Portfolio optimization for t and skewed-t returns

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Abstract

It is well-established that equity returns are not Normally distributed, but what should the portfolio manager do about this, and is it worth the effort? As we describe, there are now some good choices for multivariate modeling distributions that capture heavy tails and skewness in the data; we argue that among the best are the (Student) t and skewed t distributions. These can be efficiently calibrated to data, and show a much better fit to real data than the Normal distribution. By examining efficient frontiers computed using different distributional assumptions, we show, using for illustration 5 stocks chosen from the Dow index, that the choice of distribution has a significant effect on how much available return can be captured by an optimal portfolio on the efficient frontier.

Portfolio optimization requires balancing risk and return; for this purpose one needs to employ some precise concept of "risk". Already in 1952, Markowitz used the standard deviation (StD) of portfolio return as a risk measure, and, thinking of returns as normally distributed, described the efficient frontier of fully invested portfolios having minimum risk among those with a specified return. This concept has been extremely valuable in portfolio management because a rational portfolio manager will always choose to invest on this frontier.

The construction of an efficient frontier depends on two inputs: a choice of risk measure (such as StD, VaR, or ES, described below), and a probability distribution used to model returns.

Using StD (or equivalently, variance) as the risk measure has the drawback that it is generally insensitive to extreme events, and sometimes these are of most interest to the investor. Value at Risk (VaR) better reflects extreme events, but it does not aggregate risk in the sense of being subadditive on portfolios. This is a well-known difficulty addressed by the concept of a "coherent risk measure" in the sense of Artzner, et. al. [1999]. A popular example of a coherent risk measure is expected shortfall (ES), though VaR is still more commonly seen in practice.

Perhaps unexpectedly, the choice of risk measure has no effect on the actual efficient frontier when the underlying distribution of returns is Normal – or more generally any "elliptical" distribution. Embrechts, McNeil, and Straumann [2001] show that when returns are elliptically distributed, the minimum risk portfolio for a given return is the same whether the risk measure is standard deviation, VaR, ES, or any other positive, homogeneous, translation-invariant risk measure.

This fact suggests that the portfolio manager should pay at least as much attention to the family of probability distributions chosen to model returns as to the choice of which risk measure to use.

It is now commonly understood that the multivariate Normal distribution is a poor model of generally acknowledged "stylized facts" of equity returns:

- return distributions are fat-tailed and skewed
- volatility is time-varying and clustered

- returns are serially uncorrelated, but squared returns are serially correlated.

The purpose of this paper is to make the case that portfolio managers should consider using heavy-tailed distributions as models for equity returns – especially the multivariate Student t and skewed t distributions. Recently other authors have also argued, with different data that these distributions empirically superior, e.g. Keel, et. al. [2006], and Aas and Hobaek Haff [2006].

Does it really cost anything to determine optimal portfolios by calibrating returns data to a Normal distribution, rather than some heavier-tailed choice? The answer is yes. Not only do other distributions do a better job of modeling extreme events, but using them allows the manager to capture portfolio returns that are inaccessible using the Normal model.

We illustrate this below with a portfolio of 5 stocks using daily returns data to optimize the one-day forecast return at a fixed risk level. We use a GARCH filter to remove serial correlations of squared returns; we then fit this approximately *i.i.d.* five dimensional data using a selection of potential distributions from the Generalized Hyperbolic family, including Normal, hyperbolic (Hy), normal inverse Gaussian (NIG), variance gamma (VG), (Student) t, and skewed t (defined below). We find that the t and skewed t have the largest log likelihood, despite having fewer parameters than Hy, NIG, or VG.

After discussion of coherent risk measures, value at risk, and expected shortfall, we examine the problem of portfolio optimization for these different risk measures and returns distributions, concentrating on the t and skewed t. We show (proposition 6) that for zero skewness, these distributions produce the same efficient frontiers no matter which risk measure or degree of freedom is chosen, so long as the same means and correlations are used. Nevertheless, our data set illustrates how much potential return is lost by a manager who calibrates data to a Normal distribution when returns are in reality t or skewed t distributed.

The Multivariate Generalized Hyperbolic Distributions

The family of multivariate skewed-t distributions is a subfamily of the larger family of "generalized hyperbolic (GH) distributions", introduced by Barndorff-Nielsen [1977] and championed for financial applications in McNeil, Frey, and Embrechts [2005].

These distributions are usefully understood as examples of a nice class of distributions called normal mean-variance mixture distributions, defined as follows.

Definition 1 Normal Mean-Variance Mixture. *The d-dimensional random variable $\mathbf{X}$ is said to have a multivariate normal mean-variance mixture distribution if*

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + W\boldsymbol{\gamma} + \sqrt{W}\mathbf{Z}, \quad \textit{where} \tag{1}$$

1. *$\mathbf{Z} \sim N_k(\mathbf{0}, \Sigma)$, the k-dimensional Normal distribution with mean zero and covariance Σ (a positive semi-definite matrix),*
2. *$W \geq 0$ is a positive, scalar-valued r.v. which is independent of $\mathbf{Z}$, and*
3. *$\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are parameter vectors in $\mathbb{R}^d$.*

The mixture variable W can be interpreted as a shock which changes the volatility and mean of an underlying normal distribution. From the definition, we can see that, conditional on W, $\mathbf{X}$ is Normal:

$$\mathbf{X} \mid W \sim N_d(\boldsymbol{\mu} + W\boldsymbol{\gamma}, W\Sigma), \tag{2}$$

and

$$E(\mathbf{X}) = \boldsymbol{\mu} + E(W)\boldsymbol{\gamma} \tag{3}$$

$$COV(\mathbf{X}) = E(W)\Sigma + var(W)\boldsymbol{\gamma}\boldsymbol{\gamma}' \tag{4}$$

the latter defined when the mixture variable W has finite variance $var(W)$.

If the mixture variable W is generalized inverse gaussian (GIG) distributed (see Appendix), then $\mathbf{X}$ is said to have a generalized hyperbolic distribution (GH). As described in the Appendix, the GIG distribution has three real parameters, λ, χ, ψ, and we write $W \sim N^{-}(\lambda, \chi, \psi)$ when W is GIG.

Therefore the multivariate generalized hyperbolic distribution depends on three real paramters λ, χ, ψ, two d-dimensional parameter vectors $\boldsymbol{\mu}$ (location) and $\boldsymbol{\gamma}$ (skewness) in $\mathbb{R}^d$, and a $d \times d$ positive semidefinite matrix Σ. We then write

$$\mathbf{X} \sim GH_d(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\gamma}, \Sigma).$$

Some Special Cases

Hyperbolic distributions (Hy):

When $\lambda = 1$, we get the multivariate generalized hyperbolic distribution whose univariate margins are one-dimensional hyperbolic distributions. For $\lambda = (d+1)/2$, we get the d-dimensional hyperbolic distribution. However, its marginal distributions are no longer hyperbolic.

The one dimensional hyperbolic distribution is widely used in the modelling of univariate financial data, for example in Eberlein and Keller [1995] and Farjado and Farias [2003].

Normal Inverse Gaussian distributions (NIG):

If $\lambda = -1/2$, then the distribution is known as normal inverse gaussian (NIG). NIG is also commonly used in the modelling of univariate financial returns. Hu [2005] contains a fast calibration algorithm.

Variance Gamma distribution (VG):

If $\lambda > 0$ and $\chi = 0$, then we get a limiting case known as the variance gamma distribution. For the variance gamma distribution, we can calibrate all the parameters including λ; see Hu [2005].

Skewed t Distribution:

If $\lambda = -\nu/2, \chi = \nu$ and $\psi = 0$, we get a limiting case which is called the skewed-t distribution by Demarta and McNeil [2005], because it generalizes the usual t distribution, obtained from the skewed t by setting the skewness parameter $\boldsymbol{\gamma} = 0$. (The skewed t can also be described as a Normal mean-variance mixture distribution, where the mixture variable W is inverse gaussian $Ig(\nu/2, \nu/2)$, see McNeil, Frey, and Embrechts [2005].)

The t distribution is widely used in modelling univariate financial data since the degree of freedom measures the heaviness of heavy tails. The EM algorithm now makes practical the use of the t distribution also for multivariate data.

Without skewness, the Student t is elliptical, and therefore predicts, for example, that joint crashes have the same likelihood as joint booms. This partly motivates the introduction of skewness with the skewed-t.

For convenience, explicit density functions of the skewed t distributions are given in the Appendix. The mean and covariance of a skewed t distributed random vector $\mathbf{X}$ are

$$E(\mathbf{X}) = \boldsymbol{\mu} + \boldsymbol{\gamma}\frac{\nu}{\nu-2} \tag{5}$$

$$COV(\mathbf{X}) = \frac{\nu}{\nu-2}\Sigma + \boldsymbol{\gamma}\boldsymbol{\gamma}'\frac{2\nu^2}{(\nu-2)^2(\nu-4)} \tag{6}$$

where the covariance matrix is defined when $\nu > 4$, and the expectation when $\nu > 2$.

Furthermore, in the limit as $\boldsymbol{\gamma} \to \mathbf{0}$, we get the joint density function of the t distribution:

$$f(\boldsymbol{x}) = \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})(\pi\nu)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}(1+\frac{\boldsymbol{\rho(x)}}{\nu})^{-\frac{\nu+d}{2}} \tag{7}$$

with mean and covariance

$$E(\mathbf{X}) = \boldsymbol{\mu}, \quad COV(\mathbf{X}) = \frac{\nu}{\nu-2}\Sigma \tag{8}$$

The Portfolio Property

A great advantage of the generalized hyperbolic distributions with this parametrization is that they are closed under linear transformation. These facts are easily proved in McNeil, Frey, and Embrechts [2005]. To be precise, if

$$\boldsymbol{X} \sim GH_d(\lambda, \chi, \psi, \boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma})$$

and $\boldsymbol{Y} = B\boldsymbol{X} + \boldsymbol{b}$ for $B \in \mathbb{R}^{k\times d}$ and $\boldsymbol{b} \in \mathbb{R}^k$, then

$$\boldsymbol{Y} \sim GH_d(\lambda, \chi, \psi, B\boldsymbol{\mu} + \boldsymbol{b}, B\Sigma B', B\boldsymbol{\gamma}) \tag{9}$$

In particular, if $\boldsymbol{X} \sim SkewedT_d(\nu, \boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma})$, we have

$$\boldsymbol{Y} \sim SkewedT_k(\nu, B\boldsymbol{\mu} + \boldsymbol{b}, B\Sigma B', B\boldsymbol{\gamma}) \tag{10}$$

Forming a linear portfolio $y = \boldsymbol{\omega}^T \boldsymbol{X}$ of the components of $\boldsymbol{X}$ amounts to choosing $B = \boldsymbol{\omega}^T = (\omega_1, \cdots, \omega_d)$ and $\boldsymbol{b} = \boldsymbol{0}$. In this case,

$$y \sim GH_1(\lambda, \chi, \psi, \boldsymbol{\omega}^T\boldsymbol{\mu}, \boldsymbol{\omega}^T\Sigma\boldsymbol{\omega}, \boldsymbol{\omega}^T\boldsymbol{\gamma})$$

or, in the skewed-t case,

$$y \sim SkewedT_1(\nu, \boldsymbol{\omega}^T\boldsymbol{\mu}, \boldsymbol{\omega}^T\Sigma\boldsymbol{\omega}, \boldsymbol{\omega}^T\boldsymbol{\gamma}) \tag{11}$$

That is, all portfolios share the same degree of freedom ν.

This also shows that the marginal distributions are automatically known once we have calibrated the multivariate generalized hyperbolic distributions, i.e., $X_i \sim SkewedT_1(\nu, \mu_i, \Sigma_{ii}, \gamma_i)$.

Calibration of t and Skewed t Distributions Using the EM Algorithm

The mean-variance representation of the multivariate skewed t distribution has the great advantage that the EM algorithm is directly applicable. See McNeil, Frey, and Embrechts [2005] for a general discussion of this algorithm for calibrating generalized hyperbolic distributions.

The EM (expectation-maximization) algorithm is a two-step iterative process in which (the E-step) an expected log likelihood function is calculated using current parameter values, and then (the M-step) this function is maximized to produce updated parameter values. After each E and M step, the log likelihood is increased, and the method converges to a maximum log likelihood estimate of the distribution parameters.

What helps this along is that the skewed t distribution can be represented as a conditional normal distribution, so most of the parameters $(\Sigma, \boldsymbol{\mu}, \boldsymbol{\gamma})$ can be calibrated, conditional on W, like a Gaussian distribution. See Hu [2005] for details of our implementation and comparisons with other versions.

Data Sets and Calibration

For illustration, we consider portfolios composed of the following 5 stocks: WALT DISNEY, EXXON MOBIL, PFIZER, ALTRIA GROUP and INTEL, and use adjusted closing prices for the period 7/1/2002 to 08/04/2005. The daily close data are converted to log returns. Exhibit 1 illustrates the relative price movements of each stock using the most recent 750 returns. The initial price of each stock is rescaled to one to facilitate the comparison of relative performance.

From Exhibit 2, we can see that squared returns series show some evidence of serial correlation.

We use a $GARCH(1,1)$ model with Gaussian innovations to remove the serial return dependence for each stock.

That is, we fit parameters α_0, α_1 and β_1 in the following $GARCH(1,1)$ model of the return series X_t:

$$X_t = \sigma_t Z_t \text{ where } Z_t \sim N(0,1) \quad i.i.d., \tag{12}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \tag{13}$$

We then think of Z_t as a "filtered return" which we hope is $i.i.d.$

From Exhibit 3, we can see that squared filtered returns series show no evidence of serial correlation; Exhibit 4 shows that heteroscedasticity clearly exists in 5 stocks.

After we get the approximately $i.i.d.$ training data, we can estimate the multivariate density. Note that in the $GARCH$ fitting we assume filtered marginal returns are Gaussian in order to arrive at best fit *pseudo-maximum-likelihood* $GARCH$ parameters (see Bradley and Taqqu [2002]), but the multivariate distribution that best fits the filtered data need not be *a posteriori* Gaussian.

From QQ-plots versus normal for those 5 stocks in Exhibit 5, we can see that a normal distribution is not a good fit in the tails. Therefore we consider several distributions in the generalized hyperbolic family to model the multivariate density.

Exhibit 6 shows the maximized log likelihood for fitting the filtered returns to various distributions. It shows again that all the generalized hyperbolic distributions we examine have higher log likelihood than the Normal distribution, and the skewed t has the highest log likelihood, with the t close behind.

Risk Measures and Portfolio Optimization

Suppose $\boldsymbol{\omega}^T = (\omega_1, \cdots, \omega_d)$ is the capital amount invested in each risky security in a portfolio, and $\mathbf{X}^T = (X_1, \cdots, X_d)$ is the return of each risky security. Let

$$L(\boldsymbol{\omega}, \mathbf{X}) = -\sum_{i=1}^{d} \omega_i X_i = -\boldsymbol{\omega}^T \mathbf{X}$$

denote the loss of this portfolio over a fixed time interval Δ and F_L its distribution function. (Usually, the time interval Δ is one, ten, or 30 days for equity portfolio management.)

From the portfolio property, if $\boldsymbol{X}$ has distribution $N_d(\boldsymbol{\mu}, \Sigma)$ (Normal), $t_d(\nu, \boldsymbol{\mu}, \Sigma)$ (Student t), or $SkewedT_d(\nu, \boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma})$ (skewed t), then the loss $L(\boldsymbol{\omega}, \boldsymbol{X})$ has distribution

$$L \sim N_1(\text{-}\boldsymbol{\omega}^T \boldsymbol{\mu}, \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega}) \tag{14}$$

$$L \sim t_1(\nu, -\boldsymbol{\omega}^T \boldsymbol{\mu}, \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega}) \tag{15}$$

or

$$L \sim SkewedT_1(\nu, -\boldsymbol{\omega}^T \boldsymbol{\mu}, \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega}, -\boldsymbol{\omega}^T \boldsymbol{\gamma}) \tag{16}$$

respectively.

Whatever the model distribution of the loss random variable L, we independently need to choose a risk measure that associates L with some numerical measure of risk.

Definition 2 Value at Risk *Given a confidence level α between 0 and 1 (such as 99% or 95%), the VaR at confidence level α is the smallest value l such that the probability that the loss L exceeds l is no larger than $(1-\alpha)$. In other words,*

$$VaR_\alpha = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}$$

For the Normal and t distributions, the following explicit VaR formulas are easy to verify. When the loss L is Normally distributed with mean μ and variance σ^2, then

$$VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha) \tag{17}$$

where Φ denotes the standard normal distribution function. When the loss L is t distributed, $L \sim t_1(\nu, \mu, \sigma^2)$, then

$$VaR_\alpha = \mu + \sigma t_\nu^{-1}(\alpha) \tag{18}$$

where t_ν denotes the distribution function of the standard t with degree of freedom ν.

It's helpful to consider more generally some desirable properties for a risk measure.

Definition 3 Coherent Risk Measure *(Artzner et. al. [1999]). A real valued function ρ of a random variable is a coherent risk measure if it satisfies the following properties,*

1. **Subadditivity**. *For any two random variables X and Y, $\rho(X+Y) \leq \rho(X) + \rho(Y)$.*
2. **Monotonicity**. *For any two random variables $X \geq Y$, $\rho(X) \geq \rho(Y)$.*
3. **Positive homogeneity**. *For $\lambda \geq 0$, $\rho(\lambda X) = \lambda\rho(X)$.*
4. **Translation invariance**. *For any $a \in \mathbb{R}$, $\rho(a + X) = a + \rho(X)$.*

In the language above, StD is not a coherent risk measure; VaR is a coherent measure if the underlying distribution is elliptical, but not generally. Expected shortfall (ES), also called Conditional Value at Risk, introduced by Rockafellar and Uryasev [2001], is always coherent.

Definition 4 Expected Shortfall (ES). *For a continuous loss distribution with $\int_{\mathbb{R}} |l| dF_L(l) < \infty$, the ES_α at confidence level $\alpha \in (0,1)$ for loss L of a security or a portfolio is defined to be*

$$ES_\alpha = E(L|L \geq VaR_\alpha) = \frac{\int_{VaR_\alpha}^{\infty} l dF_L(l)}{1-\alpha} \tag{19}$$

$$= \frac{\int I_{\{-(\boldsymbol{\omega}^T \mathbf{x}) \geq VaR_\alpha\}}[-(\boldsymbol{\omega}^T \mathbf{x})] f(\mathbf{x}) d\mathbf{x}}{1-\alpha} \tag{20}$$

ES can also be computed explicitly for some loss distributions: if L is normally distributed $N(\mu, \sigma^2)$, then

$$ES_\alpha = \mu + \sigma \frac{\psi(\Phi^{-1}(\alpha))}{1-\alpha} \tag{21}$$

where ψ is the density of standard normal distribution. If L is t distributed $t(\nu, \mu, \sigma^2)$, then

$$ES_\alpha = \mu + \sigma \frac{f_\nu(t_\nu^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1} \right) \tag{22}$$

where f_ν is the density function of the standard t with degree of freedom ν.

For skewed t, there is no closed formula for or VaR or ES. To calculate VaR or ES, we use numerical integration and a zero-finder routine, or Monte Carlo simulation by using equation (20).

Next we need the concept of an *elliptical distribution*. Briefly, an elliptical distribution is an affine transformation of a spherical distribution; a spherical distribution is one which is invariant under rotations and reflections (that is, spherically symmetric). Explicit definitions are available from many sources, e.g. Bradley and Taqqu [2002]. The normal and t distributions are elliptical; the skewed t is not when $\boldsymbol{\gamma} \neq \mathbf{0}$.

Proposition 5 Efficient Frontier for Elliptical Distributions. *(Embrechts, McNeil, and Straumann [2001]). Suppose* $\mathbf{X}$ *is elliptically distributed and all univariate marginals have finite variance. For any* $r \in \mathbb{R}$*, let*

$$\mathcal{Q} = \{Z = \sum_{i=1}^{d} \omega_i X_i | \omega_i \in \mathbb{R}, \sum_{i=1}^{d} \omega_i = 1, E(Z) = r\}$$

be the set of all fully invested portfolio returns with expectation r*. Then for any positively homogeneous, translation invariant risk measure* ρ*,*

$$argmin_{Z \in \mathcal{Q}} \rho(Z) = argmin_{Z \in \mathcal{Q}} \sigma_Z^2.$$

This proposition means that if we assume that the underlying distribution is elliptical, then the Markowitz minimum variance portfolio, for a given return, will be the same as the optimized portfolio obtained by minimizing any other translation invariant and positively homogeneous risk measure, such as VaR or ES. That is, the portfolio allocation does not depend on the choice of risk measure (or confidence level), but only on the choice of distribution.

The skewed t distribution is not elliptical if $\boldsymbol{\gamma} \neq \mathbf{0}$. In this case we see in practice that the efficient portfolios do depend on the choice of confidence level, and on the whether we use VaR, ES, or StD. The practitioner might view this as a disadvantage of using distributions with skewness – she will have to decide whether the data show enough skewness to justify the need to confront these extra choices.

A practical disadvantage of skewed distributions is that we do not have a closed form formula for VaR or ES. Instead, we turn to Monte Carlo simulation to minimize ES at confidence level α by sampling the multivariate distribution of returns. (This method also can be applied to the elliptical distributions mentioned above.)

More specifically, from (20), we can rewrite the definition of expected shortfall as follows,

$$ES_\alpha = VaR_\alpha + \frac{\int [-(\boldsymbol{\omega}^T \mathbf{x}) - VaR_\alpha]^+ f(\mathbf{x}) d\mathbf{x}}{1-\alpha},$$

where $[x]^+ := max(x, 0)$.

We get a new objective function by replacing VaR by p,

$$F_\alpha(\boldsymbol{\omega}, p) = p + \frac{\int [-(\boldsymbol{\omega}^T \mathbf{x}) - p]^+ f(\mathbf{x}) d\mathbf{x}}{1-\alpha}. \tag{23}$$

Rockafellar and Uryasev [2001] showed that ES can be computed by minimizing this function with respect to $\boldsymbol{\omega}$ and p. If the minimum is $(\boldsymbol{\omega}^*, p^*)$, then $\boldsymbol{\omega}^*$ is the optimized portfolio composition and $p*$ is the corresponding portfolio's VaR at confidence level α.

Below, we sample the multivariate density by Monte Carlo simulation to estimate $F_\alpha(\boldsymbol{\omega}, p)$ by

$$\hat{F}_{\alpha}(\boldsymbol{\omega}, p) = p + \frac{\sum_{k=1}^{n}[-(\boldsymbol{\omega}^T \mathbf{x}_k) - p]^+}{n(1-\alpha)}, \tag{24}$$

where $\mathbf{x}_k$ is the k-th sample from some distribution and n is the number of samples.

Efficient Frontier Analysis

We now study the possible efficient frontiers, for this 5-stock universe, as we vary the risk measure (StD, 99% VaR, 99% ES) and the modeling distribution (Normal, Student t, skewed t).

Suppose we are standing at August 4, 2005, the last date in our data set, and the holding period is one day. 750 sample data are used in the calibration. The one day ahead forecasted $GARCH$ volatilities for all the stocks are denoted $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_5)^T$ at that date. The weight constraint condition is written as

$$\sum_{i=1}^{5} \omega_i = 1, \tag{25}$$

where we assume the initial capital is 1 and ω_i is the capital invested in risky stock i. We suppose short sales are allowed.

Suppose that the calibrated filtered expected log return of stock i is $\hat{\mu}_i$, then the de-filtered forecasted expected return is $\mu_i = \sigma_i \hat{\mu}_i$. Let

$$\boldsymbol{\mu} = (\mu_1, \cdots, \mu_5)^T \tag{26}$$

so that the expected portfolio return is $\boldsymbol{\omega}^T \boldsymbol{\mu}$. We set the expected portfolio return to be a constant c,

$$\boldsymbol{\omega}^T \boldsymbol{\mu} = c \tag{27}$$

and find the efficient frontier by minimizing StD, VaR, or ES subject to the constraints (25) and (27).

Normal Frontier

Under the modeling assumption that returns are Normal, we calibrate the mean and covariance of a multivariate Normal to our filtered data, and then compute the normal efficient frontier.

Exhibit 7 shows the filtered expected log return and the $GARCH$ volatility forecast on Aug 4, 2005; Exhibit 8 shows the best-fit covariance matrix (upper triangular), the variance (diagonal) and correlation matrix (lower triangular) for the returns of the 5 stocks.[1]

We numerically minimize the portfolio variance, 99% VaR, and 99% ES to compute the normal frontiers. Exhibit 9 shows the portfolio compositions and the corresponding standard deviation, 99% VaR and 99% ES. Because the Normal distribution is elliptical, we expect and see that these three methods all arrive at the same portfolio composition for a given return. Exhibit 10 shows three efficient frontiers plotted against 99% ES – where the objective function is either variance (StD), 99% ES, or 99% ES via Monte Carlo simulation. The three frontiers are the same because the optimal portfolios are the same. Note also that changing the confidence level of the objective function will leave the picture unchanged as long as we plot the same variable on the horizontal axis.

t Frontier

Exhibit 11 shows the expected log return for the filtered data using the t distribution. The calibrated degree of freedom is 5.87. Exhibit 12 shows the dispersion matrix (upper triangle), and correlation matrix (lower triangle) for the five stocks[2].

Since the t distribution is elliptical, we again expect the same portfolio

[1]The expected return $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\Sigma}$ are first calibrated with filtered returns. We then restore the de-filtered expected return $\boldsymbol{\mu}$ and covariance matrix Σ by $\mu_i = \hat{\mu}_i \sigma_i$ and $\Sigma = A\hat{\Sigma}A$, where A=Diag($\boldsymbol{\sigma}$).

[2]The expected return $\hat{\boldsymbol{\mu}}$ and dispersion matrix $\hat{\Sigma}$ are first calibrated using filtered returns. We then restore the de-filtered expected return $\boldsymbol{\mu}$ and dispersion matrix Σ by $\mu_i = \hat{\mu}_i \sigma_i$ and $\Sigma = A\hat{\Sigma}A$, where A=Diag($\boldsymbol{\sigma}$). For this reason, the expected log return depends on $\hat{\Sigma}$, and hence on the choice of multivariate distribution. This explains why the expected log returns differ in Exhibits 7 and 11.

compositions on the t frontier, whether we minimize StD, 99% VaR, or 99% ES. This is confirmed in Exhibit 13.

From this table, we can also see that the portfolio compositions are different than those of the Normal frontier. Exhibit 14 displays these (coincident) frontiers on 99% ES - return axes, with the normal frontier included for reference.

Normal vs. t frontiers

From the portfolio property and our explicit formulas (17), (18), (21), (22) for VaR and ES, we have

$$VaR_\alpha = \boldsymbol{\omega}^T \boldsymbol{\mu} + c_1 \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega} \tag{28}$$

where c_1 is a constant depending only on α for the Normal distribution, and a different constant depending only on α and ν for the t distribution. Similarly,

$$ES_\alpha = \boldsymbol{\omega}^T \boldsymbol{\mu} + c_2 \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega} \tag{29}$$

for c_2 depending only on α and ν.

Since $\boldsymbol{\omega}^T \boldsymbol{\mu}$ is held fixed when minimizing risk for the efficient frontier, all three risk measures StD, VaR, ES will therefore produce the same efficient portfolios for both the Normal and the t distributions, provided that we use the same $\boldsymbol{\mu}$ and Σ. Since $\boldsymbol{\mu}$ is the mean of the t distribution and, from equation (4), Σ is a scalar multiple of the covariance matrix, we can summarize this as

Proposition 6 Invariance of efficient portfolios. *If the vector of asset returns is multivariate Normal or t distributed, with correlation matrix C and mean $\boldsymbol{\mu}$, then the portfolios on the efficient frontier depend on C and $\boldsymbol{\mu}$, but do not depend on the degree of freedom ν or on whether the risk measure is chosen to be StD, VaR, or ES.*

The difference between the t and Normal frontiers in Exhibit 14 is therefore due solely to the different means and correlations that arise in calibrating the data to the best-fit Normal or t distributions.

The cost of using a Normal model in a Student t world

For a fixed level of expected return, the corresponding fully invested risk-minimizing portfolio depends on which distribution is used to model returns, because of differing calibrated means and correlations.

As a first example, suppose Adam is a traditional Markowitz mean-variance manager, using StD as his risk measure. He calibrates a multi-variable Normal distribution to his filtered returns, in effect assuming the Normal distribution is a good model for realized returns. Adam now believes his efficient frontier is as shown by the dashed line in Exhibit 15.

However, as we have shown above, the Student t distribution is in fact a better fit to the data. If we suppose that the "true" distribution is the calibrated Student t distribution, the actual efficient frontier is shown by the solid line in Exhibit 15. The circles indicate the efficient portfolios that Adam computes under his incorrect Normal assumption, where we are plotting the "true" expected log return and standard deviation based on the Student t distribution. (Note that these portfolios do not lie on the Normal frontier because Adam's computation of risk using his Normal distribution gives him the wrong answer.)

As expected, all of Adam's portfolios lie below the true frontier. The distance between the circles and the solid curve in Exhibit 15 illustrates amount of available return Adam fails to capture because his chosen portfolios do not lie on the real efficient frontier. For moderate levels of risk he could have increased his portfolio expected return by 20 or 30 percent if he had chosen porfolios on the true efficient frontier.

Suppose now that Betty is another manager who uses 99% ES as her risk measure because of its coherence properties, but for convenience she still assumes filtered returns are normal. Her normal efficient frontier is plotted as the dashed curve in Exhibit 16. If filtered returns are in fact Student t distributed, then the true efficient frontier is the solid curve in the same figure. Here, the normal frontier is actually inaccessible. As can be seen from the tables and from the plotted circles, for a fixed return, the risk-minimizing portfolio Betty chooses is actually not the true ES minimizing portfolio but is inside the accessible region. Betty is investing sub-optimally due to her choice of the Normal as modeling distribution.

Exhibit 16 also illustrates that the minimum variance portfolios are identical to the minimum ES portfolios – as expected since the Student t is an elliptical distribution.

A similar discussion, illustrated by plotting frontiers against 99% VaR in Exhibit 17, shows the results for Carol, a 99 % VaR minimizer who assumes returns are Normal.

Skewed t Frontier

Exhibit 6 showed that a fitted skewed t distribution has a slightly higher log likelihood than the Student t because of a small amount of skewness, shown in Exhibit 18. The calibrated degree of freedom is 5.93. Exhibit 19 shows the dispersion matrix (upper triangular), and correlation matrix (lower triangular) for the 5 stocks.

We use Monte Carlo simulation[3] to find the skewed t frontier by minimizing expected shortfall. Exhibit 20 shows the 99% level portfolio compositions and the corresponding 99% ES, and Exhibit 21 shows the 95% level portfolio compositions and the corresponding 95% ES. Since the skewed t distribution is not elliptical, the 99% level and 95% level produce slightly different portfolios.

In Exhibit 22 we show a comparison of the three efficient frontiers, one for each of the three distributions, against 99% ES. We also include 95% ES to illustrate that the confidence level now matters. The skewed t and t frontiers are very close for small returns. When returns are large, the two curves diverge. Note that the calibrated $\boldsymbol{\mu}$ and Σ are similar for the t and skewed t distributions, so the divergence is attributable to the skewness parameter in the skewed t distribution, which affects correlations according to equation

[3]We use the filtered returns series to calibrate skewed t distribution and then use the mean-variance mixture definition to sample the multivariate skewed t distribution to get the 1,000,000 samples $\hat{X}_{1000000\times 5}$. Specifically, in Matlab, we generate 1,000,000 multivariate normal distributed random variables with mean $\mathbf{0}$ and covariance $\hat{\Sigma}$, which is calibrated using filtered returns series, then we generate 1,000,000 $InverseGamma(\nu/2, \nu/2)$ distributed random variables, finally, we get 1,000,000 multivariate skewed t distributed random variables by using the mean-variance mixture definition. The restored samples $X = \hat{X}A$, where $A = Diag(\boldsymbol{\sigma})$. The restored mean $\mu_i = (\hat{\mu}_i + \frac{\nu}{\nu-2}\hat{\gamma}_i)\sigma_i$ where $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\gamma}}$ are location and skewness parameters respectively calibrated using filtered data.

(4). Here again, if we suppose that the true distribution of returns is skewed t, the manager who assumes skewness is zero arrives at the wrong efficient portfolios for large returns. Comparison of Exhibits 20 and 13 show that skewness has a noticeable effect on both the magnitude of the minimum ES for large returns, and on the portfolio composition itself.

Conclusion

Distributions matter. When calibrating non-normal data to a normal distribution, it is not surprising that we might see inaccurate estimates of means and correlations. This is confirmed with our daily equity price data.

The result is that the composition of optimized portfolios can be quite sensitive to the kind of modeling distribution chosen.

The t distribution forms a better fit (in the sense of log likelihood) to our equity data than does the Normal or several other common families of Generalized Hyperbolic distributions. The skewed t is slightly better. When passing from Normal to t, the calibrated filtered means and dispersion matrices (Σ) change substantially, leading to a noticeable effect on the efficient frontiers. Introducing skewness with the skewed t distribution does not change the calibrated correlations or means much, but the skewness still affects the efficient portfolios for larger values of expected return. There is some evidence of skewness in our data, but the increased log likelihood obtained by introducing a skewness parameter is small.

Calibration of the t and skewed t distributions can be accomplished with the EM algorithm. In the case of the skewed t, we lack explicit formulas for VaR or ES, so that we must use Monte Carlo simulation to compute risk in that case.

Since non-elliptical distributions are in many ways less convenient, managers may choose for simplicity to opt for the t over the skewed t distribution, especially when estimated skewness may be small. The t distribution is easy enough, compared to the Normal, that we recommend managers graduate at least to that family. They can find a much better fit to the data at the cost of only one extra parameter (ν), and, because we believe real returns are fat-tailed, capture much more of the true available expected return at a give true level of risk.

Appendix

Distribution Formulas

Definition 7 Generalized Inverse Gaussian distribution(GIG). *The random variable X is said to have a generalized inverse gaussian(GIG) distribution if its probability density function is*

$$h(x;\lambda,\chi,\psi)=\frac{\chi^{-\lambda}(\sqrt{\chi\psi})^{\lambda}}{2K_\lambda(\sqrt{\chi\psi})}x^{\lambda-1}exp\left(-\frac{1}{2}(\chi x^{-1}+\psi x)\right),x>0, \tag{30}$$

where K_λ is a modified Bessel function of the third kind with index λ,

$$K_\lambda(x)=\frac{1}{2}\int_0^\infty y^{\lambda-1}e^{-\frac{x}{2}(y+y^{-1})}dy, \quad x>0 \tag{31}$$

and the parameters satisfy

$$\begin{cases} \chi>0,\psi\geq 0 & if\lambda<0 \\ \chi>0,\psi>0 & if\lambda=0 \\ \chi\geq 0,\psi>0 & if\lambda>0 \end{cases}$$

In short, we write $X\sim N^-(\lambda,\chi,\psi)$ if X is GIG distributed.

Generalized Hyperbolic Distributions. If the mixing variable $W\sim N^-(\lambda,\chi,\psi)$, then the density of the resulting generalized hyperbolic distribution is

$$f(\boldsymbol{x})=c\frac{K_{\lambda-\frac{d}{2}}\left(\sqrt{(\chi+(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))\,(\psi+\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma})}\right)e^{(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}\boldsymbol{\gamma}}}{\left(\sqrt{(\chi+(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))\,(\psi+\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma})}\right)^{\frac{d}{2}-\lambda}}, \tag{32}$$

where the normalizing constant is

$$c=\frac{(\sqrt{\chi\psi})^{-\lambda}\psi^{\lambda}(\psi+\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma})^{\frac{d}{2}-\lambda}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}K_\lambda(\sqrt{\chi\psi})},$$

and $|\cdot|$ denotes the determinant.

Skewed t Distribution. Let $\mathbf{X}$ be skewed t distributed, and define

$$\rho(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}). \tag{33}$$

Then the joint density function of $\boldsymbol{X}$ is given by

$$f(\boldsymbol{x}) = c\frac{K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu+\rho(\boldsymbol{x}))\,(\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma})}\right)e^{(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}\boldsymbol{\gamma}}}{\left(\sqrt{(\nu+\rho(\boldsymbol{x}))\,(\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma})}\right)^{-\frac{\nu+d}{2}}(1+\frac{\rho(\boldsymbol{x})}{\nu})^{\frac{\nu+d}{2}}}, \tag{34}$$

where the normalizing constant is

$$c = \frac{2^{1-\frac{\nu+d}{2}}}{\Gamma(\frac{\nu}{2})(\pi\nu)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}.$$

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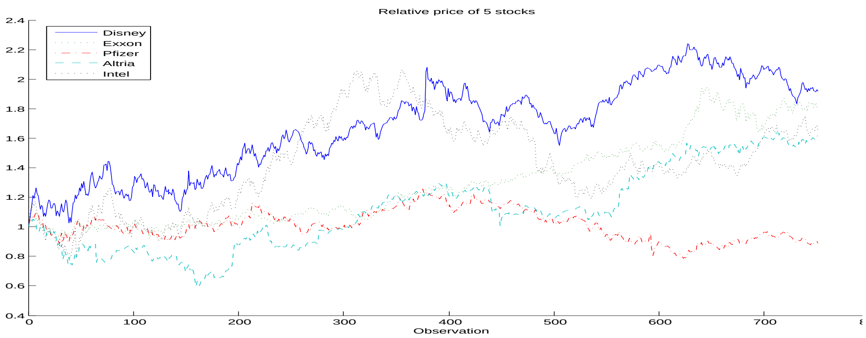

Exhibit 1: Relative price of 5 stocks: Disney, Exxon, Pfizer, Altria, and Intel

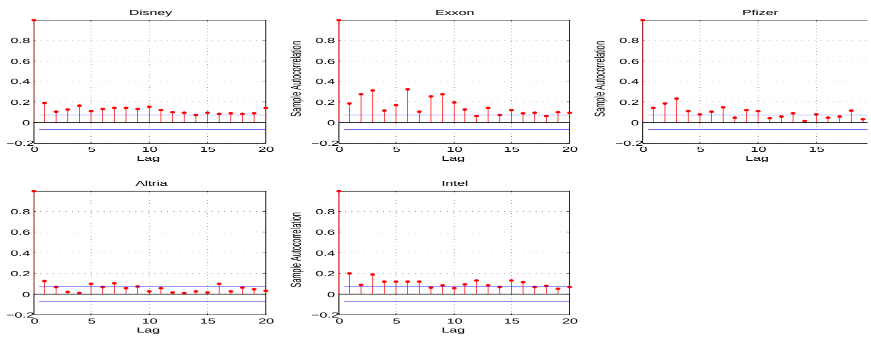

Exhibit 2: Correlograms of squared log return series for 5 stocks

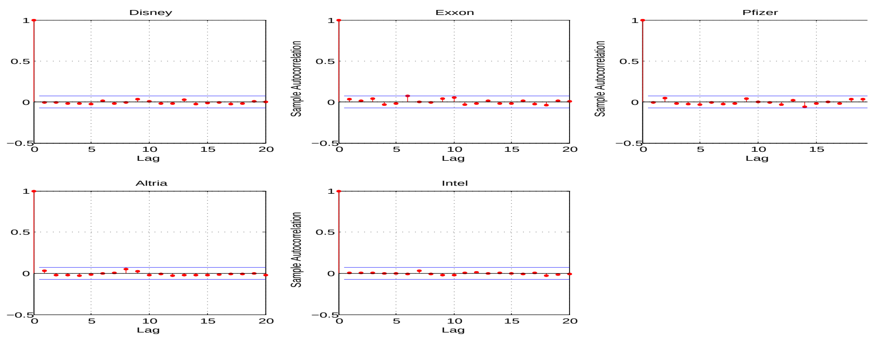

Exhibit 3: Correlograms of squared filtered log return series for 5 stocks

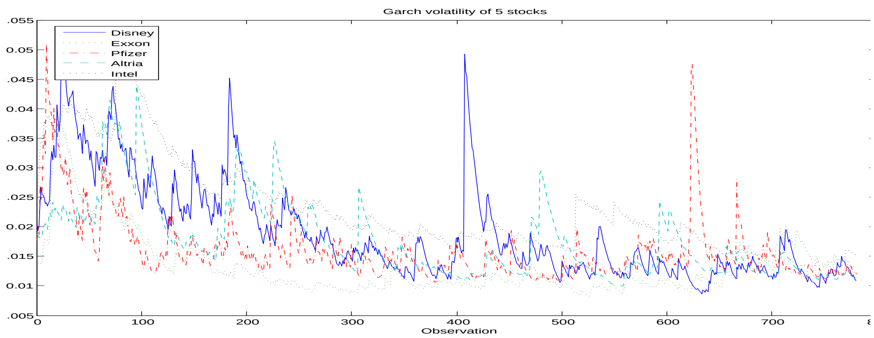

Exhibit 4: $GARCH$ volatility of log return series for 5 stocks over time

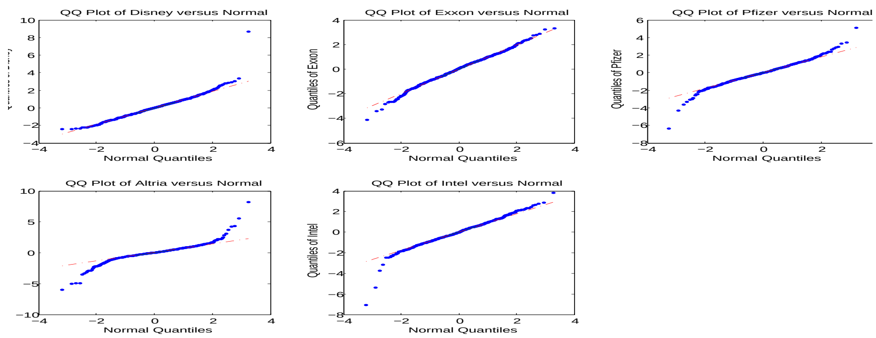

Exhibit 5: QQ-plot versus normal distribution for 5 stocks

Model	Normal	Student t	Skewed t	VG	Hyperbolic	NIG
LL	-5095.0	-4877.8	-4873.9	-4901.7	-4891.5	-4884.2

Exhibit 6: Log likelihood (LL) of estimated multivariate density for 6 distribution families

Stock	Disney	Exxon	Pfizer	Altria	Intel
Expected filtered return	0.040	0.073	-0.015	0.039	0.027
$GARCH$ volatility	0.0107	0.0128	0.0130	0.0113	0.0156

Exhibit 7: Expected filtered log return and one day ahead forecasted $GARCH$ volatility on 08/04/2005

Stock	Disney	Exxon	Pfizer	Altria	Intel
Disney	1.009	0.372	0.340	0.191	0.428
Exxon	0.367	1.015	0.364	0.199	0.309
Pfizer	0.337	0.359	1.008	0.217	0.302
Altria	0.189	0.197	0.215	1.009	0.171
Exxon	0.420	0.303	0.297	0.168	1.027

Exhibit 8: Covariance and correlation matrix obtained by calibrating filtered returns to a Normal distribution. Variances are on the diagonal, covariances above the diagonal, and correlations below.

Return	StD	99%VaR	99% ES	Disney	Exxon	Pfizer	Altria	Intel
0	0.0096	0.0223	0.0256	0.319	-0.206	0.528	0.320	0.040
0.0002	0.0084	0.0194	0.0222	0.318	-0.038	0.344	0.333	0.043
0.0004	0.0079	0.0180	0.0207	0.318	0.131	0.161	0.345	0.045
0.0006	0.0082	0.0186	0.0214	0.317	0.300	-0.023	0.358	0.048
0.0008	0.0093	0.0209	0.0241	0.317	0.468	-0.206	0.371	0.050
0.001	0.0109	0.0244	0.0281	0.316	0.637	-0.390	0.384	0.052
0.0012	0.0129	0.0287	0.0331	0.316	0.806	-0.573	0.397	0.055
0.0014	0.0150	0.0335	0.0386	0.316	0.974	-0.757	0.409	0.057
0.0016	0.0173	0.0386	0.0444	0.315	1.143	-0.940	0.422	0.060
0.0018	0.0196	0.0438	0.0505	0.315	1.312	-1.124	0.435	0.062
0.002	0.0220	0.0492	0.0567	0.314	1.480	-1.307	0.448	0.065

Exhibit 9: Portfolio composition and corresponding standard deviation, 99%VaR and 99%ES for the Normal frontier

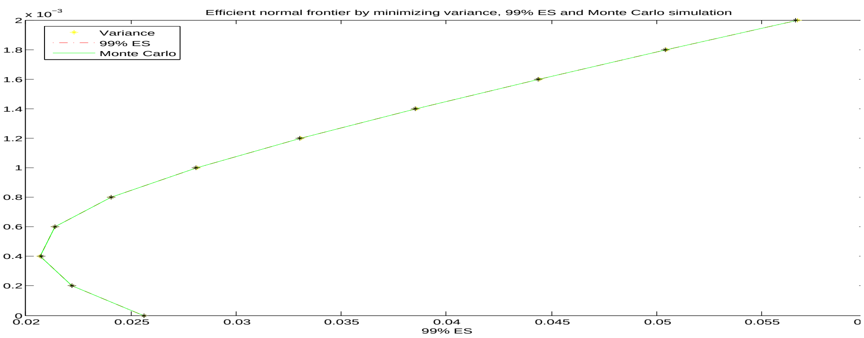

Exhibit 10: Markowitz efficient frontier vs 99% ES for the Normal distribution. Note thhat all three risk measures give the same results.

Stock	Disney	Exxon	Pfizer	Altria	Intel
Expected filtered return	0.015	0.077	-0.018	0.069	0.030
$GARCH$ volatility	0.0107	0.0128	0.0130	0.0113	0.0156

Exhibit 11: Expected filtered log return using the Student t distribution

Stock	Disney	Exxon	Pfizer	Altria	Intel
Disney	0.709	0.268	0.267	0.159	0.332
Exxon	0.363	0.771	0.274	0.170	0.244
Pfizer	0.378	0.373	0.702	0.155	0.250
Altria	0.265	0.271	0.259	0.511	0.138
Exxon	0.460	0.324	0.349	0.225	0.734

Exhibit 12: Dispersion (Σ) and correlation matrix obtained by calibrating filtered returns to a t distribution. Correlations are shown below the diagonal. (Recall that the covariance matrix is equal to $\frac{\nu}{\nu-2}\Sigma$)

Return	StD	99%VaR	99% ES	Disney	Exxon	Pfizer	Altria	Intel
0	0.0095	0.0245	0.0316	0.494	-0.153	0.447	0.247	-0.035
0.0002	0.0086	0.0218	0.0281	0.410	-0.048	0.315	0.336	-0.014
0.0004	0.0080	0.0203	0.0262	0.326	0.057	0.184	0.425	0.008
0.0006	0.0081	0.0201	0.0261	0.242	0.162	0.052	0.515	0.030
0.0008	0.0086	0.0214	0.0278	0.158	0.267	-0.080	0.604	0.051
0.001	0.0097	0.0238	0.0310	0.074	0.371	-0.211	0.693	0.073
0.0012	0.0110	0.0271	0.0352	-0.010	0.476	-0.343	0.782	0.094
0.0014	0.0126	0.0309	0.0402	-0.094	0.581	-0.474	0.871	0.116
0.0016	0.0143	0.0352	0.0457	-0.178	0.686	-0.606	0.961	0.138
0.0018	0.0161	0.0396	0.0515	-0.262	0.791	-0.737	1.050	0.159
0.002	0.0180	0.0443	0.0576	-0.347	0.895	-0.869	1.139	0.181

Exhibit 13: Portfolio composition and corresponding standard deviation, 99%VaR and 99%ES for the t frontier

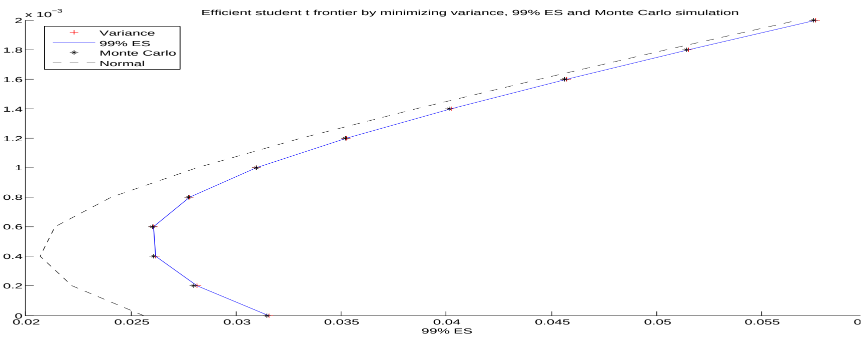

Exhibit 14: t frontier and Normal frontier versus 99% ES. The t frontier is unchanged if we use Monte Carlo simulation to compute ES, or if we minimize variance instead of ES

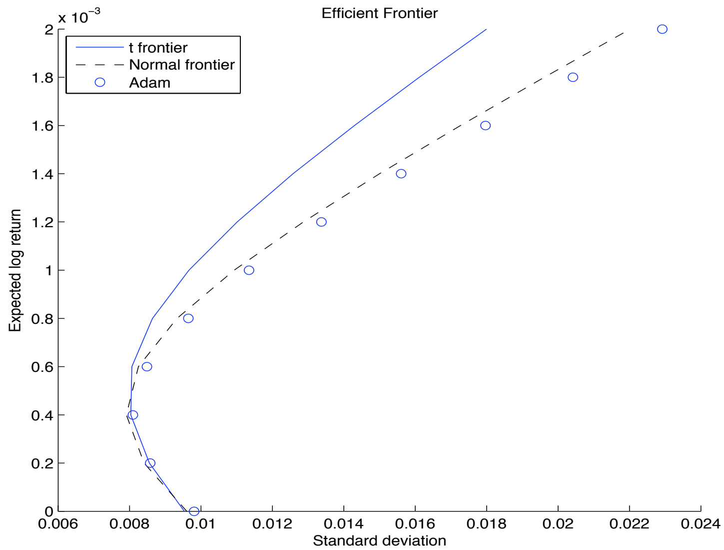

Exhibit 15: The efficient t and Normal frontiers vs StD, along with Adam's portfolio optimized under the assumption of Normality

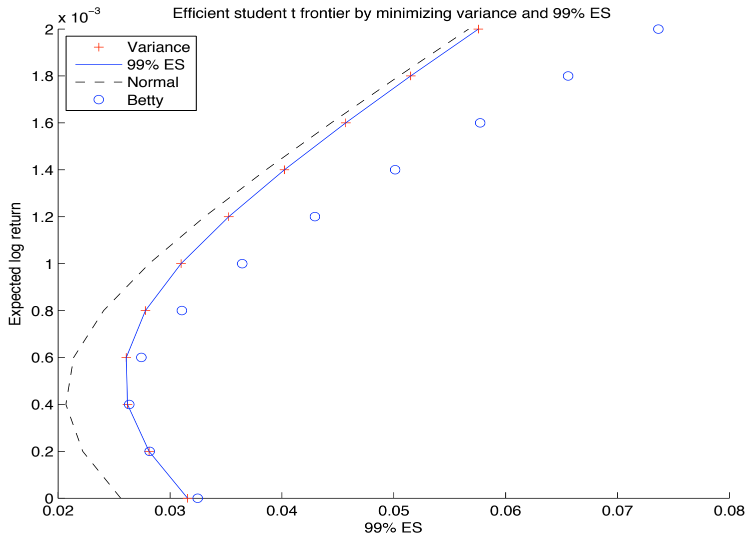

Exhibit 16: The efficient t and Normal frontiers vs 99% ES, along with Betty's portfolio optimized under the assumption of Normality

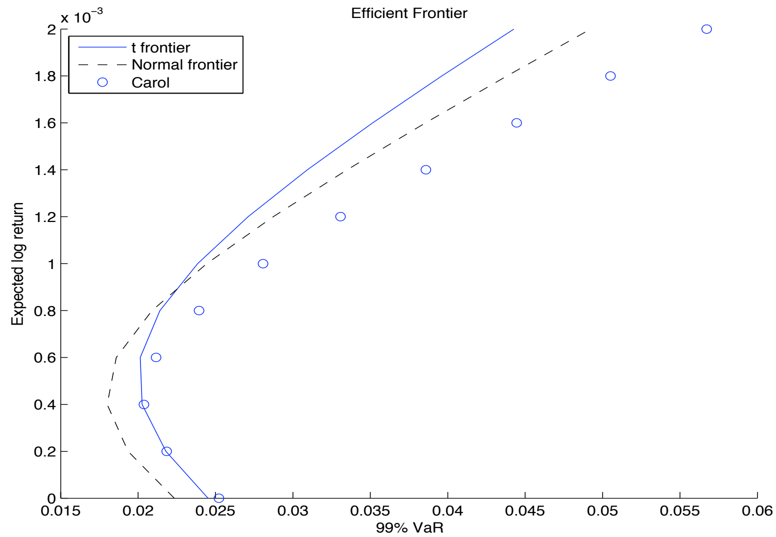

Exhibit 17: The efficient t and Normal frontiers vs 99% VaR, along with Carol's portfolio optimized under the assumption of Normality

Stock	Disney	Exxon	Pfizer	Altria	Intel
location parameters	-0.071	0.089	-0.030	0.161	0.042
skewness parameters	0.073	-0.010	0.010	-0.079	-0.010
$GARCH$ volatility	0.0107	0.0128	0.0130	0.0113	0.0156

Exhibit 18: Expected filtered log return and skewness parameters for the skewed t distribution

Stock	Disney	Exxon	Pfizer	Altria	Intel
Disney	0.706	0.269	0.267	0.164	0.333
Exxon	0.269	0.773	0.275	0.171	0.244
Pfizer	0.267	0.275	0.704	0.157	0.251
Altria	0.164	0.171	0.157	0.509	0.139
Exxon	0.333	0.244	0.251	0.139	0.736

Exhibit 19: Dispersion matrix Σ obtained by calibrating filtered returns to a skewed t distribution

Return	99% ES	Disney	Exxon	Pfizer	Altria	Intel
0	0.0320	0.393	-0.219	0.515	0.337	-0.026
0.0002	0.0280	0.383	-0.058	0.328	0.363	-0.015
0.0004	0.0263	0.374	0.101	0.139	0.389	-0.003
0.0006	0.0274	0.381	0.259	-0.051	0.406	0.006
0.0008	0.0312	0.399	0.415	-0.244	0.416	0.013
0.001	0.0367	0.428	0.573	-0.436	0.415	0.021
0.0012	0.0433	0.456	0.733	-0.626	0.413	0.024
0.0014	0.0506	0.485	0.892	-0.817	0.409	0.031
0.0016	0.0583	0.514	1.052	-1.007	0.406	0.036
0.0018	0.0662	0.549	1.209	-1.200	0.404	0.038
0.002	0.0744	0.587	1.365	-1.394	0.399	0.042

Exhibit 20: Portfolio composition and corresponding 99%ES for the skewed t frontier

Return	95% ES	Disney	Exxon	Pfizer	Altria	Intel
0	0.0215	0.354	-0.222	0.515	0.367	-0.013
0.0002	0.0187	0.348	-0.065	0.325	0.393	-0.002
0.0004	0.0175	0.349	0.094	0.136	0.415	0.006
0.0006	0.0182	0.356	0.253	-0.054	0.430	0.014
0.0008	0.0206	0.369	0.412	-0.245	0.442	0.023
0.001	0.0242	0.386	0.570	-0.435	0.449	0.029
0.0012	0.0285	0.407	0.730	-0.625	0.453	0.036
0.0014	0.0333	0.426	0.889	-0.816	0.459	0.042
0.0016	0.0383	0.447	1.047	-1.007	0.466	0.047
0.0018	0.0436	0.470	1.206	-1.197	0.468	0.053
0.002	0.0489	0.492	1.366	-1.387	0.472	0.057

Exhibit 21: Portfolio composition and corresponding 95%ES for the skewed t frontier

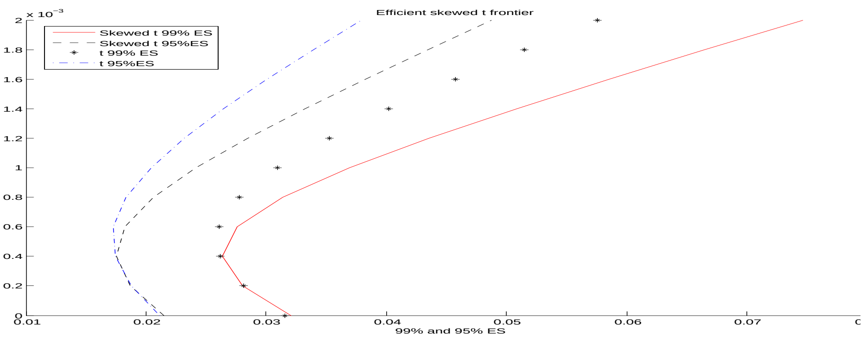

Exhibit 22: Skewed t efficient frontier at 99% ES or 95% ES versus t frontier