Coverings of 3-manifolds by three open solid tori

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1 Introduction

We consider the problem of obtaining a list of compact 3-manifolds that can be obtained as a union of given sets. The concept of an $\mathcal{A}$-category of a manifold M^n as introduced in [CP] is a generalization of the Lusternik-Schnirelmann category $cat(M^n)$. For a fixed closed connected k-manifold A, $0 \leq k \leq n-1$, a subset B in M^n is said to be *A-contractible* if there are maps $\varphi : B \longrightarrow A$ and $\alpha : A \longrightarrow M^n$ such that the inclusion map $i : B \longrightarrow M$ is homotopic to $\alpha \cdot \varphi$. The $\mathcal{A}$-category $cat_A(M^n)$ of M^n is the smallest number of sets, open and A-contractible needed to cover M^n. Thus when A is a point P, $cat_P(M^n) = cat(M^n)$. In the case $A = S^1$ it was shown in [GGH2] that the fundamental group of a closed 3-manifold M with $cat_{S^1}(M) = 2$ is cyclic and it then follows from Perelman's work [MT] that in this case M is a lens space; hence M can be covered by two open solid tori. As a first step to obtaining a list of all 3-manifolds with $cat_{S^1}(M) = 3$ we ask about minimal covers of M by three open sets, each homotopy equivalent to S^1. In particular we consider covers of M by three open solid tori.

If a closed 3-manifold M can be covered by three open balls, then M is a connected sum of S^3 and finitely many S^2-bundles over S^1. This was first shown by Hempel and McMillan [HM] and a proof of a slight generalization (allowing punctured balls) was given in [GGH1]. These proofs did not use the Poincarè Conjecture. A much shorter proof can be given by using it, since then it suffices to compute the fundamental group of M. In [GGH3] we used this approach, applying Perelman's results about the 3-dimensional spherical space-form conjecture, to obtain a classification of all closed 3-manifolds that can be covered by two open balls and one open solid torus or by one open ball and two open solid tori. This method however does not seem to be amenable to a cover by three

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open solid tori, since it would involve a computation of fundamental groups of Seifert fiber spaces.

In this paper we obtain a classification of all closed 3-manifolds that can be covered by three open solid tori. The main result is

Theorem: If M is a union of three open solid tori then M is homeomorphic to $B\#\tilde{L}(3)$ or to $B\#\tilde{S}(3)$, where B is a connected sum of S^3 and S^2-bundles over S^1 (with any number of $n \geq 0$ factors), $\tilde{L}(3)$ is a connected sum of at most 3 lens spaces, and $\tilde{S}(3)$ is a closed Seifert fiber space with at most 3 exceptional fibers over any (closed) orbit surface.

This Theorem follows from Corollary 9 (d) which is proved in section 5. Our proofs do not use Perelman's results.

In our proofs we replace unions of balls and solid tori by unions that are obtained by attaching two-handles to the balls and solid tori. This leads us to consider unions of punctured balls, punctured solid tori and punctured lens spaces and *general position decompositions* (defined in section 2). Such a decomposition of a 3-manifold M consists of a cover of M by pl 3-dimensional submanifolds such that the boundaries of any two intersect transversely and the boundaries of any three have no common intersection points. In Theorem 7 (section 4), we obtain a classification of all compact 3-manifolds (with non-empty boundary) that admit a general position decomposition into punctured balls, punctured lens spaces and one or two punctured solid tori. This is used in section 5 to obtain in Theorem 8 a classification of all closed 3-manifolds that admit a general position decomposition into punctured balls, punctured lens spaces and at most three punctured solid tori. Then Corollary 9 gives a classification of all closed 3-manifolds that are a union of three open submanifolds, each an open ball or an open solid torus.

2 Preliminaries

The following lemma, proved in [GGH1], Corollary 1(a) , allows us to work in the pl-category.

Lemma 1 *Suppose M is a closed 3-manifold covered by three open sets U_1, U_2, U_3, such that U_i is homeomorphic to the interior of a compact connected 3-manifold M_i (i=1,2,3). Then M admits a covering $M = M_1 \cup M_2 \cup M_3$ such that ∂M_1 is transverse to ∂M_2 , $\partial M_3 \subset int(M_1 \cup M_2)$, and M_1, M_2, M_3 are pl embedded.*

By an open punctured ball (resp. solid torus) we mean a manifold homeomorphic to an open ball (resp. solid torus) with a finite number of points removed.

By an (n-times) *punctured* M we mean a manifold obtained from M by removing interiors of (n) disjoint balls in $int(M)$. We allow $n = 0$. Note that a connected (*punctured* M)$= M\#H$, for some punctured ball H.

We will use the following notations throughout this paper:

For a 3-manifold M, the manifold obtained by filling in all 2-sphere boundary components by 3-balls is denoted by $\hat{M}$.
H or H_i denotes a punctured ball with finitely many punctures (with possibly no punctures).
W or W_i denotes a handlebody (orientable or non-orientable).
L or L_i denotes a (punctured) lens space (possibly $S^1 \times S^2$ or S^3).
V or V_i denotes a (punctured) solid torus.
$S(n)$ denotes a Seifert fiber space with at most n exceptional fibers and any compact orbit surface with non-empty boundary.
$\tilde{S}(n)$ denotes a closed Seifert fiber space with at most n exceptional fibers and any (closed) orbit surface.
$W(S(n))$, a *Seifert web on* $S(n)$, is obtained from $S(n)$ by attaching any finite number of 1-handles.

$\mathbb{B}$ denotes the collection of manifolds that are connected sums of S^3 and S^2-bundles over S^1 (with finitely many factors).
$\mathbb{W}$ denotes the collection of 3-manifolds $W_1\#\cdots\#W_n$, $n \geq 0$ that are connected sums of handlebodies W_i (with finitely many factors).
For collections of manifolds $\mathbb{X}$,$\mathbb{Y}$ we denote by $\mathbb{X}\#\mathbb{Y}$ the collection of manifolds of the form $X\#Y$, where $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$.

For convenience we write $M \in \mathbb{B}\#\hat{L}\#H$ instead of saying that M is a connected sum of $\hat{L}$, H, and a finite number of S^2-bundles over S^1.

It is easy to prove (e.g. Lemma 2 of [GH]) that the collection $\mathbb{B}\#\mathbb{W}\#H$ is closed under attaching 1-handles, moreover:

Lemma 2 *The family* $\mathbb{B}\#\mathbb{W}\#L_1\cdots\#L_n\#W(S(n))\#H$ *is closed under the operation of attaching* 1*-handles.*

Remark 3 *(a) Attaching a* 2*-handle to* L *results in* L *(with one more puncture).*
(b) Attaching a 2*-handle to* V *results in* V *(with one more puncture) or in* L*.*
(c) Cutting L *along a disk results in* $L \cup H$ *or, if* L *is a punctured* $S^2 \times S^1$*, possibly in* H*.*
(d) Cutting V *along a disk results in* $V \cup H$ *or* H*.*

The following lemma is well known for irreducible 3-manifolds. Noting that for an irreducible 3-manifold $\hat{M}$ every 2-sphere in M bounds a punctured ball in M, the proof in [GGH] yields:

Lemma 4 *Suppose M is a compact 3-manifold such that $\hat{M}$ is irreducible. If M contains a 2-sided compressible torus T then either T bounds a punctured solid torus or a punctured knot space R in M such that a meridian curve of ∂R bounds a disk D in $\overline{M-R}$. In particular if T is a compressible boundary component of M then M is a punctured solid torus.*

Let $M_1, \cdots, M_n$ be compact submanifolds of a 3-manifold N. We say that

$$(*) \quad N = M_1 \cup \cdots \cup M_n$$

is a *general position decomposition* of N if ∂M_i is transverse to ∂M_j for each $i \neq j, (i, j = 1, \cdots, n)$, and $\partial M_i \cap \partial M_j \cap \partial M_k = \emptyset$ for all distinct i, j, k.

The *complexity* α of the general position decomposition $(*)$ is the number of components of $\{\partial M_i \cap \partial M_j \mid i \neq j,\ i, j = 1, \cdots, n\}$.

Lemma 5 *Suppose $N = M_1 \cup \cdots \cup M_n$ is a general position decomposition of N and $N \subset int(M)$ for some compact 3-manifold M. If a component c of $\partial M_i \cap \partial M_j$ is null homotopic on ∂M_i then there is a disk D on M_i such that $\partial D = D \cap \partial M_j$ for some $i \neq j$ and $D \cap M_k = \emptyset$ or $D \subset int(M_k)$ for all $k \neq i, j$ and either*

(i) $N \approx M_1' \cup \cdots \cup M_n'$ is a general position decomposition of N with complexity $\alpha' < \alpha$, where $M_k' = M_k$ for $k \neq j$ and M_j' is obtained from M_j by attaching a 2-handle to ∂M_j with cocore D, or

(ii) $N = N'$ or N is obtained from N' by attaching a 1-handle to $\partial N'$, where $N' \approx M_1' \cup \cdots \cup M_n'$ is a general position decomposition with complexity $\alpha' < \alpha$, and where $M_k' = M_k$ for $k \neq j$ and $M_j' = M_j \backslash D$.

(Here $M_j \backslash D$ is obtained from M by cutting along the properly embedded disk D).

Proof.

The component c bounds a disk on ∂M_i. Let D be an innermost such disk on ∂M_i, i.e. $\partial D = D \cap \partial M_j$ for some $j \neq i$ and $D \cap M_k = \emptyset$ or $D \subset int(M_k)$ for all $k \neq i, j$. Let $U(D)$ be a regular neighborhood (rel ∂M_j) of D in M.

If D is not contained in M_j let $M_j' = M_j \cup U(D)$ (If $D \subset int(M_k)$ for $k \neq j$ choose $U(D) \subset int(M_k)$). Then M_j' and N are as in (i) (see Fig 1).

If $D \subset M_j$ let $M_j' = \overline{M_j - U(D)}$. Then M_j' and N' are as in (ii) (see Fig 2). If $D \cap M_k = \emptyset$ for all $k \neq i, j$ then N is obtained from N' by attaching a 1-handle with cocore D. If $U(D) \subset intM_k$ for some $k \neq i, j$ then $N = N'$.

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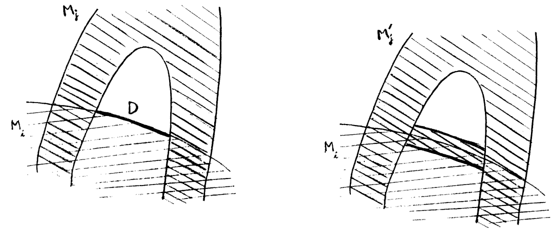

Figure 1:

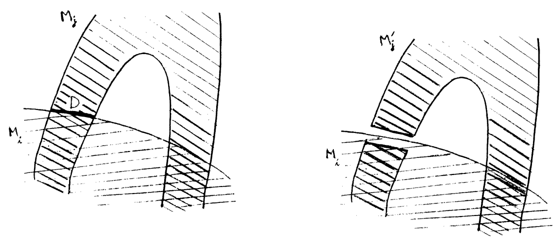

Figure 2:

3 Complexity 0.

First we consider general position decompositions of punctured balls, punctured lens spaces and at most 2 punctured solid tori with complexity 0.

Proposition 6 *Let $N \subset intM$ for some 3-manifold M.*

(a) If $N = \bigcup_i H_i \cup L_1 \cdots \cup L_n$ is a general position decomposition with complexity 0 then $N \in \mathbb{B}\#\hat{L}_1\#\cdots\#\hat{L}_n\#H$.

(b) If $N = \bigcup_i H_i \cup L_1 \cdots \cup L_n \cup V_1$ is a general position decomposition with complexity 0 then $N \in \mathbb{B}\#\hat{L}_1\#\cdots\#\hat{L}_n\#H$ or $N \in \mathbb{B}\#\hat{L}_1\#\cdots\#\hat{L}_n\#\hat{V}_1\#H$.

(c) If $N = \bigcup_i H_i \cup L_1 \cdots \cup L_n \cup V_1 \cup V_2$ is a general position decomposition with complexity 0 then
$N \in \mathbb{B}\#\hat{L}_1\#\cdots\#\hat{L}_n\#\hat{V}_i\#H$ ($i = 1$ or 2) or
$N \in \mathbb{B}\#\hat{L}_1\#\cdots\#\hat{L}_n\#\hat{V}_1\#\hat{V}_2\#H$ or
$N \in \mathbb{B}\#\hat{L}_1\#\cdots\#\hat{L}_n\#\hat{L}\#H$.

(Recall that we allow $\hat{L} = S^3$).

Proof.

Let T_i denote the torus boundary of V_i and let $\mathbf{S}$ denote the collection of 2-spheres $\bigcup_i \partial H_i \cup \partial L_1 \cdots \cup \partial L_n$, resp. $\bigcup_i \partial H_i \cup \partial L_1 \cdots \cup \partial L_n \cup (\partial V_1 - T_1)$, resp. $\bigcup_i \partial H_i \cup \partial L_1 \cdots \cup \partial L_n \cup (\partial V_1 - T_1) \cup (\partial V_2 - T_2)$. For each i, $\mathbf{S} \cap intH_i$

cuts H_i into punctured balls. Similarly, since $\hat{V}_i$ is irreducible, $\mathbf{S} \cap V_i$ cuts V_i into a punctured solid torus V_i' and punctured balls. Also $\mathbf{S} \cap L_j$ cuts L_j into a punctured lens space L_j' and punctured balls. (If L_j is a punctured $S^2 \times S^1$ then L_j' may be a punctured $S^2 \times S^1$ or a punctured ball; in any other case $\hat{L}_j' = \hat{L}_j$). Denote the collection of all the resulting punctured balls by $\{H_k'\}$.

(a) In this case $N = \bigcup_k H_k' \cup L_1' \cdots \cup L_n'$ is obtained from a disjoint collection of punctured balls H_k' and punctured lens spaces L_j' by identifying some boundary components in pairs and the result follows.

(b) In this case consider $N = \bigcup_k H_k' \cup L_1' \cdots \cup L_n' \cup V_1'$. If there is a point x in $int(H_k') \cap V_1'$ then (viewing x in V_1') each point y in V_1' can be joined to x by a path w in V_1' such that $int(w) \subset int(V_1')$, i.e. w does not cross $\partial H_k'$. Hence (viewing x in H_k'), w lies in H_k' and therefore $V_1' \subset H_k'$. Similarly, if $V_1' \cap int$ $L_j' \neq \emptyset$ we obtain $V_1' \subset L_j'$. In these cases we delete V_1' from the decomposition of N and obtain N as in case (a).

In any other case N is obtained from the disjoint collection of the punctured balls H_k', the punctured lens spaces L_j', and the punctured solid torus V_1' by identifying some boundary spheres in pairs. The result follows.

(c) We have $N = \bigcup_k H_k' \cup L_1' \cdots \cup L_n' \cup V_1' \cup V_2'$.

case (c1): T_1 is not contained in $intV_2'$.

By the argument above, considering paths in V_2', we see that if $V_2' \cap H_i' \neq \emptyset$, resp. $V_2' \cap L_i' \neq \emptyset$, resp. $V_2' \cap V_1' \neq \emptyset$, then $V_2' \subset H_i'$, resp. $V_2' \subset L_i'$, resp. $V_2' \subset V_1'$. We delete V_2' from the decomposition of N and get N as in (b) (the third term in the list 6(c)).

If V_2' is disjoint from H_i', L_i', and V_1' we obtain N from V_2' and $N' = \bigcup_k H_k' \cup L_1' \cdots \cup L_n' \cup V_1'$ by identifying 2-spheres in the boundary. Since N' is as in (b), N is as in the second and third term of 6(c).

The same argument applies if T_2 is not contained in $intV_1'$.

case (c2): $T_1 \subset intV_2'$ and $T_2 \subset intV_1'$

Then by [(3.1)(b) of [GGH]] we have $\hat{V}_1' \cup \hat{V}_2' = \hat{L}$ and $N = \bigcup_k H_k' \cup L_1' \cdots \cup L_n' \cup L$ is as in case (a) with one more lens space summand, which yields the third term of 6(c).

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4 Balls, lens spaces, and two solid tori.

We now consider the case when the complexity is strictly positive.

Theorem 7 *Let $N \subset intM$ for some 3-manifold M.*

(a) If $N = \bigcup_i H_i \cup L_1 \cdots \cup L_n$ is a general position decomposition then $N \in \mathbb{B}\#\mathbb{W}\#\hat{L}_1\# \cdots \#\hat{L}_n\#H$.

(b) If $N = \bigcup_i H_i \cup L_1 \cdots \cup L_n \cup V_1$ is a general position decomposition then $N \in \mathbb{B}\#\mathbb{W}\#\hat{L}_1\# \cdots \#\hat{L}_n\#\hat{L}_{n+1}\#H$.

(c) If $N = \bigcup_i H_i \cup L_1 \cdots \cup L_n \cup V_1 \cup V_2$ is a general position decomposition then
(c_1) $N \in \mathbb{B}\#\mathbb{W}\#\hat{L}_1\# \cdots \#\hat{L}_n\#\hat{L}_{n+1}\#\hat{L}_{n+2}\#H$ or
(c_2) $N \in \mathbb{B}\#\mathbb{W}\#\hat{L}_1\# \cdots \#\hat{L}_n\#W(S(2))\#H$.

(Recall that we allow $\hat{L}_k = S^3$).

Proof.
The proof is by induction on the complexity α of the decomposition for N. If $\alpha = 0$ we obtain N as in Proposition 6 with $\mathbb{W} = \emptyset$ or $\mathbb{W} = \hat{V}_i$ or $\mathbb{W} = \hat{V}_1\#\hat{V}_2$. Suppose $\alpha \geq 1$.

cases (a) and (b):

Let $\mathbf{C}$ denote the collection of the circles $\{\partial H_i \cap \partial H_j\ (i \neq j), \partial H_i \cap \partial L_k, \partial L_i \cap \partial L_j\}$, resp. $\{\partial H_i \cap \partial H_j\ (i \neq j), \partial H_i \cap \partial L_k, \partial L_i \cap \partial L_j, \partial H_i \cap \partial V_1, \partial L_i \cap \partial V_1\}$ and consider a circle c of $\mathbf{C}$ on ∂H_i (resp. ∂L_i). Since c is null homotopic on ∂H_i (resp. ∂L_i) we apply Lemma 5.

In case (i) of Lemma 5 we obtain a new general position decomposition of N with smaller complexity that differs from the given one only in that some H_j or L_j or V_1 is replaced by adding a 2-handle. By Remark 3 the new decomposition is of the same type, where V_1 either survives or is replaced by L_{n+1}, and the Theorem follows by induction. (If V_1 survives to the end, it becomes a part of $\mathbb{W}$).

In case (ii) we obtain N from N' by attaching a 1-handle, where N' has a general position decomposition of complexity $< \alpha$ and is as in (a) or (b) except that some H_j or L_j or V_1 is cut along a disk. By Remark 3, N' is as in (a) or (b) and the result follows by induction and Lemma 2.

case (c):

Let T_i be the torus boundary of V_i.
If there is a component of $T_1 \cap T_2$ that is null homotopic on T_1 or T_2, or if $T_1 \cap T_2 = \emptyset$, then we apply the same argument as above (using Lemma 5) to obtain N from N' by attaching 1-handles and where N' has a decomposition of the same type as N and with smaller complexity.
Thus assume that $T_1 \cap T_2 \neq \emptyset$ and every component of $T_1 \cap T_2$ is not null homotopic on T_1 and T_2. Then the components of $\partial V_1 \cap V_2$, resp. $\partial V_2 \cap V_1$, consist of annuli A_i, resp. A'_j.

If a boundary component of an A'_j, say, is null homotopic in V_1 then all the A'_j are compressible in V_1 and we find a meridian disk D of V_1 that is disjoint to all A'_j and such that ∂D is disjoint to H_i (resp. L_j) or $D \subset int(H_i)$ (resp. $D \subset int(L_j)$). A regular neighborhood of D in V_1 decomposes V_1 into two punctured balls H_0, H'_0 and we replace $V_1 \cup V_2$ in the decomposition of N by $H_0 \cup H'_0 \cup V_2$ to obtain N as in case (b). This results in $7(c_1)$.

Thus we may assume that the boundary components of A'_j, resp. A_i, are not null homotopic in V_1, resp. V_2 and it follows that the A'_j are incompressible boundary parralel annuli in $\hat{V}_1$ (see for example [Ha]) and there is a Seifert fibration of $\hat{V}_1$ such that the A_i's and A'_j's consist of fibers. The fibration of $\hat{V}_1 \cap \hat{V}_2$ extends to a Seifert fibration of $\hat{V}_2$ and hence $\hat{V}_1 \cup \hat{V}_2 = S(2)$. The result follows.

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5 Closed 3-manifolds and three solid tori.

The main result (Corollary 9) follows from the following

Theorem 8 *Let M be a closed 3-manifold.*
(a) If M admits a general position decomposition $M = \bigcup_i H_i \cup L_1 \cdots \cup L_l$ then $M \in \mathbb{B} \# \hat{L}_1 \# \cdots \# \hat{L}_l$.
(b) If M admits a general position decomposition $M = \bigcup_i H_i \cup L_1 \cdots \cup L_l \cup V_1$ then $M \in \mathbb{B} \# \hat{L}_1 \# \cdots \# \hat{L}_l \# \hat{L}_{l+1}$.
(c) If M admits a general position decomposition $M \in \bigcup_i H_i \cup L_1 \cdots \cup L_l \cup V_1 \cup V_2$ then $M \in \mathbb{B} \# \hat{L}_1 \# \cdots \# \hat{L}_l \# \hat{L}_{l+1} \# \hat{L}_{l+2}$.
(d) If M admits a general position decomposition $M = \bigcup_i H_i \cup L_1 \cdots \cup L_l \cup V_1 \cup V_2 \cup V_3$ then
(d_1) $M \in \mathbb{B} \# \hat{L}_1 \# \cdots \# \hat{L}_l \# \hat{L}_{l+1} \# \hat{L}_{l+2} \# \hat{L}_{l+3}$ or
(d_2) $M \in \mathbb{B} \# \hat{L}_1 \# \cdots \# \hat{L}_l \# \tilde{S}(3)$.

(Again recall that we allow $\hat{L}_k = S^3$).

Proof.
We only demonstrate (d), since the proofs for (a), (b) and (c) are similar and easier.

Write $M = N \cup V_3$, where $N = \bigcup_i H_i \cup L_1 \cdots \cup L_l \cup V_1 \cup V_2$ is as in Theorem $7(c_1)$ or (c_2).

In **case 7(c_1)** we represent N as

$$(**) \quad N = H \cup K_1 \cup \cdots \cup K_m \cup W_1 \cup \cdots \cup W_n \cup L_1 \cup \cdots \cup L_l \cup L_{l+1} \cup L_{l+2}$$

where H is a punctured ball, K_j is a once-punctured S^2-bundle over S^1, W_i is a once-punctured handlebody (j=1, …, m; i=1, …, n), and L_k is a once punctured lens space (k = 1, …, l+2). We may assume that $\hat{L_k} \neq S^2 \times S^1$ since otherwise it can be considered as a K_j. Moreover $K_j \cap K_i = W_j \cap W_i = K_j \cap L_k = W_i \cap L_k = \emptyset$ for $i \neq j$, and $H \cap W_i = \partial H \cap \partial W_i = C_i$ $(i = 1, \ldots, n)$, $H \cap K_j = \partial H \cap \partial K_j = C_j'$ $(j = 1, \ldots, m)$, $H \cap L_k = \partial H \cap \partial L_k = C_k''$ (k = 1, …, l+2) are 2-spheres. Let S_j be a non-separating 2-sphere in $intK_j$. We may assume that C_i, C_j',C_k'' and S_j are transverse to ∂V_3. Since M is closed, $\partial V_3 \subset int(N)$ and $\partial N \subset int(V_3)$. In particular, for each i, the non-sphere boundary component F_i of W_i is contained in $int(V_3)$ and separates V_3 into two components.

Let β denote the number of components of $\mathbf{S} = \partial V_3 \cap (\bigcup_i C_i \cup \bigcup_j C_j' \cup \bigcup_j S_j \cup \bigcup_j C_k'')$

First assume $\beta = 0$.

If for some i, no component of ∂V_3 is contained in W_i then each point of $W_i \cap V_3$ can be joined to a point of F_i by a path in W_i that misses ∂V_3; hence this path must lie entirely in V_3 and it follows that $W_i \subset V_3$. Deleting W_i from $(**)$ we may assume that $\partial V_3 \cap W_i \neq \emptyset$ for each i.

Now the torus boundary T_3 of V_3 is contained in $int(W_i)$ for some i. Since $\hat{W}_i$ is irreducible we have by Lemma 4 two cases:

(i) T_3 bounds an (at most once punctured) solid torus V in W_i
(ii) T_3 bounds an (at most once punctured) knot space R in W_i with a meridian disk D in $\overline{W_i - R}$.

Let $Q = W_i \cap V_3$.

case (i). We replace $W_i \cup V_3$ in the decomposition for $N \cup V_3$ by the (punctured) lens space $L_{l+3} = \overline{W_i - Q} \cup_{T_3} V_3 = V \cup_{T_3} V_3$ (where the union $\cup_{T_3}$ is along the common boundary T_3). (See Figure 3).

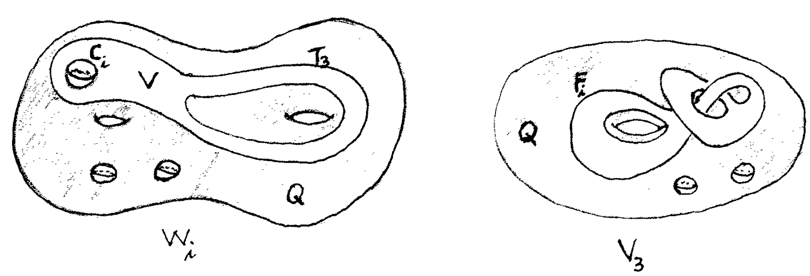

Figure 3:

Then $M = H \cup K_1 \cup \cdots \cup K_m \cup W_1 \cup \cdots W_{i-1} \cup W_{i+1} \cup W_n \cup L_1 \cup \cdots \cup L_l \cup L_{l+1} \cup L_{l+2} \cup L_{l+3}$ where $F_j \subset L_{l+3}$ for all $j \neq i$.

Suppose a 2-sphere S of ∂L_{l+3} lies in W_j for some $j \neq i$. If S bounds a ball in W_j let B be an innermost such ball. Then either $L_{l+3} \subset B$ or $L_{l+3} \cap B = \emptyset$. The first case can not happen since $C_i \subset L_{l+3}$ and $W_i \cap W_j = \emptyset$. In the second case replace L_{l+3} by $L_{l+3} \cup B$ in the decomposition of M to get an L_{l+3} with fewer punctures. Hence we may assume that all spheres of $\partial L_{l+3} - C_i$ in W_j are parallel to C_j and we may isotope these 2-spheres across C_j out of W_j into H. Thus we now have $\partial L_{l+3} \cap W_j = \emptyset$ and it follows that $W_j \subset L_{l+3}$. Deleting W_j from the decomposition of M we obtain $M = H \cup K_1 \cup \cdots \cup K_m \cup L_1 \cup \cdots \cup L_l \cup L_{l+1} \cup L_{l+2} \cup L_{l+3}$ with complexity 0.

Cutting each K_j along S_j into a (once-punctured) ball B_j we obtain a general position decomposition of $M' = M \setminus \bigcup_j S_j$ as in $7(a)$ with $\partial M'$ a collection of 2-spheres (two copies of S_j for each j). Hence M is obtained from $M' = \mathbb{B} \# \hat{L}_1 \# \cdots \# \hat{L}_l \# \hat{L}_{l+1} \# \hat{L}_{l+2} \# \hat{L}_{l+3} \# H_0$ by identifying boundary spheres of H_0 in pairs and we obtain $8(d_1)$.

case (ii). The meridian disk D in $\overline{W_i - R}$ is also a meridian disk for V_3, since ∂D is not null homotopic on $\partial R = T_3$. Since $W_i \cap W_j = \emptyset$ there is a regular neighborhood $U(D)$ in V_3 that misses W_j for all $j \neq i$. In the decomposition of $M = N \cup V_3$ replace $W_i \cup V_3$ by the two punctured balls $B_0 = R \cup U(D)$ and $B_1 = \overline{V_3 - U(D)}$, see figure 4. Note that $(\partial B_0 \cup \partial B_1) \cap W_j = \emptyset$ for $j \neq i$ and $B_0 \cup B_1 = H_0$ is a (punctured) ball. As before we can isotope each W_j

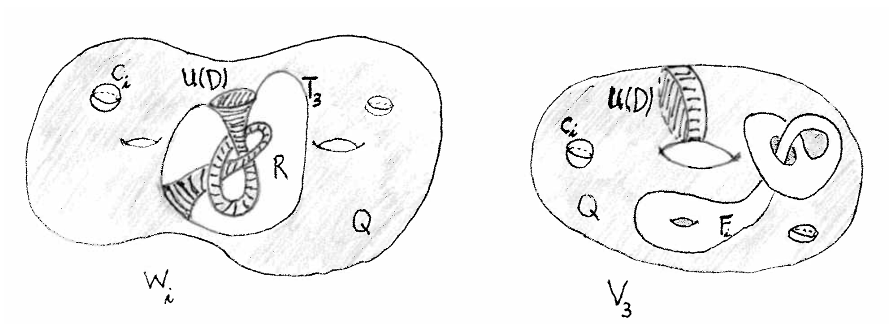

Figure 4:

into B_0 or B_1 and then delete the W_j's from the decomposition of M to obtain $M = H \cup K_1 \cup \cdots \cup K_m \cup L_1 \cup \cdots \cup L_l \cup L_{l+1} \cup L_{l+2} \cup H_0$ with complexity 0 and proceed as in case (i).

Now consider $\beta \neq 0$.

Let c be a component of **S**. If $c \subset \partial V_3 \cap C_i$, say, then c bounds a disk on C_i and there is an innermost such disk $D \subset C_i$ such that $D \cap \partial V_3 = \partial D$. Let $U(D)$ be a regular neighborhood rel ∂V_3 of D in $int(N)$. If $D \subset V_3$ we replace V_3 by $\overline{V_3 - U(D)}$ in the decomposition of M, if D is not in V_3 we replace V_3 by $V_3 \cup U(D)$. In either case we obtain a new general position decomposition of M with smaller β and with V_3 having one more puncture or V_3 being replaced by a (punctured) ball B or $V_3 \cup B$ or a punctured lens space. The result follows now by induction on β.

In **case 7(**c_2**)** we represent N as

$$(***) \quad N = H \cup K_1 \cup \cdots \cup K_m \cup W_1 \cup \cdots \cup W_n \cup L_1 \cup \cdots \cup L_l \cup W(S(2))$$

where H, K_j, W_i, L_k are as before and $W(S(2))$ is now a once punctured Seifert web on $S(2)$. As in the above case $7(c_1)$ we may delete the W_j's from the decomposition $(***)$ and assume that $T_3 \subset intW(S(2))$.

If T_3 is compressible in $W(S(2))$ then we repeat the above arguments of cases (i) and (ii) (with W_i replaced by $W(S(2))$ and noting that $W(S(2))$ is irreducible) to obtain M from $M' = \mathbb{B}\#\hat{L}_1\# \cdots \#\hat{L}_l\#\hat{L}_{l+1}\#H_0$ by identifying boundary spheres of the punctured ball H_0 in pairs. This results in $8(d_1)$.

If T_3 is incompressible in $W(S(2))$ then T_3 can be isotoped off the 1-handles into $S(2) \subset W(S(2))$. By Waldhausen's Theorem [W] (see also [Ha]), T_3 can be further isotoped to consist of fibers.

Let $Q = V_3 \cap W(S(2))$.

Suppose there is a component Q_1 of Q that does not intersect T_3. Then Q_1 is a submanifold of $W(S(2))$ such that ∂Q_1 consists of components of $\partial W(S(2))$. Hence $Q_1 = W(S(2)) \subset intV_3$, and we may delete $W(S(2))$ from $(***)$, which leads to 8(a).

Thus we now assume that Q is connected. Then Q is a submanifold of $W(S(2))$ such that ∂Q consists of T_3 and components of $\partial W(S(2))$ and we replace $W(S(2)) \cup V_3$ in $(***)$ by $W'(S(2)) \cup_{T_3} V_3$, where $W'(S(2)) = \overline{W(S(2)) - Q}$. Since T_3 is a fibered torus in $S(2) \subset W(S(2))$, the submanifold $W'(S(2))$ is again a Seifert web on some $S(2)$. Since M is closed $\partial \hat{W}'(S(2)) = T_3$, hence $\hat{W}'(S(2)) = S(2)$ and it follows from Proposition 2 of [H] that $\hat{W}'(S(2)) \cup_{T_3} \hat{V}_3 = S(2) \cup_{T_3} \hat{V}_3$ is either a connected sum of S^2-bundles over S^1 (if a fiber of $\partial S(2)$ is homotopic to a meridian of V_3) or an $S(3)$. This results in $8(d_2)$.

■

Corollary 9 *Suppose M is a closed 3-manifold.*
(a) If M is a union of three open (punctured) balls then $M \in \mathbb{B}$.
(b) If M is a union of two open (punctured) balls and an open (punctured) solid torus then $M \in \mathbb{B}\#\hat{L}$.
(c) If M is a union of an open (punctured) ball and and two open (punctured)

solid tori then $M \in \mathbb{B}\#\hat{L}_1\#\hat{L}_2$.
(d) If M *is a union of three open (punctured) solid tori then*
(d_1*)* $M \in \mathbb{B}\#\hat{L}_1\#\hat{L}_2\#\hat{L}_3$ *or*
(d_2*)* $M \in \mathbb{B}\#\tilde{S}(3)$.

Proof.
By Lemma 1 M has a general position decomposition $M = H_1 \cup H_2 \cup H_3$, resp. $M = H_1 \cup H_2 \cup V_2$, resp. $M = H_1 \cup V_1 \cup V_2$, resp. $M = V_1 \cup V_2 \cup V_3$. Since $\partial M = \emptyset$ the result follows from Theorem 8.
■

Remark: It is easy to see that the converse of Corollary 9 is also true. For example, if M is a connected sum of S^2-bundles over S^1 and three lens spaces, then we can even decompose M as a union of three solid tori such that their interiors are pairwise disjoint, see e.g. ([GHN], [G]).

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