Numerical uniformization of hyperelliptic curves *

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Dedicated to Olli Lehto on his 70^{th} birthday.

Abstract

We develop algorithms which allow us to uniformize numerically any given real hyperelliptic plane M–curve. Starting from an equation for such a curve, we get (floating point approximations) for the generators of a discontinuous group of Möbius transformations uniformizing the given M–curve. Furthermore, we show how to compute an equation for a Riemann surface given by a such a group. For real hyperelliptic plane curves having maximal number of real components this construction gives a complete answer to the problem of numerical uniformization.

Some of the algorithms described here have already been coded. The programs will be made publicly available through the WWW home page of the HCM network "Computational conformal geometry" (http://sophie.helsinki.fi/).

1 Introduction

The starting point is the fact that the category of compact Riemann surfaces is equivalent to that of projective algebraic curves.

It is our aim to be able to study that equivalence of categories in a concrete computational fashion, i.e., we would like to be able to compute:

1. equations, as a plane algebraic curve, for compact Riemann surfaces
2. discontinuous groups of Möbius transformations uniformizing a given algebraic curve

It has been, around the turn of the century and early this century, the desire of a generation of mathematicians to carry out the above computations. In spite of intensive work carried out at that time (see e.g. [1], [2], [6], [5]), practical computations have been possible only in some particular cases, like that of the Klein's Riemann surface of genus 3, which has $168 = 84(3-1)$ automorphisms (see e.g. [4]). That is the largest possible group of automorphisms of genus 3 Riemann surfaces. The existence of automorphisms has been used to faciliate the otherwise complicated computations.

In this paper we present algorithms[1] which solve both of the above problems in the case of real hyperelliptic algebraic M–curves. The solution relies on methods of P. J. Myrberg ([6], see also [5]). Recall that by a real hyperelliptic M–curve we mean a hyperelliptic algebraic curve, of some genus $p \geq 2$, defined by a real polynomial in such a way that the complex conjugation induces an antiholomorphic involution of the curve which keeps $p+1$ closed curves point–wise fixed. Such a curve can always be defined by a polynomial equation of the type

$$y^2 = P(x), \tag{1}$$

where $P(x)$ of degree $2p+2$ and has $2p+2$ distinct real zeros.

We perform two computations:

1. We compute a uniformization for a real hyperelliptic M–curve. I.e., starting with a polynomial equation defining such a curve we derive a discontinuous group of Möbius transformations G such that the original curve is $\Omega(G)/G$, where $\Omega(G)$ is the domain of discontinuity of G.
2. Conversly, if G is a discontinous group of Möbius transformations of the second kind, fixing the unit disk and such that $\Omega(G)/G$ is a hyperelliptic Riemann surface, then we compute an equation representing this Riemann surface as an algebraic plane curve.

The input data in the first algorithm consists of a list of coefficients of a single polynomial defining a real hyperelliptic M–curve in the projective plane. Starting with this data, the first algorithm generates floating point approximations for the coefficients of hyperbolic Möbius transformations

$$g_j : \hat{C} \to \hat{C},\ g(z) = \frac{a_j z + b_j}{c_j z + d_j},\ a, b, c, d \in R,\ j = 1, \ldots, p+1,$$

mapping the upper half–plane onto itself and generating a discontinuous group G, whose domain of discontinuity $\Omega(G)$

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[1] This paper contains the description of the algorithms in question and proofs for their convergence. Reasons for the facts that the algorithms give us approximations of the desired objects, are explained but, in one case (Theorem 6), proof, that is technically quite involved, is omitted. This result has actually been shown, by other methods, in an old paper of P.J. Myrberg ([6]). A direct proof will appear in another paper, presently under preparation, by Semmler and Seppälä.

extends also to the lower half–plane, in such a way that the quotient $\Omega(G)/G$ is isomorphic to the given hyperelliptic curve.

The latter algorithm is the inverse of the former one. Thus this is a complete solution to the numerical uniformization problem in the case of real hyperelliptic M–curves. This solution is, however, only numeric. The algorithms described here give only approximations and not exact data. Initial tests seem to indicate that the covergence of the first algorithm is fast while that of the latter one is not as good. At this point we cannot, however, give any estimates about the convergence of the first algorithm. Estimates regarding the convergence of the second algorithm can be derived from [7, Lemma 8].

2 Preliminaries

We need to consider an auxiliary mapping that maps the exterior of an interval in the complex plane onto the exterior of a disk. The algorithm is based on iteration of such mappings.

Let

$$\tau_j \in C,\ j = 1, 2, \tag{2}$$

be two different points in the complex plane. Let I denote the closed interval having τ_1 and τ_2 as end–points.

Let

$$m = \frac{\tau_1 + \tau_2}{2} \text{ and } s = \frac{\tau_2 - \tau_1}{4}. \tag{3}$$

We assume that the indices are so chosen that the real part of s is non–negative. In the applications considered here, the τ's will be real and we simply assume that $\tau_2 > \tau_1$.

Consider the equation

$$\frac{z - (m + 2s)}{z - (m - 2s)} = \left(\frac{w - (m + s)}{w - (m - s)}\right)^2. \tag{4}$$

Suitable choice of the square–root gives us, by equation (4), a mapping

$$\phi : C \setminus I \to C \setminus \overline{D}, \tag{5}$$

where D is a disk centered at m with radius $|s|$.

This mapping ϕ plays a central role in our construction and it is, therefore, necessary to take a closer look at it.

We observe that the following lemmata hold:

Lemma 1 *The mapping ϕ determined by equation (4) is given by*

$$\phi(x) = x + \frac{x - m}{2}\left(\sqrt{1 - 4\frac{s^2}{(x - m)^2}} - 1\right), \tag{6}$$

for $|x - m| \geq 2|s|$,
or

$$\phi(x) = x - s\left(t + t^3 + t^5 + O\left(t^7\right)\right), \tag{7}$$

where $t = s/(x - m)$ and $|x - m| \geq 2|s|$.

Lemma 2 *Assume that the τ's are real and $\tau_2 > \tau_1$. Then m is also real and $s > 0$. In this case the mapping ϕ moves points of the real axis outside of the interval $[m - 2s, m + 2s]$ towards the mid-point m. For $|x - m| \geq 2|s|$, let*

$$\delta(x) = |x - \phi(x)| \tag{8}$$

or

$$\delta(x) = |x - m|\left(t^2 + t^4 + 2t^6 + O(t^8)\right), \tag{9}$$

where $t = s/(x - m)$.

Then, for $x \geq m + 2s$, $\delta(x)$ is strictly decreasing, and, for $x \leq m - 2s$, $\delta(x)$ is strictly increasing. The function $\delta(x)$ attains its maximum for $x = m + 2s$ and for $x = m - 2s$. The maximum value is

$$\delta(m \pm 2s) = s. \tag{10}$$

Furthermore,

$$\lim_{x \to \pm\infty} \delta(x) = 0. \tag{11}$$

Both of the above lemmata are simple straightforward technical computations and proofs are left to the reader. Observe that the property (11), which is important for our construction, follows from the fact that on the right hand side of defining equation (4), $2s$ was replaced by s only.

In the above, the mapping ϕ was determined by equation (4) and by requiring that ϕ maps the complement of the interval I onto the complement of the disk D with m as center and $|s|$ as radius.

The other choice of the sign of the square root (when solving w in terms of z from equation (4)) yields a mapping $\tilde{\phi} : C \setminus I \to D$. Let

$$e(z) = \frac{mz - m^2 + s^2}{z - m} \tag{12}$$

be the elliptic rotation, with angle π, of the complex plane keeping the points $m - s$ and $m + s$ fixed. Then e maps the complement of D onto D and we have

$$\tilde{\phi} = e \circ \phi. \tag{13}$$

In the subsequent sections we use this mapping ϕ with varying parameters τ_j to perform the desired iteration.

3 Opening up the slots

What follows will later be applied to real hyperelliptic M–curves C of genus p defined by the equation

$$y^2 = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_{2p+2}), \tag{14}$$

where all the parameters λ_i are real and no two of them agree. For the considerations of this section, however, the curve C itself is irrelevant.

Here we simply consider, for a positive integer p, a set $\{\lambda_1, \lambda_2, \ldots, \lambda_{2p+2}\}$ of $2p + 2$ distinct *real* numbers. We assume that the points λ_i are numbered in such a way that

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{2p+2}.$$

Then the closed intervals $I_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \ldots, p + 1$, are disjoint. We call these intervals *slots* or, more precisely, *first generation slots.*

We start now describing our iterative method.

Let ϕ_1 be the mapping ϕ defined by the equation (4) with $\tau_1 = \lambda_1$ and $\tau_2 = \lambda_2$. The mapping ϕ_1 *opens up* the first slot I_1 replacing it by a disk D_1 and maps all other slots onto new slots $\phi_1(I_j)$.

The corresponding elliptic rotation e_1, defined by (12), maps the complement of $\overline{D}_1$ onto D_1. In particular, e_1 maps the deformed slots $\phi_1(I_2), \phi_1(I_3), \ldots$ onto new slots $e_1(\phi_1(I_j)), j = 2, 3, \ldots$ in D_1. These are the first 2^{nd}–*generation slots.* Let us denote them by I_j^2. The opening of a slot and the creation of 2^{nd} generation slots is illustrated in figure 1.

Next, let ϕ_2 be the mapping ϕ defined by the equation (4) with $\tau_1 = \phi_1(\lambda_3)$ and $\tau_2 = \phi_1(\lambda_4)$. The mapping ϕ_2

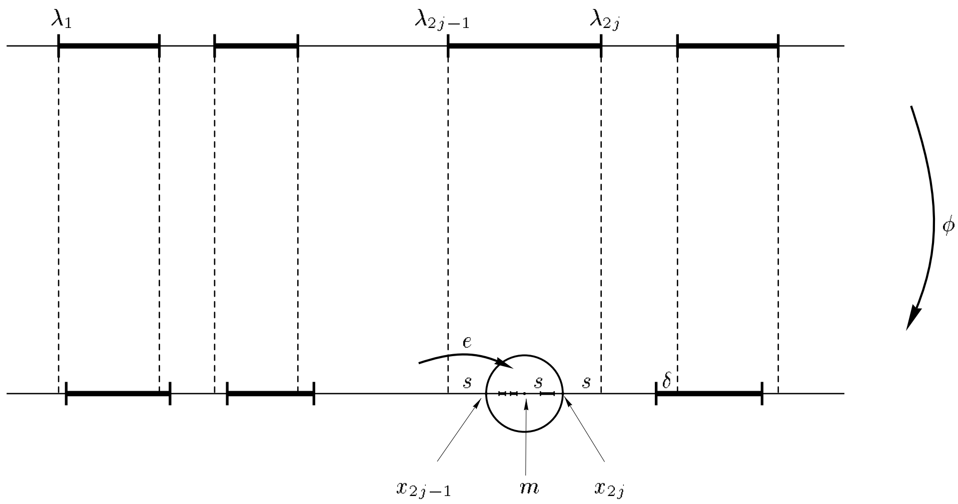

Figure 1: The mapping ϕ opens a slot of length $4s$ and replaces it by a disk D of radius s. The higher generation slots are obtained as images of the other slots under an elliptic Möbius transformation e mapping the complement of the closure of disk D onto D itself. Also the other slots move towards the mid-point m of the slot that is being opened. The function δ measures this movement. The points x_{2j-1} and x_{2j} are the intersection points of ∂D and the real axis.

- opens up the slot $\phi_1(I_2)$ and replaces it by a disk D_2
- maps the boundary of the disk D_1 onto a Jordan curve going through the points $\phi_2(\phi_1(\lambda_1))$ and $\phi_2(\phi_1(\lambda_2))$
- maps the deformed slots $\phi_1(I_j)$ onto new slots $\phi_2(\phi_1(I_j))$.

Let e_2 be the corresponding elliptic rotation, defined by (12) with

$$m = \frac{\phi_1(\lambda_3)+\phi_1(\lambda_4)}{2} \text{ and } s = \frac{\phi_1(\lambda_4)-\phi_1(\lambda_3)}{4}. \quad (15)$$

The rotation e_2 maps the slots $\phi_2(\phi_1(I_3)), \phi_2(\phi_1(I_4)), \ldots$ onto new 2^{nd}–generation slots $I_j^2, j = p+1 \ldots 2p$ inside the disk D_2. The images $e_2(\phi_2(I_j^2))$ of the first 2^{nd}–generation slots $I_j^2, j = 1, \ldots p$ inside D_2 are referred to as 3^{rd}–generation slots I_j^3.

Repeat this procedure until all the first generation slots have been opened and new higher generation slots have been formed. For the moment our main interest lies in following what happens to the points x_{2j-1}^0 and x_{2j}^0 which are defined as the intersection points of the boundaries of the disks D_j, $j = 1, \ldots, p+1$, and the real axis. Choose the indices in such a way that $x_{2j-1}^0 < x_{2j}^0$ for all values of j.

For all indices $j = 1, \ldots, p+1$, define the sequences x_{2j-1}^k and x_{2j}^k recursively by setting

$$x_{2j-1}^1 = \phi_{j+1}(x_{2j-1}^0) \text{ and } x_{2j}^1 = \phi_{j+1}(x_{2j}^0) \quad (16)$$

and

$$x_{2j-1}^2 = \phi_{j+2}(x_{2j-1}^1) \text{ and } x_{2j}^2 = \phi_{j+2}(x_{2j}^1) \quad (17)$$

and so on.

Repeating the above construction for all the openings of all generations of slots we get $2p+2$ infinite sequences x_i^k.

Lemma 3 *For all values $i = 1, 2, \ldots, 2p+2$, the corresponding sequences x_i^k converge as $k \to \infty$.*

Assume that the slots are ordered according to their generation and, within each generation, from left to right. Let us fix i and consider the sequence x_i^k, which *apriori* depends[2] on the choice of the ordering of the slots. We have to show that the limits

$$\lim_{k\to\infty} x_i^k = x_i^\infty \quad (18)$$

exist.

Let ϕ_m denote the opening of the slot I_m.

For a given sequence x_i^k where $i = 2j$ or $2j-1$ observe that the convergence of the sequence itself is equivalent to the convergence of the series

$$x_i^0 + (\phi_{j+1}(x_i^0) - x_i^0) + (\phi_{j+2}(\phi_{j+1}(x_i^0)) - \phi_{j+1}(x_i^0)) + \ldots \quad (19)$$

This series simply keeps track on how much the initial point x_i^0 moves as we continue opening up the slots. It is enough to show that the series (19) converges. That immediately implies the convergence of the sequence x_i^k. Furthermore, the limit point x_i^∞ is simply the sum of the series (19).

To prove the convergence of the series (19) observe that it is some kind of alternating series in the sense that some of its terms are positive and some are negative. By taking only positive terms or only negative terms we get two series. It suffices to show that both of them converge.

Let $\Delta x_i^0 = x_i^0$ and use the notation Δx_i^{k-1} to denote the k:th term of series (19), $k = 2, 3, \ldots$. Let $\Delta x_i^{k_n}, n = 1, 2, \ldots$ be the negative terms only.

Next comes the crucial argument. All the initial slots I_j are contained in the closed interval $[\lambda_1, \lambda_{2p+2}]$. By Lemma 2 all openings of slots I_j move the exterior points towards the mid–point of the slot I_j. Therefore all the various generation slots will be contained in the same closed interval $[\lambda_1, \lambda_{2p+2}]$.

Taking the negative terms of the series (19) only we add up the movements of the initial point to left. Such movements occur only when opening up slots that are on the left hand side of the point in question. Since, by the construction, the point x_i^0 and all its iterates are always in the

[2] Actually it can be shown that the limit points x_i^∞ do not depend on the ordering of the slots.

interval $[\lambda_1, \lambda_{2p+2}]$, it follows that the negative terms of the series (19) must form a converging series.

Same argument shows that the positive terms of the series (19) form also a converging series. We deduce therefore that the original series (19) converges absolutely. □

Lemma 4 *For $i \neq i'$, $x_i^\infty \neq x_{i'}^\infty$.*

Two limit points x_i^∞ and $x_{i'}^\infty$ may agree if and only if they have been obtained from the end–points of the same first generation slot I_j. This follows from the properties of the function $\delta(x)$ (see Lemma 2).

Let x_{2j-1} and x_{2j} be the intersection points of the real axis and the disk D_j obtained by opening up the slot I_j. Let y_1 and y_2 be the end–points of the 2^{nd} generation slots in the interval $[x_{2j-1}, x_{2j}]$ closests to x_{2j-1} and to x_{2j}, respectively. The the intervals (x_{2j-1}, y_1) and (y_2, x_{2j}) do not contain any 2^{nd}–generation slots. Nor will their deformations contain any higher generation slots. Let

$$\epsilon = |x_{2j-1} - y_1| + |y_2 - x_{2j}|$$

be the combined Euclidean length of these intervals.

Using again Lemma 2 we infer that, when opening any other slots, the intervals (x_{2j-1}, y_1) and (y_2, x_{2j}) *will get longer*. The distance between x_{2j-1} and x_{2j} and any of their deformations will, therefore, always be $> \epsilon$. This implies that the distance between the limit points x_{2j-1}^∞ and x_{2j}^∞ is also $> \epsilon$. Therefore $x_i^\infty \neq x_{i'}^\infty$ whenever $i \neq i'$. □

4 Hyperelliptic M–curves

Let $p \geq 2$ be an integer and consider an algebraic plane curve C defined by the affine equation

$$y^2 = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_{2p+2}) \tag{20}$$

where all the parameters λ_i are distinct, real and numbered in such a way that $\lambda_1 < \lambda_2 < \ldots < \lambda_{2p+2}$. Such an equation determines a real algebraic M–curve of genus p. We wish to use the considerations of the previous sections to find generators $g_1, g_2, \ldots$ for a discontinuous group G of Möbius transformations such that

$$C = \Omega(G)/G, \tag{21}$$

where $\Omega(G)$ denotes the domain of discontinuity of G.

To that end it is useful to recall the usual construction for the curve C. It is the following:

- take first two copies of the Riemann sphere
- cut them both open along the slots $I_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \ldots, p+1$.
- identify the boundary components of the first Riemann sphere, that was cut open, with the boundary components of the second one in such a way that you get a double cover of the original Riemann sphere ramified at the points λ_i.

First important observation is that the complement Ω_0 of the union of the slots I_j in the Riemann sphere $\hat{C}$, $\hat{C} \setminus \cup_{j=1}^{p+1} I_j$ can be identified, conformally, with a region C_u of the original algebraic curve C. Let $\epsilon : C \to C$ denote the hyperelliptic involution. Then $C_u \cup \epsilon(C_u)$ is the complement of $p+1$ simple closed geodesic curves[3] of the algebraic curve C.

Consider the first opening of the slot I_1. It is a conformal mapping

$$\phi_1 : \hat{C} \setminus I_1 \to \hat{C} \setminus \overline{D}_1. \tag{22}$$

Here we are using the notation of sections 2 and 3. Via this mapping,

$$U_1 = \hat{C} \setminus \left(\overline{D}_1 \cup \left(\cup_{j=2}^{p+1} I_j\right)\right) \tag{23}$$

is conformally equivalent to C_u. The elliptic rotation e_1 maps the outside of D_1 onto its inside. It follows that $e_1(U_1) \cup U_1$ is conformally equivalent to the complement of the above mentioned simple closed geodesic curves on the algebraic curve C. In other words, $e_1(U_1) \cup (U_1)$ is almost all of the Riemann surface in question. Furthermore, the mapping e_1 corresponds to the hyperelliptic involution.

Let now Ω_2 denote the complement of the union of the slots $I_2, I_3, \ldots, I_{p+1}$ and the 2^{nd}–generation slots $e_1(I_j)$, $j > 1$. The above remarks imply that we have an analytic local homeomorphism $\pi_1 : \Omega_2 \to C$.

We continue this procedure and keep up opening slots in a certain given order, for instance, by generation and in each generation from left to right. Let Ω_n denote the complement of all the various generation slots after having opened n slots (which have disappeared since they have been replaced by disks).

Since each opening of slots is a conformal mapping, we have, at each level an analytic local homeomorphism $\pi_n : \Omega_n \to C$. Once we have opened all the 1^{st}–generation slots, i.e., for $n > p+1$, the mapping π_n can already be extended to a surjective mapping. (The extension refers to the extension of the mapping π_n to the boundaries of the disks that have replaced the opened slots. This is straightforward.)

Define the set Ω^∞ by setting

$$\hat{C} \setminus \Omega^\infty = \{z \in \hat{C} | \exists z_n \in \ \hat{C} \setminus \Omega^n \text{ such that } \lim_{n\to\infty} z_n = z\}. \tag{24}$$

Consider the limits x_j^∞ of Lemma 3. Let e_j^∞ denote that elliptic Möbius transformation which keeps the point x_{2j-1}^∞ and x_{2j}^∞ fixed and rotates the infinite complex plane by angle π.

Let G be the group generated by all products $e_j^\infty e_{j'}^\infty$, $j, j' = 1, 2, \ldots, p+1$.

Lemma 5 *The group G is a Kleinian group and set Ω^∞ is the domain of discontinuity of the group G.*

The group G is a proper subgroup of the group E generated by all the above defined elliptic Möbius transformations e_j^∞, $j = 1, 2, \ldots, p+1$, having their fixed–points on the real axis. This group is, on the other hand, discontinuous by standard combination theorems. Thefore its proper subgroup G is discontinuous, i.e., kleinian.

The statement concerning the domain of discontinuity follows from the characterization

$$\Omega^\infty = \hat{C} \setminus \overline{\cup_{j=1}^{p+1} \cup_{g\in G} \{g(x_{2j-1}^\infty), g(x_{2j}^\infty)\}} \tag{25}$$

which, on the other hand, follows directly from the constructions. □

Consider next the mappings $\pi_n : \Omega^n \to C$. Proof of the following theorem relies on Lemma 3 and will be contained in

[3] When speaking of geodesic curves, we refer to the hyperbolic metric of the corresponding Riemann surface and to curves that are geodesics in that metric.

a paper, by Semmler and Seppälä, which is presently under preparation. The main point to be shown here is, that for any $z \notin R$, the sequence $\phi_k(z)$ converges. This is simply Lemma 3 extended to the whole complex plane. The details are technically involved and will appear elsewhere later. P. J. Myrberg has another proof for this ([6]).

The next theorem holds for general hyperelliptic curves. To avoid complications, however, we assume here that the hyperelliptic algebraic curve C, given by equation (20) is a *real* curve having *maximal number* of real components, i.e., we consider hyperelliptic real algebraic M–curves.

Theorem 6 *The mappings π_n form a converging sequence and the limit mapping π_∞ is an analytic regular covering mapping*

$$\pi_\infty : \Omega^\infty \to C.$$

The cover group is the Kleinian group G. Therefore, $C = \Omega^\infty/G$.

The group G can be viewed as a Fuchsian group of the second kind acting both in the upper and in the lower half–planes. Its limit set is nowhere dense on the real axis. Any of the elliptic Möbius–transformations e_j^∞ induce the hyperelliptic involution of the algebraic curve $C = \Omega^\infty/G$.

5 Equations for hyperelliptic Riemann surfaces

Let us now start conversly from a hyperelliptic Riemann surface defined by the action of a Möbius–group of the second kind having the properties of the group G in Theorem 6. For technical convenience we consider, instead of the upper half–plane, the unit disk. This is only to make certain statements and computations technically simpler.

So we assume that

- G is a Möbius–group acting in the unit disk.
- G is of the second kind and freely generated by Möbius–transformations $g_1, \ldots, g_p$ mapping the unit disk onto itself.
- If $\Omega(G)$ is the domain of discontinuity of G, then $X = \Omega(G)/G$ is a hyperelliptic Riemann surface.
- The hyperelliptic involution $X \to X$ is determined by a given elliptic Möbius–transformation e such that $e^2 \in G$.

For $j = 1, 2, \ldots, p$, let

$$h_j(z) = \frac{\overline{\zeta}_j}{1 - \overline{\zeta}_j z}, \quad \zeta_j = g_j(0), \quad 1 \le j \le p. \tag{26}$$

Consider the Poincaré series

$$\Theta(h_j)(z) = \sum_{g \in G} h_j(g(z)) \left(\frac{\partial g(z)}{\partial z} \right). \tag{27}$$

We have ([3, Corollary 1, page 208]):

Theorem 7 *The Poincaré series (27) converge uniformly on compact subsets of the domain of discontiuity of G. The limit differentials*

$$\omega_j = \{\Theta(h_j) \mid 1 \le j \le p\}$$

are holomorphic and (ω_j) is a basis for holomorphic differentials on $X = \Omega(G)/G$.

In practical computations we can use the Theorem 7 to approximate the values of base differentials by computing (finitly many) elements of the Poincaré series. A formula regarding the errors in such computations has been derived in ([7, Lemma 4]).

Theorem 7 allows us to compute numerically estimates for images of points of X under the canonical mapping $\phi_K : X \to P^{p-1}(C)$. This is what we do.

We start with the basis (ω_j) for holomorphic one forms given in Theorem 7. In order to be able to find an equation of the type (14) we study the canonical mapping with a suitable choice of basis of $D^1(X)$.

Let $\sigma(z) = 1/\overline{z}$ be the reflection in the unit circle ∂D. It acts on the space $D^1(X)$ of holomorphic one forms on X by pull–back:

$$\sigma^\#(\omega)(z) = \overline{\omega(\sigma(z))}\,\overline{\frac{\partial \sigma(z)}{\partial \overline{z}}}.$$

This is a linear transformation of the *real* vector space $D^1(X)$ onto itself. With respect to the complex multiplication, it is conjugate–linear. Replacing the generators ω_1 and ω_2 by

$$\omega_j^\sigma := \frac{\omega_j + \sigma^\#(\omega_j)}{2},$$

if needed, we may assume that the generators are invariant with respect to the action of $\sigma^\#$.[4]

The elliptic transformation e, determining the hyperelliptic involution of X, acts also on $D^1(X)$ by pull–back:

$$e^\#(\omega)(z) := \omega(e(z)) \frac{\partial e(z)}{\partial z}.$$

We have:

Lemma 8 *If (ω_j) is a basis for holomorphic one forms of the hyperelliptic Riemann surface X, then also the differentials*

$$\omega_j^e := \frac{\omega_j - e^\#(\omega_j)}{2}$$

form a basis for $D^1(X)$. This basis is anti–invariant with respect to the action of the hyperelliptic involution.

This follows from the fact that a hyperelliptic Riemann surface always has a basis for holomorphic one–forms that is anti–invariant under the action of the hyperelliptic involution.

In fact, if X is defined — as an algebraic curve — by the polynomial equation

$$y^2 = P(x)$$

where P has $2p+2$ distinct zeros, then one may consider $\sqrt{P(x)}$ as a one–valued function defined on X. It is anti–invariant with respect to the action of the hyperelliptic involution.

Let $\gamma : X \to P^1(C)$ denote the hyperelliptic projection. Then the differentials

$$\tau_j = \frac{\gamma^\#(x^j dx)}{\sqrt{P}} \tag{28}$$

form a basis for holomorphic one–forms of X. This basis is anti–invariant. This is well–known (see e.g. [8, page 55, example 1]).

[4] In this case, that is actually not needed. As defined above, the generators ω_j are already $\sigma^\#$ invariant.

The claim of the lemma follows now from the fact that

$$\tau_j^e = \frac{\tau_j - e^{\#}(\tau_j)}{2}$$

and from the fact that vectors of the basis (τ_j) can be expressed in terms of the basis (ω_j). □

5.1 Riemann surfaces of genus 2

If X is a Riemann surface of genus 2, then the above basis (ω_j) has only two elements. Assume that they are invariant with respect to the action of the reflection in ∂D and anti–invariant with respect to the action of the hyperelliptic involution.

The canonical mapping

$$\phi_K : X \to P^{p-1}(C) = P^1(C)$$

maps X onto the Riemann sphere.

X being of genus 2, the degree of the canonical divisor K satisfies $\deg K = 2(p-1) = 2$. This means that the canonical mapping takes, in the case of genus 2 Riemann surfaces, every value twice, i.e., that the canonical mapping *is* the hyperelliptic projection.

The parameters λ_j in the equation (14) are now simply those points over which the hyperelliptic projection is ramified.

There are two different ways of finding those points. The following arguments work both also in the case of Riemann surfaces of higher genus.

The hyperelliptic involution e is known. The branch–points of the hyperelliptic projection are simply those points $z \in \Omega(G)$ satisfying

$$\exists g \in G \text{ such that } e(z) = g(z). \tag{29}$$

Solving equation (29) numerically for several $g \in G$, one can find all the $2p+2 = 6$ branch–points. Oberserve that for most elements $g \in G$, the equation (29) does not have any solutions in $\Omega(G)$. The solutions, i.e. the branch points of the hyperelliptic projection will be points on the boundary of the unit disk.

Other possibility is to consider components I on $\partial D \setminus L(G)$, where $L(G)$ is the limit set of G. These arcs I project onto simple closed curves on $\Omega(G)/G$ going through the images of two of the branch–points. Projecting, by the hyperelliptic projection, further down to $P^1(C)$, the image of I is an interval of the real axes. The end–points of that interval are images of two branch–points of the hyperelliptic proecction. These images are the parameters λ_j in the equation (14) for the Riemann surface $\Omega(G)/G$.

Hence, in genus 2 case, an equation for the Riemann surface X can be readily computed in terms of the Poincaré series in Theorem 7.

5.2 Riemann surfaces of higher genus

The case of riemann surfaces of genus more than 2 is technically more involved. The canonical mapping

$$\phi_K : X \to P^{p-1}(C)$$

maps X into a higher dimensional projective space.

Using the basis (τ_j) given in (28) for a general hyperelliptic Riemann surface, one can show ([8, page 81, example 3]) that the canonical mapping ϕ_K can be presented in the form

$$\phi_K = v_{p-1} \circ \gamma$$

where γ is the hyperelliptic projection $\gamma : X \to P^1(C)$ and

$$v_{p-1} : P^1(C) \to P^{p-1}(C),$$

$$v_{p-1}(X_0, X_1) = (X_0^p X_1, X_0^{p-1} X_1^2, \ldots, X_0 X_1^p),$$

is the *Veronese mapping*. The image of the Veronese mapping is a curve V in $P^{p-1}(C)$, called the *Veronese variety* or the *Veronese curve*, and the Veronese mapping is an isomorphism between $P^1(C)$ and V. The Veronese curve in $P^{p-1}(C)$ is defined by the quadratic equations

$$Y_i Y_j = Y_k Y_l, \tag{30}$$

where $i + j = k + l$.

Let (ω_j) be any basis for $D^1(X)$. The canonical mapping of X depends on the choice of the basis. The image of X in $P^{p-1}(C)$ is, however, always isomorphic to the Veronese curve. Assume that the differentials

$$\omega_j' = \sum_{i=1}^{p} a_i^j \omega_i, \; j = 1, \ldots, p, \tag{31}$$

form an arbitrary basis of $D^1(X)$. Let $\phi_K' : X \to P^{p-1}(C)$ be the corresponding canonical mapping. Computing enough points of $\phi_K'(X)$ and requiring that these image points satisfy equtions (30) one gets a system of second order polynomial equations. Soving the coefficients a_i^j from that system of equtions, one can find, numerically, an approximation for a basis (ω_j') of $D^1(X)$ for which the corresonding canonical mapping maps the original Riemann surface X onto the Veronese curve in $P^{p-1}(C)$. We may, therefore, assume that $\phi_K(X)$ *is* the Veronese curve.

Then we may compute the parameters λ_j appearing in equation (14) for X computing again the images of the branch-points of the hyperelliptic projection under the composed mapping (which *is* the hyperelliptic projection)

$$v_{p-1}^{-1} \circ \phi_K,$$

where the canonical mapping is formed with respect to the above basis for $D^1(X)$ (i.e. such that $\phi_K(X)$ is the Veronese curve). This concludes the algorithm in the case of higher genus hyperelliptic Riemann surfaces.

6 Implementations and applications

We have implemented, using Maple, the algorithms computing uniformization for hyperelliptic curves. The Maple code is rather simple and fast. First we start with the curve defined by the polynomial equation

$$y^2 = (x^2 - 1)(x^2 - 36)(x^2 - 64), \tag{32}$$

which is of type (14) with the slots

$$\{[-8, -6], [-1, 1], [6, 8]\}.$$

Some results are listed in Table 1.

In a second example we consider the polynomials

$$y^2 = (x^2 - 1/k^2)(x^2 - 36)(x^2 - 64), \tag{33}$$

for various k which are of type (14) with the slots

$$\{[-8, -6], [-1/k, 1/k], [6, 8]\}.$$

Some results are listed in Table 2.

n	x_1^n	x_2^n	x_3^n	x_4^n	x_5^n	x_6^n	some first generation slots
0	-8	-6	-1	1	6	8	
1	-7.5	-6.5	-1.0420	.96863	5.9807	7.9833	[-6.9833, -6.9807] [-6.9686, -6.9580]
2	-7.4660	-6.4607	-.53931	.46598	5.9385	7.9517	[-6.9468,-6.9442] [-6.9320,-6.9213] [-5.0388e-03,+5.6177e-03]
3	-7.4484	-6.4418	-.50531	.50532	6.4418	7.4484	[-6.9285,-6.9259] [-6.9137,-6.9030] [3.1603e-02,+4.2316e-02]
6	-7.4484	-6.4418	-.50530	.50530	6.4418	7.4484	

Table 1: Some steps in opening slots (n is the number of opened slots). The table lists values of first terms of the various series x_j^n, $j = 1, 2, \ldots, 6$. The important point here is that the convergence of these series seems to be very fast in this case. In this table we have opened only the first generation slots. Opening the slot $[x_{2j-1}, x_{2j}]$ causes a big change in the corresponding values x_{2j-1}^n and x_{2j}^n. Opening other slots disturb these values only very little.

k	n	x_1	x_2	x_3	x_4	x_5	x_6	lower estimate for the length of the degenerating handle
1	6	-7.4484	-6.4418	-.50530	.50530	6.4418	7.4484	5.2346
2	6	-7.4741	-6.4715	-.25261	.25261	6.4715	7.4741	5.9430
3	6	-7.4789	-6.4770	-.16840	.16840	6.4770	7.4789	6.3524
4	6	-7.4805	-6.4789	-.12630	.12630	6.4789	7.4805	6.6412
5	6	-7.4813	-6.4798	-.10104	.10104	6.4798	7.4813	6.8649
6	6	-7.4817	-6.4803	-.084199	.084199	6.4803	7.4817	7.0474
7	6	-7.4820	-6.4806	-.072170	.072170	6.4806	7.4820	7.2017
8	6	-7.4821	-6.4808	-.063149	.063149	6.4808	7.4821	7.3353
9	6	-7.4822	-6.4809	-.056132	.056132	6.4809	7.4822	7.4532

Table 2: Degenerating curves (33) acquire a long handle as $k \to \infty$. Lower estimates for the lengths of such a degenerating handle, for various values of k, are given in this table.

The importance of the curves (33) lies in the fact that as $k \to \infty$, the curves (33) approach a nodal curve, i.e., as $k \to \infty$ the points, in the *moduli space* of smooth complex curves of genus 2, corresponding to the ismorphism classes of curves (33) tend to the infinity of the moduli space. This is clear and well–known.

The present algorithm gives a clear picture regarding the degeneration of the geometry of the corresponding Riemann surface. The fact that

$$\lim_{k\to\infty} \frac{1}{k} = \lim_{k\to\infty} -\frac{1}{k} = 0$$

implies that, at the limit, the curve becomes singular. Geometrically this means that a simple closed curve on the corresonding Riemann surface gets pinched to a point, or, that one of the handles of the Riemann surface becomes infinitly long at the limit. This growing of the length of the corresponding handle can now be estimated from below by this algorithm. In Table 2 such estimates for various values of k in (33) are listed.

Let e_k^j be the elliptic Möbius transformation fixing the points x_{2j-1} and x_{2j} corresponding to the value k, $k = 1, 2, \ldots, 10$, in Table 1. Then $g_k := e_k^1 \circ e_k^2$ is a hyperbolic Möbius transformation whose axis projects into a simple closed curve going through the degenerating handle. Let $m(g_k)$ denote the multiplier of g_k. Computed in the hyperbolic metric of the upper half–plane, the length of this simple closed curve is simply $2 \log m(g_k)$. Recall that the *multiplier* m of a non–trivial hyperbolic Möbius transformation g, is a real number $m > 1$ such that g can be conjugated to the Möbius transformation $z \mapsto mz$. This metric need not agree with the proper hyperbolic metric of the Riemann surface in question. In any case the hyperbolic metric of the upper half–plane can be used to derive lower estimates for the growth of the length of the handle in question.

These computations have been made by Maple and they illustrate the concrete geometric applications of the algorithms presented here.

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