ZERO-DIMENSIONAL SUBRINGS OF COMMUTATIVE RINGS

ROBERT GILMER
Department of Mathematics 3027, Florida State University, Tallahassee, FL 32306-3027, U.S.A., email: gilmer@math.fsu.edu

Abstract. We consider various conditions of the sets of zero-dimensional, Artinian, and von Neumann regular subrings of a commutative ring. Section 1 treats questions of existence of such rings, Section 2 deals with the situation in which all subrings belong to one of the three classes, and Section 3 is concerned with the behavior of the sets under intersection. In Section 4 we give a brief survey of some generalizations and extensions of results of Sections 1–3, as well as some related results.

Key words: Zero-dimensional subrings, Artinian rings, von Neumann regular rings.

Introduction

Several recent papers in the literature have dealt with aspects of zero-dimensionality in commutative rings. Work of M. Arapovic [1, 2, 3] seems to have provided the stimulus for much of this activity. Beginning with the paper [8] that was closely related to Arapovic's work, W. Heinzer and I have considered several questions concerning zero-dimensional commutative rings. The current paper is primarily a survey of some recent results on questions regarding zero-dimensional subrings — existence, behavior, etc. — and many of its results are cited from [9] and [12]. In particular, any result of my own recorded in this paper represents work done jointly with Heinzer. Before discussing results of the paper in greater detail, we introduce some terminology and notation.

We work within the category of commutative unitary rings. If R is a subring of S, we assume that the unity of S belongs to R, and hence is the unity of R. The term *dimension of* R, written $\dim(R)$ or $\dim R$, always refers to the Krull dimension of R. Hence $\dim R = n$ if there exists a chain $P_0 < P_1 < \ldots < P_n$ of prime ideals of $\mathrm{Spec}(R)$, but no longer such chain; $\dim R = \infty$ if there exist arbitrary long chains of prime ideals of R, and $\dim R = 0$ if each proper prime ideal of R is maximal in R. Numerous characterizations of zero-dimensionality are known (see, for example [19], [1], [16, Sect. 3]); one of these most useful to us is the result that R is zero-dimensional if and only if some power of each principal ideal of R is idempotent. We denote by $\mathcal{Z}$ the class of zero-dimensional commutative unitary rings and by $\mathcal{A}$ and $\mathcal{V}$, respectively, the subclasses of $\mathcal{Z}$ consisting of Artinian and von Neumann regular rings, respectively.

Section 1 of this paper is concerned with the problem of determining conditions under which a ring R contains a zero-dimensional (or Artinian or von Neumann regular) subring. A trivial case where this occurs is where $\mathrm{char}\, R \neq 0$. In the characteristic-zero case, we observe that the von Neumann regular ring $\prod\{\mathbf{Z}/p\mathbf{Z} : p$ is a prime integer$\}$ admits no Artinian subring while, for a fixed prime q, the ring $\prod_{n=1}^{\infty}(\mathbf{Z}/q^n\mathbf{Z})$ admits no zero-dimensional subring. On the other hand, Theorem 1.7 shows that a zero-dimensional semiquasilocal ring T is integral over an Artinian subring S and that T is a directed union of Artinian subrings.

In Section 2 we characterize in Theorems 2.1, 2.3, and 2.4, respectively, rings R for which each subring is zero-dimensional, Artinian, or von Neumann regular. This is, of course, opposite of the case considered in Section 1 — that is, the case where R may contain no zero-dimensional, etc. subrings. To indicate the flavor of the results of Section 2, we note that Theorem 2.1 shows that each subring of R is zero-dimensional if and only if, for each $P \in \operatorname{Spec}(R)$, R/P is a field of nonzero characteristic that is algebraic over its prime subfield.

If R is a ring that admits a zero-dimensional subring and if $\mathcal{Z}(R)$ denotes the set of zero-dimensional subrings of R, Theorem 3.2 shows that $\mathcal{Z}(R)$ is closed under arbitrary intersection. A key concept in the proof of (3.2) is that of the pointwise inverse y of an element x of R such that xR is idempotent; y is defined by the conditions $x = x^2y$ and $y = y^2x$, and it is uniqueness of y that yields closure of $\mathcal{Z}(R)$ under intersection. It follows easily (Corollary 3.3) that the set of von Neumann regular subrings of R is closed under arbitrary intersection if it is nonempty. On the other hand, the set of Artinian subrings of R need not be closed under finite intersection, and if closed under finite intersection, need not be closed under arbitrary intersection.

Section 4 is a brief survey of some extensions and generalizations of results of Sections 1–3. In addition, some related results are mentioned in Section 4.

1. Existence of Zero-Dimensional Subrings

For a ring R, denote by $\mathcal{Z}(R)$, $\mathcal{A}(R)$, and $\mathcal{V}(R)$, respectively, the sets of zero-dimensional, Artinian, and von Neumann regular subrings of R. We consider briefly in this section the three conditions:

$$(z^*)\ \mathcal{Z}(R) \neq \emptyset$$
$$(a^*)\ \mathcal{A}(R) \neq \emptyset$$
$$(v^*)\ \mathcal{V}(R) \neq \emptyset$$

Clearly (z^*) follows from either (a^*) or (v^*), and it is easy to see that there are no other implications among the three conditions. Recall that we only consider subrings of R that contain the unity of R, and hence contain π, the prime subring of R. Thus, a trivial case in which $\mathcal{A}(R)$ and $\mathcal{Z}(R)$ are nonempty is that in which π is finite — that is, $\operatorname{char} R = n > 0$; in this case $\mathcal{V}(R)$ is nonempty if and only if n is square-free. In the case of characteristic 0, the set $\mathcal{C}(R) = \{\operatorname{char}(R/M) : M$ is a maximal ideal of $R\}$ is relevant to our considerations, as indicated by the following result.

Proposition 1.1. *If R is a subring of the ring S, then $\mathcal{C}(S) - \{0\} \subseteq \mathcal{C}(R)$; the inclusion may be proper, but if R is zero-dimensional, $\mathcal{C}(S) = \mathcal{C}(R)$.*

Proof. If $p \in \mathcal{C}(S) - \{0\}$, say $p = \operatorname{char}(S/M)$ where M is maximal in S, then $M \cap R$ is contained in a maximal ideal N of R and $\mathrm{char}(R/N) = p$. Thus $\mathcal{C}(S) - \{0\} \subseteq \mathcal{C}(R)$. The example $S = \mathbf{Q}, R = \mathbf{Z}$ shows that the inclusion may be proper. Minimal primes of R are contracted from each extension ring of R, and from this it follows easily that $\mathcal{C}(S) = \mathcal{C}(R)$ if R is zero-dimensional.

Corollary 1.2. *If $\mathcal{C}(S)$ is infinite, then $\mathcal{C}(R)$ is infinite for each subring R of S. In particular, S admits no Artinian subring.*

If P denotes the set of positive primes, then Corollary 1.2 shows that $\prod_{p\in P}(\mathbf{Z}/p\mathbf{Z})$ is a von Neumann regular ring that admits no Artinian subring. We show presently (1.4) that a related ring, $\prod_{n=1}^{\infty}(\mathbf{Z}/q^n\mathbf{Z})$ for q prime, admits no zero-dimensional subring. The proof of Proposition 1.4 uses the following result from [8].

Result 1.3. ([8, Example 5]) Let q be a prime integer, let $T = \prod_{n=1}^{\infty}(\mathbf{Z}/q^n\mathbf{Z})$, and let $S = \pi + I$, where $\pi \simeq \mathbf{Z}$ is the prime subring of T and $I = \sum_{n=1}^{\infty} \oplus(\mathbf{Z}/q^n\mathbf{Z})$ is the direct sum ideal of T. Then S is not imbeddable in a zero-dimensional ring.

Proposition 1.4. ([12, Prop. 2.6]) *Let the notation be as in the statement of Result 1.3. Then T contains no zero-dimensional subring.*

Proof. Assume to the contrary that T contains a zero-dimensional subring R. The integral closure R^* of R in T is also zero-dimensional, and R^* contains each idempotent of T. However, $S = \pi + I$ is generated as an extension ring of π by the primitive idempotents e_i, where e_i is the characteristic function of $\{i\}$. Hence $S \subseteq R^*$, and this contradicts Result 1.3.

Proposition 1.5 provides another class of rings that admit no zero-dimensional subring.

Proposition 1.5. ([12, Prop. 2.2]) *If T is a finitely generated extension ring of $\mathbf{Z}$, then T admits no zero-dimensional subring.*

Proof. Suppose not, and let S be a zero-dimensional subring of T. Since the prime ideal (0) of $\mathbf{Z}$ is contracted from S, it is contracted from a maximal ideal M of S. Then the field S/M is a finitely generated ring extension of $\mathbf{Z}$; this is known to be impossible, for $\mathbf{Z}$ is a Hilbert ring [18, §1–3], [7, §31].

The rings considered in Propositions 1.4 and 1.5 admit no Artinian subrings, but in Theorem 1.7 we show that Artinian subrings abound in a zero-dimensional semiquasilocal ring. The proof of (1.7) uses a basic result that has many applications.

Result 1.6. The ring S is integral over its subring R if and only if S/P is integral over $R/(P \cap R)$ for each $P \in \mathrm{Spec}(S)$.

Proof. The condition on S/P is clearly necessary. To show that it is sufficient, suppose S is not integral over R. If $s \in S$ is not integral over R, then $N = \{f(s) :$

$f(X)$ is a monic polynomial over $R\}$ is a multiplicative set in S that does not meet the ideal (0) of S. If $P \in \mathrm{Spec}(S)$ is such that $P \cap N = \emptyset$, then $s + P$ is an element of S/P that is not integral over $R/(P \cap R)$.

Theorem 1.7. **([9, Cor. 5.5])** *If T is a zero-dimensional semiquasilocal ring, then there exists an Artinian subring S of T such that T/S is integral. The ring T is a directed union of Artinian subrings.*

Proof. The ring T is a finite direct sum of zero-dimensional quasilocal rings T_i. If we prove Theorem 1.7 for the ring T_i, the general case then follows easily. Hence we assume without loss of generality that T is quasilocal with maximal ideal M. Let $c = \mathrm{char}(T/M)$ and let π be the prime subring of T. If $c \neq 0$, then π is finite since M is the nilradical of T; if $c = 0$, then T contains an isomorphic copy of $\mathbf{Q}$ since $\pi \simeq \mathbf{Z}$ meets M in (0) and elements of $T - M$ are units of T. Thus, in either case, T contains an Artinian subring R. Because 0 and 1 are the only idempotents in T, the ring R must, in fact, be local. Let P be the maximal ideal of R. Then $M \cap R = P$ and we can choose a transcendence basis $\{x_\lambda + M\}_{\lambda \in \Lambda}$ for T/M over R/P. Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be a set of indeterminates over R and consider the R-epimorphism φ of $R[\{Y_\lambda\}]$ onto $R[\{x_\lambda\}]$ that maps Y_λ to x_λ for each $\lambda \in \Lambda$. If $f \in R[\{Y_\lambda\}]$ has unit content, then $\varphi(f) \notin M$, and hence $\varphi(f)$ is a unit of T. Let U be the multiplicative set in $R[\{Y_\lambda\}]$ consisting of all polynomials of unit content. Then φ extends to an epimorphism of $R(\{Y_\lambda\}) = R[\{Y_\lambda\}]_U$ onto $R[\{x_\lambda\}]_{\varphi(U)} = S$. It is known that $PR(\{Y_\lambda\})$ is the only maximal ideal of $R(\{Y_\lambda\})$ [20, (6.17)], and clearly each element of $PR(\{Y_\lambda\})$ is nilpotent. Therefore $R(\{Y_\lambda\})$ is a local Artinian ring with maximal ideal $PR(\{Y_\lambda\})$. Consequently, S is also local and Artinian. We show that T/S is integral. To do so, it suffices, by Result 1.6, to show that T/M is integral over $S/(M \cap S)$. This latter statement follows because $R[\{x_\lambda\}] \subseteq S$ and, by choice of $\{x_\lambda\}$, T/M is integral over $R[\{x_\lambda\}]/(M \cap R[\{x_\lambda\}])$.

The set $\{S_\alpha\}$ of finitely generated extensions of S in T is a directed family of subrings of T with union T. Each S_α is Noetherian since S is Noetherian and each S_α is zero-dimensional because S_α/S is integral. Therefore the rings S_α are Artinian.

Corollary 1.8. *Suppose T is a ring with only finitely many idempotents. If T admits a zero-dimensional subring, then T admits an Artinian subring.*

Proof. In view of Theorem 1.7, it suffices to show that a zero-dimensional ring S with only finitely many idempotents is semiquasilocal. Thus, if $M_1, \ldots, M_{n+1}$ are distinct maximal ideals of S, choose $x \in M_{n+1} - (\cup_{i=1}^n M_i)$. Since S is zero-dimensional, $x^t S = eS$ for some $t \in \mathbf{Z}^+$ and some idempotent e. Then $e \in M_{n+1} - (\cup_{i=1}^n M_i)$, and it follows that if S has k maximal ideals, then S has at least $k-1$ idempotents. Consequently, S is semiquasilocal, as we wished to show.

2. Universal Conditions on Zero-Dimensional Subrings

Suppose R is a ring. Section 1 addresses the possibility that $\mathcal{Z}(R)$, $\mathcal{A}(R)$ or $\mathcal{V}(R)$ is empty. Another extreme case is that in which each subring of R is in $\mathcal{Z}(R)$,

$\mathcal{A}(R)$ or $\mathcal{V}(R)$. In determining conditions under which this occurs, we employ the following terminology. If E is a ring-theoretic property, we say that R *has property* E *hereditarily* if each subring of R has property E. Thus, R is *hereditarily zero-dimensional* if each subring of R is zero-dimensional, R is *hereditarily Artinian* if each subring of R is Artinian, etc. Theorem 2.1 gives equivalent conditions in order that R should be hereditarily zero-dimensional. In the statement of (2.1), recall that a field F is *absolutely algebraic* if it is algebraic over its prime subfield.

Theorem 2.1. **([9; (4.2), (4.3)])** *In a ring R, the following conditions are equivalent.*

(1) R is hereditarily zero-dimensional.

(2) R/P is an absolutely algebraic field of nonzero characteristic for each $P \in \operatorname{Spec}(R)$.

(3) $\operatorname{char} R \neq 0$ and R is integral over its prime subring.

Proof. (1)$\Rightarrow$(2): Assume that R is hereditarily zero-dimensional and let π be the prime subring of R. Then π is finite since π is zero-dimensional. If $P \in \operatorname{Spec}(R)$, the field R/P is hereditarily zero-dimensional since the subrings of R/P are precisely the homomorphic images of subrings of R containing P. Clearly this implies that R/P is algebraic over its prime subfield $\pi/(P \cap \pi)$, a finite field. Hence R/P is an absolutely algebraic field of nonzero characteristic.

(2)$\Rightarrow$(3): The hypothesis of (2) implies that R is zero-dimensional, and it follows from Result 1.6 that R is integral over its prime subring π. Hence $\dim \pi = 0$, so $\operatorname{char} R \neq 0$.

The implication (3)$\Rightarrow$(1) is patent.

Corollary 2.2. *If R is hereditarily zero-dimensional, then R is a directed union of Artinian subrings.*

Proof. As in the proof of Theorem 1.7, the set of finite extensions of π in R is a directed set of Artinian subrings of R with union R.

There are easy counterexamples to the converse of Corollary 2.2. For example, an Artinian ring of nonzero characteristic, such as $\mathbf{Z}_p(X)$, need not be integral over its prime subring. A positive result in the direction of the converse is that a ring T is zero-dimensional if it is the directed union of zero-dimensional subrings.

The hereditarily Artinian rings are obviously the rings that are both hereditarily zero-dimensional and hereditarily Noetherian; the latter class of rings has been considered in [14]. Theorem 2.3 contains a characterization of hereditarily Artinian rings.

Theorem 2.3. **([9, Thm. 4.10])** *The following conditions are equivalent for a ring R.*

(1) R is hereditarily Artinian.

(2) The nilradical $N(R)$ of R is finite and $R/N(R)$ is a finite direct sum of absolutely algebraic fields of nonzero characteristic.

(3) $R = R_1 \oplus \ldots \oplus R_m \oplus S$, where each R_i is an absolutely algebraic field of nonzero characteristic and S is a finite ring.

Proof. (1)$\Rightarrow$(2): Assume that R is hereditarily Artinian. Theorem 2.1 shows that $R/N(R)$ is a finite direct sum of absolutely algebraic fields of nonzero characteristic. To show that $N = N(R)$ is finite, we assume without loss of generality that $R = \pi + N$, where π is the prime subring of R. In this case each residue field of R is finite and $R = R_1 \oplus \ldots \oplus R_k$, where each R_i is a local Artinian ring with finite residue field. Since $N(R) = N(R_1) \oplus \ldots \oplus N(R_k)$, it suffices to show that each $N(R_i)$ is finite. Let $M_i = N(R_i)$ be the maximal ideal of R_i. Then $M_i^t = (0)$ for some $t \in \mathbf{Z}^+$. For $1 \leq j \leq t-1$, M_i^j/M_i^{j+1} is a finitely generated vector space over the finite field R_i/M_i, and hence M_i^j/M_i^{j+1} is finite. Consequently, $|M_i| = \prod_{j=1}^{t-1} |M_i^j/M_i^{j+1}|$ is finite. This completes the proof that (1) implies (2).

(2)$\Rightarrow$(3): If (2) is satisfied, then $N(R)$ and $R/N(R)$ are Artinian R-modules, so R is an Artinian ring. Thus $R = R_1 \oplus \ldots \oplus R_n$ is a direct sum of Artinian local rings, each having residue field absolutely algebraic of nonzero characteristic. Let M_i be the maximal ideal of R_i. To establish (3), it suffices to show that R_i is finite if $M_i \neq (0)$. Since $M_i \subseteq N(R)$, M_i is finite. Let $R_i x$ be a minimal nonzero ideal of R_i. The annihilator of x in R_i is a maximal ideal of R_i, and hence is M_i. Therefore $R_i x$ is a one-dimensional vector space over R_i/M_i, and consequently $|R_i/M_i| = |R_i x|$ is finite since $R_i x \subseteq M_i$. Because M_i and R_i/M_i are finite, so is R_i.

(3)$\Rightarrow$(1): Under the hypothesis of (3) it is clear that S and each R_i is hereditarily Artinian. Thus, to establish (1) it suffices, by induction, to show that the direct sum of two hereditarily Artinian rings T_1 and T_2 is hereditarily Artinian. Toward this end, let U be a subring of $T_1 \oplus T_2$ and let e_i be the identity element of T_i. The ring $U[e_1, e_2]$ is a finitely generated U-module. Also, $U[e_1, e_2] = U_1 \oplus U_2$, where U_i is a subring of T_i. Thus $U[e_1, e_2]$ is Artinian. Since $U[e_1, e_2]$ is integral over U, $\dim U = 0$, and since $U[e_1, e_2]$ is a finite U-module, the Noetherian property of $U[e_1, e_2]$ is inherited by U [6, Thm. 2]. Therefore U is Artinian and $T_1 \oplus T_2$ is hereditarily Artinian.

Von Neumann regular rings are characterized as the zero-dimensional reduced rings, so the next result follows immediately from Theorem 2.1.

Theorem 2.4. *Suppose R is a von Neumann regular ring, and let* $\mathrm{Spec}(R) = \{M_a : a \in A\}$. *Each subring of R is von Neumann regular if and only if R/M_a is absolutely algebraic of nonzero characteristic for each $a \in A$.*

3. Behavior of the Family of Zero-Dimensional Subrings under Intersection

Several properties of the sets $\mathcal{Z}(R)$, $\mathcal{A}(R)$, $\mathcal{V}(R)$ have been considered in the literature. Here we focus on closure under (finite or arbitrary) intersection. We show (Theorem 3.2 and Corollary 3.3) that $\mathcal{Z}(R)$ and $\mathcal{V}(R)$ are closed under arbitrary intersection. In general $\mathcal{A}(R)$ is not closed under finite intersection, and $\mathcal{A}(R)$ may be closed under finite intersection, but not under arbitrary intersection. The proof of our first result, Theorem 3.2, uses a concept that arises from the following lemma (cf. [21, Lemma 2], [15, Lemma 4.3.9]).

Lemma 3.1. *Suppose R is a ring and $x \in R$ is such that the ideal xR is idempotent. Then there exists a unique element $y \in R$ such that $x = x^2y$ and $y = y^2x$.*

Proof. There exists an element $u \in R$ such that $x = ux^2$. Considered as a ring, the ideal Rx has identity element ux. Since $ux = u^2x^2$, x is a unit of Rx with inverse $y = u^2x$ in Rx. We have $x = x^2y$ and $y = y^2x$ because xy is the identity element of Rx. For uniqueness, suppose $w \in R$ is such that $x = x^2w$ and $w = w^2x$. Then $w \in Rx$ and wx is the identity element of Rx. Hence $w = y$, the unique inverse of x in Rx.

If x, y, and R are as in the statement of Lemma 3.1, then following the terminology of [15, p. 137], we call y the *pointwise inverse* of x in R.

Theorem 3.2. **([10, Thm. 2.4])** *Suppose T is a ring that admits a zero-dimensional subring. Then $\mathcal{Z}(T)$ is closed under arbitrary intersection.*

Proof. Let $\{R_a : a \in A\}$ be a nonempty family of zero-dimensional subrings of T and let $R = \cap_{a \in A} R_a$. We show that R is zero-dimensional by showing that, for each $r \in R$, some power of the ideal rR is idempotent. If $b \in A$, some power of rR_b is idempotent, and the same power of rT is idempotent. Choose $n \in \mathbf{Z}^+$ such that r^nT is idempotent. We show that r^nR_a is idempotent for each $a \in A$. Thus, fix $a \in A$ and choose $m \in \mathbf{Z}^+$ so that r^mR_a is idempotent. If $n \geq m$, then r^nR_a is idempotent. If $n < m$, then to show r^nR_a is idempotent, it suffices to show that $r^n \in r^mR_a$. Let e be an idempotent generator of the ideal r^mR_a. Since r^nT is idempotent, $r^n \in r^mT = eT$, say $r^n = et$, $t \in T$. Then $r^n = er^n \in eR_a = r^mR_a$, as we wished to show. Let y be the pointwise inverse of r^n in T. Since r^n has a pointwise inverse in each R_a, and since the pointwise inverse of r^n in T is unique, we conclude that $y \in \cap_{a \in A} R_a = R$. This implies that r^nR is idempotent, as we wished to show.

Corollary 3.3. *Suppose $\{R_a : a \in A\}$ is a nonempty family of zero-dimensional subrings of a ring T and let $R = \cap_{a \in A} R_a$. If R is reduced, then R is von Neumann regular; in particular, R is von Neumann regular if some R_a is von Neumann regular. The set $\mathcal{V}(T)$ is closed under arbitrary intersection.*

Because a von Neumann regular ring with only finitely many idempotents is a finite direct sum of fields, and hence is Artinian, the following result follows from Corollary 3.3.

Corollary 3.4. *Let the notation and hypothesis be as in the statement of Corollary 3.3. If R is reduced and has only finitely many idempotents, then R is Artinian. In particular, R is Artinian if some R_a is an Artinian reduced ring.*

It follows from Corollary 3.4 that $\mathcal{A}(T)$ is closed under arbitrary intersection if T is reduced and $\mathcal{A}(T)$ is nonempty. In general, however, the intersection of two Artinian subrings of a ring need not be Artinian. For example, let F be a field that admits subfields F_1 and F_2 such that $[F : F_1]$ and $[F : F_2]$ are finite, but $[F : F_1 \cap F_2]$ is infinite. (A classic example of this is where $F = K(t)$ is a simple transcendental extension of a field K of characteristic 0, $F_1 = K(t^2)$, and $F_2 = K(t^2 + t)$.) If $T = F[[X]]/(X^2)$, T is a zero-dimensional local PIR of the form $F + M$, where $M = (X)/(X^2)$ is the maximal ideal of T and, by abuse of notation, we are using F to denote the canonical image of F in T. Let $T_i = F_i + M$ for $i = 1, 2$. Because T is a finitely generated integral extension of T_i, each T_i is an Artinian local ring. However, $T_1 \cap T_2 = (F_1 \cap F_2) + M$ is not Noetherian; M is not finitely generated as an ideal of $T_1 \cap T_2$ since $[F : F_1 \cap F_2]$ is infinite.

We can generalize the construction of the preceding paragraph to obtain an example of a ring T such that $\mathcal{A}(T)$ is nonempty and closed under finite intersection, but not under arbitrary intersection. To do so, we use the concept of the Steinitz number of a field algebraic over a finite field [17, Exer. 6, p. 147]. Let $\{p_i\}_{i=1}^{\infty}$ be a labeling of the prime integers. If K is a field algebraic over the prime field with p elements, then the Steinitz number associated with the field K is the infinite tuple $(a_1, a_2, \ldots, a_n, \ldots)$, where each a_i is either a nonnegative integer or the symbol ∞, and a_i is determined as follows. If K contains a field with $p^{p_i^n}$ elements, but does not contain a field with $p^{p_i^{n+1}}$ elements, then $a_i = n$. If K contains a field with $p^{p_i^n}$ elements for each nonnegative integer n, then $a_i = \infty$. Up to isomorphism, an absolutely algebraic field of characteristic p is uniquely determined by its Steinitz number. Moreover, given any infinite tuple $(a_1, a_2, \ldots, a_n, \ldots)$, there exists an absolutely algebraic field of characteristic p having this tuple as its Steinitz number. If E and F are two absolutely algebraic fields of nonzero characteristic that are both subfields of some field, then $E \subseteq F$ if and only if $b_i \leq c_i$, for all $i \in \mathbf{Z}^+$, where $(b_1, b_2, \ldots)$ is the Steinitz number of E and $(c_1, c_2 \ldots)$ is the Steinitz number of F. Moreover, if $E \subseteq F$, then F is finite over E if and only if $b_i = c_i$ for all but finitely many positive integers i, and if $b_i < c_i$, then $c_i < \infty$ — that is, if b_i is finite, then c_i is also finite. It also readily follows that if E_1 and E_2 are absolutely algebraic fields of nonzero characteristic that are both subfields of some field and if E_i has Steinitz number $(b_{1i}, b_{2i}, \ldots)$, then the Steinitz number of $E_1 \cap E_2$ is $(d_1, d_2, \ldots)$, where $d_i = \min\{b_{i1}, b_{i2}\}$, and the Steinitz number of E_1E_2 is $(e_1, e_2, \ldots)$, where $e_i = \max\{b_{i1}, b_{i2}\}$.

Example 3.5. ([12, Thm. 4.6]) Suppose K is an absolutely algebraic field of characteristic $p > 0$, let $V = Kv$ be a one-dimensional vector space over K, and let $T = K(+)V$ be the idealization of K and V (see [20, p. 2], [16, p. 161]). Then T

is an Artinian local PIR isomorphic to $K[[X]]/(X^2)$ and, by abuse of notation, we write $T = K + V$ and we write elements of T in the form $a + bv$ with $a, b \in K$. As a first step in proving that $\mathcal{A}(T)$ is closed under finite intersection, we show that:

$$(*) \qquad \mathcal{A}(T) = \{F + W : F \text{ is a subfield of } K \text{ and } W \text{ is a finite-dimensional subspace of the } F\text{-vector space } V\}$$

First we show that each such $F + W$ is Artinian. Since T is integral over its prime subring $\pi \simeq \mathbf{Z}_p$, $F + W$ is zero-dimensional quasilocal with maximal ideal W. Because W is finite-dimensional as a vector space over F, W is finitely generated as an ideal of $F+W$ and $F+W$ is Noetherian by Cohen's Theorem. Hence each $F+W$ is Artinian. If R is any subring of T we note that $R = (R \cap K) + (R \cap V)$, where $R \cap K$ is a subfield of K and $R \cap V$ is a subspace of the $(R \cap K)$-vector space V. Thus, if $a + u \in R$, where $a \in K$ and $u \in V$, then since a is algebraic over π, $a^{p^k} = a$ for some $k \in \mathbf{Z}^+$. Therefore $(a + u)^{p^k} = a^{p^k} + u^{p^k} = a \in R$, and from this we easily conclude that $R = (R \cap K) + (R \cap V)$. Each subring of K is a field, and clearly $R \cap V$ is closed under $(R \cap K)$-multiplication. Hence R is as described. If R is Artinian, then $R \cap V$ is a finitely generated R-module, and therefore a finitely generated $(R \cap K)$-module since $(R \cap V)^2 = (0)$. Thus we have established $(*)$.

Suppose then that $R_1 = F_1 + W_1$ and $R_2 = F_2 + W_2$ are two Artinian subrings of T, where each F_i is a subfield of K and W_i is a finite-dimensional subspace of the F_i-vector space V. Since $K \cap V = (0)$, we have $R_1 \cap R_2 = (F_1 \cap F_2) + (W_1 \cap W_2)$, and to show that $R_1 \cap R_2$ is Artinian, we must show that $W_1 \cap W_2$ is finite-dimensional as a vector space over $F_1 \cap F_2$. This is a place where Steinitz numbers come into play. For $i = 1, 2$, let $\{c_{ij} v\}_{j=1}^{k_i}$ be a basis for the F_i-vector space W_i and let $E_i = F_i(\{c_{ij} : 1 \le j \le k_i\})$. Clearly $W_i \subseteq E_i v$ and $W_1 \cap W_2 \subseteq E_1 v \cap E_2 v = (E_1 \cap E_2)v$. Thus it suffices to show that $[E_1 \cap E_2 : F_1 \cap F_2]$ is finite if $[E_1 : F_1]$ and $[E_2 : F_2]$ are finite. Let $(b_{1i}, b_{2i}, \ldots)$ be the Steinitz number of F_i and let $(c_{1i}, c_{2i}, \ldots)$ be the Steinitz number of E_i. There exists a finite subset J of $\mathbf{Z}^+$ such that $b_{j1} = c_{j1}$ and $b_{j2} = c_{j2}$ for each $j \in \mathbf{Z}^+ - J$, while c_{j1} and c_{j2} are finite for each $j \in J$. Let $(d_1, d_2, \ldots)$ and $(e_1, e_2, \ldots)$ be the Steinitz numbers of $F_1 \cap F_2$ and $E_1 \cap E_2$, respectively. For $j \in \mathbf{Z}^+ - J$, $d_j = \min\{b_{j1}, b_{j2}\} = \min\{c_{j1}, c_{j2}\} = e_j$, and for $j \in J$, $e_j = \min\{c_{j1}, c_{j2}\}$ is finite. Consequently, $[E_1 \cap E_2 : F_1 \cap F_2]$ is finite, as desired. This shows that $\mathcal{A}(T)$ is closed under finite intersection for each choice of K.

On the other hand, if K is the field with Steinitz number $(1, 1, \ldots)$ and if K_i is the subfield of K with the same Steinitz number except for a coordinate of 0 in the i-th coordinate, then $T_i = K_i + V$ is an Artinian subring of T for each i, but $\cap_{i=1}^{\infty}(K_i + V) = \pi + V$ is not Artinian.

4. Generalizations, Extensions and Related Results

Because of space considerations, we have intentionally avoided in Sections 1–3 stating some results in the most general context in which they are known. Moreover, some of the known theory of zero-dimensional subrings has not been mentioned at all. In this section we briefly fill in some of the gaps in the previous coverage.

If R is a subring of the ring S, we call a subring of S containing R an S-*overring* of R. Rather than ask whether S admits a zero-dimensional (or Artinian, etc.) subring, one can ask whether some S-overring of R is zero-dimensional. This more general setting has been considered in [12, Sect. 2], where it was shown, for example, that if R is a Hilbert ring of positive dimension and S is a finitely generated extension ring of R, then R admits no zero-dimensional S-overring [12, Prop. 2.2].

In relation to Theorem 1.7, several questions concerning $\mathcal{A}(S)$ naturally arise. Is $\mathcal{A}(S)$ directed? Is it closed under taking compositum? Under what conditions is S a directed union of Artinian subrings? Such questions have been considered in [9, Sect. 5,6], particularly for rings S of the form $\prod_{a \in A} R_a$, where each R_a is zero-dimensional; they are also considered in Section 5 of [12]. Example 5.1 of [12] shows that $\mathcal{A}(S)$ need not be directed. In particular, $\mathcal{A}(S)$ need not be closed under taking compositum. A necessary condition for S to be a directed union of Artinian subrings is that S is zero-dimensional. Corollary 2.2 shows that a hereditarily zero-dimensional ring is a directed union of Artinian subrings, and Theorem 1.7 yields the same conclusion for a zero-dimensional semiquasilocal ring. Using results of Cohen [4] concerning the structure of complete local rings, Gilmer and Heinzer showed in Corollary 5.11 of [12] that $\mathcal{A}(S)$ is closed under compositum if S is a hereditarily zero-dimensional semiquasilocal ring. A question in this area raised in Remark 6.10 of [9] asks whether the property of being a directed union of Artinian subrings is inherited from $S/N(S)$, where $N(S)$ is the nilradical of S, to S.

The questions concerning $\mathcal{A}(S)$ raised in the preceding paragraph may be asked in the relative context of S-overrings of R; they may be asked for S in a restricted class of rings (the case where S is von Neumann regular is of interest); or they may be considered for the sets $\mathcal{Z}(S)$ and $\mathcal{V}(S)$. Apart from the question of what von Neumann regular rings are directed unions of Artinian subrings, these questions seem to have received little coverage in the literature.

The universal characterizations of Section 2 can also be considered in a relative context. In this connection, if R is a subring of S and if E is a ring-theoretic property, we say that (R, S) is an E-*pair* if each S-overring of R has property E. Note that the case where $R = \pi$, the prime subring of S, is the same as the universal case for S — that is, the case where S has property E hereditarily. The E-pair terminology seems to stem from [5], where integrally closed pairs of integral domains are considered. Wadsworth treats Noetherian pairs of domains in [23], and Noetherian pairs of rings are considered in [14]. In relation to results of Section 2, zero-dimensional pairs and Artinian pairs of rings are characterized in Section 4 of [9]. Section 3 of [12] contains an investigation of pairs (R, S) such that (1) R admits a zero-dimensional S-overring, and (2) each zero-dimensional S-overring of R is Artinian. The corresponding universal condition on a ring T is that in which (1) T admits a zero-dimensional subring, and (2) each zero-dimensional subring of T is Artinian.

Suppose R is a subring of the ring S. Theorem 3.1 of [10] shows that R admits a zero-dimensional S-overring if and only if some power of rS is idempotent for each $r \in R$; if this condition is satisfied, then Theorem 3.2 shows that there exists a unique minimal zero-dimensional S-overring $R^0 = R^0(S)$ of R. Properties of R^0 as an extension ring of R have been investigated in [2] and in [10]. To summarize briefly, a set of algebra generators for R^0 over R is known; a primary ideal of R^0 is uniquely determined by its contraction to R; if $P \in \mathrm{Spec}(R^0)$, R^0/P is isomorphic to the quotient field of $R/(P \cap R)$; and a prime $Q \in \mathrm{Spec}(R)$ is contracted from R^0 if and only if it is contracted from S. The ring $R^0(S)$ is not uniquely determined up to R-isomorphism, and in fact, Theorem 4.1 of [10] shows that if $\dim R > 0$ and if R is imbeddable in a zero-dimensional ring, then there exist extension rings S_1 and S_2 of R such that $R^0(S_1)$ and $R^0(S_2)$ are not isomorphic. Gilmer and Heinzer in [11] have used the rings $R^0(S)$ to prove that certain families of fields can be realized as the family of residue fields of a zero-dimensional ring.

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