Optimal intervention in the foreign exchange market when interventions affect market dynamics

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Abstract

We address the problem of optimal Central Bank intervention in the exchange rate market when interventions create feedback in the rate dynamics. In particular, we extend the work done on optimal impulse control by Cadenillas and Zapatero (1999, 2000) to incorporate temporary market reactions to Bank interventions. We obtain new explicit optimal impulse control strategies that account for these market reactions.

Keywords: optimal impulse control, quasi-variational inequalities, foreign exchange intervention.

1   Introduction

In countries dependent on foreign trade and foreign capital, the Central Bank is normally in charge of exchange rate policy. This usually means that

the Central Bank has the ability to intervene in the markets in order to keep their currency rates within a band, or close to a target rate set by the Bank's policy makers.

Intervention can take two different (compatible) forms: adjustment of domestic interest rate levels, which influences the attractiveness of foreign investments; and direct purchases or sales of foreign currency reserves in the foreign exchange market.

The first form of intervention can be modeled as a continuous (classical) control problem, and the second as an impulse control problem. In this paper we focus only on the second type of intervention, interpreting a market intervention as a way to change the exchange rate by a desired amount via sales or purchases of reserves over a time short enough to be reasonable modeled as an instantaneous impulse.

The problem is to find an optimal intervention strategy keeping the exchange rate near a target level, set by the bank, while minimizing cost of intervention. Since the exchange rate will always drift away from the target rate between interventions, one approach to control intervention costs has been to set a "target zone" or band, and act to keep the exchange rate within this band. Various papers in the economics literature have considered this problem, e.g. Krugman (1991); Froot and Obstfeld (1991); Flood and Garber (1991).

The first person to apply the theory of stochastic impulse control to this problem was Jeanblanc-Picque (1993), later extended by Korn (1997). Both considered an exogenously specified target band within which the exchange rate is to be contained, and found the optimal sizes of interventions required

when the exchange rate reaches the boundary of the target band.

An important insight was obtained by Mundaca and Oksendal (1998) and Cadenillas and Zapatero (1999, 2000), who realized that it is not necessary to exogenously set a target band, but rather that the correct target band can be derived endogenously as part of the solution to the optimization problem. The cost function is a combination of a cost of intervention and a running cost given by an increasing function of the distance between the target rate and the current rate. Mundaca and Oksendal used a standard brownian motion model for the underlying exchange rate; Cadenillas and Zapatero used a geometric brownian motion and explicitly computed, for certain examples, the optimal intervention levels and intervention amounts.

They expressly assume, however, that investors do not observe or anticipate the interventions of the Bank, so that the process driving the rate dynamics is not affected by interventions. To do otherwise, they said, would allow investors to learn and anticipate the Bank's moves, and "would yield different dynamics for the exchange rate and would probably make the model intractable" (Cadenillas and Zapatero, 2000).

However, it would be more realistic to be able to solve a model allowing for the market to notice and react to large price-moving interventions by the Central Bank. In this paper we overcome the intractability mentioned above to solve the problem with the same level of explicitness as do Cadenillas and Zapatero, but allowing for a market reaction to interventions. We assume that the rate dynamics changes to a different process for a random period of time T after each intervention (where T is assumed independent of the rate process), after which it reverts to the pre-intervention process. For

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example, we could imagine that the volatility of the exchange rate might move to a new, higher level for a period of time after an intervention, to reflect heightened market uncertainty about the path of rates in the near term.

We still must retain the assumption that investors do not anticipate interventions by the Bank. This is much milder than the assumption that investors do not *react* to interventions. Indeed, we might expect that the Bank itself will be revising the parameters used in it's exchange rate model as time passes, so that the optimal solution today would not persist due to the incorporation of new information in the Bank's model. If this happens, investors are unlikely to have much confidence in forecasts of the next Bank intervention. Hence it is not necessary to assume that interventions are invisible to the market in order to reasonably ignore the market effect of investor prediction of future intervention times.

We solve our problem by applying the theory of stochastic impulse control and, as we are able to obtain analytical results, we can compare the policies with and without a reaction period. In the spirit of Cadenillas and Zapatero, and since we can provide numerical solutions, we also include comparative statics analysis about the effects of the changes of parameters on the optimal intervention strategy.

We consider a currency with exchange rate dynamics modeled by a general Itô diffusion, such as a geometric Brownian Motion, which temporarily changes to a different Itô diffusion during a "reaction period" that lasts for a finite random time T after each intervention. The Central Bank tries to keep this exchange rate close to a given target, and there is a running cost

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associated to the difference between the exchange rate and the target. However, there are also fixed and, optionally, proportional costs associated with each intervention. In addition, we assume that the Bank is not allowed to intervene during the temporary reaction period. The objective of this paper is to find the optimal level of intervention, as well as the optimal sizes of the interventions, so as to minimize the total cost.

We shall prove that when the exchange rate lies in a specific interval - the continuation region - the Central Bank must not intervene. However, when the exchange rate reaches the boundary of that interval – when it enters the intervention region– then the Central Bank must intervene (as soon as the reaction time has expired), pushing the exchange rate to yet another interval, the preferred region inside the continuation region. (In case there are no proportional intervention costs, the preferred interval degenerates to a single point.)

The structure of the paper is as follows: in section 2 we formulate the problem by introducing the exchange rate dynamics and the Central Bank objective; in section 3 we state the quasi-variational inequalities for this problem and state sufficient conditions of optimality for this impulse control problem; in section 4 we solve the problem of Central Bank intervention, with and without recovery period, and we perform some comparative statics analysis. In section 5 we present the proof of the main theorem, and we close the paper with some conclusions.

2   Problem Statement

Let us denote by $X_{t;y} = X_y(t)$ the exchange rate process, representing domestic currency units per unit of foreign currency at time $t \in [0, \infty)$, with initial value $X_{0;y} = y$. Suppose that in the absence of intervention $X_{t;y}$ follows the stochastic process given by the time-homogeneous SDE

$$(2.1) \qquad dX_{t;y} = \mu_1(X_{t;y})dt + \sigma_1(X_{t;y})dW_t,$$

where W_t is a one-dimensional Brownian motion in a probability space $(\Omega, \mathcal{F}, P)$, and μ_1, σ_1 are Lipschitz functions on $\mathbb{R}$. If $\mu_1(x) > 0$ the currency experiences a devaluatory pressure and if $\mu_1(x) < 0$ a pressure to appreciate. As a typical example for numerical illustration, we will look at the special case of geometric brownian motion when μ_1 and σ_1 are linear, but our results apply to this more general setting.

Suppose that the dynamics of the exchange rate process reacts temporarily after an intervention takes place: for a random, bounded length of time $T \geq 0$ after an intervention, we suppose that the process is driven by a new drift μ_2 and volatility σ_2 (also assumed Lipschitz), after which time the drift and volatility revert to the original functions. We denote by $\tilde{X}_{t;y}$ the diffusion process followed during this reaction time of length T after each intervention:

$$d\tilde{X}_{t;y} = \mu_2(\tilde{X}_{t;y})\ dt + \sigma_2(\tilde{X}_{t;y})\ dW_t.$$

For simplicity we impose the restriction that new interventions are not allowed during this reaction period.

Consider the impulse control ν

$$\nu = (\tau_1, \tau_2, ...; \xi_1, \xi_2, ...),$$

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where (τ_i) is an infinite increasing sequence of stopping times, and each $\xi_i : \Omega \to \mathbb{R}$ is $\mathcal{F}_{\tau_i}$-measurable. Intuitively, τ_i indicates the time of the ith intervention of size ξ_i.

For a fixed constant $0 < \bar{T}$, the reaction times are an *iid* sequence $T_i \in [0, \bar{T}]$, assumed also independent of the process W_t. We suppose $E[T_i] > 0$ (since otherwise $T_i = 0$ and there is no market reaction at all). The controlled process $X_{t;y}^{\nu}$, starting at value y, can be written as

$$
\begin{aligned}
(2.2) \qquad dX_y^\nu(t) &= dX_y(t) & 0 \le t < \tau_1 \\
X_y^\nu(\tau_j) &= \Gamma\left(X_y^\nu(\tau_j^-), \xi_j\right) & j = 1, 2, \dots \\
dX_y^\nu(t) &= d\tilde{X}_{X_y^\nu(\tau_j)}(t) & \tau_j \le t \le \tau_j + T_j \\
dX_y^\nu(t) &= dX_{X_y^\nu(\tau_j + T)}(t) & \tau_j + T_j < t < \tau_{j+1}
\end{aligned}
$$

for a given function $\Gamma(x, \xi_j)$; in particular, we will assume $\Gamma(x, \xi) = x - \xi$, meaning that the intervention at x consists of a downward impulse of size $\xi \in \mathbb{R}$ (upward if ξ is negative). Moreover, given the nature of our problem, we will only consider controls forcing $X_x^\nu(\tau_j)$ to remain positive.

We consider the performance function

$$
J^\nu(x) = E\left[\int_0^\infty e^{-rt} f(X_x^\nu(t))dt + \sum_i e^{-r\tau_i} K\left(X_x^\nu(\tau_j^-), \xi_j\right)\right]
$$

where $K(x, \xi)$ is a given function that represents the cost of intervention depending on the state x at intervention and the intervention size ξ. (In examples we often take $K(x, \xi)$ to be either a constant K or a constant plus proportional costs $K_1 + K_2|\xi|$.) The constant r represents the discount factor, assumed fixed here, and f is a continuous running cost function, for example measuring the deviation from a target value.

The Central Bank wants to use a policy that minimizes the performance function over all possible admissible controls. Therefore, we define the Value Function as

$$V(x) = \inf_{v \in \mathcal{V}} J^v(x)$$

where $\mathcal{V}$ is the set of admissible controls. This value function depends on the precise definition of admissible – in application it is enough that $\mathcal{V}$ includes all reasonable controls that might be considered in practice. For our purposes we define $\mathcal{V}$ as follows.

Definition 2.1 *An impulse control*

$$\nu = (\tau_1, \tau_2, ...; \xi_1, \xi_2, ...),$$

is **admissible** *($\nu \in \mathcal{V}$) if*

(2.3)
$$X_x^\nu(\tau_i) > 0 \text{ for all } i,$$

(2.4)
$$\tau_{i+1} - \tau_i \geq T_i \text{ for all } i,$$

(2.5)
$$E\left[\int_0^\infty e^{-rt} f(X_x^\nu(t)) \, dt\right] < \infty,$$

and

(2.6)
$$\lim_{t \to \infty} E\left[\int_0^\infty (e^{-rt} X_x^\nu(t))^2 \, dt\right] < \infty.$$

Condition (2.4) means that the central bank will not intervene while the market is still reacting to the previous intervention. This implies that $\tau_n \to \infty$ as $n \to \infty$ almost surely: since the T_i have positive mean, the law of large numbers applies and $\tau_n \geq \sum_{i=1}^{n-1} T_i \to \infty$. Conditions (2.5) and

(2.6) are mild boundedness conditions that will be easily satisfied by any practical intervention policy and running cost function.

Additionally, let's define $\tilde{K}$ as the running cost immediately after an intervention takes place provided that the process restarts from $\bar{x}$ and that it remains under the second diffusion regime for a period of time T

$$\tilde{K}(\bar{x}) = E\left[\int_0^T e^{-rt}f(\tilde{X}_{\bar{x}}(t))dt\right].$$

(Since the sequence (T_i) is *iid* and independent of $\tilde{X}$, $\tilde{K}$ does not depend on which intervention has occurred. We write T for a generic random variable with the same distribution as T_i.)

3 Quasi-variational Inequalities

To solve the impulse control problem formulated above, We will use the quasi-variational inequalities (QVI) approach (Bensoussan and Lions, 1984), which involves constructing the value function $V(x)$ as a solution to a system of inequalities described below. The value function $V(x)$ then determines the optimal control strategy, called the QVI-control associated to V. Once we have proved that solutions of the QVI yield the optimal intervention strategy, this reduces the control problem to the much easier problem of solving the QVI. We state the general results here and illustrate their use in the next section.

First, we require an optimal intervention operator $\mathcal{M}$ adapted to our situation. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Definition 3.1 *For a function $\phi : \mathbb{R}^+ \to \mathbb{R}$, and $x \in \mathbb{R}^+, \xi \in \mathbb{R}$, define*

operators $\mathcal{M}$ and M as follows:

$$(3.1) \qquad M(\phi, x, \xi) = K(x, \xi) + \tilde{K}(x - \xi) + E\left[e^{-rT}\phi(\tilde{X}_{x-\xi}(T))\right]$$

and

$$(3.2) \qquad \mathcal{M}\phi(x) = \inf_{\xi}\{M(\phi, x, \xi) : x - \xi > 0\}.$$

whenever these are well-defined.

We also need the partial differential operator $\mathcal{L}$ given by

$$(3.3) \qquad \mathcal{L}\phi = \frac{1}{2}\sigma_1^2(x)\frac{\partial^2}{\partial x^2}\phi + \mu_1(x)\frac{\partial}{\partial x}\phi - r\phi.$$

This operator will be useful in two ways. First, Ito's formula applied to any function of the form $e^{-rt}\phi(X_x(t))$, where X is the uncontrolled process (2.1), gives, for any two times $S < U$,

$$(3.4) \qquad e^{-rt}\phi(X_x(t))|_S^U = \int_S^U e^{-rt}\mathcal{L}\phi(X_x(t))\ dt$$

$$(3.5) \qquad \qquad + \int_S^U e^{-rt}\sigma_1(X_x(t))\phi'(X_x(t))\ dW_t.$$

Second, when there are no interventions, V takes the form

$$(3.6) \qquad V(x) = E\left[\int_0^\infty e^{-rt}f(X_x(t))dt\right]$$

which, if f is bounded and continuous (e.g. Oksendal (2003), ch. 9), satisfies the equation

$$(3.7) \qquad \mathcal{L}V(x) + f(x) = 0.$$

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We will see that the solution of our impulse control problem splits the domain of x in two regions, an intervention and a continuation region. In the intervention region, where it is optimal to intervene, it should be the case that $V(x) = \mathcal{M}V(x)$. On the other hand, the interval where intervention is not optimal because $\mathcal{M}V(x) > V(x)$, is referred to as the continuation region, and we have $\mathcal{L}V(x) + f(x) = 0$ in that region. This suggests that the Value function should satisfy a set of inequalities, which are commonly known as the Quasi-Variational Inequalities.

Definition 3.2 *We say that the function ϕ **satisfies the quasi-variational inequalities (Q.V.I.)** if ϕ satisfies the following three conditions:*

$$(3.8) \qquad\qquad \mathcal{L}\phi(x) + f(x) \geq 0,$$

$$(3.9) \qquad\qquad \phi(x) \leq \mathcal{M}\phi(x),$$

$$(3.10) \qquad (\mathcal{L}\phi(x) + f(x))(\phi(x) - \mathcal{M}\phi(x)) = 0.$$

From a solution of the quasi-variational inequalities, we construct the following control:

Definition 3.3 *Let ϕ be a continuous solution of the QVI. Then, the following impulse control is called a **QVI-control associated to ϕ** (if it exists): for every $n \in N$,*

$$\tau_n := \inf\{t > \tau_{n-1} + T_{n-1} : \phi(X_x^\nu(t^-)) = \mathcal{M}\phi(X_x^\nu(t^-))\}$$

$$\xi_n := \arg\min\left\{K(X_x^\nu(\tau_n^-), \xi) + \tilde{K}(X_x^\nu(\tau_n^-) - \xi)\right.$$

$$\left. + E\left[e^{-rT}\phi(\tilde{X}_{X_x^\nu(\tau_n^-)-\xi}(T)|\mathcal{F}_{\tau_n^-}\right] : \xi \in \mathbb{R}, X_x^\nu(\tau_n^-) - \xi > 0\right\},$$

where $\tau_0 = 0$.

The arg min above might not be unique, so we do not claim there is only one QVI-control – though in practice we do observe uniqueness.

The following main theorem permits us to verify that a solution of the Q.V.I. and the admissible control attached to it solve the impulse control problem. Denote by $\mathcal{L}_2$ the operator defined by

$$(3.11) \qquad \mathcal{L}_2\phi(x) = \frac{1}{2}\sigma_2^2(x)\frac{\partial^2}{\partial x^2}\phi(x) + \mu_2(x)\frac{\partial}{\partial x}\phi(x) - r\phi(x).$$

Theorem 3.4 *Let $\phi \in C^1(\mathbb{R}^+)$ be a solution of the QVI and suppose there is a finite subset $\mathcal{N} \subset \mathbb{R}^+$ such that $\phi \in C^2(\mathbb{R}^+ - \mathcal{N})$. If ϕ satisfies the growth conditions*

$$(3.12) \qquad E\int_0^\infty (e^{-rt}\sigma_i(X_x^\nu(t))\phi'(X_x^\nu(t)))^2 \, dt < \infty, \quad i = 1, 2,$$

$$(3.13) \qquad \lim_{t\to\infty} E\left[e^{-rt}\phi(X_x^\nu(t))\right] = 0,$$

and

$$(3.14) \qquad E\left[\int_0^\infty e^{-rt}|\mathcal{L}_2\phi(X_x^\nu(t))| \, dt\right] < \infty,$$

for every process $X_x^\nu(t)$ corresponding to an admissible impulse control ν, then for every $x \in \mathbb{R}^+$

$$V(x) \geq \phi(x).$$

Moreover, if the QVI-control corresponding to ϕ is admissible then it is an optimal impulse control, and for every $x \in \mathbb{R}^+$

$$(3.15) \qquad V(x) = \phi(x).$$

The theorem is proved in Section 5.

4 Solving the Q.V.I.

We now illustrate how theorem 3.4 is used in the context of finding the Central Bank's optimal impulse control strategy, where we assume for convenience that exchange rates evolve as geometric brownian motion. As a warm-up, we first describe the case $T = 0$ when there is no market reaction period. This is the case described by Cadenillas and Zapatero (1999).

4.1 The case without market reaction

A Central Bank desires to keep the exchange rate close to a target ρ using an impulse control strategy. The admissible controls are the same as above, except that since $T = 0$ here we need to replace condition (2.4) with the admissibility condition

(4.1)
$$P\left(\lim_{i \to \infty} \tau_i \leq t \right) = 0 \ \ \forall\, t \geq 0.$$

Let X_t represent domestic currency units per unit of foreign currency at time t and suppose that the dynamics of X_t are given by

$$X_x^\nu(t) = x + \int_0^t \mu X_x^\nu(s) \ ds + \int_0^t \sigma X_x^\nu(s) \ dW_s + \sum_{i=1}^\infty 1_{\{\tau_i < t\}} \xi_i,$$

where $\nu = (\tau_i, \xi_i)_{i=1}^\infty$ is an admissible impulse control. This means that $X_x^\nu(t)$ follows a geometric Brownian motion in the absence of interventions.

The Central Bank wants to find the optimal impulse control that minimizes the following functional that depends on the control ν

$$J^\nu(x) = E\left[\int_0^\infty e^{-rt} f(X_x^\nu(t)) \ dt + \sum_i e^{-r\tau_i} g(\xi_i) 1_{\{\tau_i < \infty\}} \right],$$

where

$$f(x) = (x - \rho)^2,$$

and for ease of exposition in this illustration we take $g(\xi_i) = K$, for some positive constant K, meaning that there are only fixed intervention costs. The Value Function for this example is

$$V(x) = \inf_{\nu} J^{\nu}(x)$$

where the infimum is taken over all admissible controls.

Cadenillas and Zapatero (1999) show that the optimal control consists in forcing an intervention each time the exchange rate process hits the boundary of a band $[a, b]$, and the optimal intervention consists in jumping to a value α, where $a < \alpha < b$. (When there are proportional intervention costs as well, they show that the strategy is to jump to the boundary of a band properly contained in $[a, b]$.)

They show that the Value Function is

$$V(x) = \begin{cases} \phi(\alpha) + K & \text{if } x < a \\ \phi(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \left(\frac{1}{r - \sigma^2 - 2\mu}\right)x^2 - \frac{2\rho}{r-\mu}x + \frac{\rho^2}{r} & \text{if } a \leq x \leq b \\ \phi(\alpha) + K & \text{if } x > b \end{cases}$$

where $\gamma_{1,2} = \frac{-\mu + 0.5\sigma^2 \pm \sqrt{(\mu - 0.5\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}$. The unknown parameters A, B, a, α, b are found using continuity, optimality, and smooth pasting conditions (as we will explain in the next subsection). Note that V solves $\mathcal{L}V(x) + f(x) = 0$ in the continuation region, $a < x < b$. In the intervention region, as there are no intervention costs, we have that $\mathcal{M}V(x) = V(\alpha) + K$.

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4.2 The case with market reaction

We assume for this example that we have the same intervention and running costs as above, but now after each intervention the volatility parameter changes to σ_2, perhaps larger than σ, for a time $T > 0$, representing a temporary new market regime in reaction to the intervention. During this reaction time additional intervention is not allowed.

Our method for finding the optimal impulse control strategy is to propose the form of the optimal control ν (up to some unknown parameters), and use ν to construct a solution ϕ to the QVI for which ν is the QVI-control. If, for the proper choice of parameters, ϕ and ν satisfy the smoothness and growth conditions of theorem 3.4, and if ν is admissible, we will be able to conclude by theorem 3.4 that ϕ is the value function for the problem and ν is our desired optimal strategy.

The proposal is that we should intervene each time the exchange rate process leaves an interval (a, b), as soon as at least time T has elapsed since the last intervention, and that the optimal intervention consists in shifting the exchange rate to α, where $a < \alpha < b$. The constants a, b, and α are as yet unknown.

If indeed it is optimal to intervene only outside the interval (a, b), then we will expect ϕ to satisfy the partial differential equation (3.8) inside the interval. (The interval (a, b) is called the "continuation region" because the exchange rate freely follows the original SDE in this interval, after any reaction period.)

Therefore, solving

$$\mathcal{L}\phi(x) + f(x) = 0$$

we obtain, for $a < x < b$,

$$\phi(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \left(\frac{1}{r - \sigma^2 - 2\mu}\right)x^2 - \frac{2\rho}{r - \mu}x + \frac{\rho^2}{r}$$

where $\gamma_{1,2} = \frac{-\mu + 0.5\sigma^2 \pm \sqrt{(\mu - 0.5\sigma^2) + 2r\sigma^2}}{\sigma^2}$, and A and B are yet to be found.

We need to compute the running cost $\tilde{K}$ incurred during the reaction period T:

$$\tilde{K}(x) = E\left[\int_0^T e^{-rt}(\tilde{X}_x(t) - \rho)^2 \, dt\right].$$

Given that $\tilde{X}_x(t)$ follows a Geometric Brownian Motion with drift μ and volatility σ_2, the above integral can be computed analytically, because since T is independent of the process W_t, the expectation over T can be computed separately:

$$
\begin{aligned}
\tilde{K}(x) &= E\left[\int_0^T e^{-rt}(x^2 e^{2(\mu t - \frac{1}{2}\sigma_2^2 t + \sigma_2 W_t)} - 2e^{-rt}(e^{(\mu t - \frac{1}{2}\sigma_2^2 t + \sigma_2 W_t)}\rho x + \rho^2) \, dt\right] \\
&= E\left[\int_0^T e^{-rt}\left(x^2 e^{2\mu t + \sigma_2^2 t} - 2\rho x e^{\mu t} + \rho^2\right) dt\right]
\end{aligned}
$$

where we have used $E[e^{\sigma_2 W_t}] = e^{\frac{1}{2}\sigma_2^2 t}$.

The boundary conditions at the intervention points a, b can be found using continuity and the condition $\phi = \mathcal{M}\phi$, with the operator $\mathcal{M}$ defined by equation (3.2), and α denoting a minimizer in the definition of $\mathcal{M}$:

$$\alpha = \arg\min\left(K + \tilde{K}(\alpha) + E[e^{-rT}\phi(\tilde{X}_\alpha(T))]\right).$$

We obtain

(4.2) $$\phi(a) = K + \tilde{K}(\alpha) + E\left[e^{-rT}\phi(\tilde{X}_\alpha(T))\right]$$

where for each fixed T, $E\left[e^{-rT}\phi(\tilde{X}_\alpha(T))\right]$ is given by

$$e^{-rT}\left\{\int_a^b \phi(x)p(x;\alpha) \, dx + \phi(a)\int_{-\infty}^a p(x;\alpha) \, dx + \phi(b)\int_b^\infty p(x;\alpha) \, dx\right\}$$

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and $p(x;\alpha)$ is the probability density of a Log-Normal distribution as $\log(\tilde{X}_\alpha(T))$ is normally distributed with mean $\ln\alpha + \mu T - \frac{1}{2}\sigma_2^2 T$ and standard deviation $\sigma_2\sqrt{T}$. A similar equation applies for $\phi(b)$,

$$(4.3) \qquad \phi(b) = K + \tilde{K}(\alpha) + E\left[e^{-rT}\phi(\tilde{X}_\alpha(T))\right].$$

In addition we have the "smooth pasting requirement" (the C^1 condition), needed for the application of theorem 3.4

$$(4.4) \qquad \phi'(a) = 0,$$

$$(4.5) \qquad \phi'(b) = 0,$$

and the optimality of α, which implies solving

$$(4.6) \qquad \frac{d}{d\alpha}\left(\tilde{K}(\alpha) + E\left[e^{-rT}\int_{-\infty}^{+\infty} p(x;\alpha)\phi(x)\ dx\right]\right) = 0,$$

where the expectation here is over the distribution of T, which is an implicit parameter in $p(x;\alpha)$.

The above equations can be solved to obtain the parameters A, B, a, b and α, and the Value function has the same structure as in the case without market reaction, but with these different parameters:

$$V(x) = \begin{cases} K + \tilde{K}(\alpha) + E[e^{-rT}\phi(\tilde{X}_\alpha(T))] & if\ x < a \\ \phi(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \left(\frac{1}{r-\sigma^2-2\mu}\right)x^2 - \frac{2\rho}{r-\mu}x + \frac{\rho^2}{r} & if\ a \le x \le b \\ K + \tilde{K}(\alpha) + E[e^{-rT}\phi(\tilde{X}_\alpha(T))] & if\ x > b \end{cases}$$

Note that by construction this function is continuous at a and b, and we denote the common value by

$$\Theta := \phi(a) = \phi(b) = K + \tilde{K}(\alpha) + E[e^{-rT}\phi(\tilde{X}_\alpha(T))].$$

Finally, to complete the solution, we must verify the hypotheses of theorem 3.4 to conclude that this is indeed the Value Function, and that the proposed policy is optimal:

Theorem 4.1 *Let A, B, a, b, α be a solution of the system of equations (4.2) - (4.6) with $a < \alpha < b$, and let Θ be the constant defined above. Consider the C^1 function $V(x) : (0, \infty) \to [0, \infty)$ defined by*

$$(4.7) \quad V(x) = \phi(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \left(\frac{1}{r - \sigma^2 - 2\mu} \right) x^2 - \frac{2\rho}{r - \mu} x + \frac{\rho^2}{r}$$

for $a < x < b$, and $V(x) = \Theta$ otherwise.

If

$$(4.8) \qquad\qquad a < \rho - (r\Theta)^{1/2} \text{ and } b > \rho + (r\Theta)^{1/2},$$

and

$$(4.9) \qquad\qquad\qquad V(\alpha) < \Theta,$$

then $V(x)$ is the Value Function; namely,

$$V(x) = \inf_{\nu} J^{\nu}(x)$$

and the optimal policy is the QVI-control corresponding to V, given by

$$\tau_i = \inf\{t >= \tau_{i-1} + T_{i-1} : X^{\nu}(t) \notin (a, b)\}$$

and

$$X^{\nu}(\tau_i) = X^{\nu}(\tau_i^-) - \xi_i = \alpha.$$

Proof: We start by showing that V satisfies the Q.V.I.:

First QVI Inequality: From the definition of V,

$$\mathcal{L}V(x) + f(x) = -r\Theta + (x - \rho)^2$$

when $x < a$ or $x > b$, and

$$\mathcal{L}V(x) + f(x) = \mathcal{L}\phi(x) + (x - \rho)^2$$

when $a \le x \le b$.

By construction of ϕ, $\mathcal{L}\phi(x) + (x - \rho)^2 = 0$ in the interval $[a, b]$; outside the interval we have that $\mathcal{L}\phi(x) + (x - \rho)^2 > 0$ because of conditions (4.8).

Second QVI Inequality: From the definitions of $\mathcal{M}$ and α, we have $\mathcal{M}V(x) = K + \tilde{K}(\alpha) + E[e^{-rT}V(\alpha)] = \Theta$ for all x. Therefore, $\mathcal{M}V(x) = V(x)$ for $x \notin [a, b]$. We need to show that $V(x) \le \Theta$ for all $x \in (a, b)$, from which we can conclude $\mathcal{M}V(x) \le V(x)$.

To see this, notice that on the interval (a, b), the third derivative V''' is of the form $c_1 x^{d_1} + c_2 x^{d_2}$, which can have at most one zero on $\mathbb{R}^+$. Therefore the second derivative V'' can change sign at most twice in (a, b). Because $V(a) = V(b) = \Theta$ and $V'(a) = V'(b) = 0$, this means $V(x) - \Theta$ cannot take both signs on $[a, b]$ – this would imply at least three sign changes for the second derivative – so we must either have $V(x) \le \Theta$ or $V(x) \ge \Theta$ for all x. Because of (4.9), it must be the former.

Third QVI Inequality: Holds as a result of the above arguments.

To verify the remaining conditions of theorem 3.4, first note that $V'(x)$ is bounded, so condition (3.12) holds via the admissibility condition (2.6). Condition (3.13) is immediately satisfied because V is bounded. Condition (3.14) is satisfied because V'' is bounded, even though discontinuous at a and b.

Finally, the QVI control given above is admissible: condition (2.3) holds since X is lognormal and $\alpha > 0$; condition (2.4) holds by construction; and conditions (2.5) and (2.6) hold because X and $\tilde{X}$ are lognormal and X^{ν} is bounded inside $[a, b]$ except for possible excursions for a duration at most $\bar{T}$.

4.3 Numerical Example:

Consider $\rho = 1.4$, $r = 0.06$, $\mu = 0.1$, $\sigma = 0.3$, $\sigma_2 = 0.4$, $K = 0.5$, and a fixed recovery time $T = 1$.

A searching algorithm was implemented to obtain the parameters A, B, a, b and α that solve the system of equations (4.2) - (4.6). The key element that allows solving the expectation $E\left[e^{-rT}\phi(\tilde{X}_{\alpha}(T))\right]$ on equations (4.2), (4.3), and (4.6) is realizing that $\phi(x)$ is constant outside (a, b). For fixed T, the expectation can then be expressed as:

$$e^{-rT}\left\{\int_{a}^{b}\phi(x)p(x;\alpha)\ dx + \phi(a)\int_{-\infty}^{a}p(x;\alpha)\ dx + \phi(b)\int_{b}^{\infty}p(x;\alpha)\ dx\right\}.$$

These integrals can be computed for a given trial set of parameters A, B, a, b and α. Note that the Log-Normal probability density depends on α and T. If T is random with a given probability density, additional integration is required. Finally, it is important to mention that the evaluation of the expectation for the case of fixed and proportional transactions costs is still possible with some modifications.

Table 1 and figure 1 compare the policy results with and without the reaction period. The coefficients A and B are found to be -1.330 and -93.064 when $T = 0$ and -1.193 and -92.759 when $T = 1$. (Conditions

(4.8) are easily verified.)

The new regime after interventions requires a modification in the policy observed by the Central Bank: the intervention points are different, notice that the band is widened; in addition, the amount of intervention is also different, as the new restarting value α indicates. The presence of a market reaction leads to the need for greater patience by the Central Bank, less frequent but larger interventions, and greater optimal costs.

4.4   Comparative Statics Analysis

One of the main advantages of obtaining analytical solutions is that comparative analysis can be performed. If a Central Bank knows what type of reaction to expect from the market after performing interventions – namely, an increase (decrease) in volatility, or an increase (decrease) in the trend on the dynamics of its currency – then the optimal policy can be found.

In table 2 we present the optimal policy in four different scenarios. Both the bands and the intervention sizes depend on the nature of the reaction of the market during the reaction time after interventions. It is reasonable to suppose volatility increases after interventions and, as we showed above, this implies a wider band than in the case without market reaction. For comparison purposes we also computed the case when the volatility decreases during the reaction period; in this case the band shrinks. It could also be the case that the trend of the currency is modified as a temporary outcome of the intervention; perhaps reflecting the sentiment of the market with respect to the confidence on the Central Bank's actions. It is interesting to observe that the band widens if the drift trend increases or decreases and,

as expected, the optimal restarting point is closer to the long term target $\rho = 1.4$ when the drift is lowered during the reaction period.

In table 3 we show the effect of varying T for the case when the market reaction is a temporary volatility increase to $\sigma_2 = 0.4$. Observe that as T increases the band widens. We conclude the analysis with the optimal policy when the reaction time T is uniformly distributed between 0 and 1.

5 Proof of Theorem

Before we prove the theorem, we need the following lemma:

Lemma 5.1 *Let ν be an admissible control, and let σ be an intervention time for ν. If the function ϕ satisfies $\phi \leq \mathcal{M}\phi$ and the growth condition (3.12), then we have the inequality*

$$e^{-r\sigma} \left(\phi(X_x^\nu(\sigma^-)) - \phi(X_x^\nu(\sigma)) \right)$$

$$\leq e^{-r\sigma}K + E\left[\int_\sigma^{\sigma+T} e^{-rt}(\mathcal{L}_2\phi(X_x^\nu(t)) + f(X_x^\nu(t)) \; dt|\mathcal{F}_\sigma \right],$$

with equality if $\phi = \mathcal{M}\phi$, and where $\mathcal{L}_2$ is the operator defined by

$$(5.1) \qquad \frac{1}{2}\sigma_2^2(x)\frac{\partial^2}{\partial x^2} + \mu_2(x)\frac{\partial}{\partial x} - r.$$

Proof: Application of Ito's formula (3.4) to $e^{-rt}\phi(X_x^\nu(t))$ gives

$$e^{-r(\sigma+T)}\phi(X_x^\nu(\sigma+T)) = e^{-r\sigma}\phi(X_x^\nu(\sigma)) + \int_\sigma^{\sigma+T} e^{-rt}\mathcal{L}_2\phi(X_x^\nu(t)) \; dt$$

$$+ \int_\sigma^{\sigma+T} e^{-rt}\sigma_2(X_x^\nu(t))\phi'(X_x^\nu(t)) \; dW_t,$$

because between times σ and $\sigma+T$ the controlled process follows the second diffusion. Taking conditional expectations we obtain

$$(5.2) \qquad E\left[e^{-r(\sigma+T)}\phi(X_x^\nu(\sigma+T)) \mid \mathcal{F}_\sigma \right] = e^{-r\sigma}\phi(X_x^\nu(\sigma))$$

$$+E\left[\int_{\sigma}^{\sigma+T} e^{-rt}\mathcal{L}_2\phi(X_x^\nu(t))\ dt + \int_{\sigma}^{\sigma+T} e^{-rt}\sigma_2(X_x^\nu(t))\phi'(X_x^\nu(t))\ dW_t\mid\mathcal{F}_\sigma\right]$$

$$= e^{-r\sigma}\phi(X_x^\nu(\sigma)) + E\left[\int_{\sigma}^{\sigma+T} e^{-rt}\mathcal{L}_2\phi(X_x^\nu(t))\ dt\mid\mathcal{F}_\sigma\right]$$

where the stochastic integral vanishes by condition (3.12) and that fact that

the Ito integral is a martingale.

Now, the inequality $\phi \leq \mathcal{M}\phi$ says that for any positive z and y,

$$
\begin{aligned}
\phi(z) - E\left[e^{-rT}\phi(\tilde{X}_y^\nu(T))\right] &\leq\ K + \tilde{K}(y) \\
&=\ K + E\int_0^T e^{-rt}f(\tilde{X}_y^\nu(t))\ dt,
\end{aligned}
$$

Therefore, using $z = X_x^\nu(\sigma^-)$, $y = X_x^\nu(\sigma)$, this equation can be written

as

$$\phi(X_x^\nu(\sigma^-)) - E\left[e^{-rT}\phi(X_x^\nu(\sigma+T))\mid\mathcal{F}_\sigma\right]$$

$$\leq K + E\left[\int_{\sigma}^{\sigma+T} e^{-r(t-\sigma)}f(X_x^\nu(t))\ dt\mid\mathcal{F}_\sigma\right]$$

Multiplying by $e^{-r\sigma}$ we obtain

$$e^{-r\sigma}\phi(X_x^\nu(\sigma^-)) - E\left[e^{-r(\sigma+T)}\phi(X_x^\nu(\sigma+T))|\mathcal{F}_\sigma\right]$$

$$\leq e^{-r\sigma}K + E\left[\int_{\sigma}^{\sigma+T} e^{-rt}f(X_x^\nu(t))dt|\mathcal{F}_\sigma\right],$$

therefore using (5.2) we have

$$
\begin{aligned}
e^{-r\sigma}\phi(X_x^\nu(\sigma^-)) - e^{-r\sigma}\phi(X_x^\nu(\sigma)) &\leq\ E\left[\int_{\sigma}^{\sigma+T} e^{-rt}\mathcal{L}_2\phi(X_x^\nu(t))\ dt\mid\mathcal{F}_\sigma\right] \\
&+\ e^{-r\sigma}K + E\left[\int_{\sigma}^{\sigma+T} e^{-rt}f(X_x^\nu(t))\ dt\mid\mathcal{F}_\sigma\right],
\end{aligned}
$$

which is the desired result.

We now prove theorem 3.4.

Proof: Consider any admissible control $v = \{(\tau_n, \Delta X_n)\}_{n \in N}$, and let $\tau_0 = 0$.

Define the stopping time $\tau^*(t) = \max\{\tau_i : \tau_i \le t\}$; note $\tau^*(t) \to \infty$ as $t \to \infty$ almost surely since $\tau_i \to \infty$ a.s.

We need to estimate the quantity

(5.3)
$$e^{-r\tau^*(t)} \phi(X_x^v(\tau^*(t))) - \phi(x) = A + B$$

where A and B are the *finite* sums given by

$$A = \sum_{i=1}^{\infty} 1_{\{\tau_i \le t\}} \left(e^{-r\tau_i} \phi(X_x^v(\tau_i^-)) - e^{-r\tau_{i-1}} \phi(X_x^v(\tau_{i-1})) \right)$$

and

$$B = \sum_{i=1}^{\infty} 1_{\{\tau_i \le t\}} e^{-r\tau_i} \left(\phi(X_x^v(\tau_i)) - \phi(X_x^v(\tau_i^-)) \right).$$

For the term in the first summation A, in the event $\{\tau_i \le t\}$, when $i = 2, 3, \ldots, n$, an application of Ito's formula gives

$$
e^{-r\tau_i} \phi(X_x^v(\tau_i^-)) - e^{-r\tau_{i-1}} \phi(X_x^v(\tau_{i-1})) = \int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs} \mathcal{L}_2 \phi(X_x^v(s)) \, ds
$$
$$
+ \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs} \mathcal{L} \phi(X_x^v(s)) \, ds
$$
$$
+ \int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs} \phi'(X_x^v(s)) \sigma_2(X_x^v(s)) \, dW_s
$$
$$
+ \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs} \phi'(X_x^v(s)) \sigma_1(X_x^v(s)) \, dW_s.
$$

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Using QVI inequality (3.8), $\mathcal{L}\phi + f \geq 0$, this expression becomes

$$e^{-r\tau_i}\phi(X_x^\nu(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^\nu(\tau_{i-1})) \geq \int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs}\mathcal{L}_2\phi(X_x^\nu(s)) \ ds$$

$$+ \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs}(-f(X_x^\nu(s)) \ ds$$

$$+ \int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs}\phi'(X_x^\nu(s))\sigma_2(X_x^\nu(s)) \ dW_s$$

$$+ \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs}\phi'(X_x^\nu(s))\sigma_1(X_x^\nu(s)) \ dW_s,$$

with equality when $\mathcal{L}\phi + f = 0$. For $i = 1$ we have, similarly,

$$e^{-r\tau_i}\phi(X_x^\nu(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^\nu(\tau_{i-1}))$$

$$= \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\mathcal{L}\phi(X_x^\nu(s)) \ ds + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi'(X_x^\nu(s))\sigma_1(X_x^\nu(s)) \ dW_s,$$

$$\geq \int_{\tau_{i-1}}^{\tau_i} e^{-rs}(-f(X_x^\nu(s))) \ ds + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi'(X_x^\nu(s))\sigma_1(X_x^\nu(s)) \ dW_s.$$

For the term in the second summation B of equation (5.3), we use lemma 5.1 so that in the event $\{\tau_i \leq t\}$ we have

$$e^{-r\tau_i}\left(\phi(X_x^\nu(\tau_i) - \phi(X_x^\nu(\tau_i^-)))\right)$$

$$\geq -e^{-r\tau_i}K - E\left[\int_{\tau_i}^{\tau_i+T} e^{-rs}\left(\mathcal{L}_2\phi(X_x^\nu(s)) + f(X_x^\nu(s)) \ ds|\mathcal{F}_{\tau_i}\right],$$

with equality when $\phi = \mathcal{M}\phi$.

Therefore, reversing the sign and writing the $i = 1$ and $i > 1$ terms separately, equation (5.3) becomes

$$(5.4) \qquad \phi(x) - e^{-r\tau^*(t)}\phi(X_x^\nu(\tau^*(t))) \leq I_1 + I_2$$

where I_1 and I_2 correspond the terms $i = 1$ and $i = 2, \ldots, n$; namely,

$$I_1 = 1_{\{\tau_1 \leq t\}}\left(e^{-r\tau_1}K + E\left[\int_{\tau_1}^{\tau_1+T} e^{-rs}(\mathcal{L}_2\phi(X_x^\nu(s)) + f(X_x^\nu(s))) \ ds|\mathcal{F}_{\tau_1}\right]\right.$$

$$+ \left.\int_{\tau_0}^{\tau_1} e^{-rs}f((X_x^\nu(s))) \ ds - \int_{\tau_0}^{\tau_1} e^{-rs}\phi'(X_x^\nu(s))\sigma_1(X_x^\nu(s)) \ dW_s\right),$$

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$$I_2 = \sum_{i=2}^{\infty} 1_{\{\tau_i \leq t\}} \left\{ \left[e^{-r\tau_i} K + E \left[\int_{\tau_i}^{\tau_i + T} e^{-rs} (\mathcal{L}_2 \phi(X_x^\nu(s)) + f(X_x^\nu(s))) \, ds | \mathcal{F}_{\tau_i} \right] \right] \right.$$

$$- \int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs} \mathcal{L}_2 \phi(X_x^\nu(s)) \, ds + \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs} f(X_x^\nu(s)) \, ds$$

$$\left. - \int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs} \phi'(X_x^\nu(s)) \sigma_2(X_x^\nu(s)) \, dW_s - \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs} \phi'(X_x^\nu(s)) \sigma_1(X_x^\nu(s)) \, dW_s \right\}.$$

We now take expectations on both sides of equation (5.4)

$$(5.5) \qquad \phi(x) - E \left[e^{-r\tau^*(t)} \phi(X_x^\nu(\tau^*(t))) \right] \leq E[I_1] + E[I_2],$$

and realize that the expectations of the stochastic integrals vanish because of condition (3.12). If we collect all terms of the right hand side where the integrand is $f(X_x^\nu(t))$, we have

$$E \left[1_{\{\tau_1 \leq t\}} \left(E \left[\int_{\tau_1}^{\tau_1 + T} e^{-rs} f(X_x^\nu(s)) \, ds | \mathcal{F}_{\tau_1} \right] + \int_{\tau_0}^{\tau_1} e^{-rs} f((X_x^\nu(s))) \, ds \right) \right]$$

$$+ \quad E \left[\sum_{i=2}^{\infty} 1_{\{\tau_i \leq t\}} \left(E \left[\int_{\tau_i}^{\tau_i + T} e^{-rs} (f(X_x^\nu(s))) \, ds | \mathcal{F}_{\tau_i} \right] + \int_{\tau_{i-1}+T}^{\tau_i} e^{-rs} f(X_x^\nu(s)) \, ds \right) \right]$$

$$= \qquad E \left[\int_0^{\tau^*(t)+T} e^{-rs} f(X_x^\nu(s)) \right],$$

where we have used $\tau_0 = 0$ and

$$E \left[E[\int_{\tau_i}^{\tau_i + T} e^{-rs} (f(X_x^\nu(s))) \, ds] \mid \mathcal{F}_{\tau_i} \right] = E \left[\int_{\tau_i}^{\tau_i + T} e^{-rs} (f(X_x^\nu(s))) \, ds \right].$$

Collecting all terms where the integrand is $\mathcal{L}_2\phi(X_x^\nu(t))$ in equation (5.5) we obtain

$$1_{\{\tau_1 \leq t\}} E\left(E\left[\int_{\tau_1}^{\tau_1+T} e^{-rs}(\mathcal{L}_2\phi(X_x^\nu(s)))\ ds | \mathcal{F}_{\tau_1}\right]\right)$$

$$+ \sum_{i=2}^\infty 1_{\{\tau_i \leq t\}} \left\{ E\left(E\left[\int_{\tau_i}^{\tau_i+T} e^{-rs}(\mathcal{L}_2\phi(X_x^\nu(s)))\ ds | \mathcal{F}_{\tau_i}\right]\right) \right.$$

$$\left. - \quad E\left[\int_{\tau_{i-1}}^{\tau_{i-1}+T} e^{-rs}\mathcal{L}_2\phi(X_x^\nu(s))\ ds\right] \right\}$$

$$= \quad E\left[\int_{\tau^*(t)}^{\tau^*(t)+T} e^{-rs}\mathcal{L}_2\phi(X_x^\nu(s))\ ds\right].$$

Hence, equation 5.5 becomes

(5.6)
$$\phi(x) - E\left[e^{-r\tau^*(t)}\phi(X_x^\nu(\tau^*(t)))\right]$$

$$\leq E[\int_0^{\tau^*(t)} e^{-rs}f(X_x^\nu(s))\ ds + \int_{\tau^*(t)}^{\tau^*(t)+T} e^{-rs}(f(X_x^\nu(s)) + \mathcal{L}_2\phi(X_x^\nu(s)))\ ds$$

$$+ \sum_{i=1}^\infty 1_{\{\tau_i \leq t\}} e^{-r\tau_i}K].$$

Now, let t go to infinity, notice that condition (3.13) tells us that the left-hand side becomes

$$\lim_{t \to \infty} \phi(x) - E[e^{-r\tau^*(t)}\phi(X_x^\nu(\tau^*(t)))] = \phi(x),$$

while condition (3.14), the admissibility condition (2.5), and the dominated convergence theorem imply that

$$E\left[\int_{\tau^*(t)}^{\tau^*(t)+T} e^{-rs}(f(X_x^\nu(s)) + \mathcal{L}_2\phi(X_x^\nu(s)))\ ds\right] \to 0 \quad as \quad t \to \infty.$$

Therefore,

$$\phi(x) \leq E\left[\sum_{i=1}^\infty 1_{\{\tau_i < \infty\}} e^{-r\tau_i}K + \int_0^\infty e^{-rs}f(X_x^\nu(s))\ ds\right].$$

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Hence we have shown that

$$\phi(x) \leq J^v(x).$$

As this is true for any control v, we have

$$\phi(x) \leq V(x).$$

Now, the above inequalities become equalities for the QVI-control associated to ϕ.

6 Conclusions

We have addressed the problem of Central Bank intervention in the exchange rate market incorporating the effect of a temporary market reaction, of random duration, affecting the dynamics of the exchange rate process. The reaction time T can have any bounded non-negative distribution provided it is stationary and independent of the rate process. Using the Quasi-Variational Inequalities approach to impulse control problems, we presented a verification theorem that allows us to find the optimal policy and the Value Function for the problem. The main technical innovation is the use of a new optimal intervention operator $\mathcal{M}$ adapted to this setting.

We obtained an explicit solution of the problem for geometric brownian motion, and showed how the optimal policies are influenced by the presence of the reaction period after interventions. If volatility is assumed to jump up temporarily after interventions, the result is that the target band widens and the optimal costs increase. The band narrows if interventions are assumed to cause a decrease in volatility; a change of drift in either direction causes

the band to widen slightly. Thus, in most cases, Banks should optimally intervene a little less often than they would if interventions were completely invisible to the FX market.

It would be interesting to extend these results in various ways. Can the assumption that no interventions are allowed during the reaction period be relaxed or removed? What if the reaction time is not independent of the process? What if the drift and volatility during the reaction time are random rather than fixed? These are topics for further work.

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Table 1: Optimal Policy with and without a recovery period.

Reaction Period	a	b	α
None ($T = 0$)	0.622	2.307	1.249
$T = 1$	0.581	2.365	1.212

Table 2: Optimal policy for different market reactions.

Type of reaction	a	b	α
No reaction ($\sigma_1 = \sigma_2 = 0.30$, $\mu_2 = \mu_1 = 0.10$)	0.622	2.307	1.249
Volatility increases ($\sigma_2 = 0.40$, $\mu_2 = 0.10$)	0.581	2.365	1.212
Volatility decreases ($\sigma_2 = 0.10$, $\mu_2 = 0.10$)	0.678	2.230	1.235
Drift increases ($\sigma_2 = 0.30$, $\mu_2 = 0.15$)	0.618	2.314	1.186
Drift decreases ($\sigma_2 = 0.30$, $\mu_2 = 0.05$)	0.621	2.309	1.275

Table 3: Optimal Policy for different reaction time periods when volatility increases during the reaction time.

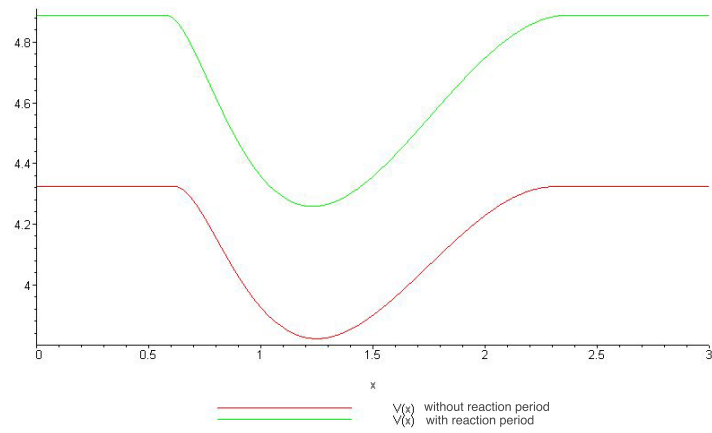| Reaction Period	a	b	α
None ($T = 0$)	0.622	2.307	1.249
$T = 1$	0.581	2.365	1.212
$T = 2$	0.516	2.461	1.072
$T \sim U[0, 1]$	0.602	2.336	1.242

Figure 1: Value function with one and two volatility regimes. The optimal strategy is more expensive if there are two volatility regimes and a positive reaction period.