COMPUTING ON RIEMANN SURFACES

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Abstract. These notes are a review on computational methods that allow us to use computers as a tool in the research of Riemann surfaces, algebraic curves and Jacobian varieties.

It is well known that compact Riemann surfaces, projective algebraic curves and Jacobian varieties are only different views to the same object, i.e., these categories are equivalent. We want to be able to put our hands on this equivalence of categories. If a Riemann surface is given, we want to compute an equation representing it as a plane algebraic curve, and we want to compute a period matrix for it.

Vice versa, we want to be able to compute the uniformization for a given algebraic plane curve, or a Riemann surface corresponding to a given Jacobian variety.

In another direction we consider tools that allow us to compute eigenvalues and eigenfunctions of the Laplace operator for Riemann surfaces. The correspondence between the Laplace spectrum of a Riemann surface and the geometry of the surface in general is intriguing. The programs to be described later give us a possibility to explore this correspondence in an explicit manner.

The above mentioned computational problems are hard and most of them are open in the general case. In certain particular cases, like that of hyperelliptic algebraic curves, interesting results are known ([7], [9],[10], [11], [13], [14], [15]). We will review some of these results and consider implementations of programs needed to make practical use of these results.

For the reader's convenience, we also review some of the basic underlying mathematical concepts. Our basic references to the theory of Riemann surfaces are [4], [6], [8] and [12].

1. Preliminaries

In this section we describe a method for defining Möbius transformations for computations by a computer. We are interested, in particular, in Möbius transformations that map either the upper half–plane or the unit disk onto itself. Such a transformation can be expressed in the form

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc=1. \tag{1}$$

Transformation (1) maps the upper–half plane U onto itself if and only if all the coefficients a, b, c, and d are real, and it maps the unit disk D onto itself if and only if $c=\overline{b}$ and $d=\overline{a}$, as is well known.

For computational purposes, we should define a Möbius transformation by simply the formula (1), i.e., by giving either exact values (rational or algebraic numbers)

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or floating point approximations for the coefficients a, b, c, and d. In the applications, that we have in mind, almost always one ends up using simply floating point approximations of the coefficients.

Formula (1) is, of course, sufficient for computing further with the Möbius transformation in question. However, formula (1) is not the geometrically natural way for defining a Möbius transformation.

Geometrically, a hyperbolic transformation

$$g(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1, \tag{2}$$

is determined by the following three parameters:

- the *attracting* fixed point $a(g) = \lim_{n\to\infty} g^n(z)$
- the *repelling* fixed point $r(g) = \lim_{n\to\infty} g^{-n}(z)$
- the *multiplier*

$$k(g) = (g(z), z, r(g), a(g)) = \frac{g(z) - r(g)}{g(z) - a(g)} \cdot \frac{z - a(g)}{z - r(g)}, \tag{3}$$

where z is any point in $\hat{\mathbb{C}}$ not fixed by g. Especially, $k(g) > 1$. The fixed points are real if $g(U) = U$. The non–Euclidean line through $a(g)$ and $r(g)$, the *axis* of g, is denoted by $ax(g)$. It has natural orientation by $r(g) \to a(g)$.

Any hyperbolic Möbius transformation g having $k = k(g)$, $k > 1$, as the multiplier and $a = a(g)$ and $r = r(g)$ as the attracting and the repelling fixed–points can be written in the form

$$g(z) = \frac{(ka - r)z - ar(k-1)}{(k-1)z + a - kr}. \tag{4}$$

The fixed–points a and r of a hyperbolic Möbius transformation determine and are determined by an oriented hyperbolic line, namely the axis. These are the first two geometric parameters that we will use when defining hyperbolic Möbius transformations for computations.

We still need to specify the multiplier k. This parameter tells us how fast the transformation moves points towards the attracting fixed–point.

A concrete geometric way to define the multiplier is offered by the *isometric circle.* The isometric circle is a circular arc which is perpendicular to ∂D and to the axis of g. Provided that ∞ is not a fixed–point of g, the isometric circle is defined by the condition that the mapping g, when restricted to the isometric circle, is an isometry of the Euclidean metric.

The point where the isometric circle intersects the axis of g is the geometric object that we are looking for.

When defining a hyperbolic Möbius transformation for computations by a computer, the point, where the isometric circle intersects the axis, can be chosen by the mouse. The point will be an interior point of D and a point of the axis of g. The point has to be closer to the repelling fixed–point than to the attracting fixed–point.

Some technical computations are involved here. Using the representation (4), the isometric circle I_g of g can be written as

$$I_g = \{z \in D \mid |(k-1)z + a - kr| = |(ka - r)(a - kr) + ar(k-1)^2|^{\frac{1}{2}}\}.$$

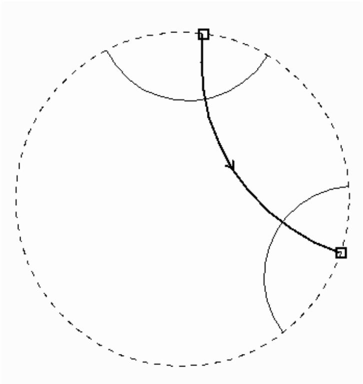

FIGURE 1. A hyperbolic Möbius transformation as seen by a computer. This picture is captured from the display of the program CARS. The picture shows: (**1**) the axis of a hyperbolic Möbius transformation, (**2**) the orientation of the axis, i.e., giving the attracting and repelling fixed points, and (**3**) the isometric circle and the image of the isometric circle under the mapping in question. All the geometric parameters – fixed–points and the isometric circle – can be modified by convenient mouse actions. The interface CARS then outputs, for computations with other programs, the parameters of the Möbius transformation. From this figure one can also immediately see the action of the defined Möbius transformation in the unit disk. The transformation maps the outside of the isometric circle onto the inside of the image of the isometric circle.

At this point we assume that the parameters a and r are known. By the mouse we choose the point z. The multiplier k is then that solution of the equation

$$|(k-1)z + a - kr| = |(ka - r)(a - kr) + ar(k-1)^2|^{\frac{1}{2}} \tag{5}$$

which is real and > 1.

Equation (5) leads to awkward computations. In the case where g acts in the unit disk we can we may assume that $a = -\overline{r}$ since this situation can always be achieved by a conjugation with an elliptic rotation fixing the origin. This rotation maps the isometric circle of the original mapping to that of the conjugated mapping since the rotation is also an isometry of the Euclidean metric. Observe that a general conjugation with a Möbius transformation does not have this property.

Then, if z is the intersection point of the isometric circle and the axis of g, then $g(z) = -\overline{z}$ by symmetry (since $a = -\overline{r}$). This gives us a simple equation from which we can compute k in terms of a, r and z, the intersection point of the isometric circle and the axis of g. Hence the Möbius transformation g becomes uniquely defined by the geometric parameters a, r and z.

Figure 1 illustrates the computer screen that shows the above geometric parameters of a hyperbolic Möbius transformation. This figure is from te display of the program CARS which allows one to define Möbius transformations conveniently and to modify them by mouse actions.

The program CARS allows one also:

- to write words of Möbius transformations,
- to display their axes and isometric circles and
- to see how these geometric parameters of a word of Möbius transformations change as we deform any of the primitive transformations that define this word.

This allows one to study, for instance, the question whether the axis of a certain word of Möbius transformations intersect the axis of some other word of Möbius transformations.

2. Riemann surfaces

We consider Riemann surfaces

$$X = D/G \tag{6}$$

where G is a Fuchsian group acting in D. Alternatively we may choose to work in the upper half–plane. We use the representation (6) to define Riemann surfaces for computations.

Hence defining a Riemann surface, on which we compute, reduces to giving a set of Möbius transformations generating the group G. From the computational point of view, these sets of Möbius transformations form simply lists of lists, specifying either the coefficients exactly (rational or algebraic) or giving floating point approximations of the coefficients of the groups in question.

In most of the interesting cases, the group G is not freely generated. Hence, in addition to the floating point approximations of the generators, one should also give the relations satisfied by the generators.

It is an open problem in general to determine when a set of Möbius transformations generates a discontinuous group. In order to ensure that one gets a Fuchsian group, one could proceed as follows. Define first a hyperbolic polygon P in D and then consider Möbius transformations defining a suitable side pairing for P. In such a way one can easily build generators for Fuchsian groups G such that the initial polygon P is a fundamental domain for G and that the Riemann surface D/G is of any desired topological type.

We have, however, chosen an alternative way which builds on the combination theorems of Maskit and others. To that end we start with considering certain simple cases first.

2.1. **Elementary cases.** Let $G = \langle g, h \rangle$ be a Fuchsian group freely generated by two hyperbolic Möbius transformations whose isometric circles *do not* intersect. We have two possible configurations:

1. the axis of g and h do not intersect (Fig. (2))
2. the axis of g and h do intersect (Fig. (3)).

This geometric configuration of the axis of g and that of h is reflected in an interesting way in the formulae regarding the product $g \circ h$. To see this we consider pairs of hyperbolic Möbius in general.

For technical convenience we consider Möbius transformations acting in the upper half plane instead of the unit disk. By a suitable conjugation one can transform all the considerations to the unit disk as well.

Let (g, h) be a pair of hyperbolic transformations fixing the upper half–plane U. Suppose that g and h have no common fixed points and denote

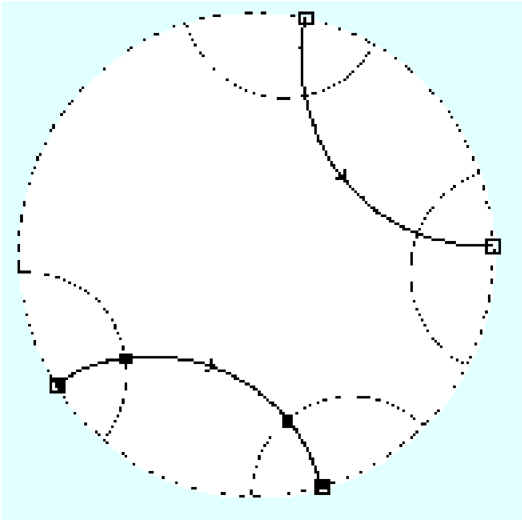

FIGURE 2. Möbius transformations g and h as in this figure generate a Fuchsian group corresponding to a Y–piece, i.e., to a pair of pants.

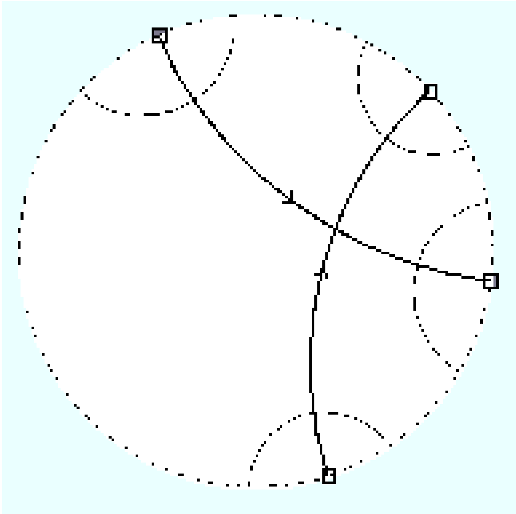

FIGURE 3. Möbius transformations g and h as in this figure generate a group corresponding to a Q–piece, i.e., to a handle.

- $t = (r(g), r(h), a(h), a(g))$,
- $k_1 = k(g)$,
- $k_2 = k(h)$,
- $k_3 = k(g \circ h)$.

In order to derive an expression for k_3 in terms of t, k_1 and k_2 we normalize by conjugation such that $r(h) = 1$, $a(h) = 0$ and $a(g) = \infty$. Then $t = r(g)$ and we have by formula (4)

$$g(z) = k_1 z - t(k_1 - 1),$$
$$h(z) = \frac{z}{(1-k_2)z + k_2}.$$

It follows that

$$f(k_3) = f(g \circ h) = |tf(k_1 k_2) + (1-t)f(k_1/k_2)|. \tag{7}$$

Here we use the notation

$$f(k) = \sqrt{k + \frac{1}{k} + 2} = \sqrt{k} + \frac{1}{\sqrt{k}}.$$

For a Möbius transformation g, we also write $f(g) = f(k(g))$, where $k(g)$ is the multiplier of g.

We divide pairs (g,h) of hyperbolic Möbius transformations into three disjoint *classes* $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{E}$ (cf. formula (7)):

$$\begin{aligned} (g,h) \in \mathcal{H} &\iff f(g \circ h) = tf(k_1k_2) + (1-t)f(k_1/k_2) \geq 2 \\ &\iff t \geq t_1 = \frac{2 - f(k_1/k_2)}{f(k_1k_2) - f(k_1/k_2)}, \\ (g,h) \in \mathcal{P} &\iff f(g \circ h) = -tf(k_1k_2) - (1-t)f(k_1/k_2) \geq 2 \\ &\iff t \leq t_2 = \frac{-2 - f(k_1/k_2)}{f(k_1k_2) - f(k_1/k_2)}, \\ (g,h) \in \mathcal{E} &\iff t_2 < t < t_1. \end{aligned}$$

Here $\mathcal{H}$ stands for "handle", $\mathcal{P}$ for "pants" and $\mathcal{E}$ for "elliptic". The Möbius transformation $g \circ h$ is either hyperbolic, parabolic or elliptic. Note that $g \circ h$ is elliptic if and only if $(g,h) \in \mathcal{E}$.

All pairs of Möbius transformations corresponding to the setting of Fig. (2) belong to the class $\mathcal{P}$ and all pairs of Fig. (3) belong to the class $\mathcal{H}$. For a detailed proof of this result and a general treatment of the geometry of pairs of Möbius transformations see [12, Section 1.5].

We have the following result ([12, Lemma 1.6.1, p. 40]):

Theorem 1. *The conjugacy class of a pair (g,h) of hyperbolic Möbius transformations belonging either to the class $\mathcal{P}$ or to the class $\mathcal{H}$ is uniquely determined by the multipliers of g, h and $g \circ h$.*

Theorem 1 has a natural geometrical meaning. Consider a pair (g,h) belonging to the class $\mathcal{P}$. The corresponding Fuchsian group $G = \langle g, h \rangle$ is of the second kind and easy considerations show that D/G is a Y–piece, i.e., a pair of pants. The hyperbolic metric of the unit disk defines a complete hyperbolic metric on D/G. The lengths of the geodesic curves freely homotopic to the three boundary curves are $\log k(g)$, $\log k(g)$ and $\log k(g \circ h)$ (provided that the axis of $g \circ h$ projects onto the third geodesic curve in question, if not, then take $g \circ h^{-1}$ instead). Hence the hyperbolic metric of a Y–piece is determined by the lengths of the three boundary curves.

The same applies to the situation where $G = \langle g, h \rangle$ defines a Q–piece, i.e., a handle. Also in this case the metric is defined by the lengths of the geodesic curves corresponding to the axes of g, h and to that of the product of g and h.

These cases are not completly analogous. It can be shown that the three parameters defining a Y–piece can be freely chosen, i.e., that the lengths of the boundary geodesics can be freely given. In the case of a Q–piece, the respective parameters correspond also to the lengths of geodesic curves. But now two of these curves intersect and their lengths cannot be freely given.

2.2. **Building Riemann surfaces out of Y–pieces and of Q–pieces.** Let Y and Y' be two Y– or Q–pieces having geodesic boundaries such that the boundary curve α of Y has the same length as the boundary curve α' of Y'. In this case we may identify α with α' using constant speed. The identification depends on one parameter, called the *gluing angle*. Apart from the gluing angle, the above identification defines a unique surface X. The hyperbolic metric of Y and that of Y' define a smooth hyperbolic metric of X hence making X into a Riemann surface.

To make this unique and our argument general, we consider, for $g \geq 2$, a collection of $g-2$ Y–pieces, $Y_1, Y_2, \ldots, Y_{g-2}$ and a collection of g Q–pieces $Q_1, Q_2, \ldots, Q_g$.

We assume that the list of Y–pieces is ordered and that the boundary geodesics of each Y–piece, Y_j, $\alpha_1^j, \alpha_2^j, \alpha_3^j$, form also an ordered list. Assume furthermore that the lengths of the boundary geodesics of the Y–pieces satisfy

$$\ell(\alpha_3^j) = \ell(\alpha_1^{j+1}) \tag{8}$$

Then we can first form a sphere Y with g holes by identifying the boundary geodesic α_3^j with α_1^{j+1} for $j = 1, \ldots, g-2$.

Next assume that the list (g_j, h_j) of generators of each of the Q–pieces Q_j is ordered and that the generators are Nielsen reduced.[1]

The axes g_l and h_l project onto intersecting geodesic curves on Q_l. Possibly replacing g_l or h_l by their inverses, we may always assume that the commutator

$$c_l = g_l \circ h_l^{-1} \circ g_l^{-1} \circ h_l \tag{9}$$

corresponds to the single boundary geodesic γ_l. of the Q–piece Q_l.

Let α_k^j be a boundary geodesic of Y. We can glue a Q–piece Q_l to α_k^j if and only if

$$\ell(\alpha_k^j) = \ell(\gamma_l). \tag{10}$$

From the ordering of the Y–pieces Y_j and from the ordering of the boundary curves α_k^j of each Y_j, we get an ordering for the boundary curves of the g–hole sphere Y. Call these boundary curves $\delta_1, \delta_2, \ldots, \delta_g$ and identify the boundary curve γ_l of Q_l with δ_l.

In order to parametrize these identifications we do the following:

1. Use the notation $\alpha_j^4 = \alpha_j^1$. Drop perpendiculars from each α_j^n to α_j^{n+1}, $j = 1, \ldots, g$, $n = 1, 2, 3$. Call y_j^n the end point of the corresponding perpendicular ending at α_j^n.
2. Drop perpendiculars from each geodesic corresponding to g_l onto the boundary geodesic γ_l of the Q–piece Q_l. Call x_l the end-point of the perpendicular on γ_l.

The points x_l and y_j are called *base points* of their respective boundary geodesics.

In the gluing process we identify pairs of boundary geodesics of Y–pieces and of Q-pieces. On the resulting genus g Riemann surfaces all these pairs of boundary geodesics correspond to a single geodesic curve. Call them $\delta_1, \ldots, \delta_{2g-1}$.

[1]Nielsen reduced generators are generators whose multipliers are as small as possible. There is an algorithm which finds Nielsen reduced generators for each Y–piece $D/\langle g, h\rangle$ starting from arbitrary generators g and h.

Next assume that the geodesic curves δ_j are all oriented and that the base point sets are also ordered so that we can speak about the first base point ξ_j^1 and of the second base point ξ_j^2 of δ_j.

Definition 1. *The* gluing angle θ_j *associated to the geodesic* δ_j *is then simply*

$$\theta_j = 2\pi \frac{\textit{distance from } \xi_j^1 \textit{ to } \xi_j^2 \textit{ along the positive direction of } \delta_j}{\textit{length of } \delta_j} \tag{11}$$

In view of Theorem 1 and the above definition of the gluing angles we have:

Theorem 2. *Let X be a genus g, $g > 1$, Riemann surface obtained by gluing $g-2$ Y–pieces and g Q–pieces together in an appropriate fashion. Let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the geodesic curves corresponding to the axes of the generators g_j and h_j of the Fuchsian groups of the Q–pieces. Let ρ_j be the geodesic curve corresponding to the axis of the product $g_j \circ h_j$. Let $\delta_1, \ldots, \delta_{2g-1}$ be the curves along which gluing has been performed and let θ_k, $k = 1, 2, \ldots, 2g-1$ be the gluing angles. Then the $7g-2$ parameters*

$$\ell(\alpha_j), \ell(\beta_j), \ell(\rho_j), \ell(\delta_k), \theta_k, j = 1, 2, \ldots, g, \; k = 1, 2, \ldots, 2g-1 \tag{12}$$

define the Riemann surface X uniquely up to an isometry.

Remark. The parameters $\ell(\rho_j)$ in the above theorem are almost superfluous in the sense that once the parameters $\ell(\alpha_j)$ and $\ell(\beta_j)$ are fixed, $\ell(\rho_j)$ can take only two different values.

Let X be a Riemann surface constructed in the above fashion and defined by the parameters of Theorem 2. Then one can *deform* X by cutting X open along any of the curves δ_j and letting one part of X slide and gluing back again. This sliding changes only the respective gluing angle θ_j and the sliding is the *Fenchel–Nielsen twist* of the Riemann surface X along the geodesic curve δ_j.

It is useful to know how the above construction and the Fenchel–Nielsen twist can be carried out on the universal cover.

To that end, consider two Y–pieces Y and Y' and the gluing of these Y–pieces together along a boundary geodesic. The general gluing and the corresponding Fenchel–Nielsen twist is performed in the same way as in this more elementary case.

Represent the Y–piece Y as $Y = U/\langle g, h\rangle$ and Y' as $Y' = U/\langle g', h'\rangle$. Assume that the fixed points of g and h both lie in the positive real axis and those of g' and h' in the negative. Assume furthermore, that the products $g \circ h$ and $g' \circ h'$ are both hyperbolic Möbius transformations fixing 0 and ∞, i.e., that they have the same axis which is the positive imaginary axis. Then $g \circ h = g' \circ h'$ and it turns out[2] that the group $\langle g, h, g', h'\rangle$ uniformizes the Riemann surface X. The Fenchel–Nielsen twist corresponds to conjugating the pair (g', h') by a hyperbolic Möbius transformation fixing 0 and ∞ and leaving the other pair (g, h) intact.

This construction can be generalized to all situations where we glue two Riemann surfaces together along *one* boundary curve ([8]). This construction has also been implemented in the program CARS , which allows one to build any Riemann surface out of Y–pieces and Q–pieces. The only restriction is that, at any step, we glue two

[2]This statement actually includes two theorems. First of all one has to show that the group generated by g, h, g' and h' is discontinuous and secondly that it uniformizes X. For a detailed treatment of these topics see Combination Theorems in [8].

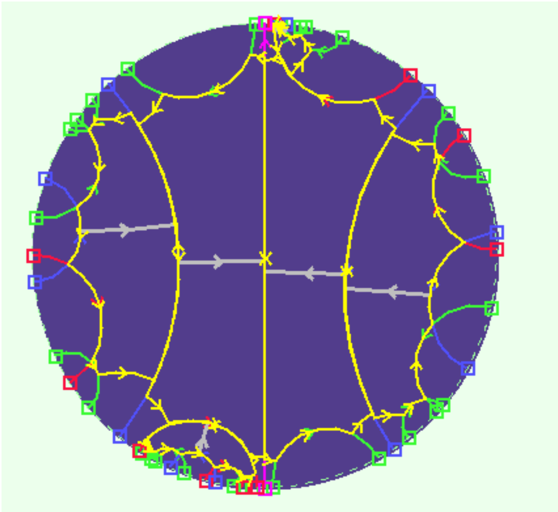

FIGURE 4. A genus 4 Riemann surface as seen by the program CARS . The bright hyperbolic geodesic arcs determine polygons that are fundamental domains for the corresponding Y-pieces or Q–pieces. Each such fundamental domain has also a geodesic arc perpendicular to two of the sides of the polygon. The purpose of these geodesic arcs is simply to fix two base–points on each of the geodesics along which gluing happens. The directed distance of these base–points determines the gluing angle. The program CARS allows one to perform the Fenchel–Nielsen twists by grabbing one of the base–points by mouse and then moving it.

Riemann surfaces together along *one* boundary curve only. This means, in particular, that we cannot realize, by the program CARS, all possible decompositions of compact Riemann surfaces into pairs of pants, i.e., into Y–pieces.

CARS allows one to perform Fenchel–Nielsen twists along any of the geodesic curves along which gluing has been performed. CARS allows also one to make these curves shorter or longer in such a generality that one can, in this way, construct any Riemann surface of any given finite type.

3. EQUATIONS FOR RIEMANN SURFACES

In sections 1 and 2 we have seen how one can conveniently define Möbius transformations and Riemann surfaces for computations. The interface CARS, described in those sections, is an input device that can be used to input Riemann surfaces for computations with other programs, most importantly with general purpose mathematics programs such as AXIOM, MAPLE or MATHEMATICA. The communication between CARS and other programs will later be provided by a communication protocol known as OPEN MATH.

Anyway, now that we have defined Riemann surfaces we want to compute with them. An important problem is to compute a polynomial equation representing a given compact Riemann surface as a projective algebraic plane curve. It is classical that compact Riemann surfaces are simply projective algebraic curves. To find a polynomial defining a plane algebraic curve that corresponds to a given compact genus g, $g > 1$, Riemann surface X we have to solve two problems:

1. approximate a *good* mapping of X into $\mathbb{P}^2(\mathbb{C})$ and compute points of the image of X in $\mathbb{P}^2(\mathbb{C})$
2. solve a system of linear equations to find a polynomial of degree at most $4g-4$ passing through the points computed above

The above simple plan involves a difficulty: The approximation of a mapping of X into $\mathbb{P}^2(\mathbb{C})$ involves an approximation of a bicanonical mapping of X into a higher dimensional projective space and then projecting down from there. The approximation of the bicanonical mapping in question is done by certain suitable Poincaré series. These series, however, converge slowly and therefore pose a serious numerical problem. We have to be able to distinguish those elements from the Poincaré series that contribute most to the value of the series.

To be more precise, assume that G is a Fuchsian group of the first kind acting in the unit disk D in such a way that D/G is a compact Riemann surface of genus p. Let $g \in G$ be a (primitive in G) Möbius transformation covering the homotopy class of a simple closed geodesic curve $\gamma \in D/G$. Assume that

$$g(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{13}$$

Let

$$\nu(g) = (\mathrm{tr}^2 g - 4)^{1/2} = \sqrt{k_g} - \frac{1}{\sqrt{k_g}} = ((a+d)^2 - 4)^{1/2}. \tag{14}$$

Here k_g is the multiplier of the hyperbolic Möbius transformation g and $\mathrm{tr}^2 g$ is the square of the trace of a matrix, of determinant $+1$, corresponding to the Möbius transformation g. Observe that such a matrix is well–defined only up to sign. The *square* of the trace is, however, always well–defined.

Using this notation define

$$\omega_g = \nu(g)(cz^2 + (d-a)z - b)^{-2}. \tag{15}$$

The importance of the formula (15) lies in the fact that

$$(\omega_g \circ g)\left(\frac{\partial g}{\partial z}\right)^2 = \omega_g \tag{16}$$

as can be verified by a direct computation. Therefore ω_g is a holomorphic 2–form of the cyclic group $\langle g \rangle$. This motivates one to look at the *Petersson–Poincaré series*

$$\theta_\gamma = \sum_{h \in \langle g \rangle \backslash G} (\omega_g \circ h)\left(\frac{\partial h}{\partial z}\right)^2 \tag{17}$$

associated to the simple closed geodesic curve γ.

Wolpert has shown the following result [16, Theorem 3.7, page 521]:

Theorem 3. *Let $\alpha_1, \ldots, \alpha_{3p-3}$ be a decomposition of the Riemann surface D/G into Y–pieces (a maximal set of non–intersecting simple closed curves). Then the Petersson–Poincaré series θ_{α_j} converge, their limits are holomorphic 2–forms of the group G, and $\{\theta_{\alpha_j}\}$ is a basis (over $\mathbb{C}$) for the space of holomorphic 2–forms of the group G.*

This result allows us to approximate the values of quadratic differentials of a compact Riemann surface at as many points as one wants. In particular, this means that we can approximate numerically the corresponding bicanonical mapping

$$\phi_{2\kappa} : X \to \mathbb{P}^{3g-4}(\mathbb{C}),\ X \ni p \mapsto (\theta_{\alpha_1}(z), \ldots, \theta_{\alpha_{3g-3}}(z)) \in \mathbb{P}^{3g-4}(\mathbb{C}), \tag{18}$$

where $z \in D$ is any point projecting onto $p \in X = D/G$.

This numerical approximation involves computing a finite part of the series (17). For numerical accuracy it is important that the elements to be computed of the series (17) are the ones which contribute most to the sum.

Loris Renggli and Klaus–Dieter Semmler have provided a solution to this problem. Let G_F be a finite part of G. Let us consider that part of the series (17) which corresponds to G_F. Let $A \subset D$ be a fundamental domain for the group G.

Let

$$\theta_\gamma^F = \sum_{h \in \langle g \rangle \backslash G_F} (\omega_g \circ h) \tag{19}$$

be a finite approximation of θ_γ corresponding to the finite part G_F of G.

Considering the usual proof for the convergence of the Poincaré series, one can estimate the error $|\theta_\gamma(z) - \theta_\gamma^F(z)|$ in terms of the euclidean area of the set

$$D \setminus \left(\cup_{g \in G_F} g(A)\right), \tag{20}$$

for *any* fundamental domain A of G in D. This leads to the observations that, when selecting elements of the finite part G_F, we should take elements g in the order defined by the Euclidean areas of the domains $g(A)$. This observation has been coded, by Loris Renggli and Klaus–Dieter Semmler, to a program producing good finite parts of a Fuchsian group G.

Further details regarding the error estimates and generalizations of this method will be contained in a forthcoming theses of Pekka Smolander (University of Joensuu, Finland).

4. Numerical uniformization

The methods described in Section 3 allows one to approximate numerically a polynomial equation defining a given compact Riemann surface as a plane algebraic curve.

The converse problem is more difficult. Starting with a complex algebraic curve C, we would like to find a group G of Möbius transformations such that $C = \Omega(G)/G$, where $\Omega(G)$ is (a component of) the domain of discontinuity of G.

In full generality this is an open problem. We can, however, solve in certain particular cases. In the following we review the considerations of P. J. Myrberg ([9]) as presented in [10], show a new uniqueness result (Lemma 8) and clarify the proof of Lemma 10 (which contained an error in [10]).

4.1. **Opening mapping.** We need to consider an auxiliary mappings, called *opening mappings*. Such a mapping maps the exterior of an interval in the complex plane onto the exterior of a disk. The algorithm is based on iteration of such mappings.

Let

$$\tau_j \in \mathbb{C},\ j = 1, 2, \tag{21}$$

be two different points in the complex plane. Let I denote the closed interval having τ_1 and τ_2 as end–points.

Let

$$m = \frac{\tau_1 + \tau_2}{2} \text{ and } s = \frac{\tau_2 - \tau_1}{4}. \tag{22}$$

We assume that the indices are so chosen that the real part of s is non–negative. In the applications considered here, the τ's will be real and we simply assume that $\tau_2 > \tau_1$.

Consider the equation

$$\frac{z - (m + 2s)}{z - (m - 2s)} = \left(\frac{w - (m + s)}{w - (m - s)} \right)^2. \tag{23}$$

Suitable choice of the square–root gives us, by equation (23), a mapping

$$\phi : \mathbb{C} \setminus I \to \mathbb{C} \setminus \overline{D}, \tag{24}$$

where D is a disk centered at m with radius $|s|$.

This mapping ϕ plays a central role in our construction and it is, therefore, necessary to take a closer look at it.

We observe that the following lemmata hold:

Lemma 4. *The mapping ϕ determined by equation (23) is given by*

$$\phi(x) = x + \frac{x - m}{2} \left(\sqrt{1 - 4\frac{s^2}{(x - m)^2}} - 1 \right), \tag{25}$$

for $|x - m| \geq 2|s|$,
or

$$\phi(x) = x - s\left(t + t^3 + t^5 + O\left(t^7\right)\right), \tag{26}$$

where $t = s/(x - m)$ and $|x - m| \geq 2|s|$.

Lemma 5. *Assume that the τ's are real and $\tau_2 > \tau_1$. Then m is also real and $s > 0$. In this case the mapping ϕ moves points of the real axis outside of the interval $[m - 2s, m + 2s]$ towards the mid-point m. For $|x - m| \geq 2|s|$, let*

$$\delta(x) = |x - \phi(x)| \tag{27}$$

or

$$\delta(x) = |x - m|\left(t^2 + t^4 + 2t^6 + O(t^8)\right), \tag{28}$$

where $t = s/(x - m)$.

Then, for $x \geq m + 2s$, $\delta(x)$ is strictly decreasing, and, for $x \leq m - 2s$, $\delta(x)$ is strictly increasing. The function $\delta(x)$ attains its maximum for $x = m + 2s$ and for $x = m - 2s$. The maximum value is

$$\delta(m \pm 2s) = s. \tag{29}$$

Furthermore,

$$\lim_{x \to \pm\infty} \delta(x) = 0. \tag{30}$$

Both of the above lemmata are simple straightforward technical computations and proofs are left to the reader. Observe that the property (30), which is important for our construction, follows from the fact that on the right hand side of defining equation (23), $2s$ was replaced by s only.

In the above, the mapping ϕ was determined by equation (23) and by requiring that ϕ maps the complement of the interval I onto the complement of the disk D with m as center and $|s|$ as radius.

The other choice of the sign of the square root (when solving w in terms of z from equation (23)) yields a mapping $\tilde{\phi} : \mathbb{C} \setminus I \to D$. Let

$$e(z) = \frac{mz - m^2 + s^2}{z - m} \tag{31}$$

be the elliptic rotation, with angle π, of the complex plane keeping the points $m - s$ and $m + s$ fixed. Then e maps the complement of D onto D and we have

$$\tilde{\phi} = e \circ \phi. \tag{32}$$

In the subsequent considerations we use this mapping ϕ with varying parameters τ_j to perform the desired iteration.

4.2. **Opening up the slots.** What follows will later be applied to real hyperelliptic M–curves C of genus p defined by the equation

$$y^2 = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_{2p+2}), \tag{33}$$

where all the parameters λ_i are real and no two of them agree. For the considerations of this section, however, the curve C itself is irrelevant.

Here we simply consider, for a positive integer p, a set $\{\lambda_1, \lambda_2, \dots, \lambda_{2p+2}\}$ of $2p + 2$ distinct *real* numbers. We assume that the points λ_i are numbered in such a way that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{2p+2}.$$

Then the closed intervals $I_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \dots, p+1$, are disjoint. We call these intervals *slots* or, more precisely, *first generation slots.*

We start now describing our iterative method.

Let ϕ_1 be the mapping ϕ defined by the equation (23) with $\tau_1 = \lambda_1$ and $\tau_2 = \lambda_2$. The mapping ϕ_1 *opens up* the first slot I_1 replacing it by a disk D_1 and maps all other slots onto new slots $\phi_1(I_j)$.

The corresponding elliptic rotation e_1, defined by (31), maps the complement of $\overline{D}_1$ onto D_1. In particular, e_1 maps the deformed slots $\phi_1(I_2), \phi_1(I_3), \dots$ onto new slots $e_1(\phi_1(I_j)), j = 2, 3, \dots$ in D_1. These are the first 2^{nd}*–generation slots.* Let us denote them by I_j^2. The opening of a slot and the creation of 2^{nd} generation slots is illustrated in figure 5.

Next, let ϕ_2 be the mapping ϕ defined by the equation (23) with $\tau_1 = \phi_1(\lambda_3)$ and $\tau_2 = \phi_1(\lambda_4)$. The mapping ϕ_2

- opens up the slot $\phi_1(I_2)$ and replaces it by a disk D_2
- maps the boundary of the disk D_1 onto a Jordan curve going through the points $\phi_2(\phi_1(\lambda_1))$ and $\phi_2(\phi_1(\lambda_2))$
- maps the deformed slots $\phi_1(I_j)$ onto new slots $\phi_2(\phi_1(I_j))$.

Let e_2 be the corresponding elliptic rotation, defined by (31) with

$$m = \frac{\phi_1(\lambda_3) + \phi_1(\lambda_4)}{2} \text{ and } s = \frac{\phi_1(\lambda_4) - \phi_1(\lambda_3)}{4}. \tag{34}$$

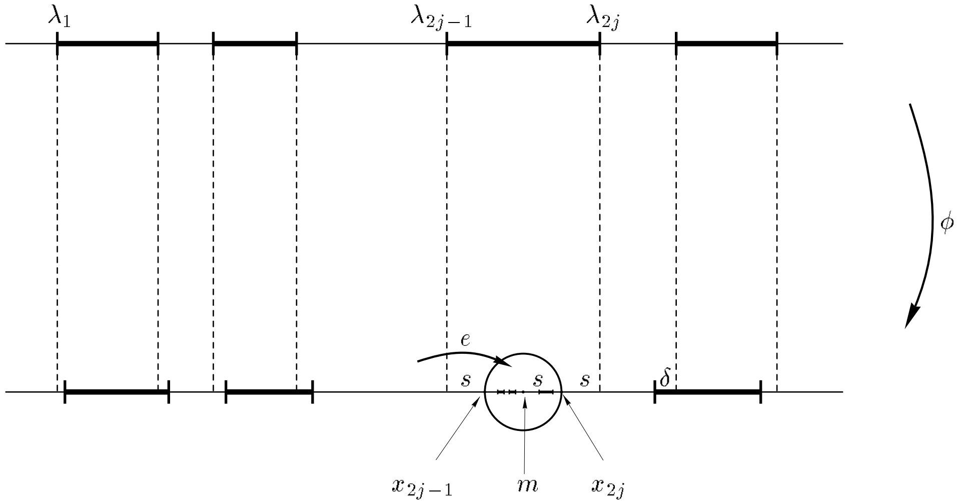

FIGURE 5. The mapping ϕ opens a slot of length $4s$ and replaces it by a disk D of radius s. The higher generation slots are obtained as images of the other slots under an elliptic Möbius transformation e mapping the complement of the closure of disk D onto D itself. Also the other slots move towards the mid-point m of the slot that is being opened. The function δ measures this movement. The points x_{2j-1} and x_{2j} are the intersection points of ∂D and the real axis.

The rotation e_2 maps the slots $\phi_2(\phi_1(I_3)), \phi_2(\phi_1(I_4)), \ldots$ onto new 2^{nd}–generation slots $I_j^2, j = p+1 \ldots 2p$ inside the disk D_2. The images $e_2(\phi_2(I_j^2))$ of the first 2^{nd}–generation slots $I_j^2, j = 1, \ldots p$ inside D_2 are referred to as 3^{rd}–generation slots I_j^3.

Repeat this procedure until all the first generation slots have been opened and new higher generation slots have been formed. For the moment our main interest lies in following what happens to the points x_{2j-1}^0 and x_{2j}^0 which are defined as the intersection points of the boundaries of the disks D_j, $j = 1, \ldots, p+1$, and the real axis. Choose the indices in such a way that $x_{2j-1}^0 < x_{2j}^0$ for all values of j.

For all indices $j = 1, \ldots, p+1$, define the sequences x_{2j-1}^k and x_{2j}^k by setting

$$x_{2j-1}^1 = \phi_{j+1}(x_{2j-1}^0) \text{ and } x_{2j}^1 = \phi_{j+1}(x_{2j}^0) \tag{35}$$

and

$$x_{2j-1}^2 = \phi_{j+2}(x_{2j-1}^1) \text{ and } x_{2j}^2 = \phi_{j+2}(x_{2j}^1) \tag{36}$$

and so on.

Repeating the above construction for all the openings of all generations of slots we get $2p+2$ infinite sequences x_i^k. The following two lemmata are taking directly from [10]. For the reader's convenience we reproduce also the proofs here.

Lemma 6. *For all values $i = 1, 2, \ldots, 2p+2$, the corresponding sequences x_i^k converge as $k \to \infty$.*

Assume that the slots are ordered according to their generation and, within each generation, from left to right. Let us fix i and consider the sequence x_i^k, which depends on the choice of the ordering of the slots. We have to show that the limits

$$\lim_{k\to\infty} x_i^k = x_i^\infty \tag{37}$$

exist.

Let ϕ_m denote the opening of the slot I_m.

For a given sequence x_i^k where $i = 2j$ or $2j-1$ observe that the convergence of the sequence itself is equivalent to the convergence of the series

$$x_i^0 + (\phi_{j+1}(x_i^0) - x_i^0) + (\phi_{j+2}(\phi_{j+1}(x_i^0)) - \phi_{j+1}(x_i^0)) + \ldots \tag{38}$$

This series simply keeps track on how much the initial point x_i^0 moves as we continue opening up the slots. It is enough to show that the series (38) converges. That immediately implies the convergence of the sequence x_i^k. Furthermore, the limit point x_i^∞ is simply the sum of the series (38).

To prove the convergence of the series (38) observe that it is some kind of alternating series in the sense that some of its terms are positive and some are negative. By taking only positive terms or only negative terms we get two series. It suffices to show that both of them converge.

Let $\Delta x_i^0 = x_i^0$ and use the notation Δx_i^{k-1} to denote the k:th term of series (38), $k = 2, 3, \ldots$. Let $\Delta x_i^{k_n}, n = 1, 2, \ldots$ be the negative terms only.

All the initial slots I_j are contained in the closed interval $[\lambda_1, \lambda_{2p+2}]$. By Lemma 5 all openings of slots I_j move the exterior points towards the mid–point of the slot I_j. Therefore all the various generation slots will be contained in the same closed interval $[\lambda_1, \lambda_{2p+2}]$.

Taking the negative terms of the series (38) only we add up the movements of the initial point to left. Such movements occur only when opening up slots that are on the left hand side of the point in question. Since, by the construction, the point x_i^0 and all its iterates are always in the interval $[\lambda_1, \lambda_{2p+2}]$, it follows that the negative terms of the series (38) must form a converging series.

Same argument shows that the positive terms of the series (38) form also a converging series. We deduce therefore that the original series (38) converges absolutely. □

Lemma 7. *For $i \neq i'$, $x_i^\infty \neq x_{i'}^\infty$.*

Two limit points x_i^∞ and $x_{i'}^\infty$ may agree if and only if they have been obtained from the end–points of the same first generation slot I_j. This follows from the properties of the function $\delta(x)$ (see Lemma 5).

Let x_{2j-1} and x_{2j} be the intersection points of the real axis and the disk D_j obtained by opening up the slot I_j. Let y_1 and y_2 be the end–points of the 2^{nd} generation slots in the interval $[x_{2j-1}, x_{2j}]$ closests to x_{2j-1} and to x_{2j}, respectively. The the intervals (x_{2j-1}, y_1) and (y_2, x_{2j}) do not contain any 2^{nd}–generation slots. Nor will their deformations contain any higher generation slots. Let

$$\epsilon = |x_{2j-1} - y_1| + |y_2 - x_{2j}|$$

be the combined Euclidean length of these intervals.

Using again Lemma 5 we infer that, when opening any other slots, the intervals (x_{2j-1}, y_1) and (y_2, x_{2j}) *will get longer*. The distance between x_{2j-1} and x_{2j} and any of their deformations will, therefore, always be $> \epsilon$. This implies that the distance between the limit points x_{2j-1}^∞ and x_{2j}^∞ is also $> \epsilon$. Therefore $x_i^\infty \neq x_{i'}^\infty$ whenever $i \neq i'$. □

Let g be a Möbius transformation such that

$$g(x_i^\infty) = i - 1 \text{ for } i = 1, 2, 3. \tag{39}$$

Replacing each the x_i^∞ by the corresponding $g(x_i^\infty)$ we get the *normalized set* of limit points x_i^∞.

The sequence that give us the limit points x_i^∞ depend on the order in which the slots are opened. We have, however, the following result:

Lemma 8. *The normalized limit points x_i^∞ are independent on the ordering of the slots.*

Proof will be based on the considerations of the next section and will therefore be postponed to Subsection 4.3.1.

4.3. **Hyperelliptic M–curves.** Let $p \geq 2$ be an integer and consider an algebraic plane curve C defined by the affine equation

$$y^2 = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_{2p+2}) \tag{40}$$

where all the parameters λ_i are distinct, real and numbered in such a way that $\lambda_1 < \lambda_2 < \dots < \lambda_{2p+2}$. Such an equation determines a real algebraic M–curve of genus p. We wish to use the considerations of the previous sections to find generators $g_1, g_2, \dots$ for a discontinuous group G of Möbius transformations such that

$$C = \Omega(G)/G, \tag{41}$$

where $\Omega(G)$ denotes the domain of discontinuity of G.

To that end it is useful to recall the usual construction for the curve C. It is the following:

- take first two copies of the Riemann sphere
- cut them both open along the slots $I_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \dots, p+1$.
- identify the boundary components of the first Riemann sphere, that was cut open, with the boundary components of the second one in such a way that you get a double cover of the original Riemann sphere ramified at the points λ_i.

First important observation is that the complement Ω_0 of the union of the slots I_j in the Riemann sphere $\hat{\mathbb{C}}$, $\hat{\mathbb{C}} \setminus \cup_{j=1}^{p+1} I_j$ can be identified, conformally, with a region C_u of the original algebraic curve C. Let $\epsilon : C \to C$ denote the hyperelliptic involution. Then $C_u \cup \epsilon(C_u)$ is the complement of $p+1$ simple closed geodesic curves[3] of the algebraic curve C.

Consider the first opening of the slot I_1. It is a conformal mapping

$$\phi_1 : \hat{\mathbb{C}} \setminus I_1 \to \hat{\mathbb{C}} \setminus \overline{D}_1. \tag{42}$$

Here we are using the notation of section 4.2. Via this mapping,

$$U_1 = \hat{\mathbb{C}} \setminus \left(\overline{D}_1 \cup \left(\cup_{j=2}^{p+1} I_j\right)\right) \tag{43}$$

is conformally equivalent to $\mathbb{C}_u$. The elliptic rotation e_1 maps the outside of D_1 onto its inside. It follows that $e_1(U_1) \cup U_1$ is conformally equivalent to the complement of the above mentioned simple closed geodesic curves on the algebraic curve C.

[3]When speaking of geodesic curves, we refer to the hyperbolic metric of the corresponding Riemann surface and to curves that are geodesics in that metric.

In other words, $e_1(U_1) \cup (U_1)$ is almost all of the Riemann surface in question. Furthermore, the mapping e_1 corresponds to the hyperelliptic involution.

Let now Ω_2 denote the complement of the union of the slots $I_2, I_3, \dots, I_{p+1}$ and the 2^{nd}–generation slots $e_1(I_j)$, $j > 1$. The above remarks imply that we have an analytic local homeomorphism $\pi_1 : \Omega_2 \to C$.

We continue this procedure and keep up opening slots in a certain given order, for instance, by generation and in each generation from left to right. Let Ω_n denote the complement of all the various generation slots after having opened n slots (which have disappeared since they have been replaced by disks).

Since each opening of slots is a conformal mapping, we have, at each level an analytic local homeomorphism $\pi_n : \Omega_n \to C$. Once we have opened all the 1^{st}–generation slots, i.e., for $n > p+1$, the mapping π_n can already be extended to a surjective mapping. (The extension refers to the extension of the mapping π_n to the boundaries of the disks that have replaced the opened slots. This is straightforward.)

Define the set Ω^∞ by setting

$$\hat{\mathbb{C}} \setminus \Omega^\infty = \{z \in \hat{\mathbb{C}} | \exists z_n \in \ \hat{\mathbb{C}} \setminus \Omega^n \text{ such that } \lim_{n\to\infty} z_n = z\}. \tag{44}$$

Theorem 9. *The mappings π_n form a converging sequence and the limit mapping π_∞ is an analytic regular covering mapping*

$$\pi_\infty : \Omega^\infty \to C. \tag{45}$$

Proof of the theorem relies on Lemma 6 and will be contained in a paper, by Semmler and Seppälä, which is presently under preparation. The main point to be shown here is, that for any $z \notin \mathbb{R}$, the sequence $\phi_k(z)$ converges. This is simply Lemma 6 extended to the whole complex plane. The details are technically involved and will appear elsewhere later. P. J. Myrberg has another proof for this ([9]).

Consider the limits x_j^∞ of Lemma 6. Let e_j^∞ denote that elliptic Möbius transformation which keeps the point x_{2j-1}^∞ and x_{2j}^∞ fixed and rotates the infinite complex plane by angle π.

Let G be the group generated by all products $e_j^\infty e_{j'}^\infty$, $j, j' = 1, 2, \dots, p+1$.

The following result follows immediately from the definitions.

Lemma 10. *The group G is the cover group of the analytic cover (45). Hence G is a kleinian group whose domain of discontinuity is Ω^∞.* □

4.3.1. *Proof of Lemma 8.* Let x_i^∞ and y_i^∞, $i = 1, 2, 3, \dots 2g+2$ be two sets of normalized limit points corresponding to the same initial data $\lambda_1, \dots, \lambda_{2g+2}$. Let $\pi_\infty : \Omega^\infty \to C$ and $\pi'_\infty : \Omega'^\infty \to C$ be the corresponding coverings of the original curve C (given by equation (40)). Now the identity mapping of the curve C induces a Möbius transformation $h : \Omega^\infty \to \Omega'^\infty$ with the property

$$h(x_i^\infty) = y_i^\infty \text{ for all } i = 1, 2, \dots, 2g+2. \tag{46}$$

since the sets $\{x_i^\infty\}$ and $\{y_i^\infty\}$ are normalized, $x_i^\infty = y_i^\infty$ for $i = 1, 2, 3$. Therefore the Möbius transformation h has three fixed points implying that h *is* the identity. Hence $x_i^\infty = y_i^\infty$ for all values of $i = 1, 2, \dots, 2g+2$. This argument proves Lemma 8.

5. Period matrices

The Torelli theorem states that a compact Riemann surface is determined by its Jacobian variety. Jacobian varieties are g–dimensional complex torii having certain special properties. This theorem provides a fasinating new point of view to Riemann surfaces and algebraic curves. It is, therefore, natural to try to produce the result of this theorem in a computable way, i.e., given a compact Riemann surface $X = \Omega(G)/G$, we would like to be able to compute the corresponding Jacobian variety. Such a variety is determined by a period matrix of the Riemann surface X. Hence the computation of the Jacobian reduces to the computation of periods of differentials on X.

5.1. **Poincaré series.** In this section we review considerations of [11].

Let $D = \{z \in \mathbf{C} \mid |z| < 1\}$ be the unit disk and G a Fuchsian group acting in D. Let $Y = D/G$.

Assume that *G is of the second kind.* This means that the domain of discontinuity of G, $\Omega(G)$, is connected. Then $X = \Omega(G)/G$ is a symmetric Riemann surface, the symmetry $\sigma : X \to X$ being the mapping induced by the reflection in the unit circle ∂D. All so called orthosymmetric[4] Riemann surfaces can be expressed in this way.

Use the notation $D^1(X)$ for the complex vector space of holomorphic Abelian differentials on X. Holomorphic differentials ω on X lift to automorphic holomorphic functions $\omega : D \to \mathbb{C}$ satisfying:

$$\omega(g(z)) \left(\frac{\partial g(z)}{\partial z} \right) = \omega(z) \tag{47}$$

for all $z \in D$ and for all $g \in G$. This is, of course, quite standard.

The limit set, $L(G)$, of a Fuchsian group G of the *second kind* is a subset of ∂D of measure 0. The Riemann surface $Y = D/G$ is an open Riemann Y surface which is the interior of a Riemann surface $Y \cup \partial Y$ *with boundary components.*

Let $f : D \to \hat{\mathbb{C}}$ be a meromorphic function. Let $m \geq 1$ be an integer. Consider *the Poincaré series*

$$\Theta^m(f)(z) = \sum_{g \in G} f(g(z)) \left(\frac{\partial g(z)}{\partial z} \right)^m . \tag{48}$$

If this series converges, then its limit is a meromorphic function ω satisfying

$$\omega(g(z)) \left(\frac{\partial g(z)}{\partial z} \right)^m = \omega(z). \tag{49}$$

Use the notation $\Theta(f)(z) = \Theta^1(f)(z)$.

Standard arguments imply the convergence of $\Theta^1(f)(z)$ in the case of Fuchsian groups of the second kind provided that the function f is not too bad ([5, Proposition 3, page 205]):

Lemma 11. *Let f be a rational function with no poles in $L(G)$. Then $\Theta(f)(z)$ is a meromorphic differential of the group G.*

This is an interesting result which allows us to approximate Abelian differentials of groups of the second kind numerically. The above result was contained already in a paper by W. Burnside ([2]) that was published in 1891.

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[4] An orthosymmetric Riemann surface is a Riemann surface X equipped with an antiholomorphic involution $\sigma : X \to X$ such that the fixed–point set of σ decomposes X into two parts.

Assume now that the group G is freely generated by the hyperbolic Möbius transformations $g_1, g_2, \ldots, g_p$ mapping the unit disk onto itself. For $j = 1, 2, \ldots, p$ let

$$h_j(z) = \frac{\overline{\zeta}_j}{1 - \overline{\zeta}_j z}, \quad \zeta_j = g_j(0), \quad 1 \le j \le p. \tag{50}$$

Lemma 12. $\{\Theta(h_j) \mid 1 \le j \le p\}$ *is a basis for holomorphic differentials on* $X = \Omega(G)/G$.

Proof. ([5, Corollary 1, page 208]).

5.2. **Estimating the periods.** Using Lemma 12 we can rather easily estimate periods of differentials. To that end we have to integrate the base differentials along simple closed curves on $\Omega(G)/G$. These curves can, furthermore, be taken to be geodesic curves. Consider such a curve α. Choose a lifting of α to the cover $\Omega(G)$. Denote this lifting also by α. It is now an arc in $\Omega(G)$, which is disjoint from the closure of the limit set $L(G)$ of G.

Lemma 13. *The integral of a base differential* $\Theta(h_j)$ *along* α *can be computed by the convergent series:*

$$\int_\alpha \Theta(h_j)dz = \sum_{g \in G} \log\left(\frac{1 - \overline{\zeta}_j g(\alpha(0))}{1 - \overline{\zeta}_j g(\alpha(1))}\right). \tag{51}$$

Proof. [11, Lemma 4].

In estimating the sum (51) we have the following problems:

1. Since we can use finite precision only, we can deal with a small part of G at one time only. This part that is available for our computations depends on the choice of the basis of G. Long words of elements of G get, as their numerical approximations, constant mappings. They do not contribute anything to the sum (51).
2. Rounding errors.

We can, however, improve our algorithm using the following construction. The arc α, along which we integrate, can often be taken to be an arc on the axis A_{g_α} of some Möbius transformation $g_\alpha \in G$. We may, furthermore, suppose that the orientation of A_{g_α}, as determined by the attracting and the repelling fixed–points a_{g_α} and r_{g_α}, agrees with that of α. For if that is not the case, then replace g_α by its inverse. We may suppose also, without restricting the generality, that:

- g_α is primitive in G, i.e., that the cyclic subgroup $G_\alpha = \langle g_\alpha \rangle$ of G generated by g_α contains all elements of G which have A_{g_α} as their axis.
- $g_\alpha(\alpha(0)) = \alpha(1)$. This follows from the fact that, in our applications, the projection of α on the Riemann surface $\Omega(G)/G$ is a simple closed curve. We may, therefore, suppose that g_α maps the starting–point of α onto its end–point.

Having made all the above assumptions, we observe now that

1. $m(g(\alpha) \cap g'(\alpha)) = 0$ for all $g, g' \in G_\alpha, g \ne g'$. Here m denotes the 1–dimensional Lebesgue measure.
2. $A_{g_\alpha} = \cup_{g \in G_\alpha} g(\alpha)$.

By the above properties we have now:

$$\sum_{g\in G_\alpha} -\int_{g(\alpha)} \frac{-\overline{\zeta}_j}{1-\overline{\zeta}_j\omega}d\omega = -\int_{A_{g_\alpha}} \frac{-\overline{\zeta}_j}{1-\overline{\zeta}_j\omega}d\omega = \log\frac{1-\overline{\zeta}_j r_{g_\alpha}}{1-\overline{\zeta}_j a_{g_\alpha}}. \tag{52}$$

Therefore, by equation (52), we may compute an infinite part of the series (51) by a single expression only. This improves the accuracy of our computations and reduces rounding errors considerably.

Let now $g \in G \setminus G_\alpha$. The arc $\beta = g(\alpha)$ lies on the axis of the Möbius–transformation $g_\beta = g \circ g_\alpha \circ g^{-1}$.

The subgroup $G_\beta = \langle g_\beta\rangle$ of G contains all transformations of G whose axis agrees with that of g_β. Also in this case we have

1. $A_{g_\beta} = \cup_{h\in G_\beta} h(\beta)$.
2. $m(h(\beta)\cap h'(\beta)) = 0$ for all $h,\, h' \in G_\beta,\, h \neq h'$.

Hence

$$\sum_{h\in G_\beta} -\int_{h(g(\alpha))} \frac{-\overline{\zeta}_j}{1-\overline{\zeta}_j\omega}d\omega = -\int_{A_{g_\beta}} \frac{-\zeta_j}{1-\overline{\zeta}_j\omega}d\omega = \log\frac{1-\overline{\zeta}_j r_{g_\beta}}{1-\overline{\zeta}_j a_{g_\beta}}. \tag{53}$$

By equations (52) and (53) we conclude that periods of differentials can be computed by the formula

$$\int_\alpha \Theta(h_j)dz = \sum_{h\in G/G_\alpha} \log\frac{1-\overline{\zeta}_j h(r_{g_\alpha})}{1-\overline{\zeta}_j h(a_{g_\alpha})}, \tag{54}$$

which is, in numerical computations, much more accurate and faster than formula (51).

5.3. **Period matrices of general compact Riemann surfaces.** A general compact genus p, $p > 1$, Riemann surface $X = D/G$ is defined for computations by giving generators for the corresponding Fuchsian group G of the first kind. These Fuchsian groups can be produced by the input device Cars described in Sections 1 and 2.

In order to be able to compute a period matrix for X we have to do the following:

1. Find a basis for the homology of X.
2. Find a basis for Abelian differentials of X.
3. Integrate the base differentials along curves in the homology basis.

From a given presentation $X = D/G$ of X in terms of its Fuchsian group (of the first kind) one can rather easily find Möbius transformations $g_1, h_1, \ldots, g_p, h_p$ corresponding[5] to a homology basis of X.

[5] This simply means that the axes of the transformations g_j and h_j project onto simple closed geodesic curves homotopic to curves defining a basis for the homology and that each g_j and each h_j is primitive in G.

The main problem is to find differentials on X. In a recent manuscript ([7]) this problem has been solved by representing X first as a plane algebraic curve and then using methods of algebraic geometry to find differentials on that curve. This allows one to approximate numerically period matrices of any compact Riemann surface X. Since the manuscript [7] is still being revised, we will not go into any more details here.

6. Riemann surfaces and the Laplacian

Many of the analytic and geometric properties of the Riemann surfaces are reflected in the properties of the Laplace operator. This operator arises, for instance, in the computation of the period matrices by means of conformal capacities. It also arises in connection with the closed geodesics.

Since the early sixties, the relationship between the Laplacian and the geometry of the underlying surface has been studied extensively by many authors. The best known connection, and nowadays a field of research of its own right, is the Selberg trace formula which relates the sequence of the eigenvalues of the Laplacian to the sequence of the closed geodesics.

It is a peculiar fact that even today no compact Riemann surface of genus $g \geq 2$ is known where one would be able to compute the eigenvalues in a closed form. This is quite in contrast to the case in genus 1 where the eigenvalue equation is separable and all eigenfunctions are given explicitly by means of trigonometric functions.

An obvious approach to the computation of the eigenvalues is to compute the lengths of the closed geodesics and to use the Selberg trace formula. Some experiments in this direction, however, yielded completely unsatisfactory results due to the exponential growth of the length spectrum and the slow convergence of the series involved in the trace formula.

We started, therefore, another approach using methods from numerical analysis. For this it was necessary to find a new triangulation algorithm which works on closed surfaces with non–trivial topology. A well known difficulty here is that, as the automatic triangulation proceeds around a non–trivial closed loop, it gets into collision with parts of the surface triangulated at earlier steps. This difficulty is sometimes described as the problem of colliding front waves. To overcome it we used techniques from the theory of sphere packings which also proved to be useful for proving theoretical results. For the details we refer to [3].

A special feature of our triangulation is that it may be adapted to the decompositions of a Riemann surface into building blocks, i.e., into Y–pieces and into Q–pieces. This makes it possible to *input* a Riemann surface using the Fenchel–Nielsen coordinates.

The numerical computation of the eigenfunctions and eigenvalues is based on the finite element method and may be carried out on a workstation yielding quite accurate results. Details with a number of explicit examples will appear in a forthcoming paper (by Buser, Renggli and Semmler).

F. Steiner (Physics Department, University of Hamburg) has used another numerical approach in genus 2. His approach is based on the use of the so–called boundary method on fundamental polygons ([1]).

This method is particularly accurate if no small closed geodesics occur. A joint paper with F. Steiner, which is presently under preparation, will bring these two methods together. This paper will provide an interface allowing one to go from the

Fenchel–Nielsen description to the description based on symmetric octagons and inversely.

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