Approximation of stationary statistical properties of dissipative dynamical systems: time discretization

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Abstract

We consider temporal approximation of stationary statistical properties of dissipative complex dynamical systems. We demonstrate that stationary statistical properties of the time discrete approximations (numerical scheme) converge to those of the underlying continuous dissipative complex dynamical system under three very natural assumptions as the time step approaches zero. The three conditions that are sufficient for the convergence of the stationary statistical properties are: (1) uniform dissipativity of the scheme (in the sense of pre-compactness of the union of the global attractors for the numerical approximations); (2) uniform (with respect to initial data from the union of the global attractors) convergence of the solutions of the numerical scheme to the solution of the continuous system on the unit time interval $[0,1]$; and (3) the uniform (with respect to initial data from the union of the global attractors) continuity of the solutions to the continuous dynamical system on the unit time interval $[0,1]$. The convergence of the global attractors is established under weaker assumptions. Application to the infinite Prandtl number model for convection is discussed.

keywords: stationary statistical property, invariant measure, global attractor, dissipative system, time discretization, uniformly dissipative scheme, infinite Prandtl number model for convection, Nusselt number

1   Introduction

Many dissipative dynamical systems arising in physical applications possess very complex behavior with abundant instability and sensitive dependence on initial data and parameters[29]. It is well-known that statistical properties of these kind of systems are much more important, physically relevant and stable than single trajectories [11, 18, 20, 22, 23, 32].

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For a given abstract autonomous continuous in time dynamical system determined by a semi-group $\{S(t), t \geq 0\}$ on a separable Banach space H, we recall that if the system reaches a statistical equilibrium in the sense that the statistics are time independent (stationary statistical properties), the probability measure μ on H that describes the stationary uncertainty (stationary statistical properties) can be characterized via either the strong (pull-back) or weak (push-forward) formulation [11, 20, 22, 32, 33].

Definition 1 (Invariant Measure (Stationary Statistical Solution)). *Let $\{S(t), t \geq 0\}$ be a continuous semi-group on a Banach space H which generates a dynamical system on H. A Borel probability measure μ on H is called an* **Invariant Measure***(Stationary Statistical Solution) of the dynamical system if*

$$\mu(E) = \mu(S^{-1}(t)(E)), \;\; \forall t \geq 0, \forall E \in \mathcal{B}(H) \tag{1}$$

where $\mathcal{B}(H)$ represents the σ-algebra of all Borel sets on H. Equivalently, the invariant measure μ can be characterized through the following push-forward weak invariance formulation

$$\int_H \Phi(\mathbf{u}) \, d\mu(\mathbf{u}) = \int_H \Phi(S(t)\mathbf{u}) \, d\mu(\mathbf{u}), \;\; \forall t \geq 0 \tag{2}$$

for all bounded continuous test functionals Φ.

Invariant measure (stationary statistical solution) for a discrete dynamical system generated by a map $S_{discrete}$ on a Banach space H is defined in a similar fashion with the continuous time t replaced by discrete time $n = 0, 1, 2, \cdots$.

Another popular object utilized below associated with long time behavior of a dynamical system is the global attractor which we recall for convenience [11, 13, 29].

Definition 2 (Global Attractor and Dissipative System). *Let $\{S(t), t \geq 0\}$ be a continuous semi-group on a Banach space H which generates a continuous dynamical system on H. A set $\mathcal{A} \subset H$ is called the global attractor of the dynamical system if the following three conditions are satisfied.*

1. *$\mathcal{A}$ is compact in H.*

2. *$\mathcal{A}$ is invariant under the flow, i.e.*

$$S(t)\mathcal{A} = \mathcal{A}, \text{for all } t \geq 0. \tag{3}$$

3. *$\mathcal{A}$ attracts all bounded sets in H, i.e., for every bounded set B in H,*

$$\lim_{t \to \infty} dist_H(S(t)B, \mathcal{A}) = 0. \tag{4}$$

Here, $dist_H$ denotes the Hausdorff semi-distance in H between two subsets which is defined as

$$dist_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_H \tag{5}$$

where $\| \cdot \|_H = \| \cdot \|$ denotes the norm on H.

The global attractor for a discrete dynamical system induced by a map $S_{discrete}$ on a Banach space H is defined in a similar fashion with the continuous time t replaced by discrete time $n = 0, 1, 2, \cdots$.

A dynamical system is called **dissipative** *if it possesses a global attractor.*

It is easy to see, thanks to the invariance and the attracting property, that the global attractor, when it exists, is unique [13, 29]. We also caution the reader that our definition of dissipativity may be slightly different from the traditional notation [13, 29].

We are usually interested in $\int_H \Phi(\mathbf{u}) \, d\mu(\mathbf{u})$ (statistical average) for various test functionals Φ (observable). Since we normally do not know the invariant measure μ a priori, one of the commonly used methods in calculating the statistical quantity is to substitute spatial average by long time average under Boltzmann's assumption of ergodicity ([11, 20, 22, 33])

$$\int_H \Phi(\mathbf{u}) \, d\mu(\mathbf{u}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(s)\mathbf{u}) \, ds.$$

Although this formulation may not make sense all the time since the long time average may not exist, one can always replace the long time limit by Banach (generalized) limit which are bounded linear functionals on the space of bounded functions that agrees with the usual long time limit on those functions whenever the long time limit exists [21]. One may show via the so-called Bogliubov-Krylov argument that these generalized long time average over trajectory leads to invariant measure (may depend on the chosen Banach limit and initial datum $\mathbf{u}$) of the system for appropriate dissipative dynamical systems such as the two dimensional incompressible Navier-Stokes system and the infinite Prandtl number model for convection system discussed below [11, 32, 36].

Due to the presumed complexity of the dynamics, it is foreseeable that physically interesting stationary statistical properties need to be calculated using numerical methods. Even under the ergodicity assumption, it is not at all clear that classical numerical schemes which provide accurate approximation on finite time interval will remain meaningful for stationary statistical properties (long time properties) since small error will be amplified and accumulated over long time except in the case that the underlying dynamics is asymptotically stable [12, 14, 19] where statistical approach is not necessary since there is no chaos. Therefore, it is of great importance and a challenge to search for numerical methods that are able to capture stationary statistical properties of infinite dimensional complex dynamical system. We will focus on dissipative system and time discretization here since long time approximation seems to be the key issue involved.

As we shall demonstrate below that if the system and the scheme possess three very natural properties (1) uniform dissipativity in the sense that the scheme possesses global

attractor for small enough time step and the union of these global attractors (for different time steps) is pre-compact in the phase space; (2) uniform convergence of the numerical scheme for data from the support of the invariant measures on the unit time interval $[0,1]$ modulo initial layer; and (3) the continuous dynamical system is uniformly continuous (for data from the support of the invariant measures) on the unit time interval $[0,1]$, then the stationary statistical properties of the scheme will converge to stationary statistical properties of the continuous dynamical system. Our result will be presented in an abstract fashion in order to clarify the central issues and to provide well-organized means for discussing the problems. We also hope that our work will stimulate further work on accurate and efficient numerical schemes for stationary statistical properties of infinite dimensional dissipative systems.

It is easy to see that the assumptions are natural. Since the underlying dynamical system is dissipative, it is natural to require that the numerical scheme inherit the dissipativity of the continuous in time system so that the scheme is uniformly dissipative (for small time steps). The uniform convergence of the numerical scheme for initial data from the global attractor on the unit time interval is also expected from most reasonable numerical schemes. The strong continuity of the underlying dynamical system on the unit time interval uniform with respect to initial data from the union of the global attractors is natural for most continuous dissipative dynamical systems. Once the desired natural conditions are discovered, the proof of the main result is relatively straightforward although verifying these three sufficient conditions for each application may be highly non-trivial (see section 3 for a relatively easy application to the infinite Prandtl number model for convection with a linear semi-implicit scheme).

Although we are not aware of any work on the convergence of stationary statistical properties of numerical schemes for chaotic PDEs except our previous work [3, 4] (see [5] for the case of map on the unit interval), there have been a lot of work on temporal approximation of dissipative dynamical systems such as the two dimensional incompressible Navier-Stokes system and the one-dimensional Kuramoto-Sivashinsky equation (see [12, 15, 17, 25, 26, 30, 9, 10] among others). These authors were mostly interested in the long time stability of the scheme in the sense of deriving uniform in time bounds on the scheme (sometimes bound in the phase space H only which is not sufficient for uniform dissipativity although it may be sufficient for the convergence of the global attractors), and none of them discussed statistical properties. Our result on the convergence of stationary statistical properties may be viewed as an abstraction and generalization of [3, 4]. See [8] for the heat bath approach to computing invariant measure for finite dimensional systems.

A by-product of the convergence analysis of the invariant measures presented here is the convergence of the global attractors of the scheme to that of the underlying system. This is also within expectation since the global attractors carry the support of the invariant measures. The convergence of the global attractors under time discretization has been discussed for the two dimensional Navier-Stokes system, reaction-diffusion equation, and for finite-dimensional dynamical systems [25, 27, 15] among others. Therefore our result on the convergence of global attractors may be viewed as a generalization and abstraction

of these results. However, we would like to point out that the convergence of the global attractors can be established under much weaker assumption. One only needs the uniform boundedness of the union of the global attractors K, instead of the pre-compactness (plus finite time uniform convergence for data from K). Because of this important distinction, *it is possible to have schemes that are able to capture the global attractor asymptotically but not necessarily the stationary statistical properties (invariant measures).* There are also interesting works on persistence under approximation of various invariant sets (such as steady state, time periodic orbit, inertial manifold etc) both for PDEs and ODEs under appropriate assumptions (such as spectral gap condition that is usually associated with inertial manifold theory, see [27, 16, 28, 29] and the references therein). We would also like to point out that the convergence of invariant sets and the convergence of stationary statistical properties are two related but very different issues associated with the long time behavior. It is easy to construct two dynamical systems with exactly the same global attractor or inertial manifold but with totally different dynamics or stationary statistical properties.

The rest of the manuscript is organized as follows: in section 2 we prove the main results, namely the convergence of stationary statistical properties and the global attractors under the three natural hypotheses; in section 3 we discuss an application of the main results to the infinite Prandtl number model for convection. The choice of application is both for its physical significance and mathematical simplicity so that an essentially self-contained short exposition is possible and we believe that our main abstract theorem applies to many other dissipative systems and schemes. We then provide conclusion and remarks in the fourth/last section.

2 Main results: abstract formulation

Here we show our main results, namely, uniform dissipativity plus finite time uniform convergence of the time discrete approximation together with the finite time uniform continuity of the underlying dynamical system imply convergence of the stationary statistical properties / invariant measures.

Throughout this section, all semigroups are assumed to be continuous in the sense that $S(t), t \geq 0$ and S_k are continuous operators on H.

Theorem 1 (Convergence of Stationary Statistical Properties). *Let $\{S(t), t \geq 0\}$ be a continuous semi-group on a separable Hilbert space H which generates a continuous dissipative dynamical system (in the sense of possessing a compact global attractor $\mathcal{A}$) on H. Let $\{S_k, 0 < k \leq k_0\}$ be a family of continuous maps on H which generates a family of discrete dissipative dynamical system (with global attractor $\mathcal{A}_k$) on H. Suppose that the following three conditions are satisfied.*

H1: *[Uniform dissipativity] There exists a $k_1 \in (0, k_0)$ such that $\{S_k, 0 < k \leq k_1\}$ is uniformly dissipative in the sense that*

$$K = \bigcup_{0 < k \leq k_1} \mathcal{A}_k \tag{6}$$

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is pre-compact in H.

H2: *[Uniform convergence on the unit time interval] S_k uniformly converges to S on the unit time interval (modulo an initial layer) and uniformly for initial data from the global attractor of S_k in the sense that for any $t_0 \in (0, 1)$*

$$\lim_{k \to 0} \sup_{\mathbf{u} \in \mathcal{A}_k, nk \in [t_0, 1]} \|S_k^n \mathbf{u} - S(nk)\mathbf{u}\| = 0. \tag{7}$$

H3: *[Uniform continuity of the continuous system] $\{S(t), t \geq 0\}$ is uniformly continuous on K on the unit time interval in the sense that for any $T^* \in [0, 1]$*

$$\lim_{t \to T^*} \sup_{\mathbf{u} \in K} \|S(t)\mathbf{u} - S(T^*)\mathbf{u}\| = 0. \tag{8}$$

Then the invariant measures of the discrete dynamical system $\{S_k, 0 < k \leq k_0\}$ converge to invariant measures of the continuous dynamical system S. More precisely, let $\mu_k \in \mathcal{IM}_k$ where $\mathcal{IM}_k$ denotes the set of all invariant measures of S_k. There must exist a subsequence, still denoted $\{\mu_k\}$, and $\mu \in \mathcal{IM}$ (an invariant measure of $S(t)$), such that μ_k weakly converges to μ, i.e.,

$$\mu_k \rightharpoonup \mu, \ \ as \ k \to 0. \tag{9}$$

Moreover, extremal statistics converge in upper-semi-continuous fashion in the sense that for any bounded continuous functional Φ on the phase space H, there exist ergodic invariant measures $\mu_k \in \mathcal{IM}_k$ and an ergodic invariant measure $\mu \in \mathcal{IM}$, such that

$$\sup_{\mathbf{u}_0 \in H} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Phi(S_k^n(\mathbf{u}_0)) \ = \ \int_H \Phi(\mathbf{u}) \, d\mu_k(\mathbf{u}), \forall k, \tag{10}$$

$$\sup_{\mathbf{u}_0 \in H} \limsup_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \Phi(S(t)\mathbf{u}_0) \, dt \ = \ \int_H \Phi(\mathbf{u}) \, d\mu(\mathbf{u}), \tag{11}$$

$$\limsup_{k \to 0} \sup_{\mathbf{u}_0 \in H} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Phi(S_k^n(\mathbf{u}_0)) \ \leq \ \sup_{\mathbf{u}_0 \in H} \limsup_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \Phi(S(t)\mathbf{u}_0) \, dt. \tag{12}$$

Proof: Since $K = \bigcup_{0 < k \leq k_1} \mathcal{A}_k$ is pre-compact in H and since all invariant measures are supported on the global attractor [11, 36] and $\mu_k \in \mathcal{IM}_k$, we see that $\{\mu_k\}$ is tight in the space of all Borel probability measures on H thanks to Prokhorov's theorem [1, 21, 11]. Hence it must contain a weakly convergent subsequence (still denoted $\{\mu_k\}$) which weakly converges to a Borel probability measure μ on H, i.e.

$$\int_H \varphi(\mathbf{u}) \, d\mu_k(\mathbf{u}) \to \int_H \varphi(\mathbf{u}) \, d\mu(\mathbf{u}), \ as \ k \to 0.$$

Our goal is to show that μ is invariant under $S(t)$.

Now we fix a $T^* \in (0,1]$ and let $n_k = \lfloor \frac{T^*}{k} \rfloor$ be the floor of $\frac{T^*}{k}$ (the largest integer dominated by $\frac{T^*}{k}$), and let φ be any smooth (C^1) test functional (observable) with compact support.

Since $\mu_k \in \mathcal{IM}_k$, we have by the weak invariance of μ_k under S_k and weak convergence

$$\int_H \varphi(S_k^{n_k}\mathbf{u}) \, d\mu_k(\mathbf{u}) = \int_H \varphi(\mathbf{u}) \, d\mu_k(\mathbf{u}) \to \int_H \varphi(\mathbf{u}) \, d\mu(\mathbf{u}), \text{ as } k \to 0.$$

On the other hand,

$$\int_H \varphi(S_k^{n_k}\mathbf{u}) \, d\mu_k(\mathbf{u})$$
$$= \int_H \varphi(S(T^*)\mathbf{u}) \, d\mu_k(\mathbf{u}) + \int_H (\varphi(S(n_k k)\mathbf{u}) - \varphi(S(T^*)\mathbf{u})) \, d\mu_k(\mathbf{u}) + \int_H (\varphi(S_k^{n_k}\mathbf{u}) - \varphi(S(n_k k)\mathbf{u})) \, d\mu_k(\mathbf{u})$$

For the first term, since $S(t)$ is continuous, $\varphi(S(T^*)\mathbf{u})$ is bounded and continuous in $\mathbf{u}$. Hence we have

$$\int_H \varphi(S(T^*)\mathbf{u}) \, d\mu_k(\mathbf{u}) \to \int_H \varphi(S(T^*)\mathbf{u}) \, d\mu(\mathbf{u}), \text{ as } k \to 0.$$

We also have, thanks to the uniform convergence assumption, the mean value theorem, the uniform boundedness of the derivative of the smooth (C^1) test functional (observable) with compact support, for $\mathbf{u} \in \mathcal{A}_k$,

$$\begin{aligned}
|\varphi(S_k^{n_k}\mathbf{u}) - \varphi(S(n_k k)\mathbf{u})| &\leq \sup_{\mathbf{u} \in H} \|\varphi'(\mathbf{u})\| \sup_{\mathbf{u} \in \mathcal{A}_k} \|S_k^{n_k}\mathbf{u} - S(n_k k)\mathbf{u}\| \\
&\leq \sup_{\mathbf{u} \in H} \|\varphi'(\mathbf{u})\| \sup_{\mathbf{u} \in \mathcal{A}_k, nk \in [\frac{T^*}{2}, T^*]} \|S_k^n\mathbf{u} - S(nk)\mathbf{u}\| \\
&\to 0, \text{ as } k \to 0,
\end{aligned}$$

[1] and hence, since the support of invariant measure μ_k is included in the global attractor $\mathcal{A}_k$, [11, 36]

$$\int_H (\varphi(S_k^{n_k}\mathbf{u}) - \varphi(S(n_k k)\mathbf{u})) \, d\mu_k(\mathbf{u}) \to 0, \text{ as } k \to 0.$$

Moreover,

$$|\varphi(S(n_k k)\mathbf{u}) - \varphi(S(T^*)\mathbf{u})| \leq \sup_{\mathbf{u} \in H} \|\varphi'(\mathbf{u})\| \sup_{\mathbf{u} \in \mathcal{A}_k} \|S(n_k k)\mathbf{u} - S(T^*)\mathbf{u}\| \to 0, \text{ as } k \to 0$$

by the uniform continuity of $S(t)$ at $t = T^*$ and $\mathbf{u} \in K$ since $n_k k = \lfloor \frac{T^*}{k} \rfloor k \to T^*$ as $k \to 0$. This further implies,

$$\int_H (\varphi(S(n_k k)\mathbf{u}) - \varphi(S(T^*)\mathbf{u})) \, d\mu_k(\mathbf{u}) \to 0, \text{ as } k \to 0.$$

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[1] We only need $\lim_{k \to 0} \sup_{\mathbf{u} \in \mathcal{A}_k} \sup_{T^* - \frac{1}{k} \leq nk \leq T^*} \|S_k^n \mathbf{u} - S(nk)\mathbf{u}\| = 0, \forall T^* \in (0,1]$.

Combining the estimates above, we have

$$\int_H \varphi(S_k^{n_k}\mathbf{u})\,d\mu_k(\mathbf{u}) \to \int_H \varphi(S(T^*)\mathbf{u})\,d\mu(\mathbf{u}), \text{ as } k \to 0.$$

Therefore

$$\int_H \varphi(S(T^*)\mathbf{u})\,d\mu(\mathbf{u}) = \int_H \varphi(\mathbf{u})\,d\mu(\mathbf{u})$$

which is exactly the weak invariance (2) for the smooth (C^1) test functional (observable) with compact support and $T^* \in (0,1]$.

For a general bounded continuous test functional φ, we can first approximate it by finite dimensional test functional of the form $\varphi \circ P_m$ where P_m is the orthogonal projection onto the m-dimensional subspace spanned by the first m elements of a given (fixed) orthonormal basis of H (this is where we need H to be a separable Hilbert space). We can then approximate $\varphi \circ P_m$ by smooth test functionals with compact support using mollifiers and truncation since only the value of $\varphi \circ P_m$ on the compact global attractor is relevant for statistics. This proves short time weak invariance (2) for any bounded continuous test functional ϕ and $T^* \in (0,1]$.

Now for a general $T^{**} > 1$, we exists a unique positive integer n and $T_* \in (0,1]$ such that $T^{**} = n + T_*$. Hence

$$
\begin{aligned}
\int_H \varphi(S(T^{**})\mathbf{u})\,d\mu(\mathbf{u}) &= \int_H \varphi(S^n(1)S(T_*)\mathbf{u})\,d\mu(\mathbf{u}) \\
&= \int_H \varphi(S(T_*)\mathbf{u})\,d\mu(\mathbf{u}) \\
&= \int_H \varphi(\mathbf{u})\,d\mu(\mathbf{u})
\end{aligned}
$$

where we have utilized the semi-group property of $S(t)$, the strong continuity of $S(t)$ and the short time weak invariance that we proved above with $T^* = 1$ n times and $T^* = T_*$ one time.

This ends the proof of the convergence of the invariant measures.

As for the extremal statistics, it is easy to see that due to the uniform dissipativity, for any given bounded continuous test functional Φ, there exist ergodic invariant measures μ_k and μ such that the extreme of temporal averages are equivalent to spatial averages with respect to these ergodic invariant measures [36, 11, 33], as stated in (10, 11). This is a manifestation of the fact that the extremal points of the set of invariant measures must be ergodic. As for the upper semi-convergence of the extremal statistics stated in (12), we have, thanks to the uniform dissipativity and Prokhorov's theorem, there exists a subsequence (still denoted $\{\mu_k\}$) and $\nu \in \mathcal{IM}$ such that

$$\mu_k \to \nu, \text{ as } k \to 0,$$

$$\limsup_{k\to 0} \sup_{\mathbf{u}_0 \in H} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Phi(S_k^n(\mathbf{u}_0)) = \limsup_{k\to 0} \int_H \Phi(\mathbf{u})\,d\mu_k.$$

Since $\mathcal{IM}$ is compact in the space of Borel probability measures on H, there exists an ergodic invariant measure $\nu_{max} \in \mathcal{IM}$ such that $\sup_{\tilde{\mu} \in \mathcal{IM}} \int_H \Phi(\mathbf{u}) \, d\tilde{\mu} = \int_H \Phi(\mathbf{u}) \, d\nu_{max}$ [36]. Therefore

$$
\begin{aligned}
\limsup_{k \to 0} \sup_{\mathbf{u}_0 \in H} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Phi(S_k^n(\mathbf{u}_0)) &= \limsup_{k \to 0} \int_H \Phi(\mathbf{u}) \, d\mu_k \\
&= \int_H \Phi(\mathbf{u}) \, d\nu \\
&\leq \sup_{\tilde{\mu} \in \mathcal{IM}} \int_H \Phi(\mathbf{u}) \, d\tilde{\mu} \\
&= \int_H \Phi(\mathbf{u}) \, d\nu_{max} \\
&= \lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \Phi(S(t)\mathbf{u}_0) \, dt \\
&\leq \sup_{\mathbf{u}_0 \in H} \limsup_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \Phi(S(t)\mathbf{u}_0) \, dt.
\end{aligned}
$$

This completes the proof of the theorem. $\qquad\square$

Next, we show the convergence of the global attractors under weaker assumptions, namely the uniform boundedness of K (the union of the global attractors), and uniform convergence on finite time interval (modulo an arbitrary initial layer).

Proposition 1 (Convergence of Global Attractors). *Let $\{S(t), t \geq 0\}$ be a continuous semigroup on a Banach space H which generates a dissipative dynamical system (in the sense of possessing a compact global attractor $\mathcal{A}$) on H. Let $\{S_k, 0 < k \leq k_0\}$ be a family of continuous maps on H which generates a family of discrete dissipative dynamical system (with global attractor $\mathcal{A}_k$) on H. Suppose that the following two conditions are satisfied.*

H4: [Uniform boundedness] There exists a $k_1 \in (0, k_0]$ such that $\{S_k, 0 < k \leq k_1\}$ is uniformly bounded in the sense that

$$
K = \bigcup_{0 < k \leq k_1} \mathcal{A}_k \tag{13}
$$

is bounded in H.

H5: [Finite time uniform convergence] S_k uniformly converges to S on any finite time interval (modulo any initial layer) and uniformly for initial data from the global attractor of the scheme in the sense that there exists $t_0 > 0$ such that for any $T^* > t_0 > 0$

$$
\lim_{k \to 0} \sup_{\mathbf{u} \in \mathcal{A}_k, nk \in [t_0, T^*]} \| S_k^n \mathbf{u} - S(nk)\mathbf{u} \| = 0. \tag{14}
$$

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Then the global attractors converge in the sense of Hausdorff semi-distance, i.e.

$$\lim_{k \to 0} dist_H(\mathcal{A}_k, \mathcal{A}) = 0. \tag{15}$$

Proof: Since K is bounded, for any given $\epsilon > 0$, there exists a $T_\epsilon > t_0 > 0$ such that

$$\mathrm{dist}_H(S(t)K, \mathcal{A}) < \frac{\epsilon}{2}, \forall t \geq T_\epsilon$$

because the global attractor $\mathcal{A}$ attracts all bounded set, in particular K.

Now let $\mathbf{u}_k \in \mathcal{A}_k$. Since the global attractor $\mathcal{A}_k$ is invariant under S_k, there exists a $\mathbf{v}_k \in \mathcal{A}_k$ such that $\mathbf{u}_k = S_k^{n_k} \mathbf{v}_k$ where $n_k = \lfloor \frac{T_\epsilon + 1}{k} \rfloor$.

Thanks to the uniform convergence on $[T_\epsilon, T_\epsilon + 1]$ and $\mathbf{u} \in K$, we have, there exists $k_\epsilon > 0$ such that

$$\|\mathbf{u}_k - S(n_k k)\mathbf{v}_k\| = \|S_k^{n_k} \mathbf{v}_k - S(n_k k)\mathbf{v}_k\| < \frac{\epsilon}{2}, \ k \leq k_\epsilon.$$

This implies that

$$
\begin{aligned}
\mathrm{dist}_H(\mathcal{A}_k, \mathcal{A}) &= \sup_{\mathbf{u}_k \in \mathcal{A}_k} \mathrm{dist}(\mathbf{u}_k, \mathcal{A}) \\
&\leq \sup_{\mathbf{u}_k \in \mathcal{A}_k} (\|\mathbf{u}_k - S(n_k k)\mathbf{v}_k\| + \mathrm{dist}(S(n_k k)\mathbf{v}_k, \mathcal{A})) \\
&\leq \epsilon, \ k \leq k_\epsilon.
\end{aligned}
$$

This completes the proof for the convergence of the global attractors. $\square$

We would like to reiterate the point that the uniform boundedness assumption H4 is much weaker that the uniform dissipativity assumption H1 for infinite dimensional systems although they are equivalent for finite dimensional systems. This is an important difference and hence it is theoretically possible to have schemes that are able to capture the global attractor asymptotically but not the invariant measures necessarily. Conditions H5 and H2 are roughly equivalent in some sense. They are usually valid for reasonable numerical schemes (see below for an example on the infinite Prandtl number model). Therefore convergence of the global attractors is usually easier to establish than the convergence of the invariant measures (stationary statistical properties).

In application, the discrete dynamical systems $\{S_k\}$ are usually generated by one time step discretization (numerical scheme) with time step k. In another word, $\mathbf{u}^{n+1} = S_k(\mathbf{u}^n)$ is the solution to the numerical scheme. The uniform dissipativity of the numerical scheme is customary established via the existence of a uniform (in time step) absorbing ball in another separable Hilbert space V which is compactly imbedded in H in the case of strongly dissipative system (see the next section for an example). However, this may not be feasible for weakly dissipative systems such as the Darcy-Boussinesq system for convection in fluid saturated porous media, or weakly damped driven Schrödinger equation. The finite time uniform convergence comes with classical numerical analysis for reasonable schemes (see next section for an example). The uniform continuity of the underlying continuous dynamical system is also easily verified for reasonable systems.

3 Application to the infinite Prandtl number model for convection

Here we illustrate an application of the main result to the following infinite Prandtl number for convection [2, 31, 6, 34, 7]

$$\nabla p = \Delta \mathbf{u} + Ra\,\mathbf{k}T, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{z=0,1} = 0, \tag{16}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T, \quad T|_{z=0} = 1, T|_{z=1} = 0 \tag{17}$$

where $\mathbf{u}$ is the Eulerian velocity of the fluid, p represents the kinematic pressure of the fluid, T is the temperature of the fluid, $\mathbf{k}$ is a unit vector in the z direction, Ra is the Rayleigh number measuring the ratio of differential heating over overall dissipation, and we assume that the fluids occupy the (non-dimensionalized) region $\Omega = [0, L_x] \times [0, L_y] \times [0, 1]$ with periodicity imposed in the horizontal directions for simplicity.

The choice of this example is both for its physical significance and for its mathematical simplicity so that we may have an essentially self-contained short exposition. It is well-known that this system possesses turbulent behavior at large Rayleigh number [2, 6, 18, 31]. The convergence of the stationary statistical properties for another class of semi-implicit schemes was discussed in [3, 4] utilizing Liouville type equation approach. The scheme that we study here has an added advantage over the schemes that we investigated earlier in the sense that the discretization is independent of any ad-hoc background temperature profile τ (to be introduced in (22)).

One of the most important statistical quantities in convection is the Nusselt number [2, 6, 7, 18, 35] quantifying the heat transport in the vertical direction which is the statistics corresponding to the functional

$$\Phi(\theta) = Ra \int_\Omega A^{-1}(\mathbf{k}\theta)_3 \theta \, d\mathbf{x} = \int_\Omega u_3 \theta \, d\mathbf{x}$$

where A denotes the Stokes operator with the associated boundary conditions and viscosity one, and $A^{-1}(\mathbf{k}\theta)_3$ represents the third component (vertical velocity) of $A^{-1}(\mathbf{k}\theta)$. In another word, $\mathbf{u} = A^{-1}\mathbf{f}$ solves the following Stokes system

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}|_{z=0,1} = 0, \quad \mathbf{u} \text{ periodic in } x, y.$$

Although this Φ is not bounded, for any pre-compact set K (in particular the union of the global attractors of the scheme), we can easily replace it by a bounded continuous functional $\tilde{\Phi}$ which agrees with Φ on K and hence the statistical averages of Φ and $\tilde{\Phi}$ are the same since the support of all interested invariant measures is included in K. The replacement is possible due to the pre-compactness of K (presumed, to be verified below). Our result on the convergence of the Nusselt number, i.e. (12) with the Φ given above, complements the well-known variational approach [6] since our result indicates that the numerics will provide an asymptotic lower bound for the maximal Nusselt number while the variational approach provides a rigorous upper bound.

3.1 A semi-implicit scheme

Notice that the infinite Prandtl number model can be written as a non-local non-linear advection diffusion equation for the temperature field

$$\frac{\partial T}{\partial t} + Ra \, A^{-1}(\mathbf{k}T) \cdot \nabla T = \Delta T, \quad T|_{z=0} = 1, T|_{z=1} = 0 \tag{18}$$

where A denotes the Stokes operator with the associated boundary conditions and viscosity one.

We then propose the following semi-implicit scheme for the infinite Prandtl number model

$$\frac{T_k^{n+1} - T_k^n}{k} + Ra \, A^{-1}(\mathbf{k}T_k^n) \cdot \nabla T_k^{n+1} = \Delta T_k^{n+1} \tag{19}$$

where T_k^n denotes the approximate solution at time kn with k being the time step.

Our goal now is to show that the stationary statistical properties of this scheme converge to those of the infinite Prandtl number model as the time step k approaches zero.

In order to accomplish this goal utilizing our main result proved in the previous section, we rewrite the scheme in the perturbative variable θ as usual so that the phase space is the Hilbert space $H = L^2(\Omega)$

$$\frac{\theta_k^{n+1} - \theta_k^n}{k} + Ra \, A^{-1}(\mathbf{k}\theta_k^n) \cdot \nabla \theta_k^{n+1} + Ra \, A^{-1}(\mathbf{k}\theta_k^n)_3 \tau'(z) = \Delta \theta_k^{n+1} + \tau''(z) \tag{20}$$

where $\theta_k^n = T_k^n - \tau$ with $\tau(z)$ being an add-hoc background temperature profile (to be specified below) which satisfies the non-homogeneous Dirichlet boundary condition of T_k^n, and we are searching for solution in the space $H_{0,per}^1$ (the subspace of H^1 with zero trace/value at $z = 0, 1$ and periodic in the horizontal directions). Here $A^{-1}(\mathbf{k}\theta)_3$ represents the third component (vertical velocity) of $A^{-1}(\mathbf{k}\theta)$.

The infinite Prandtl number model (18) can be written in the perturbative variable θ as well

$$\frac{\partial \theta}{\partial t} + Ra \, A^{-1}(\mathbf{k}\theta) \cdot \nabla \theta + Ra \, A^{-1}(\mathbf{k}\theta)_3 \tau'(z) = \Delta \theta + \tau''(z). \tag{21}$$

Following [6], we set the background temperature profile τ to be a locally smoothed (mollified) version of the following piecewise linear function

$$\tau(z) = \begin{cases} 1 - \frac{z}{\delta}, & 0 \le z \le \delta, \\ 0, & \delta \le z \le 1. \end{cases} \tag{22}$$

The choice of the parameter δ will be specified later in (26).

We remark that the scheme (20) is different from the scheme that we investigated earlier [3, 4] where the scheme $\frac{\theta_k^{n+1} - \theta_k^n}{k} + Ra \, A^{-1}(\mathbf{k}\theta_k^n) \cdot \nabla \theta_k^{n+1} + Ra \, A^{-1}(\mathbf{k}\theta_k^{n+1})_3 \tau'(z) = \Delta \theta_k^{n+1} + \tau''(z)$ was studied. The new scheme (20) has the advantage that it is independent of the ad-hoc

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background profile τ since it can be written in terms of the temperature field T only as is clear in (19).

It is easy to see that the well-posedness of the scheme (20) follows from the weak formulation

$$\int_\Omega (\frac{1}{k}\theta_k^{n+1}+Ra\,A^{-1}(\mathbf{k}\theta_k^n)\cdot\nabla\theta_k^{n+1})\psi+\int_\Omega \nabla\theta_k^{n+1}\cdot\nabla\psi = \int_\Omega (\frac{1}{k}\theta_k^n+\tau''-Ra\,A^{-1}(\mathbf{k}\theta_k^n)_3\tau'(z))\psi,\ \forall\psi \in H_{0,per}^1,$$
(23)

the identity $\int_\Omega A^{-1}(\mathbf{k}\theta_k^n) \cdot \nabla\theta_k^{n+1}\,\theta_k^{n+1} = 0, \forall\theta_k^{n+1} \in H_{0,per}^1, \theta_k^n \in L^2$, and the Lax-Milgram theorem [21]. Therefore, we have a discrete dynamical system S_k defined on the Hilbert space

$$H = L^2(\Omega) \tag{24}$$

given by

$$S_k(\theta_k^n) = \theta_k^{n+1}. \tag{25}$$

We notice that S_k in fact maps H into $H_{0,per}^1$ which is a direct consequence of the weak formulation.

In order to apply the main result proved in the previous theorem to the current situation, we will verify the three conditions postulated in the main theorem. These will be the goal of the next three subsections.

3.2 Uniform dissipativity

Here we demonstrate the uniform dissipativity, i.e., H1, of the scheme (20) with appropriate choice of δ.

Here and below, the c_js denote generic constants independent of k, n and the initial data unless otherwise stated (but which may depend on the Rayleigh number). Here and elsewhere $\|\theta\| = \sqrt{\int_\Omega |\theta|^2}$ denotes the spatial L^2 norm of θ, and $\|\theta\|_\infty = \mathrm{esssup}_\Omega |\theta|$ denotes the spatial L^∞ norm of θ.

We first derive a uniform bound in the L^2 space. For this purpose we take the inner product of the scheme (20) with $\psi = \theta_k^{n+1}$ and utilize the identity $(a - b, a) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2)$, and we have

$$\frac{1}{2k}(\|\theta_k^{n+1}\|^2 - \|\theta_k^n\|^2 + \|\theta_k^{n+1} - \theta_k^n\|^2) + \|\nabla\theta_k^{n+1}\|^2$$

$$\leq\ \|\tau'\|\|\nabla\theta_k^{n+1}\| + Ra|\int_\Omega \tau'(z)(A^{-1}(\mathbf{k}\theta_k^n))_3\theta_k^{n+1}|$$

$$\leq\ \|\tau'\|^2 + \frac{1}{4}\|\nabla\theta_k^{n+1}\|^2 + Ra|\int_\Omega \tau'(z)(A^{-1}(\mathbf{k}\theta_k^n))_3\theta_k^{n+1}|$$

where we have used the identity $\int_\Omega A^{-1}(\mathbf{k}\theta_k^n) \cdot \nabla\theta_k^{n+1}\,\theta_k^{n+1} = 0$ one more time.

As for the nonlinear term, we have

$$
\begin{aligned}
Ra\left| \int_\Omega \tau'(z)(A^{-1}(\mathbf{k}\theta_k^n))_3\theta_k^{n+1} \right| &\leq \frac{Ra}{\delta}\|A^{-1}(\mathbf{k}\theta_k^n)_3\|_{L^2(z\leq\delta)}\|\theta_k^{n+1}\|_{L^2(z\leq\delta)} \\
&\leq Ra\delta\|\frac{\partial}{\partial z}A^{-1}(\mathbf{k}\theta_k^n)_3\|_{L^2(z\leq\delta)}\|\frac{\partial}{\partial z}\theta_k^{n+1}\|_{L^2(z\leq\delta)} \\
&\leq Ra\delta^2\|\frac{\partial^2}{\partial z^2}A^{-1}(\mathbf{k}\theta_k^n)_3\|_{L^2(z\leq\delta)}\|\frac{\partial}{\partial z}\theta_k^{n+1}\|_{L^2(z\leq\delta)} \\
&\leq c_1Ra\delta^2\|\theta_k^n\|\|\frac{\partial}{\partial z}\theta_k^{n+1}\| \\
&\leq c_1Ra\delta^2(\|\theta_k^n - \theta_k^{n+1}\| + \|\theta_k^{n+1}\|)\|\nabla\theta_k^{n+1}\| \\
&\leq c_1Ra\delta^2\|\theta_k^n - \theta_k^{n+1}\|\|\nabla\theta_k^{n+1}\| + c_1Ra\delta^2\|\nabla\theta_k^{n+1}\|^2 \\
&\leq \frac{c_1Ra\delta^2}{4}\|\theta_k^n - \theta_k^{n+1}\|^2 + 2c_1Ra\delta^2\|\nabla\theta_k^{n+1}\|^2 \\
&\leq \frac{1}{4k}\|\theta_k^n - \theta_k^{n+1}\|^2 + \frac{1}{4}\|\nabla\theta_k^{n+1}\|^2
\end{aligned}
$$

provided that we choose the background temperature profile(through δ) and the time step k in the following fashion

$$
\delta = (8c_1Ra)^{-\frac{1}{2}}, \quad k \leq 8, \tag{26}
$$

where we have utilized the specific form of the background profile τ (22), the homogeneous boundary conditions for θ_k^n, θ_k^{n+1}, $A^{-1}(\mathbf{k}\theta_k^n)_3$, $\frac{\partial}{\partial z}A^{-1}(\mathbf{k}\theta^n)_3$, elliptic regularity (for the Stokes operator A) and the Poincaré inequality.

Therefore, under the specific choice of background temperature profile (22, 26), there exists a constant c_2, such that

$$
\frac{1}{k}(\|\theta_k^{n+1}\|^2 - \|\theta_k^n\|^2 + \frac{1}{2}\|\theta_k^{n+1} - \theta_k^n\|^2) + \|\nabla\theta_k^{n+1}\|^2 \leq 2\|\tau'\|^2 \leq c_2Ra^{\frac{1}{2}} \tag{27}
$$

which further implies, thanks to Poincaré inequality,

$$
(1+k)\|\theta_k^{n+1}\|^2 \leq \|\theta_k^n\|^2 + c_2kRa^{\frac{1}{2}}. \tag{28}
$$

This leads to, with the help of a simple iteration,

$$
\|\theta_k^{n+1}\|^2 \leq (1+k)^{-(n+1)}\|\theta_0\|^2 + c_2Ra^{\frac{1}{2}}. \tag{29}
$$

This is a uniform estimate in the L^2 space since for any $\theta_0 \in B_R(L^2)$ (ball centered at the origin with radius R in L^2), there exists an integer $N_0(R,k)$ such that

$$
(1+k)^{-n}\|\theta_0\|^2 \leq c_2Ra^{\frac{1}{2}}, \ \forall n \geq N_0.
$$

Hence

$$
\|\theta_k^n\| \leq \sqrt{2c_2}Ra^{\frac{1}{4}}, \ \forall n \geq N_0(R,k). \tag{30}
$$

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Indeed, we have

$$N_0(R,k) \approx \frac{\ln \frac{R^2}{c_2 Ra^{\frac{1}{2}}}}{\ln(1+k)} \approx \frac{\ln \frac{R^2}{c_2 Ra^{\frac{1}{2}}}}{k} \tag{31}$$

so that the time needed to enter the absorbing ball of radius $\sqrt{2c_2} Ra^{\frac{1}{4}}$ is roughly

$$k \, N_0(R,k) \approx \ln \frac{R^2}{c_2 Ra^{\frac{1}{2}}} \tag{32}$$

which is independent of k, n.

There are two byproducts of the inequality (27)

$$\frac{1}{N} \sum_{n=0}^{N} \|\nabla \theta_k^{n+1}\|^2 \leq \frac{\|\theta_0\|^2}{kN} + c_2 Ra^{\frac{1}{2}}, \tag{33}$$

$$\sum_{n=0}^{N} \|\theta_k^{n+1} - \theta_k^n\|^2 \leq 2\|\theta_0\|^2 + 2c_2 kN Ra^{\frac{1}{2}}. \tag{34}$$

The first is a bound in $L^2(H^1)$ for the scheme and the second is a bound on the difference of the solution at adjacent time steps.

Our next immediate goal is to obtain uniform estimates in H^1 for the solution of the scheme (20). The uniform in H^1 estimate will guarantee the uniform dissipativity (H1) since boundedness in H^1 implies pre-compactness in L^2 by Rellich compactness theorem. For this purpose, we take the inner product of the scheme with $\psi = -\Delta \theta_k^{n+1}$ and we have

$$
\begin{aligned}
& \frac{1}{2k}(\|\nabla \theta_k^{n+1}\|^2 - \|\nabla \theta_k^n\|^2 + \|\nabla(\theta_k^{n+1} - \theta_k^n)\|^2) + \|\Delta \theta_k^{n+1}\|^2 \\
\leq\ & \|\tau''\|\|\Delta \theta_k^{n+1}\| + Ra\|\tau'\|\|A^{-1}(\mathbf{k}\theta_k^n)\|_\infty\|\Delta \theta_k^{n+1}\| + Ra\|A^{-1}(\mathbf{k}\theta_k^n)\|_\infty\|\nabla \theta_k^{n+1}\|\|\Delta \theta_k^{n+1}\| \\
\leq\ & \|\tau''\|\|\Delta \theta_k^{n+1}\| + c_3 Ra(\|\tau'\|\|\theta_k^n\|\|\Delta \theta_k^{n+1}\| + \|\theta_k^n\|\|\theta_k^{n+1}\|^{\frac{1}{2}}\|\Delta \theta_k^{n+1}\|^{\frac{3}{2}}) \\
\leq\ & c_4(1 + (1+k)^{-n}\|\theta_0\|^2)^{\frac{1}{2}}\|\Delta \theta_k^{n+1}\| + c_5(1 + (1+k)^{-n}\|\theta_0\|^2)^{\frac{3}{2}}\|\Delta \theta_k^{n+1}\|^{\frac{3}{2}}) \\
\leq\ & \frac{1}{2}\|\Delta \theta_k^{n+1}\|^2 + c_6(1 + (1+k)^{-n}\|\theta_0\|^2)^6 \tag{35}
\end{aligned}
$$

where we have applied the regularity result for the Stokes operator A, the Sobolev imbedding of H^2 into L^∞, interpolation inequality, the uniform L^2 estimate (29) and Hölder type inequality.

This implies, when combined with (30, 33), there exists $N_1(R,k) \geq N_0(R,k)$, and $c_7 \geq 2c_2 Ra^{\frac{1}{2}}$, such that

$$\|\nabla \theta_k^{n+1}\|^2 - \|\nabla \theta_k^n\|^2 + \|\nabla(\theta_k^{n+1} - \theta_k^n)\|^2 + k\|\Delta \theta_k^{n+1}\|^2 \leq kc_7, \forall n \geq N_1 \tag{36}$$

$$\|\nabla \theta_k^{N_1}\|^2 \leq c_7, \tag{37}$$

where N_1 can be estimated as

$$N_1(R,k) \approx N_0 + \frac{2}{k}.$$

This implies the following uniform H^1 estimate with initial data from $B_R(L^2)$

$$\|\nabla\theta_k^{n+1}\|^2 \le c_7, \ \forall n \ge N_1. \tag{38}$$

This further leads to the existence of an absorbing ball in H^1 of radius $\sqrt{c_7}$ such that trajectories starting from $B_R(L^2)$ enter this absorbing ball in H^1 within approximately $N_1 \approx N_0 + \frac{2}{k}$ steps. This ends the proof of the uniform dissipativity of the approximations (6) by the compact imbedding of H^1 into L^2. This uniform bound also implies the existence of a compact global attractor $\mathcal{A}_k \subset H^1_{0,per}$ for the scheme for each time step $k \le 8$ and

$$\sup_{\theta \in K} \|\nabla\theta\| \le \sqrt{c_7}.$$

There are another three consequences of (35) which are an $L^2(H^2)$ estimate on the solution, an estimate on the difference of the solution in H^1 norm at adjacent steps, and a uniform in time H^1 norm estimate for initial data in H^1 .

$$\frac{1}{N}\sum_{n=0}^{N}\|\Delta\theta_k^{n+1}\|^2 \ \le \ \frac{\|\nabla\theta_0\|^2}{kN} + 2c_6(1+R^2)^6, \tag{39}$$

$$\sum_{n=0}^{N}\|\nabla(\theta_k^{n+1} - \theta_k^n)\|^2 \ \le \ \|\nabla\theta_0\|^2 + 2c_6(1+R^2)^6 kN, \tag{40}$$

$$\|\nabla\theta_k^{n+1}\|^2 \ \le \ (1+k)^{-n+1}\|\nabla\theta_0\|^2 + 2c_6(1+R^2)^6. \tag{41}$$

These estimates will be useful in our uniform convergence proof.

Uniform estimates in Sobolev spaces of higher orders can be derived just as in the case of continuous in time system. Here we sketch the proof that the H^2 norm of the solution is asymptotically uniformly bounded in time, i.e., there is an absorbing ball in H^2 which attracts all solutions with L^2 initial data uniformly for all k.

For this purpose we apply Δ to both sides of the scheme (20) and then multiply the scheme by $\Delta\theta_k^{n+1}$ and integrate over the domain. This leads to the following

$$\frac{1}{2k}(\|\Delta\theta_k^{n+1}\|^2 - \|\Delta\theta_k^n\|^2 + \|\Delta(\theta_k^{n+1} - \theta_k^n)\|^2) + \|\nabla\Delta\theta_k^{n+1}\|^2$$

$$\le \ \|\tau^{(4)}\|\|\Delta\theta_k^{n+1}\| + Ra(\|\Delta(A^{-1}(\mathbf{k}\theta_k^n))\|_{L^6}\|\nabla\theta_k^{n+1}\|_{L^3} + 2\|\nabla A^{-1}(\mathbf{k}\theta_k^n)\|_{L^\infty}\|\nabla^2\theta_k^{n+1}\|)\|\Delta\theta_k^{n+1}\|$$

$$+ Ra(\|\Delta(A^{-1}(\mathbf{k}\theta_k^n))\|\|\tau'\|_{L^\infty} + 2\|\nabla(A^{-1}(\mathbf{k}\theta_k^n))\|\|\nabla\tau'\|_{L^\infty} + \|A^{-1}(\mathbf{k}\theta_k^n)\|\|\Delta\tau'\|_{L^\infty})\|\Delta\theta_k^{n+1}\|$$

$$\le \ c_8(\|\Delta\theta_k^{n+1}\| + \|\Delta\theta_k^{n+1}\|^2)$$

$$\le \ c_9(\|\Delta\theta_k^{n+1}\| + \|\nabla\Delta\theta_k^{n+1}\|\|\nabla\theta_k^{n+1}\|)$$

$$\le \ \frac{1}{2}\|\nabla\Delta\theta_k^{n+1}\|^2 + c_{10}$$

where we have applied the identify $\int_\Omega A^{-1}(\mathbf{k}\theta_k^n)\nabla\Delta\theta_k^{n+1}\Delta\theta_k^{n+1} = 0$, the fact that $\Delta\theta_k^{n+1} = 0$ at $z = 0, 1$ (which follows from (20, 22)), Hölder's inequality, elliptic regularity, Sobolev imbedding, Cauchy-Schwarz, interpolation inequality, and the H^1 uniform estimate (38).

This leads to uniform H^2 estimate.

To summarize, we have the following result

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Proposition 2 (Uniform dissipativity). *There exists a constant c_{10} independent of the time step k such that the scheme (20) possess an absorbing ball in H^1 and H^2 with radius $2\sqrt{c_{10}}$ which attracts all bounded sets in L^2. In particular, we have*

$$\sup_{\mathbf{u}\in K}\|\mathbf{u}\|_{H^2} \leq 2\sqrt{c_{10}} \tag{42}$$

where K is the union of the global attractors for the scheme with different time step k.

3.3 Finite time uniform convergence

We now verify the second condition (H2) in the main theorem. In fact, we will prove slightly more general version which covers both H2 and H5, i.e., the convergence of the numerical scheme (20) on any finite time interval uniformly for initial data from $K = \bigcup \mathcal{A}_k$ (which is uniformly bounded in H^1 thanks to (38)).

For this purpose, we rewrite the scheme (20) as

$$\frac{\partial\tilde{\theta}_k(t)}{\partial t} + Ra\, A^{-1}(\mathbf{k}\theta_k(t))\cdot\nabla\theta_k(t+k) + Ra\, A^{-1}(\mathbf{k}\theta_k(t))_3\tau'(z) = \Delta\theta_k(t+k) + \tau''(z), \quad (43)$$

where

$$\theta_k(t) \;=\; \theta_k^n,\; t\in[nk,(n+1)k), \tag{44}$$

$$\tilde{\theta}_k(t) \;=\; \theta_k^n + \frac{t-nk}{k}(\theta_k^{n+1} - \theta_k^n),\; t\in[nk,(n+1)k). \tag{45}$$

For any fixed $T^* > 0$, the estimates (41, 39) imply that θ_k and $\tilde{\theta}_k$ are uniformly (in k) bounded in $L^\infty(0,T^*;H^1)$ and $L^2(0,T^*;H^2)$ and the bounds are uniform in k and $\theta_0 \in K$.

We can rephrase the differential form of the scheme as

$$\frac{\partial\tilde{\theta}_k(t)}{\partial t} + Ra\, A^{-1}(\mathbf{k}\tilde{\theta}_k(t))\cdot\nabla\tilde{\theta}_k(t) + Ra\, A^{-1}(\mathbf{k}\tilde{\theta}_k(t))_3\tau'(z) = \Delta\tilde{\theta}_k(t) + \tau''(z) + f_k(t), \quad (46)$$

where

$$\begin{aligned}
f_k(t) \;=\; & Ra\, A^{-1}(\mathbf{k}\tilde{\theta}_k(t))\cdot\nabla\tilde{\theta}_k(t) - Ra\, A^{-1}(\mathbf{k}\theta_k(t))\cdot\nabla\theta_k(t+k) \\
& + Ra\, A^{-1}(\mathbf{k}(\tilde{\theta}_k(t) - \theta_k(t)))_3\tau'(z) - \Delta(\tilde{\theta}_k(t) - \theta_k(t+k)).
\end{aligned} \tag{47}$$

It is easy to see that f_k is small due to our estimate on the difference of the approximate solution at adjacent time steps (40). Indeed, we have, for $t\in[nk,(n+1)k)$

$$\tilde{\theta}_k(t) - \theta_k(t+k) \;=\; \frac{t-(n+1)k}{k}(\theta_k^{n+1} - \theta_k^n), \tag{48}$$

$$\tilde{\theta}_k(t) - \theta_k(t) \;=\; \frac{t-nk}{k}(\theta_k^{n+1} - \theta_k^n) \tag{49}$$

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and therefore, for $\theta_0 \in K$,

$$
\begin{aligned}
Ra\|A^{-1}(\mathbf{k}\tilde{\theta}_k(t)) \cdot \nabla(\tilde{\theta}_k(t) - \theta_k(t+k))\|_{H^{-1}} &\leq Ra\|A^{-1}(\mathbf{k}\tilde{\theta}_k(t))\|_{L^\infty}\|\theta_k^{n+1} - \theta_k^n\| \\
&\leq c_{11}\|\theta_k^{n+1} - \theta_k^n\|, \\
Ra\|A^{-1}(\mathbf{k}(\tilde{\theta}_k(t) - \theta_k(t))) \cdot \nabla\theta_k(t+k)\|_{H^{-1}} &\leq Ra\|A^{-1}(\mathbf{k}(\tilde{\theta}_k(t) - \theta_k(t)))\|_{L^\infty}\|\theta_k(t+k)\| \\
&\leq c_{12}\|\theta_k^{n+1} - \theta_k^n\|, \\
Ra\|A^{-1}(\mathbf{k}(\tilde{\theta}_k(t) - \theta_k(t))_3\tau'\|_{H^{-1}} &\leq c_{13}\|\theta_k^{n+1} - \theta_k^n\|, \\
\|\Delta(\tilde{\theta}_k(t) - \theta_k(t+k))\|_{H^{-1}} &\leq c_{14}\|\nabla(\theta_k^{n+1} - \theta_k^n)\|,
\end{aligned}
$$

which further leads to

$$
\|f_k(t)\|_{H^{-1}} \leq c_{15}\|\nabla(\theta_k^{n+1} - \theta_k^n)\|, t \in [nk, (n+1)k) \tag{50}
$$

and hence, when combined with the estimate on time difference estimate (40)

$$
\|f_k\|_{L^2(0,T^*;H^{-1})} \leq c_{15}\sqrt{\sum_{n=0}^{\frac{T^*}{k}} k\|\nabla(\theta_k^{n+1} - \theta_k^n)\|^2} \leq c_{16}\sqrt{k}. \tag{51}
$$

Taking the difference of the infinite Prandtl number model (21) and the differential form of the scheme (46), denoting $\xi_k(t) = \theta(t) - \tilde{\theta}_k(t)$, we have

$$
\frac{\partial \xi_k(t)}{\partial t} + Ra\,A^{-1}(\mathbf{k}\theta(t))\cdot\nabla\xi_k(t) + Ra\,A^{-1}(\mathbf{k}\xi_k(t))\cdot\nabla\tilde{\theta}_k(t) + Ra\,A^{-1}(\mathbf{k}\xi_k(t))_3\tau'(z) = \Delta\xi_k(t) - f_k(t). \tag{52}
$$

Multiplying this equation by ξ_k and integrating over Ω we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\|\xi_k(t)\|^2 + \|\nabla\xi_k(t)\|^2 \\
&\leq Ra\|A^{-1}(\mathbf{k}\xi_k(t))\|_{L^\infty}\|\nabla\tilde{\theta}_k(t)\|\|\xi_k(t)\| + Ra\|A^{-1}(\mathbf{k}\xi_k(t))_3\|_{L^\infty}\|\tau'\|\|\xi_k(t)\| + \|f_k(t)\|_{H^{-1}}\|\nabla\xi_k(t)\| \\
&\leq c_{17}\|\xi_k(t)\|^2 + \frac{1}{2}\|f_k(t)\|^2_{H^{-1}} + \frac{1}{2}\|\nabla\xi_k(t)\|^2.
\end{aligned}
$$

Therefore we have

$$
\frac{d}{dt}\|\xi_k(t)\|^2 \leq 2c_{17}\|\xi_k(t)\|^2 + \|f_k(t)\|^2_{H^{-1}}, \quad \|\xi_k(0)\| = 0, \tag{53}
$$

which leads to

$$
\|\theta - \tilde{\theta}_k\|_{L^\infty(0,T^*;L^2)} = \|\xi_k\|_{L^\infty(0,T^*;L^2)} \leq c_{18}\|f_k\|_{L^2(0,T^*;H^{-1})} \leq c_{19}\sqrt{k} \to 0 \tag{54}
$$

uniformly for $\theta_0 \in K$.

This ends the proof of the finite time uniform convergence which further implies H2 and H5. To summarize, we have the following result

Proposition 3 (Finite time uniform convergence)**.** *For any $T^* > 0$, there exists a constant c_{19} independent of the time step k such that*

$$
\|\theta(nk) - \theta_k^n\| \leq c_{19}\sqrt{k}, \forall\theta_0 \in K, \forall nk \leq T^*, \tag{55}
$$

i.e., assumptions H2 (7) and H5 (14) are valid for the scheme (20) with $t_0 = 0$.

3.4 Finite time uniform continuity

Now we verify the finite time uniform continuity of the infinite Prandtl number model for initial data starting from K, the union of the global attractors of the scheme with different step size k.

It is easy to check that the L^2 norm of the infinite Prandtl number model (21) is uniformly bounded for θ_0 belonging to a bounded set in L^2 (see for instance [7], and the discrete version (29)). A uniform H^1 norm estimate can be also derived (see (38) for the discrete version). Indeed, multiplying the infinite Prandtl number model (21) by $-\Delta\theta$, integrating over Ω we deduce, for $\theta_0 \in K$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2 + \|\Delta\theta(t)\|^2 \\
\leq\ & \|\tau"\|\|\Delta\theta(t)\| + Ra\|A^{-1}(\mathbf{k}\theta(t))\|_{L^\infty}\|\nabla\theta(t)\|\|\Delta\theta(t)\| + Ra\|A^{-1}(\mathbf{k}\theta(t))_3\|_{L^\infty}\|\tau'\|\|\Delta\theta(t)\| \\
\leq\ & c_{20}(1 + \|\theta(t)\| + \|\theta(t)\|\|\nabla\theta(t)\|)\|\Delta\theta(t)\| \\
\leq\ & \frac{1}{2}\|\Delta\theta(t)\|^2 + c_{21}(1 + \|\nabla\theta(t)\|^2).
\end{aligned}
$$

This leads to the following estimates

$$
\begin{aligned}
\|\theta\|_{L^\infty(0,T^*;H^1)} &\leq c_{22}, & (56) \\
\|\theta\|_{L^2(0,T^*;H^2)} &\leq c_{22}. & (57)
\end{aligned}
$$

Now, integrating the infinite Prandtl number model (21) in time from t to T^* we have, for $\theta_0 \in K$,

$$
\begin{aligned}
&\|S(t)\theta_0 - S(T^*)\theta_0\| \\
\leq\ & \left|\int_t^{T^*} (\|\Delta\theta(s)\| + Ra\|A^{-1}(\mathbf{k}\theta(s))\|_{L^\infty}\|\nabla\theta(s)\| + Ra\|A^{-1}(\mathbf{k}\theta(s))_3\|_{L^\infty}\|\tau'\| + \|\tau"\|)\,ds\right| \\
\leq\ & \left|\int_t^{T^*} (\|\Delta\theta(s)\| + c_{23}(1 + \|\theta(s)\|\|\nabla\theta(s)\| + \|\theta(s)\|))\,ds\right| \\
\leq\ & \left|\int_t^{T^*} (\|\Delta\theta(s)\| + c_{24})\,ds\right| \\
\leq\ & c_{25}\sqrt{|T^* - t|} & (58)
\end{aligned}
$$

where we have used elliptic regularity, the uniform H^1 estimate, and the $L^2(H^2)$ estimate.

This completes the proof of the uniform continuity of the infinite Prandtl number model, i.e., H3. To summarize, we have the following result

Proposition 4 (Finite time uniform continuity). *For any $T^* > 0$, the infinite Prandtl number model (21) is continuous on the time interval $[0, T^*]$ uniformly for initial data from the union of the global attractors K defined in (6).*

4 Conclusions and Remarks

We have presented a general/abstract result on the convergence of stationary statistical properties of time approximation of infinite dimensional dissipative dynamical systems. The three natural conditions that guarantee the convergence of the stationary statistical properties of the time approximation to those of the underlying continuous in time dynamical system are uniform dissipativity of the scheme for small enough time step; the convergence of the solution to the scheme to the continuous dynamical system on the unit time interval $[0, 1]$ uniformly for initial data from the union of the global attractors of the scheme (for different time steps); and the uniform (for initial data from the union of the global attractors of the scheme) continuity of the underlying continuous dynamical system on the unit time interval $[0, 1]$. We hope that this work will stimulate further work on numerical schemes that are able to capture stationary statistical properties of infinite dimensional dissipative systems.

The convergence of global attractors can be derived under the weaker assumption of having a bounded union of the global attractors for the scheme. This indicates that there might exist numerical schemes that are able to capture the global attractor asymptotically but not necessarily the stationary statistical properties.

We have also illustrated the application of our main result to the infinite Prandtl number model for convection and we believe that the abstract main result presented here is applicable to many other dissipative systems and associated schemes. Fully discretized approximation can be studied similarly for Galerkin type spatial approximation that enjoys the three properties postulated (see [25, 26, 10] for fully discrete long time stable schemes for other equations). Numerical implementation in physically relevant regimes is non-trivial. Although the numerical scheme that we proposed here is linear which is advantageous over nonlinear schemes such as the one induced by the fully implicit Euler scheme proposed by many authors, the linear equation (the matrix in the fully discretized case) changes at each time step due to the presence of the convection term $Ra\, A^{-1}(\mathbf{k}\theta_k^n) \cdot \nabla\theta_k^{n+1}$. This together with the need for long time simulation combined with the presence of physically expected small spatial scales of the order of $Ra^{-\frac{1}{3}}$ or $Ra^{-\frac{1}{2}}$ [2, 6, 18] which needs to be resolved makes it a challenge to simulate the physically interesting large Rayleigh number regime. One of the immediately goal is to design more efficient numerical scheme than the one presented here. We will report this and our numerical result elsewhere.

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