MACPHERSON'S AND FULTON'S CHERN CLASSES OF HYPERSURFACES

PAOLO ALUFFI

1. Introduction

In this note we compare two notions of Chern class of an algebraic scheme X (over $\mathbb{C}$) specializing to the Chern class of the tangent bundle $c(TX) \cap [X]$ when X is nonsingular. The first of such notions is MacPherson's Chern class, defined by means of Mather-Chern classes and local Euler obstructions [5]. MacPherson's Chern class is functorial with respect to a push-forward defined via topological Euler characteristics of fibers; in particular, mapping to a point shows that the degree of the zero-dimensional component of MacPherson's Chern class of a complete variety X equals the Euler characteristic $\chi(X)$ of X. We denote MacPherson's Chern class of X by $c_{MP}(X)$. The second notion is Fulton's intrinsic class of schemes X' that can be embedded in a nonsingular variety M: Fulton shows ([3], Example 4.2.6) that the class

$$c_F(X') = c(TM) \cap s(X', M)$$

is independent of the choice of embedding of X'. This class has the advantage of being defined over arbitrary fields and in a completely algebraic fashion, but does not satisfy at first sight nice functorial properties: cf. [3], p. 377. (MacPherson's class can also be defined algebraically over any field of characteristic 0: this is done in [4].)

To state our result we need to remind the reader that if W is a scheme supported on a Cartier divisor X of a nonsingular variety M, then the Segre class of W in M can be written in terms of the Segre class of X and the Segre class of the residual scheme J to X in W: for a precise statement of this fact, see [3], Proposition 9.2, or section 2 below. By modifying this expression, we can make sense of the "Segre class" in M of an object "$X \setminus J$" in which J is intuitively speaking "removed" from X. Since this object has a Segre class, we can define its Fulton–Chern class as above. Here is our result:

Theorem 1. *Let X be a section of a very ample line bundle on a nonsingular complex variety M, and let J be its singular subscheme. Then*

$$c_{MP}(X) \doteq c_F(X \setminus J)$$

Here $\doteq$ means that the classes equal after push-forward via the map to a projective space determined by $\mathcal{L} = \mathcal{O}(\mathcal{X})$. We strongly suspect that the classes are actually equal in the Chow group of X, and that the hypothesis on $\mathcal{L}$ is unnecessary (in fact, our proof works whenever $\mathcal{L}$ is globally generated and the corresponding map to projective space is gen. finite); and that a suitable generalization should hold for

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arbitrary schemes over an algebraically closed field; but the methods we use in this note can only go so far. On the other hand, our proof of this theorem is remarkably simple (once granted the results of [2]), and is enough for example to imply:

Corollary 1. *Under the hypotheses of the theorem,*

$$\chi(X) = \int c_F(X \setminus J)$$

where $\int$ denotes degree.

The statement of theorem 1 is philosophically satisfying in that it highlights precisely in what c_{MP} and c_F must differ: Fulton's class equals MacPherson's after the scheme is 'corrected' for the presence of singularities. At the moment we take this corrected "$X \setminus J$" purely as a formal object, although we wonder whether a more concrete geometric meaning can be attached to it (after all this object has a well-defined Chern class!)

Section 2 in the paper defines $c_F(X \setminus J)$ precisely, and introduces notations that we found helpful in these computations. The proof of the theorem is in section 3, and a simple example illustrating the result is in section 4.

2. $c_F(X \setminus J)$

Let X be a Cartier divisor of a nonsingular proper n-dimensional variety M (over an algebraically closed field), and let J be a subscheme of M whose support is contained in X. Our task in this section is to define a class $c_F(X \setminus J)$ in the Chow group $A_*(X)$ of X. This class can be written explicitly in terms of the Segre classes of X and J in M:

$$c_F(X \setminus J) = c(TM) \cap s(X \setminus J, M) \quad ,$$

where the term of dimension m of $s(X \setminus J, M)$ is defined to be

$$s(X \setminus J, M)_m = s(X, M)_m + (-1)^{n-m} \sum_{j=0}^{n-m} \binom{n-m}{j} X^j \cdot s(J, M)_{m+j} \quad .$$

However, we feel we should motivate this definition; in doing so we will also introduce notations that will be useful in §3.

Let $\mathcal{I}$, $\mathcal{J}$ be respectively the ideal sheaves of X and J in M. For any nonnegative integer t we may consider the subscheme $W(t)$ of M with ideal sheaf $\mathcal{I}\mathcal{J}^{\sqcup}$: that is, $W(t)$ is a subscheme of M containing X and such that the residual scheme to X in $W(t)$ is the subscheme with ideal sheaf $\mathcal{J}^{\sqcup}$.

Definition 1. *For t a nonnegative integer, define*

$$p(X, J, t) = c_F(W(t))$$

where c_F denotes Fulton's intrinsic class (cf. section 1).

Lemma 1. *$p(X, J, t)$ is a polynomial in t (with coefficients in $A_*(X)$).*

The constant term of this polynomial will be

$$c_F(X) = p(X, J, 0) \quad ,$$

Fulton's Chern class of X. Given lemma 1, we can define

$$c_F(X \setminus J) = p(X, J, -1) \quad :$$

intuitively, just as $p(X, J, t)$ evaluates (for $t \geq 0$) Fulton's Chern class of a scheme supported on X and with an embedded component along J 'counted t times', this $c_F(X \setminus J)$ should stand for Fulton's Chern class of an object obtained by 'removing' J from X. Of course the notation $X \setminus J$ is not to be intended set-theoretically; we do not know how to interpret this object 'geometrically'.

Lemma 1 follows immediately from writing the class explicitly in terms of the Segre classes of X and J in M: for this we could just quote [3], Proposition 9.2. We prefer to introduce some notations which work as a good shorthand in writing and manipulating formulas such as the raw expression for $c_F(X \setminus J)$ given above; these notations will also save us some time in section 3. For completeness, we will rewrite and prove Proposition 9.2 from [3] in terms of these notations.

Suppose A is a rational equivalence class on a scheme S, and write $A = a^0 + a^1 + \ldots$ with $a^i \in A^i S$ (that is, the a^i are indexed by codimension).

Definition 2. *(1) The 'dual' of A, denoted $A^\vee$, is the class defined by*

$$A^\vee = \sum_{i \geq 0} (-1)^i a^i \quad .$$

(2) More generally, the 'd-th Adams' of A, denoted $A^{(d)}$, is the class defined by

$$A^{(d)} = \sum_{i \geq 0} d^i a^i \quad .$$

(3) For a line bundle $\mathcal{L}$ on S, the 'tensor of A by $\mathcal{L}$', denoted $A \otimes \mathcal{L}$, is the class defined by

$$A \otimes \mathcal{L} = \sum_{\rangle \geq \imath} \frac{\dashv^\rangle}{\rfloor(\mathcal{L})^\rangle} \quad .$$

It is clear that the operations introduced in definition 2 are linear in A; further, these definitions are compatible with corresponding vector bundle operations. For a start, it is clear that if $\mathcal{E}$ is a vector bundle on S, then

$$(c(\mathcal{E}^\vee) \cap \mathcal{A}) = (c(\mathcal{E}) \cap \mathcal{A}^\vee)^\vee \quad ;$$

$(A \otimes \mathcal{L})^\vee = \mathcal{A}^\vee \otimes \mathcal{L}^\vee$ should be equally clear from the definitions.

Next, there are compatibilities with tensoring after capping with Chern classes:

Proposition 1. *If $\mathcal{E}$ is a rank-r vector bundle on S, then*

$$\Big(c(\mathcal{E}) \cap \mathcal{A}\Big) \otimes \mathcal{L} = \frac{\infty}{\rfloor(\mathcal{L})^\nabla} \rfloor(\mathcal{E} \otimes \mathcal{L}) \cap (\mathcal{A} \otimes \mathcal{L})$$

and

$$\Big(c(\mathcal{E})^{-\infty} \cap \mathcal{A}\Big) \otimes \mathcal{L} = \rfloor(\mathcal{L})^\nabla \rfloor(\mathcal{E} \otimes \mathcal{L})^{-\infty} \cap (\mathcal{A} \otimes \mathcal{L})$$

Proof. For the first formula, we may assume by linearity that $A = a^j$. If $c_i = c_i(\mathcal{E})$, we have

$$\big(c(\mathcal{E})\cap\mathcal{A}\big)\otimes\mathcal{L} = \left(\sum_i c_i\cap a^j\right)\otimes\mathcal{L} = \sum_\rangle \frac{\rfloor_\rangle\cap\dashv^|}{\rfloor(\mathcal{L})^{\rangle+|}} = \sum_\rangle \frac{\rfloor_\rangle}{\rfloor(\mathcal{L})^\rangle}\cap\frac{\dashv^|}{\rfloor(\mathcal{L})^|}$$
$$= \frac{1}{c(\mathcal{L})^\nabla}c(\mathcal{E}\otimes\mathcal{L})\cap(\mathcal{A}\otimes\mathcal{L})$$

for example by [3], Remark 3.2.3 (b)).

For the second formula, simply replace A by $c(\mathcal{E})^{-\infty}\cap\mathcal{A}$ in the first. □

Also, the notation is fully compatible with tensoring with line bundles:

Proposition 2. *If $\mathcal{M}$ is another line bundle on S, then*

$$(A\otimes\mathcal{L})\otimes\mathcal{M} = \mathcal{A}\otimes(\mathcal{L}\otimes\mathcal{M})$$

Proof. By linearity we may assume $A = a^j$. Also, let $\ell = c_1(\mathcal{L}), \Updownarrow = \rfloor_\infty(\mathcal{M})$; then we have

$$(A\otimes\mathcal{L})\otimes\mathcal{M} = \frac{a^j}{(1+\ell)^j}\otimes\mathcal{M} = \left(\sum_\rangle \binom{\rangle+|-\infty}{\rangle}(-\infty)^\rangle\ell^\rangle\cap\dashv^|\right)\otimes\mathcal{M}$$
$$= \sum_i\binom{i+j-1}{i}(-1)^i\frac{\ell^i\cap a^j}{(1+m)^{i+j}}$$
$$= \left(\sum_i\binom{i+j-1}{i}(-1)^i\frac{\ell^i}{(1+m)^i}\right)\cap\frac{a^j}{(1+m)^j}$$
$$= \frac{1}{(1+\frac{\ell}{1+m})^j}\cap\frac{a^j}{(1+m)^j} = \frac{a^j}{(1+\ell+m)^j}$$
$$= A\otimes(\mathcal{L}\otimes\mathcal{M})$$

as needed. □

Also, it is clear from the definition that if $S_1 \xrightarrow{\pi} S_2$ is a proper map, A is a class on S_1, and $\mathcal{L}$ is a line bundle on S_2, then

$$\pi_*(A\otimes\pi^*\mathcal{L}) = \rfloor(\mathcal{L})^{\dim\mathcal{S}_\in-\dim\mathcal{S}_\infty}\left((\pi_*\mathcal{A})\otimes\mathcal{L}\right)$$

Finally, note that if D is a Cartier divisor on S, then the Segre class of D in S can be written in terms of $\otimes$:

$$s(D,V) = \frac{[D]}{1+D} = [D]\otimes\mathcal{O}(\mathcal{D})$$

(we are abusing notations a little here: the $\otimes$ is taken in S, while the result is a class on D.) And note that if J is defined by the ideal $\mathcal{J}$ in S, and $J^{(d)}$ denotes the subscheme defined by $\mathcal{J}^\lceil$, then the segre class of $J^{(d)}$ in S is the d-th Adams of $s(J,S)$.

Here is a restatement of Proposition 9.2 from [3] in terms of our notations:

Proposition 3. *Let $X \subset W \subset M$ be closed embeddings, with X a Cartier divisor on M. Let J be the residual scheme to X in W, and $\mathcal{L} = \mathcal{O}(\mathcal{X})$. Then*

$$s(W,M) = s(X,M) + c(\mathcal{L})^{-\infty} \cap (\int(\mathcal{J},\mathcal{M}) \otimes \mathcal{L})$$

And here is the standard argument, written in our notations:

Proof. If $W = M$, the statement amounts to the definition of $s(X,M)$.

If $W \neq M$, let $\pi : \widetilde{M} \to M$ be the blow-up of M along J, and let $\widetilde{W} = \pi^{-1}(W)$, $\tilde{J} = \pi^{-1}(J)$ and $\widetilde{X} = \pi^{-1}(X)$: then $\widetilde{W} = \widetilde{X} + \tilde{J}$ as Cartier divisors on $\widetilde{M}$. Let η be the induced morphism from $\widetilde{W}$ to W. By the birational invariance of Segre classes and the remarks preceding the statement:

$$s(W,M) = \eta_* s(\widetilde{W},\widetilde{M}) = \eta_* \left(([\widetilde{X}] + [\tilde{J}]) \otimes \mathcal{O}(\widetilde{\mathcal{X}} + \widetilde{\mathcal{J}})\right)$$

Letting $\tilde{\mathcal{L}} = \mathcal{O}(\widetilde{\mathcal{X}}) = \pi^*\mathcal{L}$ and $\widetilde{\mathcal{R}} = \mathcal{O}(\widetilde{\mathcal{J}})$, and applying propositions 1 and 2,

$$\begin{aligned}
([\widetilde{X}] + [\tilde{J}]) \otimes \mathcal{O}(\widetilde{\mathcal{X}} + \widetilde{\mathcal{J}}) &= ([\widetilde{\mathcal{X}}] \otimes \widetilde{\mathcal{R}} + [\widetilde{\mathcal{J}}] \otimes \widetilde{\mathcal{R}}) \otimes \tilde{\mathcal{L}} \\
&= (c(\widetilde{\mathcal{R}})^{-1} \cap [\widetilde{X}] + s(\tilde{J},\widetilde{M})) \otimes \tilde{\mathcal{L}} \\
&= ([\widetilde{X}] - \widetilde{X} \cdot s(\tilde{J},\widetilde{M}) + s(\tilde{J},\widetilde{M})) \otimes \tilde{\mathcal{L}} \\
&= s(\widetilde{X},\widetilde{M}) + (c(\tilde{\mathcal{L}}^\vee) \cap s(\tilde{J},\widetilde{M})) \otimes \tilde{\mathcal{L}} \\
&= s(\widetilde{X},\widetilde{M}) + c(\tilde{\mathcal{L}})^{-1} \cap (s(\tilde{J},\widetilde{M}) \otimes \tilde{\mathcal{L}}) \quad .
\end{aligned}$$

Pushing forward by η gives the statement. □

Proposition 3 yields an explicit expression for $p(X,J,t)$: we have already observed that the Segre class of the scheme $J^{(t)}$ defined by $\mathcal{J}^{\sqcup}$ is $s(J,M)^{(t)}$, so

$$s(W(t),M) = s(X,M) + c(\mathcal{L})^{-\infty} \cap (\int(\mathcal{J},\mathcal{M})^{(\sqcup)} \otimes \mathcal{L}) \quad ,$$

and $p(X,J,t)$ equals the class $c_F(W(t),M) = c(TM) \cap s(W(t),M)$. In particular, $p(X,J,t)$ is a polynomial over $A_*(X)$, as claimed in lemma 1, since $s(J,M)^{(t)}$ is.

We can now again write $c_F(X \setminus J)$ explicitly; our hope is that at this point this definition will look more insightful than the (equivalent) expression given at the beginning of this section:

Definition 3. *We set $c_F(X \setminus J) = p(X,J,-1)$, that is*

$$c_F(X \setminus J) = c(TM) \cap \left(s(X,M) + c(\mathcal{L})^{-\infty} \cap (\int(\mathcal{J},\mathcal{M})^\vee \otimes \mathcal{L})\right) \quad .$$

Our goal in this note is to show that if we work over $\mathbb{C}$ and choose J to be the *singular subscheme* of X, then this class agrees with MacPherson's Chern class of X after push-forward by the map defined by $\mathcal{L}$. This is done in the next section.

3. Proof of theorem 1

The statement again: if X is a hypersurface of a nonsingular variety M, and J is its singular subscheme (that is: if F is a local equation of X and $x_1,\dots,x_n$ are local parameters on M, J is the subscheme defined locally by the ideal $(\frac{\partial F}{\partial x_1},\dots,\frac{\partial F}{\partial x_n})$), then

$$c_{MP}(X) \doteq c_F(X \setminus J)$$

where $c_{MP}(X)$ is MacPherson's Chern class of X, $c_F(X \setminus J)$ was defined in section 2, and $\doteq$ denotes equality after push-forward by the map defined by the linear system $|X|$, which we are assuming to be very ample.

In other words, we have to check that for all $j \geq 0$:

$$\int c_1(\mathcal{L})^{|} \cap \rfloor_{\mathcal{MP}}(\mathcal{X}) = \int \rfloor_\infty(\mathcal{L})^{|} \cap \rfloor_{\mathcal{F}}(\mathcal{X} \setminus \mathcal{J})$$

where $\mathcal{L} = \mathcal{O}(\mathcal{X})$.

Our tool will be the μ-class of J with respect to $\mathcal{L}$, introduced in [2]: this is the class

$$\mu_{\mathcal{L}}(J) = c(T^\vee M \otimes \mathcal{L}) \cap \int(\mathcal{J}, \mathcal{M}) \quad ,$$

where $T^\vee M$ denotes the cotangent bundle of M.

Lemma 2. *For all $j \geq 0$, and letting $n = \dim M$:*

$$\int c(\mathcal{L})^{-|} \rfloor_\infty(\mathcal{L})^{|} \cap (\rfloor_{\mathcal{MP}}(\mathcal{X}) - \rfloor_{\mathcal{F}}(\mathcal{X})) = (-\infty)^{\backslash - |} \int \rfloor_\infty(\mathcal{L})^{|} \cap \mu_{\mathcal{L}}(\mathcal{J})$$

Proof. For $j \geq 0$, let M_j denote the intersection of j general sections of $\mathcal{L}$ (with $M_0 = M$), and let $X_j = M_j \cap X$. By Bertini's theorem the M_j are all non-singular; X_j are hypersurfaces of M_j, of class $\mathcal{L} = \mathcal{L}|_{\mathcal{M}_|}$. We also let J_j be the singular subschemes of the X_j.

Claim 1.

$$c_{MP}(X_j) = c_1(\mathcal{L})^{|} \cap \left(\rfloor(\mathcal{L})^{-|} \cap \rfloor_{\mathcal{MP}}(\mathcal{X}) \right) \tag{1}$$

$$c_F(X_j) = c_1(\mathcal{L})^{|} \cap \left(\rfloor(\mathcal{L})^{-|} \cap \rfloor_{\mathcal{F}}(\mathcal{X}) \right) \tag{2}$$

$$\mu_{\mathcal{L}}(J_j) = c_1(\mathcal{L})^{|} \cap \mu_{\mathcal{L}}(\mathcal{J}) \tag{3}$$

(here and elsewhere we omit writing push-forwards implied by the context).

(1) follows from the compatibility of Nash blowups and Euler obstructions with general sections, cf. for example [7], Lemmas 2.1 and 2.3.

For (2), $c_F(X_j) = c(TM_j) \cap s(X_j, M_j)$ by definition. Now M_j is embedded in M with normal bundle $\mathcal{L}^{\oplus |}$, so $c(TM_j) = c(\mathcal{L})^{-|} \rfloor(\mathcal{TM})$; and $s(X_j, M_j) = c_1(\mathcal{L})^{|} \cap \int(\mathcal{X}, \mathcal{M})$ by repeated applications of Lemma A.3 from [1].

As for (3), this follows from Proposition 1.3 in [2].

Putting (1), (2) and (3) together we see that proving the statement of the lemma amounts to showing that

$$\int c_{MP}(X_j) - c_F(X_j) = (-1)^{n-j} \int \mu_{\mathcal{L}}(J_j)$$

for all $j \geq 0$. Now recall that $\int c_{MP}(X_j)$ equals the topological Euler characteristic of X_j; while $\int c_F(X_j)$ equals

$$\int c(TM_j) \cap s(X_j, M_j) = \int c(TM_j) c(\mathcal{L})^{-\infty} \cap [\mathcal{X}_|] = \int \rfloor(\mathcal{TM}_|) \rfloor(\mathcal{L})^{-\infty} \cap [\mathcal{M}_{|+\infty}]$$

since $[X_j] = [M_{j+1}]$ as divisors in M_j; since $c(TM_j)c(\mathcal{L})^{-\infty} = \rfloor(\mathcal{TM}_{|+\infty})$, we see that $\int c_F(X_j)$ equals the topological Euler characteristic of M_{j+1}, that is of the general section of $\mathcal{L}$ in M_j.

So the left-hand-side of the formula equals the difference

$$\chi(X_j) - \chi(M_{j+1})$$

of the Euler characteristics of the special section X_j and the general section M_{j+1} of $\mathcal{L}$ on M_j. In [6], Corollary 1.7, Parusiǹski proves that this equals $(-1)^{\dim M_j}\mu(M_j, X_j)$, where $\mu(M_j, X_j)$ is his generalization to non-isolated singularities of the Milnor number. But this latter equals $\int \mu_{\mathcal{L}}(J_j)$ by Proposition 2.1 in [2], so the above formula holds. $\square$

Next we use lemma 2 to obtain the class of $c_{MP}(X) - c_F(X)$ (more precisely, of its push-forward by the map defined by $\mathcal{L}$); the result is best expressed in terms of the notations introduced in definition 2:

Lemma 3.

$$c_{MP}(X) - c_F(X) \doteq c(\mathcal{L})^{\backslash-\infty} \cap (\mu_{\mathcal{L}}(\mathcal{J})^\vee \otimes \mathcal{L})$$

Proof. If A is a class on M, and $a_{n-j} \in \mathbb{Q}$ denotes

$$\frac{\int c_1(\mathcal{L})^| \cap \mathcal{A}}{\int \rfloor_\infty(\mathcal{L})^\backslash \cap [\mathcal{M}]} \quad ,$$

then

$$A \doteq \sum_{i\geq 0} a_i c_1(\mathcal{L})^\rangle \cap [\mathcal{M}] \quad .$$

We let then $\ell^i = c_1(\mathcal{L})^\rangle \cap [\mathcal{M}]$, and write

$$\begin{gathered} c_{MP}(X) - c_F(X) \doteq A = a_0 + a_1\ell + a_2\ell^2 + \dots \\ \mu_{\mathcal{L}}(J) \doteq B = b_0 + b_1\ell + b_2\ell^2 + \dots \end{gathered}$$

Lemma 2 then can be restated as:

$$\begin{aligned} b_i &= (-1)^i \cdot \text{ coefficient of } \ell^i \text{ in } \frac{a_0 + a_1\ell + a_1\ell^2 + \dots}{(1+\ell)^{n-i}} \\ &= (-1)^i \sum_{k=0}^{i} \binom{n-k-1}{i-k} (-1)^{i-k} a_k \quad , \end{aligned}$$

so we have

$$\begin{aligned}
B &= \sum_{i=0}^{n}(-1)^i \sum_{k=0}^{i}\binom{n-k-1}{i-k}(-1)^{i-k}a_k\ell^i\\
&= \sum_{k\geq 0}(-1)^k\left(\sum_{i=k}^{n}\binom{n-k-1}{i-k}\ell^i\right)a_k\\
&= \sum_{k\geq 0}(-1)^k\left(\sum_{j=0}^{n-k}\binom{n-k-1}{j}\ell^{j+k}\right)a_k\\
&= \sum_{k\geq 0}(-1)^k(1+\ell)^{n-k-1}a_k\ell^k\\
&= (1+\ell)^{n-1}\sum_{k\geq 0}\frac{(-1)^k a_k\ell^k}{(1+\ell)^k}\\
&= c(\mathcal{L})^{\backslash-\infty}\cap(\mathcal{A}^\vee\otimes\mathcal{L}) \quad .
\end{aligned}$$

To get the statement of the lemma, we just need to "solve this for A": start from

$$c(\mathcal{L})^{\backslash-\infty}\cap(\mathcal{A}^\vee\otimes\mathcal{L}) = \mathcal{B} \quad ;$$

cap by $c(\mathcal{L})^{-(\backslash-\infty)}$:

$$A^\vee\otimes\mathcal{L} = \rfloor(\mathcal{L})^{-(\backslash-\infty)}\cap\mathcal{B} \quad ;$$

tensor by $\mathcal{L}^\vee$ and apply propositions 1 and 2:

$$\begin{aligned}
A^\vee &= (c(\mathcal{L})^{-(\backslash-\infty)}\cap\mathcal{B})\otimes\mathcal{L}^\vee = \rfloor(\mathcal{L}^\vee)^{\backslash-\infty}\cap(\rfloor(\mathcal{L}\otimes\mathcal{L}^\vee)^{-(\backslash-\infty)}\cap(\mathcal{B}\otimes\mathcal{L}^\vee))\\
&= c(\mathcal{L}^\vee)^{n-1}\cap(B\otimes\mathcal{L}^\vee)
\end{aligned}$$

Taking duals gives the statement. □

Theorem 1 follows now easily from the last lemma:

$$c_{MP}(X) \doteq c_F(X) + c(\mathcal{L})^{\backslash-\infty}\cap(\mu_{\mathcal{L}}(\mathcal{J})^\vee\otimes\mathcal{L})$$

by lemma 3; expanding the right-hand-side gives:

$$\begin{aligned}
&c(TM)\cap s(X,M) + c(\mathcal{L})^{\backslash-\infty}\cap((\rfloor(\mathcal{T}^\vee\mathcal{M}\otimes\mathcal{L})\cap\int(\mathcal{J},\mathcal{M}))^\vee\otimes\mathcal{L})\\
&\qquad = c(TM)\cap s(X,M) + c(\mathcal{L})^{\backslash-\infty}\cap((\rfloor(\mathcal{T}\mathcal{M}\otimes\mathcal{L}^\vee)\cap\int(\mathcal{J},\mathcal{M})^\vee)\otimes\mathcal{L})\\
&\qquad = c(TM)\cap s(X,M) + c(\mathcal{L})^{-\infty}\rfloor(\mathcal{T}\mathcal{M})\cap(\int(\mathcal{J},\mathcal{M})^\vee\otimes\mathcal{L})
\end{aligned}$$

by proposition 1,

$$\begin{aligned}
&= c(TM)\cap\Big(s(X,M) + c(\mathcal{L})^{-\infty}\cap(\int(\mathcal{J},\mathcal{M})^\vee\otimes\mathcal{L})\Big)\\
&= c_F(X\setminus J)
\end{aligned}$$

by the expression obtained in section 2. This concludes the proof of theorem 1.

4. Example

We conclude with an explicit computation illustrating the result. Let X be a surface in $M = \mathbb{P}^{\nvDash}$, with ordinary singularities: the singular locus is a curve Y, and X has a certain number τ of triple points and a number ν of pinch points along Y. More precisely, we assume that the completion of the local ring of X is isomorphic to:

$$\frac{\mathbb{C}[[\curvearrowleft,\curvearrowright,F]]}{(\curvearrowleft\curvearrowright)} \qquad \text{at a general point of } Y$$

$$\frac{\mathbb{C}[[\curvearrowleft,\curvearrowright,F]]}{(\curvearrowleft\curvearrowright F)} \qquad \text{at a triple point}$$

$$\frac{\mathbb{C}[[\curvearrowleft,\curvearrowright,F]]}{(F^{\nvDash}-\curvearrowleft^{\nvDash}\curvearrowright)} \qquad \text{at a pinch point}$$

Let d be the degree of Y in $\mathbb{P}^{\nvDash}$, and g the genus of its normalization. It is not hard to compute that each pinch point "contributes 1 point" to the Segre class of the singular subscheme J (supported on Y) in $\mathbb{P}^{\nvDash}$, and each triple point "contributes -4 points"; that is,

$$s(J,\mathbb{P}^{\nvDash}) \doteq \eqsim^{\nvDash} + (\nvDash - \nvDash\eth - \nsupseteq - \nsupseteq\tau + \nu)\eqsim^{\nvDash} \quad ,$$

where h denotes the hyperplane class in $\mathbb{P}^{\nvDash}$.

On the other hand, it is easy to see that in this situation one has necessarily

$$g = 1 - 2d + \frac{dm}{2} - \frac{\nu}{4} - \frac{3\tau}{2} \quad :$$

for example one may compute the μ-class of J with respect to $\mathcal{O}(\Updownarrow\langle)$ both extrinsically, using the above expression for $s(J,\mathbb{P}^{\nvDash})$, and intrinsically by using Theorem 6 in [2]; comparing the two expressions gives the above condition on g. Or see [8], p. 29. Therefore

$$s(J,\mathbb{P}^{\nvDash}) \doteq \eqsim^{\nvDash} + \left(-\gtrdot + \frac{\nvDash\nu}{\nvDash} - \tau\right)\eqsim^{\nvDash} \quad .$$

From this we get the polynomial introduced in §2:

$$\begin{aligned} p(X,J,t) &= c(TM) \cap \Big(s(X,M) + c(\mathcal{L})^{-\infty} \cap (\int(\mathcal{J},\mathcal{M})^{(\sqcup)} \otimes \mathcal{L})\Big) \\ &\doteq mh + (4m - m^2 + dt^2)h^2 + \\ &\qquad \left(6m - 4m^2 + m^3 + (4d - 3dm)t^2 + \left(-dm + \frac{3\nu}{2} - \tau\right)t^3\right)h^3 \end{aligned}$$

For $t \geq 0$ this is (the push-forward to $\mathbb{P}^{\nvDash}$ of) Fulton's Chern class of a scheme consisting of X with an embedded copy of the 't-th thickening' of its singular subscheme. Evaluating at $t = -1$ gives

$$c(X \setminus J) \doteq mh + (d + 4m - m^2)h^2 + \left(6m - 4m^2 + m^3 - 2dm + 4d - \frac{3}{2}\nu + \tau\right)h^3 \quad ;$$

by theorem 1, this is the push-forward to $\mathbb{P}^{\nvDash}$ of MacPherson's Chern class of X. The coefficient of h^3 computes its Euler characteristic, in agreement with [8], p. 29.

References

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Florida State University, Tallahassee FL 32306
E-mail address: aluffi@math.fsu.edu