ON COMPLETE INTEGRAL CLOSURE AND ARCHIMEDEAN VALUATION DOMAINS

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Abstract

Suppose D is an integral domain with quotient field K and that L is an extension field of K. We show in Theorem 4 that if the complete integral closure of D is an intersection of Archimedean valuation domains on K, then the complete integral closure of D in L is an intersection of Archimedean valuation domains on L; this answers a question raised by Gilmer and Heinzer in 1965.

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All rings considered in this paper are assumed to be commutative and to contain a unity element. If R is a subring of S, we assume that the unity element of S belongs to R, and hence is the unity element of R.

If R is a subring of S and $s \in S$, then as is well-known, s is integral over R if and only if $R[s]$ is a finitely generated R-module. On the other hand, if $R[s]$ is contained in a finitely generated R-submodule of S, then s is said to be *almost integral* over R. Clearly s is almost integral over R if it is integral over R; the converse holds if R is Noetherian, but not in general. We denote by $C(R, S)$ the set of elements of S that are almost integral over R; $C(R, S)$ is a subring of S containing R, and is called the *complete integral closure* of R in S. If $C(R, S) = S$, we say that S is *almost integral* over R, and on the opposite extreme where $C(R, S) = R$, we say that R is *completely integrally closed* in S. If S is the total quotient ring of R, then $C(R, S)$ is the *complete integral closure* of R, and if $C(R, S) = R$, then R is said to be *completely integrally closed.*

The concept of almost integrality was first considered by Krull[1] in his famous 1932 paper *Allgemeine Bewertungstheorie* [5], in which he introduced the notion of a general valuation and proved, among other things, that an integral domain is integrally closed if and only if it is an intersection of valuation domains [5, Satz 7]. Krull also proved [5, Satz 8] that a valuation domain V is completely

[1] More precisely, Krull considers the concept in the case where R is an integral domain and S is the quotient field of R. In this case it is easy to show that $s \in S$ is almost integral over R if and only if there exists a nonzero element $r \in R$ such that $rs^k \in R$ for each positive integer k.

integrally closed if and only if the rank of V is at most 1. (More generally, the complete integral closure of V is either V_P (if V has a height-one prime ideal P) or the quotient field F of V (if V has no height-one prime).) This led Krull to conjecture [5, p. 170] that a domain is completely integrally closed if and only if it is an intersection of valuation domains of rank at most 1 — that is, of Archimedean valuation domains. Nakayama disproved Krull's conjecture in [7], [8], and a simpler counterexample due to J. Ohm appears as Example 19.12 of [2]. On the other hand, Krull's conjecture is known to be correct for certain classes of domains; for example, it is valid for Noetherian domains [6, Theorem 33.10] and for domains of finite character [3, Proposition 5], [1, Theorem 5]. This paper answers a question related to the discussion above that arose in work that W. Heinzer and I did in connection with [3]. The question is the following:

> **(Q)** *Suppose D is an integral domain with quotient field K, and assume that the complete integral closure of D is an intersection of Archimedean valuation domains on K. If L is an extension field of K, is the complete integral closure of D in L an intersection of Archimedean valuation domains on L?*

We show in Theorem 4 that (Q) has an affirmative answer. Our general reference for results concerning almost integrality is Section 13 of [2]. We first resolve the case of (Q) where L is algebraic over K; Lemma 1 and Theorem 2 deal with the algebraic case.

LEMMA 1. *Let $\{D_a\}_{a\in A}$ be a family of integrally closed domains with common quotient field K, and assume that $D = \bigcap_{a\in A} D_a$ also has quotient field K. Assume that L is an algebraic extension field of K, and use bars to denote integral closure in L. Then $\overline{D} = \bigcap_{a\in A} \overline{D_a}$.*

PROOF. The inclusion $\overline{D} \subseteq \bigcap_a \overline{D_a}$ is clear. Suppose $t \in \bigcap \overline{D_a}$ and let $f(x)$ be the minimal polynomial for t over K. Since D_a is integrally closed for $a \in A$, it is known that $f(x) \in D_a[x]$ [2, Theorem 10.5], and hence $f(x) \in \bigcap_{a\in A} D_a[x] = (\bigcap_a D_a)[x] = D[x]$. Therefore $t \in \overline{D}$, as we wished to show. □

THEOREM 2. *Let D be an integral domain with quotient field K and let L be an algebraic extension field of K. If the complete integral closure E of D is an intersection of Archimedean valuation domains on K, then the complete integral closure $C(D,L)$ of D in L is an intersection of Archimedean valuation domains on L.*

PROOF. Let $\overline{E}$ be the integral closure of E in L. Since E is completely integrally closed, $\overline{E}$ is the complete integral closure of E in L and is a completely integrally closed domain [2, Theorem 13.8]. Clearly $C(D,L) \subseteq \overline{E}$, but since $\overline{E}$ is integral over E it follows that $\overline{E}$ is almost integral over D [2, Corollary 13.2], and hence $\overline{E} \subseteq C(D,L)$. Therefore $C(D,L) = \overline{E}$. By hypothesis $E = \bigcap_{a\in A} V_a$, where each V_a is an Archimedean valuation domain on K, and Lemma 1 shows that $\overline{E} = \bigcap\limits_a \overline{V_a}$, where bars indicate integral closure in L. Now $\overline{V_a}$ is a Prüfer domain [2, Theorem 20.1] and $\dim \overline{V_a} = \dim V_a \leq 1$. Consequently, $\overline{V_a}$ is an intersection of Archimedean valuation domains on L, so $\overline{E} = \bigcap\limits_a \overline{V_a}$ is also an intersection of Archimedean valuation domains on L. This completes the proof of Theorem 2. □

Suppose R is a subring of S and S is a subring of T. In contrast with the situation for integrality, almost integrality is not transitive — that is, T need not be almost integral over R even if T/S and S/R are almost integral. For example, if D is an integral domain with quotient field K, the complete integral closure $D^{(1)}$ of D need not be completely integrally closed [3, Example 1]; in fact, Hill [4] has given an example of a Bezout domain D such that $D^{(n)} < D^{(n+1)}$ for each positive integer n, where $D^{(n+1)}$ is defined inductively as the complete integral closure of $D^{(n)}$. Another contrast between the properties of integrality and almost integrality is that the inclusion $C(R,S) \subseteq C(R,T) \bigcap S$ may be proper [3, Example 2]. However, it follows from part (b) of [3, Proposition 2] that $C(R,S) = C(R,T) \bigcap S$ if S is a field. We use this equality in the proof of Proposition 3.

Proposition 3. *Suppose D is an integral domain with quotient field K and T is an extension ring of K. If Δ is the integral closure of K in T, then $C(D,T) = C(D,\Delta)$.*

Proof. By the result from [3] just cited, $C(D,\Delta) = C(D,T) \bigcap \Delta$. Hence, to prove Proposition 3, it suffices to show that $C(D,T) \subseteq \Delta$. This is straightforward: if $t \in C(D,T)$ then clearly $t \in C(K,T)$, and since K is Noetherian, t is integral over K — that is, $t \in \Delta$. □

Theorem 4. *Let D be an integral domain with quotient field K and let L be an extension field of K. If the complete integral closure of D is an intersection of Archimedean valuation domains on K, then the complete integral closure of D in L is an intersection of Archimedean valuation domains on L.*

Proof. Let Δ be the algebraic closure of K in L. Proposition 3 shows that $C(D,L) = C(D,\Delta)$, and Theorem 2 shows that $C(D,\Delta)$ is an intersection of Archimedean valuation domains on Δ. Hence the proof is complete if $\Delta = L$. If $\Delta < L$, we consider two cases.

Case 1. $C(D,L) = \Delta$. In this case we show that if $t \in L - \Delta$, then there exists a rank-one valuation domain on L that contains Δ, but not t. Thus, since t^{-1} is transcendental over Δ, then $V = \Delta[t^{-1}]_{t^{-1}V[t^{-1}]}$ is a rank-one valuation domain on $\Delta(t)$ that does not contain t. The valuation domain V admits an extension to a rank-one valuation domain W on L [2, Proposition 20.5], and $t \notin W$ since $t \notin W \bigcap \Delta(t)$. Therefore Δ is the intersection of a family of rank-one valuation domains on L. This resolves Case 1.

Case 2. $C(D, L) < \Delta$. In this case there exists a family $\{V_a\}_{a \in A}$ of rank-one valuation domains on Δ such that $C(D, L) = \bigcap_{a \in A} V_a$. For each $a \in A$, let W_a be a rank-one extension of V_a to L and, by Case 1, let $\{U_b\}_{b \in B}$ be a family of rank-one valuation domains on L such that $\Delta = \bigcap_{b \in B} U_b$. If $\mathcal{F} = \{W_a\}_a \bigcup \{U_b\}_b$, then $\mathcal{F}$ is a family of rank-one valuation domains on L and $\bigcap \mathcal{F} = (\bigcap\limits_a W_a) \bigcap \Delta = \bigcap\limits_a (W_a \bigcap \Delta) = \bigcap\limits_a V_a = C(D, L)$. □

We remark that the converse of Theorem 4 is also valid: If $C(D, L) = \bigcap_{b \in B} W_b$, where each W_b is an Archimedean valuation domain on L, then $C(D, K) = C(D, L) \bigcap K = \bigcap_{b \in B}(W_b \bigcap K)$, where each $W_b \bigcap K$ is an Archimedean valuation domain on K [2, Theorem 19.16]. Thus, for any extension field L of K, the condition that $C(D, L)$ is an intersection of Archimedean valuation domains on L depends only upon D, not on L.

References

[1] H.S. Butts and W.W. Smith, 'On the integral closure of a domain', *J. Sci. Hiroshima Univ. Ser. A–I* **30** (1966), 117–122.

[2] R. Gilmer, *Multiplicative ideal theory* (Queen's Univ., Kingston, 1992).

[3] R. Gilmer and W. Heinzer, 'On the complete integral closure of an integral domain', *J. Austral. Math. Soc.* **6** (1966), 351–361.

[4] P. Hill, 'On the complete integral closure of a domain', *Proc. Amer. Math. Soc.* **36** (1972), 26–30.

[5] W. Krull, 'Allgemeine Bewertungstheorie', *J. Reine Angew. Math.* **167** (1932), 160–196.

[6] M. Nagata, *Local rings* (Wiley Interscience, New York, 1962).

[7] T. Nakayama, 'On Krull's conjecture concerning completely integrally closed integrity domains I, II', *Proc. Imperial Acad. Tokyo* **18** (1942), 185–187, 233–236.

[8] ____, 'On Krull's conjecture concerning completely integrally closed integrity domains III', *Proc. Japan Acad.* **22** (1946), 249–250.

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