Stationary Statistical Properties of Rayleigh-Bénard Convection at large Prandtl number

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Abstract

This is the third in a sequel in our study of Rayleigh-Bénard convection at large Prandtl number. More specifically we investigate if stationary statistical properties of the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number are related to that of the infinite Prandtl number model for convection which is formally derived from the Boussinesq system via setting the Prandtl number to infinity. We study asymptotic behavior of stationary statistical solutions, or invariant measures, to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. In particular, we show that the invariant measures of the Boussinesq system for Rayleigh-Bénard convection converge to that of the infinite Prandtl number model for convection as the Prandtl number approaches infinity. We also show that specific statistical properties such as the Nusselt number for the Boussinesq system is asymptotically bounded by the Nusselt number of the infinite Prandtl number model for convection at large Prandtl number. We discover that the Nusselt numbers are saturated by ergodic invariant measures. Moreover, we derive a new upper bound on the Nusselt number for the Boussinesq system at large Prandtl number of the form $Ra^{\frac{1}{3}}(\ln Ra)^{\frac{1}{3}} + c\frac{Ra^{\frac{7}{2}}\ln Ra}{Pr^2}$ which asymptotically agrees

with the (optimal) upper bound on Nusselt number for the infinite Prandtl number model for convection.

keywords: *Rayleigh-Bénard convection, Boussinesq equations, infinite Prandtl number model, Prandtl number, Rayleigh number, Nusselt number, invariant measures, stationary statistical solutions*

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1 Introduction

We continue our study of the asymptotic behavior of solutions at large Prandtl number of the following
Boussinesq system for Rayleigh-Bénard convection (non-dimensional):

$$\begin{aligned}
\frac{1}{Pr}(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p &= \Delta\mathbf{u} + Ra\,\mathbf{k}\theta, \quad \nabla\cdot\mathbf{u} = 0, && (1)\\
\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta - u_3 &= \Delta\theta, && (2)\\
\mathbf{u}|_{z=0,1} &= 0, && (3)\\
\theta|_{z=0,1} &= 0, && (4)\\
\mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad & \theta|_{t=0} = \theta_0, && (5)
\end{aligned}$$

where $\mathbf{u}$ is the fluid velocity field, p is the modified pressure, θ is the deviation of the temperature field from the pure conduction state $1-z$, $\mathbf{k}$ is the unit upward vector, Ra is the Rayleigh number, Pr is the Prandtl number, and the fluids occupy the (non-dimensionalized) region

$$\Omega = [0, L_x] \times [0, L_y] \times [0, 1] \qquad (6)$$

with periodicity in the horizontal directions assumed for simplicity.

At very large Prandtl number, we may formally set the Prandtl number to infinity and arrive at the following
infinite Prandtl number model (non-dimensional)

$$\begin{aligned}
\nabla p^0 &= \Delta\mathbf{u}^0 + Ra\,\mathbf{k}\theta^0, \quad \nabla\cdot\mathbf{u}^0 = 0, && (7)\\
\frac{\partial \theta^0}{\partial t} + \mathbf{u}^0\cdot\nabla\theta^0 - u_3^0 &= \Delta\theta^0, && (8)\\
\mathbf{u}^0|_{z=0,1} &= 0, && (9)\\
\theta^0|_{z=0,1} &= 0. && (10)
\end{aligned}$$

which is relevant for fluids such as silicone oil and the earth's mantle as well as many gases under high pressure [4, 19, 3]. One observes that the Navier-Stokes equations in the Boussinesq system has been replaced by the Stokes system in the infinite Prandtl number model.

The fact that the velocity field is linearly *slaved* by the temperature field has been exploited in several recent very interesting works on rigorous estimates on the rate of heat convection in this infinite Prandtl number setting (see [14, 9, 11, 16] and the references therein, as well as the work of [4, 22]).

An important natural question is whether such an approximation is valid.

In our previous works [38, 40], we have shown that the infinite Prandtl number model is a reasonable model for convection at large Prandtl number in the sense that suitable weak solutions to the Boussinesq system converge to those of the infinite Prandtl number model on any fixed finite time interval ([38]) and the global attractors of the Boussinesq system converge to that of the infinite Prandtl number model ([40]) as the Prandtl number approaches infinity.

It is well-known that for complex systems such as the Boussinesq system where turbulent/chaotic behavior abound (see for instance [7, 19, 23, 3, 31]), the statistical properties for such systems are much more important and physically relevant than single trajectories [28, 18, 27]. In particular, if the complex system reaches a statistical equilibrium state, it is the stationary statistical properties which are calibrated using invariant measures that are important[18, 27, 35]. Hence it is natural and essential to ask if the stationary statistical properties (in terms of invariant measures) remain close.

Recall that the invariant measures are supported on the global attractors. Therefore, the upper semi-continuity of the global attractors that we derived earlier [40] gives us indication that the statistical properties of the Boussinesq system may be close to that of the infinite Prandtl number model even for this singular perturbation problem.

The main purpose of this manuscript is to show that general stationary statistical properties (in terms of invariant measures) of the Boussinesq system are close to general stationary statistical properties (invariant measures) of the infinite Prandtl number model at large Prandtl number. Specific statistical properties such as time averaged heat transport in the vertical direction characterized by the long time averaged Nusselt number are also related in the sense that the Nusselt number for the Boussinesq system is asymptotically bounded by the Nusselt number of the infinite Prandtl number model at large Prandtl number. Moreover, we found that the Nusselt numbers are saturated by ergodic invariant measures. We also derive an upper bound on the Nusselt number for the Boussinesq system of the form $Ra^{\frac{1}{3}}(\ln Ra)^{\frac{1}{3}} + c\frac{Ra^{\frac{7}{2}}\ln Ra}{Pr^2}$. This bound asymptotically agrees with the best known (and physically relevant) bound for the infinite Prandtl number model ($cRa^{\frac{1}{3}}$ [9, 16, 29]). These results further justify the infinite Prandtl number model for convection as an approximate model for convection at large Prandtl number.

Throughout this manuscript, we assume the physically important case of high Rayleigh number

$$Ra \geq 1 \tag{11}$$

so that we may have non-trivial dynamics.

We also follow the mathematical tradition of denoting our small parameter as ε, i.e.

$$\varepsilon = \frac{1}{Pr}. \tag{12}$$

The rest of the manuscript is organized as follows. In section 2 we introduce the definition of stationary statistical solutions (invariant measures) to the Boussinesq and infinite Prandtl number system. We prove that invariant measures for the Boussinesq system must contain subsequences that converge to invariant measures of the infinite Prandtl number model as the Prandtl number approaches infinity. In section 3, we show that the Nusselt numbers for the Boussinesq system is asymptotically bounded by that of the infinite Prandtl number model. In section 4, we derive a new upper bound on the Nusselt number for the Boussinesq system at large Prandtl number which agrees with the (optimal) upper bound on Nusselt number for the infinite Prandtl number model for convection. In section 5 we offer concluding remarks.

2 Upper semi-continuity of invariant measures

For convenience, we recall the following function spaces that are standard for the mathematical treatment of Boussinesq equations.

We denote the phase space of the Boussinesq system as

$$X = H \times L^2 \tag{13}$$

where H is the divergence free subspace of $(L^2)^3$ with zero normal component in the vertical direction and periodic in the horizontal directions ([40]). The phase space for the infinite Prandtl number model is simply L^2.

We also denote

$$Y = V \times H^1_{0,per} \tag{14}$$

where $H^1_{0,per}$ is the subspace of H^1 that is zero at $z = 0, 1$ and periodic in the horizontal directions, V is the divergence free subspace of $(H^1_{0,per})^3$

Denoting

$$\mathbf{F}_\varepsilon((\mathbf{u}, \theta)) = (-B(\mathbf{u}, \mathbf{u}) + \frac{1}{\varepsilon}(-A\mathbf{u} + Ra\, P(\mathbf{k}\theta)), -\mathbf{u} \cdot \nabla\theta + u_3 - \Delta\theta) \tag{15}$$

where $B(\mathbf{u}, \mathbf{u}) = P((\mathbf{u} \cdot \nabla)\mathbf{u})$ is the standard bilinear term in the analysis of incompressible fluids [10, 15, 34, 18, 26], P represents the Leray-Hopf projection, A denotes the Stokes operator with the associated boundary condition; we then rewrite the Boussinesq system as an abstract dynamical system

$$\frac{d}{dt}(\mathbf{u}, \theta) = \mathbf{F}_\varepsilon((\mathbf{u}, \theta)). \tag{16}$$

Similarly, denoting

$$F_0(\theta) = -Ra\, A^{-1}(\mathbf{k}\theta) \cdot \nabla\theta + Ra(A^{-1}(\mathbf{k}\theta))_3 - \Delta\theta \tag{17}$$

we can rewrite the infinite Prandtl number as

$$\frac{d}{dt}\theta = F_0(\theta). \tag{18}$$

We now introduce the concept of stationary invariant measures for the Boussinesq system and the infinite Prandtl number model which are similar to the case of Navier-Stokes system [18, 35].

Definition 1 *A stationary statistical solution for the Boussinesq system (with Prandtl number $Pr = \frac{1}{\varepsilon}$) is a probability measure μ_ε on the phase space X such that*

1.

$$\int_X \|(\mathbf{u}, \theta)\|_Y^2\, d\mu_\varepsilon((\mathbf{u}, \theta)) < \infty, \tag{19}$$

2.

$$\int_X (\mathbf{F}_\varepsilon((\mathbf{u}, \theta)), \Phi'((\mathbf{u}, \theta)))\, d\mu_\varepsilon((\mathbf{u}, \theta)) = 0 \tag{20}$$

for any test functional Φ that is bounded on bounded sets of X, Fréchet differentiable for $(\mathbf{u}, \theta) \in Y$ with $\Phi'((\mathbf{u}, \theta)) \in Y$ and the derivative is continuous and bounded as a function from Y to Y.

3.

$$\int_X \{|\nabla \mathbf{u}|_{L^2}^2 - Ra\,\theta u_3\}\, d\mu_\varepsilon((\mathbf{u},\theta)) \;\le\; 0, \tag{21}$$

$$\int_X \{|\nabla \theta|_{L^2}^2 - \theta u_3\}\, d\mu_\varepsilon((\mathbf{u},\theta)) \;\le\; 0. \tag{22}$$

The set of all stationary statistical solutions for the Boussinesq system at Prandtl number $Pr = \frac{1}{\varepsilon}$ is denoted $\mathcal{IM}_\varepsilon$.

Likewise, a stationary statistical solution for the infinite Prandtl number model is a probability measure μ_0 on the phase space L^2 such that

1.

$$\int_{L^2} \|\theta\|_{H^1_{0,per}}^2\, d\mu_0(\theta) < \infty, \tag{23}$$

2.

$$\int_{L^2} (F_0(\theta), \Phi_0'(\theta))\, d\mu_0(\theta) = 0 \tag{24}$$

for any test functional Φ_0 that is bounded on bounded sets of L^2, Fréchet differentiable for $\theta \in H^1_{0,per}$ with $\Phi'(\theta) \in H^1_{0,per}$ and the derivative is continuous and bounded as a function from $H^1_{0,per}$ to $H^1_{0,per}$.

3.

$$\int_{L^2} \{|\nabla \theta|_{L^2}^2 - Ra(A^{-1}(\mathbf{k}\theta))_3\theta\}\, d\mu_0(\theta) \le 0. \tag{25}$$

The set of all stationary statistical solutions for the infinite Prandtl number model is denoted $\mathcal{IM}_0$.

Roughly speaking, condition 1 expresses the fact that the stationary statistical solutions are supported on a smaller and smoother space; condition 2 is a functional weak formulation of the time invariance of stationary statistical solutions; condition 3 is a version of energy estimates.

It is easy to see that both $\mathcal{IM}_\varepsilon$ and $\mathcal{IM}_0$ contain more than one element since Dirac delta measures concentrated at any steady state solution is an invariant measure of the underlying (generalized) dynamical system, and we know that both the Boussinesq system and the infinite Prandtl number model contain multiple steady states [30, 25].

Recall that the well-posedness of the Boussinesq system is a major unsettled open problem. Hence stationary statistical solutions may not be invariant measures just as in the case of 3D Navier-Stokes system [18]. Nevertheless, we have eventual regularity for the Boussinesq system and there exists a global attractor at large Prandtl number, and the system is wellposed on the global attractor [39, 40]. Therefore, we may modify the proof for the 2D Navier-Stokes system [18] and show that stationary statistical solutions to the Boussinesq system at large Prandtl number are equivalent to invariant measures, i.e., measures that are invariant under the (generalized) flow. This justifies our notation of $\mathcal{IM}$. As usual, the support of these invariant measures /stationary statistical solutions is included in the global attractor just as in the 3D Navier-Stokes system case [18]. We will provide details elsewhere.

Since the infinite Prandtl number model can be viewed as the (singular) limit of the Boussinesq system as the Prandtl number approaches infinity, and since the dynamics is believed to be chaotic [3, 19, 29, 31], we naturally inquire if the statistical properties of the Boussinesq system are related to the statistical properties of the infinite Prandtl number model. The objects that captures the statistically stationary properties of the systems are stationary statistical solutions (or invariant measures) [18, 27]. Therefore we are interested in if the (stationary) statistical solutions for the Boussinesq system and the infinite Prandtl number model are related. Similar issues for the inviscid limit of the Navier-Stokes systems in terms of time dependent statistical solutions can be found in [6, 5, 12, 35].

Since the Boussinesq system and the infinite Prandtl number model possess different natural phase spaces, the convergence of invariant measures has to be discussed after we either take the marginal distribution of the invariant measures for the Boussinesq system onto the perturbative temperature field only, or lift the invariant measures for the infinite Prandtl number model to the product space of velocity and perturbative temperature.

Our goal here is to show the following result

Theorem 1 *Let $\mu_\varepsilon \in \mathcal{IM}_\varepsilon, 0 < \varepsilon \le \varepsilon_0$ be stationary statistical solutions (invariant measures) of the Boussinesq system at Prandtl number $Pr = \frac{1}{\varepsilon}$. Then there exists a subsequence which weakly converges to the lift $\mathcal{L}\mu_0$ of an invariant measure μ_0 of the infinite Prandtl number model as $\varepsilon \to 0$.*

Proof: There are two main ingredients in the proof: a compactness result which ensures the existence of a convergent subsequence, and an argument indicating that the limit must be an invariant measure of the limit system.

As an intermediate step, we first proof the upper semi-continuity in the projected sense.

Recall that [40] we have the following a priori estimates for solutions on the union of all global attractors ($\bigcup_{\varepsilon\le\varepsilon_0}\mathcal{A}_\varepsilon$) for the Boussinesq system at large Prandtl number

$$\begin{aligned}
|\frac{\partial}{\partial t}\mathbf{u}|_{L^2} &\le cRa^{\frac{9}{4}}, && (26)\\
\frac{1}{t}\int_{t_0}^{t_0+t}|\nabla\frac{\partial}{\partial t}\mathbf{u}|_{L^2}^2 &\le cRa^{\frac{9}{2}}, && (27)\\
|\mathbf{u}(t)|_{H^1} &\le cRa, && (28)\\
|\mathbf{u}(t)|_{H^2} &\le cRa^{\frac{5}{2}}, && (29)\\
\frac{1}{t}\int_{t_0}^{t_0+t}|\mathbf{u}(s)|_{H^2}^2\,ds &\le cRa^2, && (30)\\
|\theta(t)|_{H^2} &\le cRa^8, && (31)\\
|\frac{\partial\theta}{\partial t}(t)|_{L^2} &\le cRa^8. && (32)
\end{aligned}$$

as long as

$$\varepsilon Ra = \frac{Ra}{Pr} \le c_0 \tag{33}$$

where c_0 is an absolute constant.

Therefore, thanks to Prohorov's compactness theorem [2, 24], the set $\{\mu_\varepsilon, 0<\varepsilon\le\varepsilon_0\}$ is weakly pre-compact in the space $\mathcal{PM}(X)$ of all probability measures on the phase space X.

Next, we plan to show that appropriate marginal distributions of $\{\mu_\varepsilon, 0<\varepsilon\le\varepsilon_0\}$ converge weakly to an invariant measure μ_0 of the infinite Prandtl number model.

For this purpose, we make a change of variable for the Boussinesq system and introduce the following new variable

$$\mathbf{v} = \mathbf{u} - Ra\,A^{-1}(\mathbf{k}\theta) \tag{34}$$

which measures the deviation of the velocity component of the Boussinesq system from that of the infinite Prandtl number model.

We then have a new set of measures $\tilde{\mu}_\varepsilon$ on the $(\mathbf{v},\theta)$ space defined as

$$\int\Phi(\mathbf{u},\theta)\,d\mu_\varepsilon((\mathbf{u},\theta)) = \int\Phi(\mathbf{v}+Ra\,A^{-1}(\mathbf{k}\theta),\theta)\,d\tilde{\mu}_\varepsilon((\mathbf{v},\theta)), \tag{35}$$

for all appropriate test functional Φ.

The pre-compactness of the set $\{\mu_\varepsilon, 0<\varepsilon\le\varepsilon_0\}$ implies the pre-compactness of the set $\{\tilde\mu_\varepsilon, 0<\varepsilon\le\varepsilon_0\}$. Hence, the marginal distribution of $\tilde\mu_\varepsilon$ in θ, denoted $M\tilde\mu_\varepsilon$ and defined by

$$\int \Phi_0(\theta)\, dM\tilde\mu_\varepsilon(\theta) = \int \Phi_0(\theta)\, d\tilde\mu_\varepsilon((\mathbf{v},\theta)) = \int \Phi_0(\theta)\, d\mu_\varepsilon((\mathbf{u},\theta)) \tag{36}$$

is also pre-compact in the space $\mathcal{PM}(L^2)$ of all probability measures on L^2.

Therefore, without loss of generality we may assume

$$M\tilde\mu_\varepsilon \rightharpoonup \mu_0 \tag{37}$$

in $\mathcal{PM}(L^2)$.

Our goal now is to show that μ_0 must be a member of $\mathcal{IM}_0$, the set of stationary statistical solutions (invariant measures) for the infinite Prandtl number model.

We need to verify the three conditions in the definition of $\mathcal{IM}_0$.

It is easy to see that the first condition is satisfied thanks to the uniform a priori estimates.

It is easy to see that for any point $(\mathbf{u},\theta)\in\mathcal{A}_\varepsilon$ and the associated trajectory passing through this given point we have

$$A\mathbf{v} = -\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t} + B(\mathbf{u},\mathbf{u})\right). \tag{38}$$

Hence, thanks to the uniform a priori estimates, the regularity of the Stokes operator and Sobolev inequalities

$$\begin{aligned} |\mathbf{v}|_{H^2} &\le c_2\varepsilon\left(\left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2} + |\mathbf{u}|_{H^2}|\nabla\mathbf{u}|_{L^2}\right) \\ &\le c_3\varepsilon. \end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned} &\int_{L^2}\{|\nabla\theta|^2_{L^2} - Ra(A^{-1}(\mathbf{k}\theta))_3\theta\}\, d\mu_0(\theta) \\ = &\lim_{\varepsilon\to 0}\int_{L^2}\{|\nabla\theta|^2_{L^2} - Ra(A^{-1}(\mathbf{k}\theta))_3\theta\}\, dM\tilde\mu_\varepsilon(\theta) \\ = &\lim_{\varepsilon\to 0}\int_{X}\{|\nabla\theta|^2_{L^2} - Ra(A^{-1}(\mathbf{k}\theta))_3\theta\}\, d\mu_\varepsilon((\mathbf{u},\theta)) \end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{\varepsilon\to 0}\int_X \{|\nabla\theta|_{L^2}^2 - u_3\theta\}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\quad + \limsup_{\varepsilon\to 0}\int_X \{(u_3 - Ra(A^{-1}(\mathbf{k}\theta))_3)\theta\}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\leq \limsup_{\varepsilon\to 0}\int_X |\mathbf{v}|_{L^2}|\theta|_{L^2}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
&= 0 \qquad (40)
\end{aligned}
$$

where we have utilized the weak convergence of $M\tilde{\mu}_\varepsilon$, the definition of marginal distribution with $\Phi_0(\theta) = |\nabla\theta|_{L^2}^2 - Ra(A^{-1}(\mathbf{k}\theta))_3\theta$, the assumption that μ_ε is an invariant measure for the Boussinesq system, and the uniform estimates on $\mathbf{v}$.

Hence we have verified condition 3.

As for the second condition, we take special test functionals

$$\Phi((\mathbf{u},\theta)) = \Phi_0(\theta) \qquad (41)$$

for some test functional Φ_0 for the infinite Prandtl number model.

We then have

$$
\begin{aligned}
& |\int_{L^2}(F_0(\theta), \Phi_0'(\theta))\, d\mu_0(\theta)| \\
= & |\lim_{\varepsilon\to 0}\int_{L^2}(F_0(\theta), \Phi_0'(\theta))\, d(M\tilde{\mu}_\varepsilon)(\theta)| \\
= & |\lim_{\varepsilon\to 0}\int_X (F_0(\theta), \Phi_0'(\theta))\, d\mu_\varepsilon((\mathbf{u},\theta))| \\
\leq & |\limsup_{\varepsilon\to 0}\int_X (-\mathbf{u}\cdot\nabla\theta + u_3 - \Delta\theta, \Phi_0'(\theta))\, d\mu_\varepsilon((\mathbf{u},\theta))| \\
& +|\limsup_{\varepsilon\to 0}\int_X (Ra(A^{-1}(\mathbf{k}\theta))_3 - u_3, \Phi_0'(\theta))\, d\mu_\varepsilon((\mathbf{u},\theta))| \\
& +|\limsup_{\varepsilon\to 0}\int_X ((-(\mathbf{u} - Ra\, A^{-1}(\mathbf{k}\theta))\cdot\nabla\theta, \Phi_0'(\theta))\, d\mu_\varepsilon((\mathbf{u},\theta))| \\
\leq & \lim_{\varepsilon\to 0}\int_X |\mathbf{v}|_{L^2}\|\Phi_0'(\theta)\|_{L^2}\, d\mu_\varepsilon((\mathbf{u},\theta)) + \lim_{\varepsilon\to 0}\int_X |\mathbf{v}|_{L^\infty}|\nabla\theta|_{L^2}\|\Phi_0'(\theta)\|_{L^2}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
= & 0. \qquad (42)
\end{aligned}
$$

This verifies condition 2 and thus completes the proof of the upper semi-continuity in the projected sense.

Next, we show the upper semi-continuity in the lifted sense.

For this purpose, we define, for any $\mu_0 \in \mathcal{PM}(L^2)$, its lift $\mathcal{L}\mu_0 \in \mathcal{PM}(X)$ through the following relation

$$\int_X \Phi(\mathbf{u},\theta)\, d(\mathcal{L}\mu_0)((\mathbf{u},\theta)) = \int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_0(\theta) \qquad (43)$$

for every suitable test functional Φ.

It is easy to check that $\mathcal{L}\mu_0 \in \mathcal{PM}(X)$ and the marginal distribution of $\mathcal{L}\mu_0 \in \mathcal{PM}(X)$ defined above (through $\mathbf{v}$) is μ_0.

Now fix a test functional Φ which is bounded on bounded set of X, Fréchet differentiable for $(\mathbf{u}, \theta) \in Y$ with $\Phi'((\mathbf{u}, \theta)) \in Y$ and the derivative is continuous and bounded as a function from Y to Y, we have

$$\begin{aligned}
& |\int_X \Phi(\mathbf{u}, \theta)\, d\mu_\varepsilon((\mathbf{u}, \theta)) - \int_X \Phi(\mathbf{u}, \theta)\, d(\mathcal{L}\mu_0)((\mathbf{u}, \theta))| \\
= \; & |\int_X \Phi(\mathbf{u}, \theta)\, d\mu_\varepsilon((\mathbf{u}, \theta)) - \int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_0(\theta)| \\
\leq \; & |\int_X \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_\varepsilon((\mathbf{u}, \theta)) - \int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_0(\theta)| \\
& + |\int_X (\Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta) - \Phi(\mathbf{u}, \theta))\, d\mu_\varepsilon((\mathbf{u}, \theta))| \\
= \; & |\int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d(M\tilde{\mu}_\varepsilon)(\theta) - \int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_0(\theta)| \\
& + |\int_X (\Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta) - \Phi(\mathbf{u}, \theta))\, d\mu_\varepsilon((\mathbf{u}, \theta))| \\
\leq \; & |\int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d(M\tilde{\mu}_\varepsilon)(\theta) - \int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_0(\theta)| \\
& + |\int_X \sup_{(\mathbf{u},\theta)\in \mathrm{supp}\mu_\varepsilon} \|\Phi'(\mathbf{u}, \theta)\|_Y |\mathbf{v}|_{H^1}\, d\mu_\varepsilon((\mathbf{u}, \theta))| \\
\leq \; & |\int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d(M\tilde{\mu}_\varepsilon)(\theta) - \int_{L^2} \Phi(Ra\, A^{-1}(\mathbf{k}\theta), \theta)\, d\mu_0(\theta)| \\
& + c\varepsilon \sup_{(\mathbf{u},\theta)\in \mathrm{supp}\mu_\varepsilon} \|\Phi'(\mathbf{u}, \theta)\|_Y \\
\rightarrow \; & 0, \quad \text{as } \varepsilon \rightarrow 0
\end{aligned} \tag{44}$$

where we have used the weak convergence of the marginal distribution proved in the first half, the mean value property, the boundedness of the Fréchet derivative of Φ and the a priori estimates on $\mathbf{v}$.

This completes the proof of the theorem.

What we have shown here is the upper semi-continuity of the set of invariant measures in this singular limit setting. This is reminiscent to the upper semi-continuity of the global attractors [40] for the same problem as well as well-known results on upper semi-continuity of global attractors for regular perturbations of dissipative dynamical systems [33]. We do not expect continuity in general due to possible hysteresis [40].

We could also formulate a general theorem on upper semi-continuity of statistical solutions for two time scale problems of relaxation type just as we did for the upper semi-continuity of the global attractors [40]. However, we refrain from this since it is necessary to write the statistical analog of the energy inequality (condition 3 in the definition of stationary statistical solutions) which depends on specific structure of the equation.

3 Upper semi-continuity of the Nusselt number

Among all statistical properties of the Boussinesq system for Rayleigh-Bénard convection, one of the most prominent is the long time averaged Nusselt number measuring the heat transport in the vertical direction

$$\begin{aligned} Nu_\varepsilon &= \sup_{(\mathbf{u}_0,\theta_0)\in X} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t\int_\Omega |\nabla T(\mathbf{x},s)|^2\,d\mathbf{x}ds, \\ &= 1+\sup_{(\mathbf{u}_0,\theta_0)\in X} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t\int_\Omega u_3(\mathbf{x},s)T(\mathbf{x},s)\,d\mathbf{x}ds, \\ &= 1+\sup_{(\mathbf{u}_0,\theta_0)\in X} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t\int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\,d\mathbf{x}ds \end{aligned} \tag{45}$$

where $T = 1 - z + \theta$ is the temperature field.

The Nusselt number for the infinite Prandtl number is defined similarly as

$$Nu_0 = 1+\sup_{\theta_0\in L^2} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t\int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3(\mathbf{x},s)\theta(\mathbf{x},s)\,d\mathbf{x}ds. \tag{46}$$

A natural question to ask is whether the Nusselt numbers of the Boussinesq system is related to the Nusselt number of the infinite Prandtl number model.

The first observation is the following lemma which states that the Nusselt number defined above is a statistical property of the system with respect to certain ergodic invariant measure.

Lemma 1 *There exists at least one ergodic invariant measure $\nu_\varepsilon \in \mathcal{IM}_\varepsilon$ for each $\varepsilon \in [0, \varepsilon_0]$ such that*

$$Nu_\varepsilon = 1+\frac{1}{L_xL_y}\int_X\int_\Omega u_3\theta\,d\mathbf{x}\,d\nu_\varepsilon((\mathbf{u},\theta))$$

$$
\begin{aligned}
&= 1+\sup_{\mu\in\mathcal{IM}_\varepsilon}\frac{1}{L_xL_y}\int_X\int_\Omega u_3\theta\,d\mathbf{x}\,d\mu((\mathbf{u},\theta))\\
&= 1+\lim_{t\to\infty}\frac{1}{tL_xL_y}\int_0^t\int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\,d\mathbf{x}ds,\quad (\mathbf{u}_0,\theta_0)\in supp\nu_\varepsilon, \qquad (47)\\
Nu_0 &= 1+\frac{1}{L_xL_y}\int_{L^2}\int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3\theta\,d\mathbf{x}\,d\nu_0(\theta)\\
&= 1+\sup_{\mu\in\mathcal{IM}_0}\frac{1}{L_xL_y}\int_X\int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3\theta\,d\mathbf{x}\,d\mu(\theta)\\
&= 1+\lim_{t\to\infty}\frac{1}{tL_xL_y}\int_0^t\int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3(\mathbf{x},s)\theta(\mathbf{x},s)\,d\mathbf{x}ds,\quad \theta_0\in supp\nu_0. \qquad (48)
\end{aligned}
$$

Proof: We only show the case of $0<\varepsilon\le\varepsilon_0$.

For each fixed $\varepsilon\in(0,\varepsilon_0]$, there exists $(\mathbf{u}_{0j},\theta_{0j})\in\mathcal{A}_\varepsilon\subset X$ such that

$$
Nu_\varepsilon = 1+\lim_{j\to\infty}\limsup_{t\to\infty}\frac{1}{tL_xL_y}\int_0^t\int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\,d\mathbf{x}ds,\quad (\mathbf{u}(\mathbf{x},0),\theta(\mathbf{x},0))=(\mathbf{u}_{0j},\theta_{0j}). \tag{49}
$$

For each fixed orbit (corresponding to suitable weak solutions [39, 40]), the long time average is a statistical property in the sense that any fixed choice of generalized limit of time average is equivalent to spatial average with respect to a suitable stationary statistical solution by an application of the Krylov-Bogliubov theory (see [18, 35] for the case of Navier-Stokes system). In particular, after an application of the Hahn-Banach theorem, there is a special generalized limit of the time averaging which is consistent with lim sup on the particular functional and particular orbit. Therefore, there exists $\mu_{\varepsilon,(\mathbf{u}_{0j},\theta_{0j})}\in\mathcal{IM}_\varepsilon$ (this is an abuse of notation since there might be many orbits corresponding to the same initial data) such that

$$
\limsup_{t\to\infty}\frac{1}{tL_xL_y}\int_0^t\int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\,d\mathbf{x}ds=\frac{1}{L_xL_y}\int_X\int_\Omega u_3\theta\,d\mathbf{x}\,d\mu_{\varepsilon,(\mathbf{u}_{0j},\theta_{0j})}(\mathbf{u},\theta). \tag{50}
$$

We leave the detail to the interested reader.

These stationary statistical solutions are weakly compact in $\mathcal{PM}(X)$ for each fixed $\varepsilon\in(0,\varepsilon_0]$ due to the uniform a priori estimates and Prohorov's theorem [2, 24]. Therefore, without loss of generality we may assume,

$$
\mu_{\varepsilon,(\mathbf{u}_{0j},\theta_{0j})}\rightharpoonup\mu_\varepsilon \tag{51}
$$

for some $\mu_\varepsilon\in\mathcal{IM}_\varepsilon$.

This implies

$$
\begin{aligned}
Nu_\varepsilon &= 1 + \lim_{j\to\infty} \limsup_{t\to\infty} \frac{1}{tL_xL_y} \int_0^t \int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\, d\mathbf{x}ds \\
&= 1 + \lim_{j\to\infty} \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu_{\varepsilon,(\mathbf{u}_{0j},\theta_{0j})}(\mathbf{u},\theta) \\
&= 1 + \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu_\varepsilon((\mathbf{u},\theta)). \qquad (52)
\end{aligned}
$$

Now consider the following extremal subset of $\mathcal{IM}_\varepsilon$

$$
\mathcal{SIM}_\varepsilon = \{\mu \in \mathcal{IM}_\varepsilon \Big| \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu = \sup_{\nu\in\mathcal{IM}_\varepsilon} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu\} \qquad (53)
$$

The subset $\mathcal{SIM}_\varepsilon$ is non-empty by the uniform a priori estimates and Prohorov's theorem. Indeed, suppose we have $\nu_{\varepsilon,j} \in \mathcal{IM}_\varepsilon$ such that

$$
\lim_{j\to\infty} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu_{\varepsilon,j} = \sup_{\nu\in\mathcal{IM}_\varepsilon} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu, \qquad (54)
$$

(the supremum is finite due to a priori estimates), the set $\{\nu_{\varepsilon,j}, j \geq 1\}$ must be weakly pre-compact in $\mathcal{PM}(X)$ thanks to the uniform a priori estimates. Hence it must contain a subsequence that converges to some $\nu_\varepsilon \in \mathcal{PM}(X)$. It is then easy to see that $\nu_\varepsilon \in \mathcal{SIM}_\varepsilon$.

Notice that $\mathcal{M}(X)$, the space of all finite Borel measures on X form a locally convex topological space with the topology generated by weak convergence, and notice that $\mathcal{SIM}_\varepsilon$ is a compact subset of $\mathcal{M}(X)$. Therefore, the extremal set[1] of $\mathcal{SIM}_\varepsilon$ is non-empty thanks to the Krein-Milman Theorem [24, 18]. Let ν_ε be an extremal point of $\mathcal{SIM}_\varepsilon$. Then ν_ε is necessarily an extremal point of $\mathcal{IM}_\varepsilon$ which further implies, after repeating the proof of a well-known result for dynamical system [36] and utilizing the eventual regularity of the Boussinesq system [39, 40], that ν_ε must be ergodic. Therefore,

$$
\begin{aligned}
Nu_\varepsilon &= 1 + \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu_\varepsilon(\mathbf{u},\theta) \\
&\leq 1 + \sup_{\mu\in\mathcal{IM}_\varepsilon} \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu((\mathbf{u},\theta))
\end{aligned}
$$

[1]A extremal point of a set is a point that cannot be expressed as proper convex combination of another two (distinct) points in the set.

$$
\begin{aligned}
&= 1 + \frac{1}{L_x L_y}\int_X \int_\Omega u_3\theta \, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u},\theta) \\
&= 1 + \lim_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\, d\mathbf{x}ds, \quad (\mathbf{u}_0,\theta_0)\in \mathrm{supp}\nu_\varepsilon \\
&\le Nu_\varepsilon. \qquad (55)
\end{aligned}
$$

This completes the proof of the lemma.

Although the lemma establishes that the Nusselt number is a statistical property with respect to appropriate ergodic invariant measures, the limit of the Nusselt number may not be directly related to the Nusselt number of the limit system (infinite Prandtl number model) since those invariant measures corresponding to the Nusselt number are special and their limit may not be invariant measures corresponding to the Nusselt number for the limit system. In another word, the weak limits of the set $S\mathcal{IM}_\varepsilon$ may not be associated with $S\mathcal{IM}_0$. Nevertheless, we are still able to establish the following relationship which can be interpreted as upper semi-continuity of the Nusselt number in this singular limit.

Theorem 2

$$
\limsup_{\varepsilon\to 0} Nu_\varepsilon \le Nu_0. \qquad (56)
$$

Proof: Let $\nu_\varepsilon \in \mathcal{IM}_\varepsilon, \varepsilon\in[0,\varepsilon_0]$ be ergodic invariant measures corresponding to the Nusselt number that we discussed in lemma 1. Thanks to Theorem 1, we know that the set of $\{M\tilde{\nu}_\varepsilon, 0<\varepsilon\le\varepsilon_0\}$ is weakly pre-compact in $\mathcal{PM}(L^2)$. Without loss of generality we assume it weakly converges to some $\mu_0\in\mathcal{IM}_0$, i.e., $M\tilde{\nu}_\varepsilon \rightharpoonup \mu_0$. This implies

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} Nu_\varepsilon &= 1 + \limsup_{\varepsilon\to 0}\frac{1}{L_xL_y}\int_X\int_\Omega u_3\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u},\theta) \\
&\le 1 + \limsup_{\varepsilon\to 0}\frac{1}{L_xL_y}\int_X\int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u},\theta) \\
&\quad + \limsup_{\varepsilon\to 0}\frac{1}{L_xL_y}\int_X\int_\Omega (u_3 - Ra(A^{-1}(\mathbf{k}\theta))_3)\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u},\theta) \\
&\le 1 + \limsup_{\varepsilon\to 0}\frac{1}{L_xL_y}\int_{L^2}\int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3\theta\, d\mathbf{x}\, dM\tilde{\nu}_\varepsilon(\theta) \\
&\quad + \limsup_{\varepsilon\to 0}\frac{1}{L_xL_y}\int_X |v_3|_{L^2}|\theta|_{L^2}\, d\nu_\varepsilon(\mathbf{u},\theta)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \frac{1}{L_x L_y} \int_{L^2} \int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3 \theta \, d\mathbf{x} \, d\mu_0(\theta) \\
&\leq 1 + \frac{1}{L_x L_y} \int_{L^2} \int_\Omega Ra(A^{-1}(\mathbf{k}\theta))_3 \theta \, d\mathbf{x} \, d\nu_0(\theta) \\
&= Nu_0. \qquad (57)
\end{aligned}
$$

This ends the proof of the theorem.

It is worthwhile to point out that we do not expect continuity of statistical properties on parameters for general dynamical systems. In fact it is easy to find ODE counter-examples where we have bifurcation. However, there is hope that at large Rayleigh number, both the Boussinesq system and the infinite Prandtl number model possess enough mixing so that the invariant measures that saturate the Nusselt number, i.e. ν_ε, are unique for each $\varepsilon \in [0, \varepsilon_0]$ since we believe the Nusselt number will be saturated by turbulent trajectories (the set of invariant measures itself is never unique for large Rayleigh number due to existence of multiple steady states). If this assumption is valid, we then may have continuity of the Nusselt number. Still we do not have rate of convergence. Rate of convergence can be derived if we look at upper bounds on the Nusselt number instead. This is the goal of the next section.

4 Estimates on Nusselt number

The semi-continuity result in the previous section indicates that the Nusselt number for the limit system (infinite Prandtl number model) is an asymptotic bound for the Nusselt number associated with the Boussinesq system at large Prandtl number. However, no explicit convergence rate is given. The goal of this section is to derive a convergence rate result for the upper bound of the Nusselt numbers associated with the Boussinesq and infinite Prandtl number models. More specifically, we intend to derive an upper bound on the Nusselt number Nu_ε which is consistent with the best known upper bound [16, 9, 29, 1] on the Nusselt number Nu_0 for the infinite Prandtl number model.

The approach we take here is to view the Boussinesq system at large Prandtl number as a perturbation of the infinite Prandtl number model for convection. More specifically, we write the Boussinesq system as

$$
\nabla p = \Delta \mathbf{u} + Ra\, \mathbf{k} T + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \qquad (58)
$$

$$\begin{aligned} \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \Delta T, && (59) \\ \mathbf{u}|_{z=0,1} &= 0, && (60) \\ T|_{z=0} = 1, \qquad & T|_{z=1} = 0, && (61) \end{aligned}$$

where

$$\mathbf{f} = -\varepsilon(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}). \tag{62}$$

We follow the background temperature profile method for the Boussinesq system proposed by Constantin and Doering [8, 9, 16] which is a generalization of E. Hopf's original idea [34]. One difference here is we *do not enforce the spectral constraint* in choosing the background temperature profile. We will choose the background profile that "almost" satisfy the spectral constraint.

Let $\tau = \tau(z)$ be a background temperature profile that satisfies the boundary condition for the temperature field and let

$$\theta = T - \tau. \tag{63}$$

Then the perturbative temperature field [2] θ satisfies the equation

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = -u_3 \tau' + \Delta\theta + \tau''. \tag{64}$$

Therefore, the Nusselt number can be written as

$$\begin{aligned} Nu_\varepsilon &= \sup < |\nabla T|^2 > \\ &= \int_0^1 (\tau')^2 dz + \sup[< |\nabla\theta|^2 > -2 < \theta\tau'' >] \\ &= \int_0^1 (\tau')^2 dz - \inf < |\nabla\theta|^2 + 2\tau' u_3 \theta > \\ &= \int_0^1 (\tau')^2 dz - \inf < Q^{(\tau)}(\theta) >, && (65) \end{aligned}$$

here

$$Q^{(\tau)}(\theta) = |\nabla\theta|^2 + 2\tau' u_3 \theta \tag{66}$$

[2] This perturbative temperature field away from the background temperature field τ is different from the perturbative temperature field introduced in the previous sections which is the perturbation away from the pure conduction state.

where $< \cdot >$ denotes the space-time average defined as

$$< g >= \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega g(\mathbf{x},s)\,d\mathbf{x}ds. \tag{67}$$

It is easy to calculate that the vertical velocity field u_3 and the perturbative temperature field θ satisfy the following equation

$$\Delta^2 u_3 = -Ra\,\Delta_H\theta - \Delta_H f_3 + \frac{\partial^2 f_1}{\partial x\partial z} + \frac{\partial^2 f_2}{\partial y\partial z}, \tag{68}$$

$$u_3\Big|_{z=0,1} = 0, \frac{\partial u_3}{\partial z}\Big|_{z=0,1} = 0, \theta\Big|_{z=0,1} = 0, \tag{69}$$

where $\Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplace operator. This relationship implies that velocity field is almost slaved by the perturbative temperature field. In terms of the horizontal Fourier coefficients $\hat{\theta}_{\mathbf{l}}$, $\hat{u}_{3\mathbf{l}}$ and $\hat{f}_{j\mathbf{l}}, j = 1,2,3$ where $\mathbf{l} = (l_1, l_2)$ is the horizontal Fourier wave number, the relationship between the vertical velocity and the perturbative temperature can be written as

$$(\mathbf{l}^2 - \frac{d^2}{dz^2})^2\hat{u}_{3\mathbf{l}} = Ra\,\mathbf{l}^2\hat{\theta}_{\mathbf{l}} + \mathbf{l}^2\hat{f}_{3\mathbf{l}} + il_1\frac{d}{dz}\hat{f}_{1\mathbf{l}} + il_2\frac{d}{dz}\hat{f}_{2\mathbf{l}}, \tag{70}$$

$$\hat{u}_{3\mathbf{l}}\Big|_{z=0,1} = 0, \frac{d}{dz}\hat{u}_{3\mathbf{l}}\Big|_{z=0,1} = 0, \hat{\theta}_{\mathbf{l}}\Big|_{z=0,1} = 0 \tag{71}$$

where $\mathbf{l}^2 = l_1^2 + l_2^2$ as usual.

Therefore, we have the following lemma which is a modification/generalization of the proposition in [16].

Lemma 2 *The following inequality holds for all* $\mathbf{l}$

$$Re\int_0^1 \frac{\hat{\theta}_{\mathbf{l}}\hat{u}_{3\mathbf{l}}^*}{z}\,dz \geq \frac{1}{Ra}\int_0^1 \frac{|\hat{u}_{3\mathbf{l}}|^2}{z^3}\,dz - \frac{3}{2Ra}(|\hat{f}_{3\mathbf{l}}|_{L^2}^2 + \frac{|\hat{f}_{1\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{l}}|_{L^2} + |\hat{f}_{2\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{l}}|_{L^2}}{\mathbf{l}^2}). \tag{72}$$

Proof of Lemma 2: We multiply (70) by $\zeta = \frac{\hat{u}_{3\mathbf{l}}^*}{z}$ (here $*$ denotes complex conjugation) and integrate over $[0,1]$, we deduce

$$\mathrm{Re}\int_0^1 \frac{\hat{\theta}_{\mathbf{l}}\hat{u}_{3\mathbf{l}}^*}{z}\,dz = \mathrm{Re}\int_0^1\{\frac{(\mathbf{l}^4\hat{u}_{3\mathbf{l}} - 2\mathbf{l}^2\hat{u}_{3\mathbf{l}}'' + \hat{u}_{3\mathbf{l}}'''')\,\hat{u}_{3\mathbf{l}}^*}{Ra\,\mathbf{l}^2\,z} - \frac{\hat{f}_{3\mathbf{l}}\hat{u}_{3\mathbf{l}}^*}{Raz} - \frac{il_1\hat{f}_{1\mathbf{l}}'\hat{u}_{3\mathbf{l}}^*}{\mathbf{l}^2Ra\,z} - \frac{il_2\hat{f}_{2\mathbf{l}}'\hat{u}_{3\mathbf{l}}^*}{\mathbf{l}^2Ra\,z}\}\,dz$$

$$
\begin{aligned}
\geq\ & \frac{\mathbf{l}^2}{Ra}|\frac{\hat{u}_{3\mathbf{l}}}{z^{1/2}}|_{L^2}^2 - \frac{2}{Ra}\mathrm{Re}\int_0^1 \frac{\hat{u}_{3\mathbf{l}}''\hat{u}_{3\mathbf{l}}^*}{z}\,dz + \frac{1}{Ra\mathbf{l}^2}\mathrm{Re}\int_0^1 \frac{\hat{u}_{3\mathbf{l}}''''\hat{u}_{3\mathbf{l}}^*}{z}\,dz \\
& -\frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} - \frac{1}{Ra|\mathbf{l}|}(|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2})(|\frac{\hat{f}_{1\mathbf{l}}}{z^{1/2}}|_{L^2} + |\frac{\hat{f}_{2\mathbf{l}}}{z^{1/2}}|_{L^2}) \\
=\ & \frac{\mathbf{l}^2}{Ra}|z^{1/2}\zeta|_{L^2}^2 + \frac{2}{Ra}|z^{1/2}\zeta'|_{L^2}^2 + \frac{1}{Ra\mathbf{l}^2}|z^{1/2}\zeta''|_{L^2}^2 \\
& -\frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} - \frac{1}{Ra|\mathbf{l}|}(|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2})(|\frac{\hat{f}_{1\mathbf{l}}}{z^{1/2}}|_{L^2} + |\frac{\hat{f}_{2\mathbf{l}}}{z^{1/2}}|_{L^2}) \\
\geq\ & \frac{2}{Ra}|z^{1/2}\zeta'|_{L^2}^2 + \frac{2}{Ra}|z^{1/2}\zeta|_{L^2}|z^{1/2}\zeta''|_{L^2} \\
& -\frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} - \frac{1}{Ra|\mathbf{l}|}(|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2})(|\frac{\hat{f}_{1\mathbf{l}}}{z^{1/2}}|_{L^2} + |\frac{\hat{f}_{2\mathbf{l}}}{z^{1/2}}|_{L^2}) \\
\geq\ & \frac{2}{Ra}|z^{1/2}\zeta'|_{L^2}^2 + \frac{2}{Ra}|\frac{\zeta}{z^{1/2}}|_{L^2}^2 \\
& -\frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} - \frac{1}{Ra|\mathbf{l}|}(|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2})(|\frac{\hat{f}_{1\mathbf{l}}}{z^{1/2}}|_{L^2} + |\frac{\hat{f}_{2\mathbf{l}}}{z^{1/2}}|_{L^2}) \\
=\ & \frac{2}{Ra}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}^2 \\
& -\frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} - \frac{1}{Ra|\mathbf{l}|}(|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2})(|\frac{\hat{f}_{1\mathbf{l}}}{z^{1/2}}|_{L^2} + |\frac{\hat{f}_{2\mathbf{l}}}{z^{1/2}}|_{L^2}) \\
\geq\ & \frac{2}{Ra}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}^2 - \frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} \\
& -\frac{1}{Ra|\mathbf{l}|}(|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2})(|\frac{\hat{f}_{1\mathbf{l}}}{z^{1/2}}|_{L^2} + |\frac{\hat{f}_{2\mathbf{l}}}{z^{1/2}}|_{L^2}) \\
\geq\ & \frac{2}{Ra}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}^2 - \frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} \\
& -\frac{2}{Ra|\mathbf{l}|}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}(|\hat{f}_{1\mathbf{l}}|_{L^2}^{1/2}|\frac{\hat{f}_{1\mathbf{l}}}{z}|_{L^2}^{1/2} + |\hat{f}_{2\mathbf{l}}|_{L^2}^{1/2}|\frac{\hat{f}_{2\mathbf{l}}}{z}|_{L^2}^{1/2}) \\
\geq\ & \frac{2}{Ra}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}^2 - \frac{1}{Ra}|\hat{f}_{3\mathbf{l}}|_{L^2}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2} \\
& -\frac{2}{Ra|\mathbf{l}|}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}(|\hat{f}_{1\mathbf{l}}|_{L^2}^{1/2}|\hat{f}_{1\mathbf{l}}'|_{L^2}^{1/2} + |\hat{f}_{2\mathbf{l}}|_{L^2}^{1/2}|\hat{f}_{2\mathbf{l}}'|_{L^2}^{1/2}) \\
\geq\ & \frac{1}{Ra}|\frac{\hat{u}_{3\mathbf{l}}'}{z^{1/2}}|_{L^2}^2 - \frac{3}{2Ra}(|\hat{f}_{3\mathbf{l}}|_{L^2}^2 + \frac{1}{|\mathbf{l}|^2}(|\hat{f}_{1\mathbf{l}}|_{L^2}|\hat{f}_{1\mathbf{l}}'|_{L^2} + |\hat{f}_{2\mathbf{l}}|_{L^2}|\hat{f}_{2\mathbf{l}}'|_{L^2}))
\end{aligned}
$$

$$\geq \frac{1}{Ra}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2}^2 - \frac{3}{2Ra}(|\hat{f}_{3\mathbf{l}}|_{L^2}^2 + \frac{1}{|\mathbf{l}|^2}(|\hat{f}_{1\mathbf{l}}|_{L^2}|\hat{f}'_{1\mathbf{l}}|_{L^2} + |\hat{f}_{2\mathbf{l}}|_{L^2}|\hat{f}'_{2\mathbf{l}}|_{L^2})) \quad (73)$$

where we have performed integration by parts, applied the following calculus lemmas from [16] (lemma 1: for $w(0) = w(1) = w'(0) = w'(1) = 0$,

$$-\mathrm{Re}\int_0^1 \frac{w'' w^*}{z} dz = \int_0^1 z\ |(\frac{w}{z})'|^2\ dz, \quad \mathrm{Re}\int_0^1 \frac{w'''' w*}{z}\ dz = \int_0^1 z\ |(\frac{w}{z})''|^2\ dz$$

and lemma 2: for $\zeta(0) = \zeta(1) = \zeta'(0) = \zeta'(1) = 0$,

$$|\frac{\zeta}{z^{1/2}}|_{L^2}^2 \leq |z^{1/2}\zeta|_{L^2}|z^{1/2}\zeta''|_{L^2}$$

) as well as Hardy's inequality (

$$\int_0^1 \frac{g^2}{z^2} \leq \int_0^1 g'^2,\ g(0) = 0$$

) and Cauchy Schwarz inequality.

This completes the proof of lemma 2.

Next we borrow an idea [16] and introduce the following (non-monotonic) background flow τ for $\delta \in (0, 1/2)$,

$$\tau(z) = \begin{cases} 1 - z/\delta, & 0 \leq z \leq \delta, \\ \frac{1}{2} + \lambda(\delta)\ln\frac{z}{1-z}, & \delta \leq z \leq 1-\delta, \\ (1-z)/\delta & 1-\delta \leq z \leq 1, \end{cases} \quad (74)$$

with

$$\lambda(\delta) = \frac{1}{2\ln\frac{1-\delta}{\delta}}. \quad (75)$$

Now in terms of the Fourier coefficients, we have

$$\begin{aligned} & Q^{(\tau)}(\theta) \\ = & \sum_{\mathbf{l}}(|\hat{\theta}'_{\mathbf{l}}|^2 + |\mathbf{l}|^2|\hat{\theta}_{\mathbf{l}}|^2 + \tau'(\hat{u}_{3\mathbf{l}}\hat{\theta}^*_{\mathbf{l}} + \hat{u}^*_{3\mathbf{l}}\hat{\theta}_{\mathbf{l}})) \\ = & \sum_{\mathbf{l}}(\underbrace{\int_0^{1/2}(|\hat{\theta}'_{\mathbf{l}}|^2 + \mathbf{l}^2|\hat{\theta}_{\mathbf{l}}|^2)dz + 2\lambda\int_0^1 \frac{\mathrm{Re}[\hat{\theta}_{\mathbf{l}}\hat{u}^*_{3\mathbf{l}}]}{z}dz - 2\int_0^\delta\left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)\mathrm{Re}[\hat{\theta}_{\mathbf{l}}\hat{u}^*_{3\mathbf{l}}]\ dz} \\ & \underbrace{+\int_{1/2}^1(|\hat{\theta}'_{\mathbf{l}}|^2 + \mathbf{l}^2|\hat{\theta}_{\mathbf{l}}|^2)dz + 2\lambda\int_0^1 \frac{\mathrm{Re}[\hat{\theta}_{\mathbf{l}}\hat{u}^*_{3\mathbf{l}}]}{1-z}dz - 2\int_{1-\delta}^1\left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)\mathrm{Re}[\hat{\theta}_{\mathbf{l}}\hat{u}^*_{3\mathbf{l}}]\ dz}\ ), \\ = & \sum_{\mathbf{l}}(Q^{(\tau)}_{\mathbf{l},lower} + Q^{(\tau)}_{\mathbf{l},upper}) \end{aligned} \quad (76)$$

where we have re-written the terms so that the stable stratification of $\tau(z)$ in the bulk may help to asymptotically dominate the negative contributions to $Q^{(\tau)}$ from the boundary layer.

It is easy to check

$$\begin{aligned}
Q^{(\tau)}_{\mathbf{l},lower} &= \int_0^{1/2}(|\hat{\theta}'_{\mathbf{l}}|^2+\mathbf{l}^2|\hat{\theta}_{\mathbf{l}}|^2)dz + 2\lambda\int_0^1 \frac{\mathrm{Re}[\hat{\theta}_{\mathbf{l}}\hat{u}^*_{3\mathbf{l}}]}{z}dz - 2\int_0^\delta \left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{1-z}\right)\,\mathrm{Re}[\hat{\theta}_{\mathbf{l}}\hat{u}^*_{3\mathbf{l}}]\,dz \\
&\geq \int_0^{1/2}|\hat{\theta}'_{\mathbf{l}}|^2\,dz + \frac{2\lambda}{Ra}\int_0^1 \frac{|\hat{u}_{3\mathbf{l}}|^2}{z^3}\,dz - 2\int_0^\delta\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{1-z}\right)|\hat{\theta}_{\mathbf{l}}|\;|\hat{u}_{3\mathbf{l}}|\,dz \\
&\quad -\frac{3\lambda}{Ra}(|\hat{f}_{3\mathbf{l}}|^2_{L^2}+\frac{|\hat{f}_{1\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{l}}|_{L^2}+|\hat{f}_{2\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{l}}|_{L^2}}{\mathbf{l}^2}) \\
&\geq \int_0^{1/2}|\hat{\theta}'_{\mathbf{l}}|^2\,dz + \frac{2\lambda}{Ra}\int_0^1 \frac{|\hat{u}_{3\mathbf{l}}|^2}{z^3}\,dz - 2(\int_0^\delta z^4\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{1-z}\right)^2dz)^{\frac{1}{2}}|\frac{\hat{\theta}_{\mathbf{l}}}{z^{1/2}}|_{L^\infty(0,\frac{1}{2})}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}| \\
&\quad -\frac{3\lambda}{Ra}(|\hat{f}_{3\mathbf{l}}|^2_{L^2}+\frac{|\hat{f}_{1\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{l}}|_{L^2}+|\hat{f}_{2\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{l}}|_{L^2}}{\mathbf{l}^2}) \\
&\geq \int_0^{1/2}|\hat{\theta}'_{\mathbf{l}}|^2\,dz + \frac{2\lambda}{Ra}\int_0^1 \frac{|\hat{u}_{3\mathbf{l}}|^2}{z^3}\,dz - 2(\int_0^\delta z^4\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{1-z}\right)^2dz)^{\frac{1}{2}}|\hat{\theta}'_{\mathbf{l}}|_{L^2(0,\frac{1}{2})}|\frac{\hat{u}_{3\mathbf{l}}}{z^{3/2}}|_{L^2} \\
&\quad -\frac{3\lambda}{Ra}(|\hat{f}_{3\mathbf{l}}|^2_{L^2}+\frac{|\hat{f}_{1\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{l}}|_{L^2}+|\hat{f}_{2\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{l}}|_{L^2}}{\mathbf{l}^2}) \\
&\geq \left[\frac{2\lambda}{Ra}-\int_0^\delta z^4\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{1-z}\right)^2dz\right]\times\int_0^1\frac{|\hat{u}_{3\mathbf{l}}|^2}{z^3}\,dz \\
&\quad -\frac{3\lambda}{Ra}(|\hat{f}_{3\mathbf{l}}|^2_{L^2}+\frac{|\hat{f}_{1\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{l}}|_{L^2}+|\hat{f}_{2\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{l}}|_{L^2}}{\mathbf{l}^2})
\end{aligned} \quad (7$$

where we have used lemma 2 and Cauchy-Schwarz inequality as well as the one dimensional Sobolev inequality (calculus inequality) $|\frac{\hat{\theta}_{\mathbf{l}}}{z^{1/2}}|_{L^\infty(0,\frac{1}{2})} \leq |\hat{\theta}'_{\mathbf{l}}|_{L^2(0,\frac{1}{2})}$.

Noting that

$$\int_0^\delta z^4\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{1-z}\right)^2dz \;=\; \frac{\delta^3}{5}\;\times\left\{1+\mathcal{O}\left(\frac{1}{|\ln\delta|}\right)\right\}\quad\text{as }\delta\to 0, \quad (78)$$

a sufficient asymptotic condition for the non-negativity of the principle part (the part not involving $\mathbf{f}$) of $Q^{(\tau)}_{lower}$ (and also $Q^{(\tau)}_{upper}$ and hence $Q^{(\tau)})$[3] is

$$Ra\,\delta^3 \;=\; 10\,\lambda \;=\; \frac{5}{\ln\frac{1-\delta}{\delta}}. \tag{79}$$

This is satisfied asymptotically by

$$\delta \;\sim\; \left(\frac{15}{Ra\;\ln Ra}\right)^{1/3}, \quad \lambda \;\sim\; 15\ln Ra \;\;\text{as } Ra \to \infty. \tag{80}$$

Inserting this into the previous upper bound for the Nusselt number we arrive at

$$\begin{aligned} Nu_\varepsilon \;&\le\; \int_0^1 (\tau')^2\,dz + \frac{6\lambda}{Ra}\sum_{\mathbf{l}} < |\hat{f}_{3\mathbf{l}}|^2_{L^2} + \frac{|\hat{f}_{1\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{l}}|_{L^2} + |\hat{f}_{2\mathbf{l}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{l}}|_{L^2}}{\mathbf{l}^2} > \\ &\sim\; \frac{2}{\delta} + \frac{6\lambda}{Ra} < |\mathbf{f}|_{L^2}|\mathbf{f}|_{H^1} > \\ &\sim\; 2\times\left(\frac{Ra\;\ln Ra}{15}\right)^{1/3} + \frac{6\lambda}{Ra} < |\mathbf{f}|_{L^2}|\mathbf{f}|_{H^1} > \end{aligned} \tag{81}$$

It is then an easy exercise to check, thanks to the a priori estimates on $\mathbf{u}$,

$$< |\mathbf{f}|_{L^2}|\mathbf{f}|_{H^1} > \le c\varepsilon^2 Ra^{\frac{9}{2}}. \tag{82}$$

Combining the estimates above, and the uniform bound of $Ra^{1/2}$ [8], we have the following result

Theorem 3 *There exists constant c independent of Ra and Pr such that*

$$Nu_\varepsilon \le Ra^{\frac{1}{3}}(\ln Ra)^{\frac{1}{3}} + c\frac{Ra^{\frac{7}{2}}\ln Ra}{Pr^2}, \tag{83}$$

and

$$Nu_\varepsilon \le \min\{Ra^{\frac{1}{3}}(\ln Ra)^{\frac{1}{3}} + c\frac{Ra^{\frac{7}{2}}\ln Ra}{Pr^2}, Ra^{\frac{1}{2}}\}. \tag{84}$$

The upper bound above fits the common believe that the Nusselt number at large Rayleigh number should be eventually independent of the Prandtl

[3]Note here we deviate from the Constantin-Doering approach in the sense that the spectral constraint is not enforced exactly, but only asymptotically (modulus the part involving $\mathbf{f}$).

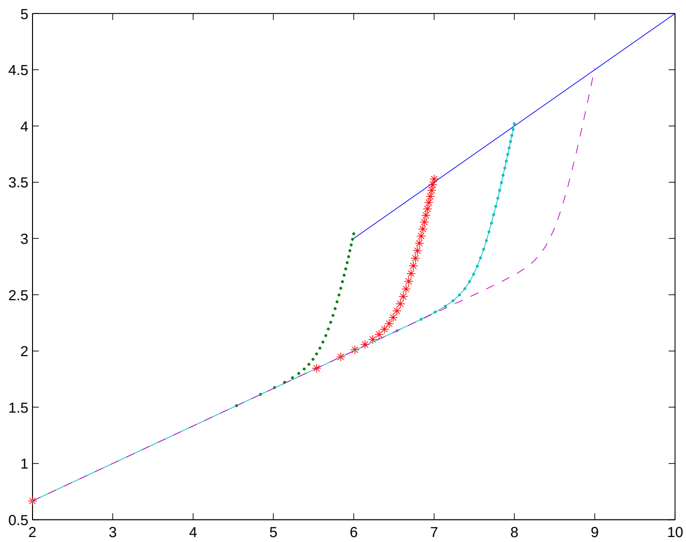

Figure 1: Schematic log-log plot of the new upper bound on the Nusselt number (84) versus Rayleigh number for different values of Prandtl number

number and the Nusselt number should scale like $Ra^{1/3}$ for large Rayleigh number. In fact there are even evidence of uniform in Prandtl number scaling of $Ra^{\frac{1}{3}}$ for the Nusselt number [1]. However, the correction term here is not very satisfactory (of the order of $Ra^{7/2}/Pr^2$) which grows faster (for fixed Prandtl number) than the known uniform in Prandtl number bound of $Ra^{1/2}$ [8] at large Rayleigh number. Although the correction term can be improved by refining the estimates from [40], we are not able to derive a bound that is consistent with the uniform $Ra^{1/2}$ yet [4].

5 Concluding Remarks

We have demonstrated that the infinite Prandtl number model is a good effective model for the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number in terms of stationary statistical properties.

More specifically we have established the upper semi-continuity of the set of invariant measures for the Boussinesq system as the Prandtl number ap-

[4] Here a consistent correction term means a term of the form Ra^α/Pr^β, with $\alpha \le 1/2$ and $\beta > 0$

proaches infinity (with the limit being the infinite Prandtl number model). Therefore, equilibrium statistics of the Boussinesq system can be asymptotically dominated by equilibrium statistics of the infinite Prandtl number model. This complements our result on the upper semi-continuity of the global attractors [40]. We are not able to show the continuity at this point since the set of invariant measures may contain multiple elements and we may experience hysteresis type phenomena. One way to obtain continuity is by adding appropriate additive white noise as these white noise will connect different branches of the attractor and render the uniqueness of invariant measure [17, 13]. Uniqueness of invariant measure (at any fixed Prandtl number) leads to the continuity in the Prandtl number (including the singular limit of $Pr \to \infty, \varepsilon \to 0$) of invariant measure. The noise may be justified as accounting for neglected small effects of various physical mechanisms not represented in the system.

We have also established the upper semi-continuity of the Nusselt number as the Prandtl number approaches infinity. This implies that the Nusselt number for the infinite Prandtl number model asymptotically bounds the Nusselt number for the Boussinesq system at large Prandtl number. This is not a direct consequence of the upper semi-continuity of the set of invariant measures since the limit of the sequence of invariant measures corresponding to the Nusselt numbers for the Boussinesq system as the Prandtl number approaches infinity may not be an invariant measure of the infinite Prandtl number model corresponding to the Nusselt number of the limit system. Again, we do not have continuity of the Nusselt number. Yet, we strongly believe that continuity is true at large Rayleigh number since we expect a unique strongly mixing trajectory/invariant measure that saturates the Nusselt number at large Rayleigh number. Of course adding appropriate noise leads to uniqueness of invariant measure which further leads to the continuity of Nusselt numbers in terms of dependence on the Prandtl number. A byproduct that we derived here is that the Nusselt numbers are saturated by ergodic invariant measures.

A more concrete result that we obtained here is an upper bound on the Nusselt number for the Boussinesq system of the form $Ra^{\frac{1}{3}}(\ln Ra)^{\frac{1}{3}} + c\frac{Ra^{\frac{7}{2}}\ln Ra}{Pr^2}$. This bound asymptotically agrees with the optimal bound for the Nusselt number of the infinite Prandtl number model ($Ra^{\frac{1}{3}}$) to the leading order at large Prandtl number. This is the first result of this kind.

Finally we remark that the results/tchniques derived here may be applied to many other systems with two disparate time scales of relaxation type.

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