Bending Waves in Flattened Stellar Systems [a]

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1. INTRODUCTION

Warping of disk galaxies is much more evident in the gas than in stars. The warp in the H1 layer of the Milky Way was detected from 21cm observations in the 1950's.[1,2] Then, in the 1970's, it became apparent that warps are a common feature of the H1 layers of external galaxies.[3,4] Warps of H1 layers become strong only where they extend far beyond the optical image[5], and the optical evidence for the occurrence of warps in stellar disks is much less strong. There are clear-cut cases such as M31[6], but van der Kruit and Searle[7−9] detected only one very marginal warp in the seven edge-on galaxies that they observed. Although Sánchez-Saavedra *et.al*[10] claimed to detect warps in 42 of the 86 highly inclined spirals that they studied, they did classify 23 of the 42 as barely perceptible.

Two of the mechanisms that have been proposed for explaining warps, an intergalactic wind[11] or an infall of matter[12], are likely to affect gaseous disks more strongly than stellar disks. Other proposed mechanisms apply equally well to stellar as to gaseous disks. It was clear to the first detectors of the warp in the Milky Way that this warp is much too large to be attributed to the tide currently being raised by any near neighbor. This explanation is similarly unsatisfactory for the warps in many external galaxies. Past close encounters could raise larger tides. Both this, and other possibilities such as Lynden-Bell's[13] proposal that the warp is a free mode of oscillation, require an understanding of dynamics of bent disks, and the waves that they can support. It is this dynamics, rather than the explanation of galactic warps, that is the topic of this paper. Massive dark halos may very well play a major role in galactic warps.[14,15,16] As Binney[17] has remarked in a recent review:

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Early theoretical work on warped galaxies, especially the classic paper by Hunter & Toomre (1969), was concerned with the dynamics of isolated disks. It is now generally supposed that even late-type galaxies, which appear to consist of a disk alone, are embedded in a dynamically dominant dark halo. This is fortunate since it is in some ways easier to understand the dynamics of a massive disk sitting in an externally generated gravitational field than to get to grips with the dynamics of an isolated disk.

I believe that the dynamics of an isolated massive disk is still a topic of astrophysical relevance, even if it is not the most astrophysically interesting one related to warps. The difficulty of detecting warps in stellar disks suggests that they are generally stable against bending perturbations. I think that an improved understanding is needed of what it takes to keep a disk stable against bending perturbations. If dark halos are indeed the major cause of the warping of the gas beyond the optical disk, it is unlikely that they would also help stabilize the stellar disk. And why should it be so difficult to come to grips with the dynamics of isolated massive disks, in view of the relative simplicity of the system with which we are dealing, one in which particles move solely under the influence of gravity?

2. DYNAMICS OF THE BENDING OF RAZOR-THIN DISKS

2.1. Equation of bending motion

The simplest possible dynamical model, and one that is still widely used even when coupled with much more elaborate halo models[18,19], is that of a razor-thin flat disk in which matter rotates in circular orbits about a center, with angular velocity $\Omega(R)$. We use polar coordinates (R, θ) in the $z = 0$ plane of the unperturbed disk. Perturbations are described by the displacement $h(R, \theta, t)$ in the z-direction, which we suppose to be small. Taking account of the convective effect due to the circular rotation, the equation of motion in the z-direction is[20]

$$\left[\frac{\partial}{\partial t} + \Omega(R)\frac{\partial}{\partial \theta}\right]^2 h = F_z = F_{\rm disk} + F_{\rm ext}. \tag{1}$$

Here F_z is the force per unit mass in the upward z-direction, with contributions $F_{\rm disk}$ from the disk itself, and $F_{\rm ext}$ from external forces.

The gravitational force due to the disk itself is

$$F_{\rm disk} = G \iint_{\text{Disk surface } S'} \frac{\Sigma(R')[h(R',\theta',t) - h(R,\theta,t)]}{|\mathbb{R} - \mathbb{R}'|^3} dS'. \tag{2}$$

$\Sigma(R)$ is the surface density of the unperturbed disk, while $\mathbb{R}$ and $\mathbb{R}'$ are position vectors in the $z = 0$ plane. The perturbing force $F_{\rm disk}$ can be regarded as the force at height h due to an array of gravitational dipoles of spatial density $\Sigma(R)h(R,\theta,t)$. The dipole nature of this force comes from the fact that it is due to a combination of positive elements given by the perturbed disk which is displaced from the $z = 0$ plane, and negative elements arising from the original unperturbed disk. The force $F_{\rm disk}$ of equation (2) can be split into the two components given by the two numerator terms. The first component is then the vertical force caused by the dipoles, while the second component is that due to the force being evaluated at height h.[16] The splitting must be done carefully so as to avoid singular integrals because the improper integral (2) is convergent by virtue of the vanishing of the numerator at $\mathbb{R} = \mathbb{R}'$. The correct procedure is described in Section A.1 of the Appendix. When it is followed, the second component from the split is

$$-G \mathop{\not\!\!\!\iint}_{\text{Disk surface } S'} \frac{\Sigma(R')h(R,\theta,t)dS'}{|\mathbb{R} - \mathbb{R}'|^3} = -\nu^2_{\rm disk}(R)h(R,\theta,t), \tag{3}$$

where the line through the integral signs denotes that the finite part of this singular integral, as defined in equation (26), is to be used. It measures the z-gravitational force due to the unperturbed disk at height h, after the discontinuous self-force due to the local density has been subtracted, and $\nu_{\rm disk}(R)$ is the natural frequency of vertical oscillations at radius R in this gravity field. It can be shown[20] (see also the Appendix) that

$$\nu^2_{\rm disk} = 2\Omega^2 - \kappa^2, \tag{4}$$

where $\Omega(R)$ is the circular angular velocity and $\kappa(R)$ is the epicyclic frequency of small non-circular motions in the gravity field of the disk. Note that the positivity of Σ is no guarantee of the positivity of the $\nu^2_{\rm disk}$ given by the finite part integral of equation (3). For example, $\nu^2_{\rm disk}$ is negative in the case of uniform rotation for which $\kappa^2 = 4\Omega^2$. If external forces, such as those due to a z-symmetric static halo, are also taken into account, then the natural frequency ν of vertical oscillations has to be modified to

$$\nu^2 = \nu_{\rm disk}^2 + \nu_{\rm ext}^2, \tag{5}$$

where $F_{\rm ext} = -\nu_{\rm ext}^2 h$.

2.2. Kinematic bending waves

When we ignore the self-gravitational forces arising from the bent disk, so that the force $F_{\rm disk}$ is given entirely by the component (3), the dynamics is described by the simple equation[21]

$$\left[\frac{\partial}{\partial t} + \Omega(R)\frac{\partial}{\partial \theta}\right]^2 h = -\nu^2(R)h. \tag{6}$$

Assuming dependence on $e^{i(m\theta-\omega t)}$, equation (6) gives the dispersion relation

$$(\omega - m\Omega)^2 = \nu^2, \tag{7}$$

which allows two possible frequencies. There is a faster direct wave which propagates with angular velocity $\omega/m = \Omega + (\nu/m)$, and a slower, and often retrograde, wave which propagates with angular velocity $\omega/m = \Omega - (\nu/m)$. When either Ω or ν vary with R, the wave velocity varies with radius, and there is no single coherent wave. The winding dilemma, of a wave shape which becomes continually more tightly coiled, is avoided in the important $m = 1$ case for a disk in a spherical galaxy because then $\nu = \Omega$. However, the dilemma generally arises with mass distributions that are flattened in z whose gravitational fields give $\nu > \Omega$.

2.3. Bending waves with self-gravity

The gravitational forces induced by the bending of the disk provide one mechanism which gives coherence between different radii, and allows the possibility of global and non-winding waves. Numerical methods generally are necessary for finding such solutions of equation (1) with the full expression (2) for the gravitational $F_{\rm disk}$ force. However, an approximate dispersion relation of

$$(\omega - m\Omega)^2 = \nu^2 + 2\pi G\Sigma k, \tag{8}$$

can be obtained for waves of short and positive radial wavenumber k by using a short wavelength approximation to the local gravitational forces exerted by the bent disk. This relation confirms explicitly that self-gravity exerts an extra stabilizing influence. Toomre and I[20] never found any unstable solutions in our investigations, though we were only able to prove the non-existence of instabilities for the angular wavenumbers $m = 0$ and $m = 1$.[b] The dispersion relation (8) shows that there are still two kinds of waves, either fast with $\omega > m\Omega + \nu$ at each radius or slow with $\omega < m\Omega - \nu$, and with a wavenumber that varies with radius.

Although the modes of a finite disk that has a sharp edge form a discrete spectrum, most of these modes run together to form a continuous spectrum when the density tapers gradually to zero at the outer edge $R = R_{\rm edge}$ of the disk.[20] A lack of clearly defined discrete modes is a difficulty for modal explanations of warps. The continuum ranges of frequencies then consist of the two discrete bands, one fast and one slow, allowed by the limiting conditions at the outer edge of the disk:

$$\begin{aligned} \omega &> m\Omega(R_{\rm edge}) + \nu(R_{\rm edge}), \\ \omega &< m\Omega(R_{\rm edge}) - \nu(R_{\rm edge}). \end{aligned} \tag{9}$$

As equation (8) correctly predicts, continuum modes fluctuate extremely rapidly as the outer edge, where the surface density Σ tends to zero, is approached. Their radial wavenumber k becomes infinite while their phase velocity ω/k slows to zero in such a manner that they never reach the outer edge in finite time. Our numerical studies essentially found that no non-trivial discrete $m = 0$ or 1 modes remain in the presence of the continuum, though some non-trivial discrete modes do so remain for $m \geq 2$.[22] However, Sparke and Casertano[16] have shown that a finite disk can have discrete $m = 1$ warping modes even

[b] At the meeting, Jeremy Goodman reported that he had found numerical evidence for instabilities at $m>1$.

when the density tapers to zero at the outer edge, provided that it does so in a region that is narrow enough so as to leave a sufficient cavity in which the discrete modes can be trapped.

2.4. Bending instabilities arising from non-circular motions

The dynamics is changed radically when random motions in the plane of the disk, which also add to the coherence between different radii, are present. We suppose in this model that internal forces normal to the disk are strong enough to maintain its razor-thin structure. Inertial effects now modify the left-hand side of the equation of motion (1) which becomes

$$\left[\frac{\partial}{\partial t}+\overline{\Omega}(R)\frac{\partial}{\partial\theta}\right]^2 h+\frac{1}{R\Sigma(R)}\frac{\partial}{\partial R}\left[R\Sigma(R)\sigma_R^2\frac{\partial h}{\partial R}\right]+\frac{\sigma_\theta^2}{R^2}\frac{\partial^2 h}{\partial\theta^2}=F_z \tag{10}$$

Here, $\overline{\Omega}(R)$ is the mean angular velocity of the circular motion about the center, and σ_R^2 and σ_θ^2 are the radial and azimuthal velocity dispersions respectively. Equation (10) has been named the *membrane equation* by Polyachenko and Shukhman[23] who first gave its full form, though Toomre[24] had earlier realized the significant effects that random motions in the plane of the disk can have. The approximate dispersion relation given by equation (10) for the short wavelength limit is

$$(\omega-m\,\overline{\Omega})^2=\nu^2+2\pi G\Sigma k-k^2\sigma_R^2-m^2\sigma_\theta^2/R^2. \tag{11}$$

It shows that the non-circular motions are destabilizing, and increasingly so at shorter wavelengths. The reason is that this is no ordinary membrane. The random motions endow it with a pressure, not necessarily isotropic, and it is in a state of compression, rather than tension. This compression causes it to tend to buckle when it is bent, instead of pulling it back to its unbent state as a tension would.

The few known analytical solutions of the membrane equation (10) confirm the instability. There is a brief mention in Hunter and Toomre[20] of the fact that two equal superposed and oppositely circularly rotating Maclaurin disks are unstable for $m\geq 2$. The recent discovery of counterrotating stars in the disk of NGC 2550[25] has made this model seem somewhat more realistic that it did in 1969. For two equal superposed and oppositely circularly rotating disks, $\overline{\Omega} = \sigma_R = 0$ and $\sigma_\theta^2 = R^2\Omega_C^2$, where $\Omega_C(R)$ is the angular velocity for circular motion. The dispersion relation (11) therefore reduces to

$$\omega^2 = \nu^2 + 2\pi G\Sigma k - m^2\Omega_C^2, \tag{12}$$

which does indeed predict instability at large enough m.

Polyachenko[26] derived the exact dispersion relation for Maclaurin disks with Kalnajs[27] distribution functions. The latter have isotropic velocity dispersions and mean angular velocities that are a constant fraction of the constant Ω_C. As equation (11) then correctly predicts, these models have instabilities at short enough wavelengths for all angular wavenumbers m. Polyachenko's paper ends with the suggestion that these buckling instabilities, or the hose-pipe instabilities as he called them, "might have some bearing on the notorious problem of the minimum flattening of elliptical galaxies." This problem has yet to be resolved, and Polyachenko's suggestion remains an important one.[28,29,30]

Malkov and his associates[31,32] used a similar model to investigate some global $m = 2$ bending modes of elliptical Freeman disks[33]. They found that many of those that are free of bar instability are prone to bending instability. The latter phenomenon is understandable in the light of the fact that self-gravity acts to stabilize, while in-plane random motions act to destabilize, the bending of a thin disk. These roles vis-à-vis stability are the reverse of what they are for density waves in the plane of a disk. This reversal of roles is apparent when the bending dispersion relation (11) is compared with the dispersion relation[21]

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\Sigma k + c^2k^2, \tag{13}$$

for density waves in an isotropic one-component fluid model with velocity dispersion c. Here Jeans instability occurs at sufficient long wavelengths unless the velocity dispersion is large enough (i.e., meets the Toomre criterion $c > \pi G\Sigma/\kappa$[34]).

Merritt and Sellwood[35] computed solutions of the membrane equation (10) for a class of Kuzmin-Toomre disks with two equal counter-rotating stellar populations. They found infinite continuous spectra of unstable modes, the frequencies of which they could fit well with a theoretical formula. They also found a few discrete oscillatory modes for the $m = 0$ and $m = 1$ cases, though none for the $m = 2$ case.

2.5. N-body simulations

Sellwood and Merritt[36] also carried out N-body simulations of two initially equal counter-rotating stellar populations, simulations that detected waves and instabilities in the plane as well as out of it. Their major findings concerning bending instabilities are that thin radially hot models are disrupted by $m = 0$ axisymmetric bell modes. The dominant bending instability in cooler thin models is an $m = 2$ saddle mode. They were able to detect an $m = 1$ warp, but it always grew more slowly than either the bell or the saddle modes. These instabilities are weakened, though mostly rather slowly, by increasing disk thickness. Nevertheless, they did find that models with an intermediate radial pressure become, after minor adjustments, almost axisymmetric, apparently stable, moderately thin, disks. Their results therefore suggest strongly that the study of disk systems of small, though finite, thickness is a topic worth pursuing.

3. BENDING WAVES IN SYSTEMS OF FINITE THICKNESS

In this section, we explore how bending waves in systems of finite thickness differ from those in razor-thin disks. In particular our interest is in understanding how the destabilizing effect of non–circular motions in the plane is overcome.

3.1. Effect of finite thickness on self-gravity

The effectiveness of the restoring force of gravity, which counteracts bendings of razor-thin disks and causes such bendings to give rise to stable waves in the absence of non-circular motions, is diminished by a finite thickness of the disk. To justify and quantify this last statement, we consider a small wavy displacement $h_0 e^{ikx}$ in the z-direction of a thin layer of gravitating matter that is stratified in the z-direction but is uniform in other directions. Suppose that this displacement shifts, but does not otherwise alter, the internal density profile $\rho_0(z)$ of the layer. Then the total force per unit mass acting in the z-direction on the element at $x = x_0$ can be calculated (see Appendix) to be

$$F_z = \frac{-2\pi G k h_0 e^{ikx_0}}{\Sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_0(z)\rho_0(s) e^{-k|z-s|} dz\, ds, \tag{14}$$

where $\Sigma = \int_{-\infty}^{\infty} \rho_0(z) dz$ is now the integrated surface density of the layer. This force simplifies to $-2\pi G\Sigma k h_0 e^{ikx_0} = -2\pi G \Sigma k h$ for a razor-thin disk for which $\rho_0(z) = \Sigma\delta(z)$, and the force appropriate to the thin disk dispersion relation (8) is then recovered. More generally however, the force F_z is described by the relation

$$F_z = -2\pi G\Sigma k\, \mathcal{F}\, h, \tag{15}$$

and contains an additional reduction factor $\mathcal{F}$. This reduction factor depends upon the wavenumber scaled with the scale-height H of the layer, and the explicit expression for it is

$$\mathcal{F}(kH) = \frac{1}{\Sigma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_0(z)\rho_0(s) e^{-k|z-s|} dz\, ds. \tag{16}$$

It diminishes steadily as the scaled wavenumber kH increases; i.e., as the thickness of the layer increases, or as the ratio of wavelength of the bending becomes short compared with this thickness. The reduction factor can be evaluated explicitly for a top hat density profile of width $2H$ as

$$\mathcal{F}_{TH}(kH) = \frac{2kH - 1 + e^{-2kH}}{2(kH)^2}. \tag{17}$$

It tends to 1 as the scale height $H \to 0$, but decays to zero as $(1/kH)$ as $kH \to \infty$ The latter type of decay is characteristic of other density profiles because the reduction factor (16) can be approximated asymptotically, for large kH and smooth peaked density profiles, by

$$\mathcal{F}(kH) \sim \frac{2}{k\Sigma^2} \int_{-\infty}^{\infty} [\rho_0(s)]^2 ds. \tag{18}$$

Hence, unless the bending causes the interval density profile to be readjusted in some especially favorable manner, self gravity cannot be the mechanism by which an unstable razor-thin disk achieves stability at finite-thickness. For an isothermal density profile

$$\rho_0(z) = \frac{\Sigma}{2H} \mathrm{sech}^2\left(\frac{z}{H}\right), \tag{19}$$

the limit (18) is easily calculated to be $\mathcal{F}(kH) \sim 2/(3kH)$. The general expression for $\mathcal{F}$ has the more complicated form

$$\mathcal{F}(kH) = \tfrac{1}{2}(kH)^2\psi'(kH/2) - 1 - kH, \tag{20}$$

where ψ is the digamma function.[c]

3.2. Dynamics of slabs of finite thickness

The altered dynamics of systems of finite thickness is the only other mechanism for achieving stability. We consider first the simpler case of gaseous systems, as dynamicists are often wont to do, because of the greater complexity of stellar systems described by the collisionless Boltzmann equation. In both instances, our analysis again refers to systems that are stratified only in the z-direction, and are uniform in the other directions, and are therefore more properly referred to as slabs, rather than disks.

3.2.1. Dynamics of gaseous slabs

Although there have been several studies of flat gaseous systems, most of them have focused on the circumstances under which instabilities of Jeans type occur. Ledoux[37], who analyzed the stability of a static slab of isothermal gas with the density profile (19), did also consider bending perturbations, and showed that there were no bending instabilities. Goldreich and Lynden-Bell[38] included bending perturbations in their study of the combined effects of uniform rotation and finite thickness on the stability of gaseous slabs. They were able to show that a polytropic slab with $\gamma = 2$, and an incompressible slab with a top-hat density profile, as well as an isothermal slab, are stable against bending. For the incompressible slab they obtained full dispersion relations. That for bending waves is, in our notation,

$$\tilde{\omega}^2 = 2\pi G\Sigma k\,\mathcal{F}_{TH}(kH)\left[\frac{nkH}{\tanh nkH}\right], \tag{21}$$

where $n = \tilde{\omega}/\sqrt{\tilde{\omega}^2 - 4\Omega^2}$ and $\tilde{\omega}$ is the frequency as measured in a frame that follows the rotation. This shifted frequency $\tilde{\omega}$ is akin to our $(\omega - m\Omega)$, and the dispersion relation

[c] This result is due to Jeremy Goodman (private communication).

(21) can be contrasted with the dispersion relation (8) for a razor-thin disk. Relation (21) has no ν term only because Goldreich and Lynden-Bell did not include any background gravitational field in the z-direction. It also contains the reduction factor $\mathcal{F}_{TH}(kH)$ of equation (17) by which gravitational restoring forces for the top-hat density profile are less than those for a razor-thin disk. However, the fluid nature of the slab, in this case an incompressible fluid with a sharp boundary on which there are surface waves, contributes an extra factor $(nkH/\tanh nkH)$ to the dispersion relation (21). This extra factor grows as kH for kH large and exactly counters the decay of the reduction factor $\mathcal{F}$, and adds to the stability. In fact, the dispersion relation (21) becomes $\tilde{\omega}^2 = ngk$ in the limit of large kH, where $g = 2\pi G\Sigma$ is the acceleration due to gravity felt at the surface. This is exactly the dispersion relation for surface waves on a deep rotating ocean. Although Fridman and Polyachenko[39] find a similar result for the non-bending z-symmetric oscillations of an incompressible layer, it is unlikely that this limit has much relevance in the present context.

3.2.2. Dynamics of stellar slabs

In view of the incompleteness of our present understanding of the dynamics of the bending of gaseous disks, it is scarcely surprising that the dynamics of the bending of stellar disks is also incompletely understood. The stellar dynamic work that is most relevant in the present context is that of Toomre[24] and Araki[40]. They discussed the stability of a stellar slab, stratified only in the z-direction as in Section 3.1, and with a distribution function that is Maxwellian in both x and z, but with different velocity dispersions σ_x and σ_z respectively. They found that bending instabilities are suppressed provided that the ratio σ_z/σ_x of the velocity dispersions exceeds 0.293. Their neutral stability curve has the form of a sheared hump in the $(\lambda/\lambda_J, \sigma_z/\sigma_x)$-plane, and is plotted in FIGURE 6 of Merritt and Sellwood.[35] Here $\lambda = 2\pi/k$ is the radial wavelength, and $\lambda_J = \sigma_x^2/G\Sigma$ is the Jeans length for instabilities in the plane. A band of unstable wavelengths, intermediate between short and long and lying within the hump, occurs for values of

σ_z/σ_x less than 0.293. A remarkably similar neutral stability curve is obtained from Polyachenko and Shukhman's[39,41] analysis of bending waves in a uniformly dense stellar slab of finite thickness, and with a Maxwellian distribution of x-velocities. Polyachenko and Shukhman's neutral stability is a somewhat inflated version of Toomre and Araki's sheared hump, and peaks at $\sigma_z/\sigma_x = 0.370$, rather than 0.293. The difference between the two minimum values of σ_z/σ_x needed to avoid instability is relatively modest in view of the extreme differences between the z-components of the distribution functions in the two cases, one a decaying exponential in the partial energy E_z and the other with a singular $(E_0 - E_z)^{-1/2}$ dependence, where E_0 is the partial energy of the most energetic particles which are marginally confined within the slab. In both cases, the results are consistent with the instability at short wavelengths ($\lambda < \lambda_J$) of razor-thin disks with random motions in the plane of the disk only, for which $\sigma_z/\sigma_x = 0$. They are also consistent with the stability of the gaseous disks discussed in the previous section which, to some extent, approximate the $\sigma_z/\sigma_x = 1$ case of isotropic velocity dispersions which is well clear of the unstable regions. The longer wavelength branches of the two neutral stability curves both bend initially to the right with increasing σ_z/σ_x, indicating that a small amount of z-velocity dispersion is initially destabilizing. This feature is barely present in Merritt and Sellwood's approximation [their eq.(26)] to this curve, which is also shown in their FIGURE 6. The form $(\omega - m\Omega)^2 = 2\pi G\Sigma k + (\sigma_z^2 - \sigma_x^2)k^2 = (4\pi^2\sigma_x^2/\lambda^2)(\sigma_z^2/\sigma_x^2 - 1 + \lambda/\lambda_J)$ proposed by Bertin and Mark[18] for the dispersion relation for bending waves in a stellar disk [their eq.(D.1)] does not come close to giving even a qualitatively correct neutral stability curve.

The dynamics of even one-dimensional z-motions of a collision stellar slab is complicated. Weinberg[42] found that such a slab can undergo purely oscillatory and undamped modes. These modes have frequencies in the gaps between the bands $[2n\pi/T_{\rm max}, 2n\pi/T_{\rm min}]$, formed by the orbital frequencies ($n = 1$) and their harmonics ($n > 1$), where $T_{\rm max}$ and $T_{\rm min}$ are the longest and shortest orbital periods. What is of greatest interest to the present discussion is that Weinberg did also consider bending disturbances with wavenumber k in the x-direction perpendicular to the stratification. also assuming a Maxwellian stellar distribution function in x-velocities. He found such bending disturbances to be damped, and that the damping is strong except in the case of the fundamental *sloshing* mode, for which it is much weaker. The fundamental sloshing mode, in which the slab moves up and down as a whole, is precisely the one that is of most interest in the present context.

Louis[43] was able to reproduce some of Weinberg's low-order one-dimensional modes using a simple truncated systems of moment equations. He found that discrete oscillation modes could be tracked into the continuum bands, where they are damped, initially only slightly. He was also able to identify the modes in N-body simulations. He did not study bending modes, remarking only that: "the lowest order sloshing mode is somewhat boring ... unless one takes into account the horizontal structure of the galactic disk which then leads to the study of bending modes." Of course it is precisely that structure which makes the sloshing mode, rather than higher-order modes with more complex z-structure, the most interesting of all and not the least bit boring. The most important implication of Louis's work is that simple truncated systems of moment equations may be capable of giving good descriptions of the bending oscillations of stellar disks, and allow one to avoid dealing directly with the collisionless Boltzmann equation.

4. DISCUSSION

The understanding of the dynamics of the bending of flattened stellar systems is significantly incomplete. It is most complete for the razor–thin disks discussed in Section 2. However, the severe instabilities predicted by Polyachenko and Shukhman's[23] membrane equation (11) when there are random velocities in the plane, as there surely are in flat stellar systems, shows that that model can have only a limited range of validity. Nature, and the N-body simulations of Sellwood and Merritt[36] suggest that it is not difficult to build rather flat rotating stellar systems that are free of bending instabilities. Presumably the consequence of the bending instability that results if a system becomes too flat is that it thickens. The work of Toomre[24], Polyachenko and Shukhman[41], and Araki[40] predicts that a relatively low level of random velocities perpendicular to the plane of the disk is sufficient to achieve stability. Since gravity is too weak to cure bending instabilities of a thin disk in the presence of random motions, and since gravity's restoring force weakens as the disk thickens, it must be inertial effects that induce the stability. Merritt and Sellwood[35] suggest that the most important mechanism by which a finite thickening succeeds in quenching bending instabilities, is the out-of-phase response of stars when forced at a frequency greater than their natural frequency. Although promising, a fully convincing dynamical demonstration of their suggestion is lacking, and their application of it illustrated in their FIGURE 6, which also incorporates the reduction factor $\mathcal{F}$ of equation (20), is not quite right.

Another important suggestion that Merritt and Sellwood make is that inhomogeneities play a major role, and that infinite slab analyses, on which all of the discussion of Section 3 is based, are deceptive in overestimating the stabilizing effect of both self-gravity and z-velocity dispersion. They found that bending modes in inhomogeneous disks can remain unstable at very long wavelengths. In N-body experiments, they found instability to bending even when the velocity anisotropy is much less extreme than the critical value for instability in an infinite slab. These phenomena have yet to be explained, and deserve an explanation.

Finally, there remains the question of whether it is possible to construct a simplified model of a flattened stellar system that includes the major phenomena of ordered and random motions, small but finite thickness, and inhomogeneities. Louis[43] showed, in a simpler context, that it is possible to construct relatively simple, yet useful, models of low-order modes of oscillation of stellar slabs using simple truncated systems of moment equations, and our interest here is also in modelling low-order modes. If it isn't, then this would appear to be another field that theoreticians may have to abandon to the N-body computationalists.

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APPENDIX

A.1. The finite part integral for the gravitational force due to a dipole array

The gravitational potential at a general point of space arising from a distribution of dipoles of strength $\mu(x,y)$ distributed on a region S of the (x,y)-plane is given by

$$\Phi_d(x,y,z) = -G \iint\limits_{S'} \frac{z\mu(x',y')\,dx'\,dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}. \tag{22}$$

This potential, which is odd in z, is discontinuous at the plane in which the dipoles lie, where it has the one-sided limits[44]

$$\lim_{z\to 0+} \Phi_d(x,y,z) = -2\pi G\mu(x,y), \qquad \lim_{z\to 0-} \Phi_d(x,y,z) = +2\pi G\mu(x,y). \tag{23}$$

The z-derivative of this potential, which we need for the upward force exerted by the dipoles, must be evaluated with some care. It is an even function of z and its two one-sided limits $z \to \pm 0$ are identical. However, the direct differentiation of equation (22), with its discontinuity, gives rise to a delta function component.

To isolate this singular behavior, we first split equation (22) into the three parts:

$$\begin{aligned}\Phi_d(x,y,z) = &-G \iint\limits_{S'-S_\delta} \frac{z\mu(x',y')\,dx'\,dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} \\ &- G \iint\limits_{S_\delta} \frac{z[\mu(x',y')-\mu(x,y)]\,dx'\,dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} \\ &- G \iint\limits_{S_\delta} \frac{z\mu(x,y)\,dx'\,dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}},\end{aligned} \tag{24}$$

where S_δ is a small circle in the (x,y)-plane, center $(x,y,0)$ and radius δ. The third integral is elementary and can be evaluated as

$$2\pi G\mu(x,y)\left\{\frac{z}{(\delta^2+z^2)^{1/2}} - \operatorname{sgn}(z)\right\}. \tag{25}$$

This is the component that gives rise to the jump discontinuity (23). We now differentiate with respect to z, evaluate at $z = \pm 0$ so as to avoid the delta function, and take the limit

$\delta \to 0$, which makes the second component of equation (24) negligible provided that the dipole density μ is sufficiently smooth, e.g. twice continuously differentiable.[44] We then obtain the upward force exerted by the dipole distribution as

$$F_z = -\frac{\partial \Phi_d}{\partial z}\bigg|_{z=\pm 0} = \lim_{\delta \to 0} \left\{ G \iint\limits_{S'-S_\delta} \frac{\mu(x',y')\,dx'\,dy'}{[(x-x')^2+(y-y')^2]^{3/2}} - \frac{2\pi G \mu(x,y)}{\delta} \right\}. \tag{26}$$

The right-hand side here defines the finite part integral that is used in equation (3) of the text. It is this finite part integral that is computed by the expansion methods described in Section II(d) of Hunter and Toomre[20], or the Fourier-Bessel methods of Sparke and Casertano.[16]

Equation (4) is a consequence of the fact that the dipole potential (22), with μ replaced by Σ, is the z-derivative of the simple source potential

$$\Phi_s(x,y,z) = -G \iint\limits_{S'} \frac{\Sigma(x',y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}. \tag{27}$$

It follows from our earlier discussion that the ν^2_{disk} of equation (3) is simply $\partial^2\Phi_s/\partial z^2\big|_{z=\pm 0}$. Because the potential Φ_s satisfies Laplace's equation away from the disk, it follows that

$$\nu^2_{\mathrm{disk}} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_s(x,y,0). \tag{28}$$

Equation (4) follows when the derivatives of Φ_s are related to the circular velocity via the equation $\Omega^2(R) = \partial\Phi_s/R\partial R$.

A.2. The reduction factor for a disk of finite thickness

The small wavy displacement considered in Section 3.1 induces a density field $\rho_0(z - h_0 e^{ikx})$, and hence a density perturbation $-h_0 e^{ikx}\rho_0'(z)$ and, by Poisson's equation, a gravitational potential $\Phi_1(z)e^{ikx}$, such that

$$\Phi_1''(z) - k^2\Phi_1(z) = -4\pi G h_0 \rho_0'(z). \tag{29}$$

The relevant solution of this equation is

$$\Phi_1(z) = (2\pi G h_0/k)\int_{-\infty}^{\infty} \rho_0'(s) e^{-k|z-s|}\,ds, \tag{30}$$

which gives

$$\Phi_1'(z) = 2\pi G h_0 \{-2\rho_0(z) + k \int_{-\infty}^{\infty} \rho_0(s) e^{-k|z-s|}\, ds\}. \tag{31}$$

The total z-force per unit length that acts on the surface density is

$$\begin{aligned} \Sigma F_z &= \int_{-\infty}^{\infty} \rho_0(z - h_0 e^{ikx})[-\Phi_0'(z) - \Phi_1'(z) e^{ikx_0}]\, dz, \\ &\simeq e^{ikx_0} \int_{-\infty}^{\infty} [h_0 \Phi_0'(z)\rho_0'(z) - \Phi_1'(z)\rho_0(z)]\, dz, \end{aligned} \tag{32}$$

where $\Phi_0(z)$ is the gravitational potential of the unperturbed state. Integration by parts of the $\rho_o'(z)$ of the first component of equation (32), followed by the use of Poisson's equation for $\Phi_0''(z)$ and equation (31) for $\Phi_1'(z)$ then gives equation (14).

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