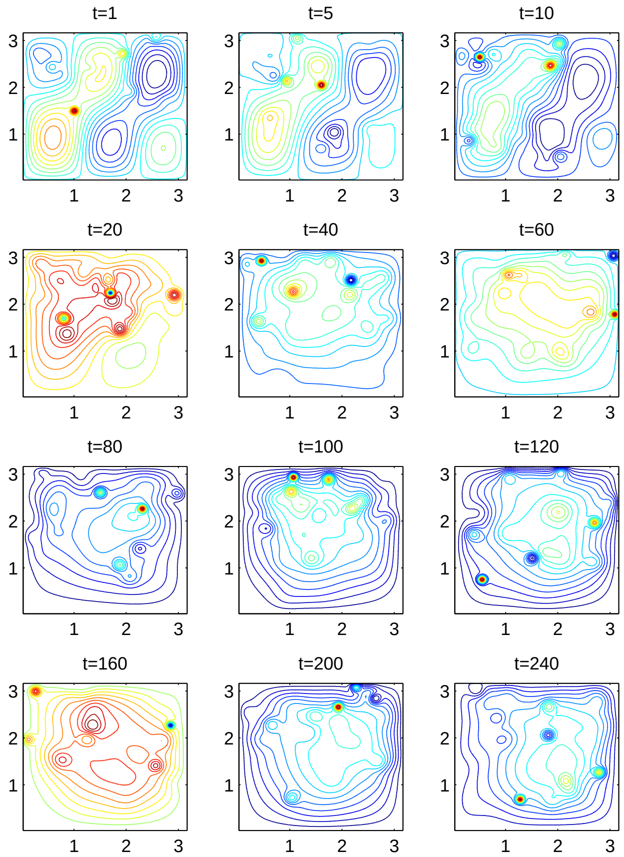

Figure 1: Contour plot of the vorticity field with initial data $0.1 \sin x \sin(3y) + 0.15 \sin(2x) \sin(2y)$, $d = 0$, $p = \frac{3}{4}$, and $d\mu = \frac{2(x-r_0)}{(\pi-2r_0)^3}\, dxdy$.