MALLIAVIN CALCULUS FOR LEVY MARKETS AND NEW SENSITIVITIES

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Abstract. We present a method to apply the Malliavin calculus to calculate sensitivities for exponential Levy models built from the Variance Gamma and Normal Inverse Gaussian processes. We also present new sensitivities for these processes. The calculation of the sensitivities is based on a finite dimensional Malliavin calculus and we compare the results with finite difference calculations. This is done using Monte Carlo methods. For European call and digital options we compare the simulation results with exact calculation of sensitivities using Fourier transform methods. The Malliavin method outperforms the finite difference method especially when payoff has serious discontinuities.

Key words. Mallivin calculus, Variance Gamma process, Normal Inverse Gaussian Process, sensitivity analysis, inverse Fourier transform method.

AMS subject classifications. 60H07, 91G60, 91G80.

1. Introduction. The sensitivity analysis of options are as important as the pricing in option theory since they are used for hedging strategies, hence for risk management purposes. In general, these sensitivities are restricted within the Black-Scholes' model parameter set and called as *the Greeks*. In this paper, we will introduce new sensitivities for the options whose underlying is driven by exponential Lévy processes, specifically Variance Gamma and Normal Inverse Gaussian processes. Then, we will show that how Malliavin calculus can be used in order to calculate the Greeks and new sensitivities that we introduce. We will give explicit formulas that can be used in Monte Carlo simulations, directly.

The sensitivities are calculated by using the finite difference method if the pricing has to be done by Monte-Carlo simulation. However, when the payoff function is not smooth enough the finite difference method may perform poorly. Malliavin calculus introduces an additional term, H^1, in the expectation while removing the derivative operator. To be precise let us consider the delta of a European call option, i.e.,

$$\frac{\partial}{\partial S_0} E[\phi(S_T)|\mathcal{F}_0] = E[\phi(S_T)H(S_T, \frac{S_T}{S_0})|\mathcal{F}_0].$$

When ϕ is the payoff function of a digital option, Malliavin weight "smooths" the expectation. Hence, the expectation of the resulting expression is calculated efficiently by Monte-Carlo simulation. We also use this approach when the derivative is taken with respect to the parameters of NIG and VG distributions after introducing variations of Malliavin weight by using new localization methods together with the one used in [12]. Our results show that Malliavin approach performs better in terms of speed of convergence as we calculate the sensitivities of European call and digital options written on S&P 500 index when the index is modelled by an exponential NIG process, [3] as opposed to an exponential VG process, [16]. In order to get the risk neutral parameters of the NIG and VG distributions we calibrated the option models to the price of S&P 500 European put/call options' market data. We used fast Fourier transform method during the calibration process as discussed in [8]. A similar idea is used for the sensitivity calculations of the options. At this point, a natural question

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[1]We will call this term as Malliavin weight throughout the paper.

arises. Why do we apply Malliavin calculus if we can calculate the sensitivities via fast Fourier transform method? For most of the path dependent financial contracts such as Asian options, look-back options, etc., a closed form solution through the characteristic function may not exist. Hence, the Malliavin approach can be effectively used in sensitivity calculation whether the payoff function is smooth or not, as it reduces the number of path generation in the finite difference method.

We will now explain the history that has brought us to this point in our research. Following the pioneering papers [12] and [13] the use of Malliavin calculus in sensitivity calculations has been extended to the models with Lévy processes from the models of lognormal type diffusion processes. In [6] following the results in [12], findings of the optimal Malliavin weight in sensitivity calculations under the lognormal diffusion models are presented. Results that are related to the finite dimensional Wiener space are given in [15]. Lately, application of Malliavin calculus realized after the extensive use of market models where the underlying is modelled by a jump diffusion or in general by a Lévy process. Some of the important applications are discussed in [10], [23] and [21], [4]. In [19], a generalized approach for Lévy processes is introduced by using the fact that Lévy processes can be decomposed into Wiener process and a Poisson random measure part. However, this assumption fails for the Variance Gamma (VG) and Normal Inverse Gaussian (NIG) processes since their decomposition only involves the Poisson random measure. In [5], we used the compound Poisson approximation of VG process and gave closed form solutions for the deltas' of call and digital options by using Malliavin calculus in a finite dimensional space introduced in [4]. This paper provides a method which is more efficient than the approximation technique as we presented in [5]. Sensitivity calculations of options that are related to the distributional parameters of the Lévy processes have not been considered in the literature. In this paper, we will introduce new sensitivity parameters for an exponential Lévy market model by considering effects of the distributional parameters of VG and NIG processes.

The paper is organized as follows. We introduce, in Section 2, the related properties of VG and NIG processes and present the formulas for sensitivities of European call and digital options, that are obtained by using the inverse Fouriér transform. Also, we introduce the new sensitivities and their surface plots with respect to moneyness and time to expiry of the options. In Section 3, the fundamentals of Malliavin calculus for simple functionals in the frame of [4] are discussed. We generalize their results in terms of the computation of the Malliavin weight. Then, in Section 4, we give the detailed calculations of Malliavin weight for different sensitivities including new ones that we introduce and the examples of *Greeks* such as delta, gamma, and rho. We also formalize the weight so that it can be used in simulations directly. Section 5 is devoted to new localization methods that we introduce together with the variations of weights that are obtained in Section 4. In the last section we introduce the option data that we use in our simulations and in the calibration of option pricing models, and we present the results of the simulations in terms of figures that show the comparison of the performance of the different approaches in calculation of sensitivities via Monte-Carlo simulations.

2. Calculating the Greeks by Inverse Fourier Transform Method. In this section we review the approach that is introduced in [8] for European style option pricing by using Fourier transform method. We use this method to calculate the sensitivities of European style call and digital options. Before we discuss the details in the chapter we introduce the characteristic functions for different market models.

Let X_t be a subordinated Brownian motion and $S_t = S_0 e^{(r-q+w)t+X_t}$, $t \geq 0$ be the risk neutral price process of the underlying with $w = -\log(\phi_{X_T}(-i))$. Here $Y_T = \log(S_T) = \log(S_0) + (r-q+w)T + X_T$ is the log-price of the underlying at time T with the risk neutral density $\rho_T(y)$.

If X_t is a Variance Gamma process, it can be written as the subordination of a Brownian motion by Gamma process as follows:

$$X_t = \theta G_t + \sigma W_{G_t}, \tag{2.1}$$

where θ and σ are the drift and the volatility of the Brownian motion, respectively. The parameter κ is the variance of the subordinator [8]. Using the same parametrization, its Lévy measure is given by

$$\nu(dx) = \frac{1}{\kappa|x|} e^{Ax - B|x|} dx, \tag{2.2}$$

where $A = \frac{\theta}{\sigma^2}$ and $B = \frac{\sqrt{\theta^2 + \sigma^2/\kappa}}{\sigma^2}$. Hence, the characteristic function of Y_T is

$$\phi^{VG}_{Y_T}(u) = \frac{e^{iu[\log(S_0)+(r-q+w)T]}}{(1 - iu\theta\kappa + \frac{\sigma^2\kappa u^2}{2})^{T/\kappa}}, \tag{2.3}$$

where $w = \dfrac{1}{\kappa}\ln(1 - \theta\kappa - \sigma^2\dfrac{\kappa}{2})$.

If X_t is an NIG process, it can be written as the subordination of a Brownian motion by Inverse Gaussian process as follows:

$$X_t = \beta\delta^2 IG_t + \delta W_{IG_t}, \tag{2.4}$$

where $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$. The density of the Inverse Gaussian, $IG(a,b)$, law is given by $f(x; a, b) = \dfrac{a}{\sqrt{2\pi}} \exp(ab) x^{-3/2} \exp(-\dfrac{1}{2}(a^2 x^{-1}) + b^2 x))$, $x > 0$. The variance of the $IG(a,b)$ distributed random variable is a/b^3. In above parametrization, $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$. Using the same parametrization, its Lévy measure is given by

$$\nu_{NIG}(dx) = \frac{\delta\alpha}{\pi} \frac{\exp(\beta x) K_1(\alpha|x|)}{|x|} dx, \tag{2.5}$$

where $K_\lambda(x)$ is the modified Bessel function of the third kind with index λ, [2] and the characteristic function of Y_T is given by

$$\phi^{NIG}_{Y_T}(u) = e^{iu[\log(S_0)+(r-q+w)T] - \delta T(\sqrt{\alpha^2-(\beta+iu)^2} - \sqrt{\alpha^2-\beta^2})}, \tag{2.6}$$

where $w = \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2})$. The (α, β, δ) parametrization of the Lévy measure is commonly used in the literature,[3], [20], [22]. Thus, we will exercise all of the calculations related to NIG with respect to this parametrization throughout this paper. However, this parametrization does not lead to a Brownian subordination in terms of the drift, volatility and variance of the subordinator as we have for VG process. In [9] a similar parametrization of the NIG process is given. The Lévy measure of the NIG process is given by

$$\nu_{NIG}(dx) = \frac{C}{|x|} e^{Ax} K_1(B|x|) dx, \tag{2.7}$$

where $A = \frac{\theta}{\sigma^2}$, $B = \frac{\sqrt{\theta^2+\sigma^2/\kappa}}{\sigma^2}$, $C = \frac{\theta^2+\sigma^2/\kappa}{\pi\sigma\sqrt{\kappa}}$ and $K_\lambda(x)$ is the modified Bessel function of the third kind with index λ. Therefore, using the same parametrization, an NIG process X_t is given by the following subordination of a Brownian motion by Inverse Gaussian process as follows:

$$X_t = \theta IG_t + \sigma W_{IG_t}, \tag{2.8}$$

where θ and σ are the drift and the volatility of the Brownian motion, respectively. The parameter κ is the variance of the subordinator. Since the options's prices are invariant under these parametrizations i.e.,

$$C(\alpha,\beta,\delta) = C(\theta(\alpha,\beta,\delta),\sigma(\alpha,\beta,\delta),\kappa(\alpha,\beta,\delta)) \tag{2.9}$$

we write following relation between the derivatives of an option's price with respect to the paramters defining the Lévy measures given in (2.5) and (2.7).

$$\begin{aligned}
\frac{\partial C}{\partial \alpha} &= \frac{\partial C}{\partial \theta}\frac{\partial \theta}{\partial \alpha} + \frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\alpha} + \frac{\partial C}{\partial \kappa}\frac{\partial \kappa}{\partial \alpha}\\
\frac{\partial C}{\partial \beta} &= \frac{\partial C}{\partial \theta}\frac{\partial \theta}{\partial \beta} + \frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\partial \beta} + \frac{\partial C}{\partial \kappa}\frac{\partial \kappa}{\partial \beta}\\
\frac{\partial C}{\partial \delta} &= \frac{\partial C}{\partial \theta}\frac{\partial \theta}{\partial \delta} + \frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\partial \delta} + \frac{\partial C}{\partial \kappa}\frac{\partial \kappa}{\partial \delta}.
\end{aligned} \tag{2.10}$$

Also,

$$\begin{vmatrix} \frac{\partial \theta}{\partial \alpha} & \frac{\partial \sigma}{\partial \alpha} & \frac{\partial \kappa}{\partial \alpha}\\ \frac{\partial \theta}{\partial \beta} & \frac{\partial \sigma}{\partial \beta} & \frac{\partial \kappa}{\partial \beta}\\ \frac{\partial \theta}{\partial \delta} & \frac{\partial \sigma}{\partial \delta} & \frac{\partial \kappa}{\partial \delta} \end{vmatrix} = \frac{-\alpha}{\sqrt{\delta}(\alpha^2-\beta^2)^{7/4}} \neq 0. \tag{2.11}$$

This enables us to calculate the sensitivities in terms of the $\{\theta,\sigma,\kappa\}$ parametrization in NIG case as well.

2.1. Sensitivities of Call and Digital Options. Let S_t be the price of the underlying at time t. If $Y_T = \log(S_T)$ is the log-price of the underlying at time T with the risk neutral density $\rho_T(y)$, then the characteristic function, $\phi_{Y_T}(u)$, of Y_T is defined by the Fourier transform of the density function as follows:

$$\phi_{Y_T}(u) = \int_{-\infty}^{\infty} e^{iuy}\rho_T(y)dy.$$

Thus, the price of a European call option is given by

$$C_T(k) = \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{\phi_{Y_T}(\vartheta(\nu))}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}d\nu, \tag{2.12}$$

where $\vartheta(\nu) = \nu-(\alpha+1)i$. In [8] the details of this result are discussed and since then it has been used in the literature extensively. In (2.12), the parameter α is called the dumping parameter, k is the log of the strike price, r is the constant interest rate and T is the time to maturity. Following the same idea, we give price function of an European digital option[2] as follows:

$$D_T(k) = \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty \frac{e^{-i\nu k}}{\alpha+i\nu}\phi_{Y_T}(\nu-\alpha i)d\nu. \tag{2.13}$$

[2]The pay-off function of the digital option is $\phi(S_T) = \mathbb{1}_{\{S_T>K\}}$.

2.1.1. Call Option Sensitivities: Delta, Gamma, Rho, Theta. Delta is the sensitivity of the option's price with respect to the initial value of the underlying asset. It measures the change in option price with respect to a small change in S_0. Thus, it is calculated by taking the partial derivative of the option price with respect to S_0.

$$\begin{aligned}\Delta_{\text{Call}}^{\text{VG, NIG}}(k) &= \frac{\partial C_T(k)}{\partial S_0} \\ &= \frac{e^{-(\alpha k+rT)}}{S_0\pi}\int_0^\infty e^{-i\nu k}\frac{\phi_{Y_T}^{\text{VG, NIG}}(\vartheta(\nu))}{\alpha+i\nu}d\nu. \end{aligned} \tag{2.14}$$

In Figures 2.1(a), 2.1(c) and 2.1(e) the surface plots of delta for BS, VG and NIG models, respectively are given. It can be seen from these figures that for deep out of the money options delta gets close to zero, while it approaches one for the deep in the money options. The delta of the deep in the money and the deep out of the money options, with the longer maturity, tend to approach zero and unity slower than the ones with shorter maturity, respectively. In other words, a \$1 change in the price of the underlying increases the price of the in the money option that is close to maturity more than the one that has more time to expiry.

Gamma is the sensitivity of the option's delta with respect to the initial value of the underlying asset. It measures the change in delta with respect to the small change in S_0.

$$\begin{aligned}\Gamma_{\text{Call}}^{\text{VG, NIG}}(k) &= \frac{\partial \Delta_{\text{Call}}}{\partial S_0} \\ &= \frac{e^{-(\alpha k+rT)}}{S_0^2\pi}\int_0^\infty e^{-i\nu k}\phi_{Y_T}^{\text{VG, NIG}}(\vartheta(\nu))d\nu. \end{aligned} \tag{2.15}$$

It is important for the hedging strategies that are used when the options are traded. Together with delta, gamma is commonly used for so called delta-gamma hedging, [7]. Figures 2.1(b), 2.1(d) and 2.1(f) shows how gamma changes with time and moneyness. Gamma has its peak for the at the money options. As moneyness of the option deviates from ATM, gamma becomes smaller. For deep in the money and the deep out of the money options gamma is larger for those with more time to expiry. This is reasonable since as the option has more time to expiry the delta of the option has more time to vary.

Rho is the sensitivity of the option with respect to the interest rate, r. It measures how much the option's price changes with the small changes in the interest rate.

$$\begin{aligned}\rho_{\text{Call}}^{\text{VG, NIG}}(k) &= \frac{\partial C_T(k)}{\partial r} \\ &= \frac{Te^{-(\alpha k+rT)}}{\pi}\int_0^\infty \frac{e^{-i\nu k}}{\alpha+1+\nu i}\phi_{Y_T}^{\text{VG, NIG}}(\vartheta(\nu))d\nu. \end{aligned} \tag{2.16}$$

It tends to approach to zero faster as the option becomes out of the money and increases with time to maturity as it is seen in Figures 2.2(a), 2.2(c) and 2.2(e).

Theta is the sensitivity of the option with respect to time to maturity, T. Except for deep in the money options, Θ is negative for all models. The reason for that is the positive dividend as it is seen in Figures 2.2(b), 2.2(d) and 2.2(f). As long as $q > 0$, Θ gets positive values as strike price approaches zero. Also, theta becomes smaller in absolute value as the option has more time to expiry until it becomes positive.

Theta is obtained by taking the partial derivative of the option price with respect to T. Hence,

$$\begin{aligned}\Theta^{\mathrm{VG}}_{\mathrm{Call}}(k) &= \frac{\partial C_T(k)}{\partial T}\\ &= \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\left\{\frac{(\alpha+i\nu)r+(\alpha+1+i\nu)(w-q)}{(\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu)}\right.\\ &\quad-\left.\frac{\frac{1}{\kappa}\log\left(1-(\alpha+1+i\nu)\theta\kappa-\frac{\sigma^2}{2}\kappa(\alpha+1+i\nu)^2\right)}{(\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu)}\right\}\phi^{\mathrm{VG}}_{Y_T}(\vartheta(\nu))d\nu\end{aligned} \tag{2.17}$$

and under NIG model

$$\begin{aligned}\Theta^{\mathrm{NIG}}_{\mathrm{Call}}(k) &= \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\left\{\frac{(\alpha+i\nu)r+(\alpha+1+i\nu)(w-q)}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}\right.\\ &\quad-\left.\frac{\delta\left(\sqrt{\alpha_{nig}^2-(\beta+\alpha+1+i\nu)^2}-\sqrt{\alpha_{nig}^2-\beta^2}\right)}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}\right\}\phi^{\mathrm{NIG}}_{Y_T}(\vartheta(\nu))d\nu.\end{aligned} \tag{2.18}$$

2.1.2. Call Option Sensitivities: Drift, Vega1, Vega2. Before we introduce these sensitivities we give the partial derivatives of the call option with respect to the parameters $\{\alpha,\beta,\delta\}$ when the underlying is modelled by an exponential NIG process.

$$\frac{\partial C_T(k)}{\partial\beta} = \frac{Te^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{(\alpha+1+i\nu)\tilde{\mathcal{U}}(-i)-\tilde{\mathcal{U}}(\vartheta(\nu))}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}\phi^{\mathrm{NIG}}_{Y_T}(\vartheta(\nu))d\nu, \tag{2.19}$$

where $\tilde{\mathcal{U}}(y)=\delta\left(\frac{-(\beta+iy)}{\sqrt{\alpha_{nig}^2-(\beta+iy)^2}}-\frac{-\beta}{\sqrt{\alpha_{nig}^2-\beta^2}}\right)$.

$$\frac{\partial C_T(k)}{\partial\delta} = \frac{Te^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{(\alpha+1+i\nu)\mathcal{U}(-i)-\mathcal{U}(\vartheta(\nu))}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}\phi^{\mathrm{NIG}}_{Y_T}(\vartheta(\nu))d\nu, \tag{2.20}$$

where $\mathcal{U}(y)=\sqrt{\alpha_{nig}^2-(\beta+iy)^2}-\sqrt{\alpha_{nig}^2-\beta^2}$.

$$\frac{\partial C_T(k)}{\partial\alpha_{nig}} = \frac{Te^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{(\alpha+1+i\nu)\hat{\mathcal{U}}(-i)-\hat{\mathcal{U}}(\vartheta(\nu))}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}\phi^{\mathrm{NIG}}_{Y_T}(\vartheta(\nu))d\nu, \tag{2.21}$$

where $\hat{\mathcal{U}}(y)=\delta\alpha_{nig}\left(\frac{1}{\sqrt{\alpha_{nig}^2-(\beta+iy)^2}}-\frac{1}{\sqrt{\alpha_{nig}^2-\beta^2}}\right)$.

Drift is a new sensitivity that we introduce as the sensitivity of the option with respect to the parameter θ of the Variance Gamma and Normal Inverse Gaussian processes.

$$\begin{aligned}\mathcal{D}^{\mathrm{VG}}_{\mathrm{Call}}(k) &= \frac{\partial C_T(k)}{\partial\theta}\\ &= \frac{-Te^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{\mathcal{U}(-i)-\mathcal{U}(\vartheta(\nu))}{\alpha+i\nu}\phi^{\mathrm{VG}}_{Y_T}(\vartheta(\nu))d\nu,\end{aligned} \tag{2.22}$$

where $\mathcal{U}(y) = 1/(1 - iy\theta\kappa + \frac{\sigma^2}{2}\kappa y^2)$. $\mathcal{D}^{\text{NIG}}_{\text{Call}}(k)$ is given by the solution of the linear system given in (2.10). For the counter parties of an option trade, the effect of the skewness of the underlying asset in the option price is important. When θ is zero the distributions of VG and NIG processes are symmetric. Thus, as θ deviates from zero the skewness of the distribution of the underlying asset changes. This is an important fact for the option pricing. For instance, positive skewness implies higher values of the underlying. Hence, the call option is going to be more expensive. In Figures 2.3(a) and 2.3(b), we observe that drift tends to decrease as maturity increases. Thus, for a positive change in θ, the value of the option decreases. We obtained negative values after we calibrated VG and NIG models on date August 06, 2008. Thus, since θ is negative, the positive change in the absolute value of theta will result in more negatively skewed distribution. Hence, it is more likely that the value of the underlying becomes less than the strike price and this causes a decline in the price of the option. Despite this general view, in Figure 2.3(b) for very short term options we observe that the drift has positive value for the near the money options with $K/S_0 > 1$. When θ is positive both figures get positive values. However, drift in NIG model becomes negative for very short term options that are near the money with $K/S_0 < 1$.

Vega is another new sensitivity that we introduce as the sensitivity of the option with respect to the parameter σ of the VG and NIG processes. It measures the change in the option's price with respect to the small changes in σ. When we compare the Figures 2.3(c), 2.3(d) and 2.4 we see that vegas of the NIG and BS models are close to each other. Vega of the VG model, however, has much smaller values.

$$
\begin{aligned}
\mathcal{V}^{\text{VG}}_{\text{Call}}(k) &= \frac{\partial C_T(k)}{\partial \sigma} \\
&= \frac{-\sigma T e^{-(\alpha k + rT)}}{\pi} \int_0^\infty e^{-i\nu k} \left\{ \frac{\mathcal{U}(-i) - (\alpha + 1 + i\nu)\mathcal{U}(\vartheta(\nu))}{\alpha + i\nu} \right\} \phi^{\text{VG}}_{Y_T}(\vartheta(\nu)) d\nu
\end{aligned}
\tag{2.23}
$$

Even though we use the Black-Scholes name for this sensitivity it is slightly different. In the Black-Scholes model σ is the only source of the volatility. However, in the VG and NIG processes, σ is a partial source of the total volatility. The other source of the volatility is κ. Thus, we define another sensitivity of the option with respect to the parameter κ of the VG and NIG processes.

Vega$_2$ measures the change in the option's price with the small changes in the parameter κ, the variance of the subordinator.

$$
\begin{aligned}
\mathcal{V}2^{\text{VG}}_{\text{Call}}(k) &= \frac{\partial C_T(k)}{\partial \kappa} \\
&= \frac{T e^{-(\alpha k + rT)}}{\pi} \int_0^\infty e^{-i\nu k} \left\{ \frac{(\alpha + 1 + i\nu)\mathcal{G}(-i) - \mathcal{G}(\vartheta(\nu))}{(\alpha + i\nu)(\alpha + 1 + i\nu)} \right\} \phi^{\text{VG}}_{Y_T}(\vartheta(\nu)) d\nu,
\end{aligned}
\tag{2.24}
$$

where $\mathcal{G}(y) = \frac{(-i\theta y + \frac{\sigma^2 y^2}{2})\kappa - \tilde{\mathcal{U}}(y)\log(\tilde{\mathcal{U}}(y))}{\kappa^2 \tilde{\mathcal{U}}(y)}$ with $\tilde{\mathcal{U}}(y) = \dfrac{1}{\mathcal{U}(y)}$. $\mathcal{V}2^{\text{NIG}}_{\text{Call}}(k)$ is given by the solution of the linear system given in (2.10). In Figures 2.3(e), 2.3(f), Vega$_2$ gets smaller for out of the money and in the money options and it increases with the time to expiry. However, Vega$_2$ of NIG model takes negative values that increase in absolute value as near and at the money options get close to expiry. This implies that as the variance of time increases in the arrival of the new information the option price decreases as it gets close to expiry.

2.1.3. Digital Option Sensitivities. We calculate the sensitivities of a digital option by taking the partial derivatives of the price function given in Equation (2.13) with respect to the related parameter.

$$\begin{aligned}\Delta^{\text{VG, NIG}}_{\text{Digital}}(k) &= \frac{\partial D_T(k)}{\partial S_0}\\ &= \frac{e^{-(\alpha k+rT)}}{S_0\pi}\int_0^\infty e^{-i\nu k}\phi^{\text{VG, NIG}}_{Y_T}(\nu-\alpha i)d\nu.\end{aligned} \tag{2.25}$$

$$\begin{aligned}\Gamma^{\text{VG, NIG}}_{\text{Digital}}(k) &= \frac{\partial \Delta_{\text{Digital}}}{\partial S_0}\\ &= \frac{e^{-(\alpha k+rT)}}{S_0^2\pi}\int_0^\infty e^{-i\nu k}(\alpha+i\nu-1)\phi^{\text{VG, NIG}}_{Y_T}(\nu-\alpha i)d\nu.\end{aligned} \tag{2.26}$$

$$\begin{aligned}\rho^{\text{VG, NIG}}_{\text{Digital}}(k) &= \frac{\partial D_T(k)}{\partial r}\\ &= \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}T\frac{(\alpha+i\nu-1)\phi^{\text{VG, NIG}}_{Y_T}(\nu-\alpha i)}{\alpha+i\nu}d\nu.\end{aligned} \tag{2.27}$$

$$\begin{aligned}\Theta^{\text{VG}}_{\text{Digital}}(k) &= \frac{\partial D_T(k)}{\partial T}\\ &= \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\left\{\frac{(\alpha+i\nu-1)r+(\alpha+i\nu)w}{\alpha+i\nu}\right.\\ &\quad\left.-\frac{\frac{1}{\kappa}\log\left(1-(\alpha+i\nu)\theta\kappa-\frac{\sigma^2}{2}\kappa(\alpha+i\nu)^2\right)}{\alpha+i\nu}\right\}\phi^{\text{VG}}_{Y_T}(\nu-\alpha i)d\nu.\end{aligned} \tag{2.28}$$

$$\begin{aligned}\Theta^{\text{NIG}}_{\text{Digital}}(k) &= \frac{\partial D_T(k)}{\partial T}\\ &= \frac{e^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\left\{\frac{(\alpha+i\nu-1)r+(\alpha+i\nu)w}{\alpha+i\nu}\right.\\ &\quad\left.-\frac{\delta\left(\sqrt{\alpha_{nig}^2-(\beta+\alpha+i\nu)^2}-\sqrt{\alpha_{nig}^2-\beta^2}\right)}{\alpha+i\nu}\right\}\phi^{\text{NIG}}_{Y_T}(\nu-\alpha i)d\nu.\end{aligned} \tag{2.29}$$

$$\begin{aligned}\mathcal{D}^{\text{VG}}_{\text{Digital}}(k) &= \frac{\partial D_T(k)}{\partial \theta}\\ &= \frac{-Te^{-(\alpha k+rT)}}{\pi}\int_0^\infty e^{-i\nu k}\left(\mathcal{U}(-i)-\mathcal{U}(\nu-\alpha i)\right)\phi^{\text{VG}}_{Y_T}(\nu-\alpha i)d\nu.\end{aligned} \tag{2.30}$$

where $\mathcal{U}(y)=1/(1-iy\theta\kappa+\frac{\sigma^2}{2}\kappa y^2)$.

$$\begin{aligned}\mathcal{V}^{\mathrm{VG}}_{\mathrm{Digital}}(k) &= \frac{\partial D_T(k)}{\partial \sigma} \qquad (2.31)\\ &= \frac{-\sigma T e^{-(\alpha k + rT)}}{\pi}\int_0^\infty e^{-i\nu k}(\mathcal{U}(-i) - (\alpha + i\nu)\mathcal{U}(\nu - \alpha i))\phi^{\mathrm{VG}}_{Y_T}(\nu - \alpha i)d\nu.\end{aligned}$$

$$\begin{aligned}\mathcal{V}2^{\mathrm{VG}}_{\mathrm{Digital}}(k) &= \frac{\partial D_T(k)}{\partial \kappa} \qquad (2.32)\\ &= \frac{T e^{-(\alpha k + rT)}}{\pi}\int_0^\infty e^{-i\nu k}\left\{\frac{(\alpha + i\nu)\mathcal{G}(-i) - \mathcal{G}(\nu - \alpha i)}{(\alpha + i\nu)}\right\}\phi^{\mathrm{VG}}_{Y_T}(\nu - \alpha i)d\nu,\end{aligned}$$

where $\mathcal{G}(y) = \frac{(-i\theta y + \frac{\sigma^2 y^2}{2})\kappa - \tilde{\mathcal{U}}(y)\log(\tilde{\mathcal{U}}(y))}{\kappa^2 \tilde{\mathcal{U}}(y)}$ with $\tilde{\mathcal{U}}(y) = \dfrac{1}{\mathcal{U}(y)}$.

$$\frac{\partial D_T(k)}{\partial \beta} = \frac{T e^{-(\alpha k + rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{(\alpha + i\nu)\tilde{\mathcal{U}}(-i) - \tilde{\mathcal{U}}(\nu - \alpha i)}{\alpha + i\nu}\phi^{\mathrm{NIG}}_{Y_T}(\nu - \alpha i)d\nu \qquad (2.33)$$

where $\tilde{\mathcal{U}}(y) = \delta\left(\frac{-(\beta + iy)}{\sqrt{\alpha_{nig}^2 - (\beta + iy)^2}} - \frac{-\beta}{\sqrt{\alpha_{nig}^2 - \beta^2}}\right)$.

$$\frac{\partial D_T(k)}{\partial \delta} = \frac{T e^{-(\alpha k + rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{(\alpha + i\nu)\mathcal{U}(-i) - \mathcal{U}(\nu - \alpha i)}{\alpha + i\nu}\phi^{\mathrm{NIG}}_{Y_T}(\nu - \alpha i)d\nu \qquad (2.34)$$

where $\mathcal{U}(y) = \sqrt{\alpha_{nig}^2 - (\beta + iy)^2} - \sqrt{\alpha_{nig}^2 - \beta^2}$.

$$\frac{\partial D_T(k)}{\partial \alpha_{nig}} = \frac{T e^{-(\alpha k + rT)}}{\pi}\int_0^\infty e^{-i\nu k}\frac{(\alpha + i\nu)\hat{\mathcal{U}}(-i) - \hat{\mathcal{U}}(\nu - \alpha i)}{\alpha + i\nu}\phi^{\mathrm{NIG}}_{Y_T}(\nu - \alpha i)d\nu \qquad (2.35)$$

where $\hat{\mathcal{U}}(y) = \delta\alpha_{nig}\left(\frac{1}{\sqrt{\alpha_{nig}^2 - (\beta + iy)^2}} - \frac{1}{\sqrt{\alpha_{nig}^2 - \beta^2}}\right)$.

3. Malliavin Calculus for Simple Functionals. In this section we will give the definitions and theorems of finite dimensional Malliavin calculus. The notations that we use follow from [4]. However, we generalize definitions and theorems for k such that $1 \leq k \leq n$ and for $k = n$ we obtain the results of [4]. Consider a sequence of independent random variables $(V_n, n \in \mathbb{N}^\star)$ on a probability space $(\Omega, \mathcal{F}, P)$ such that for all $n \geq 1$, V_n has moments of any order. We assume that V_n is absolutely continuous with respect to the Lebesgue measure and has density ρ_n which is continuously differentiable on $\mathbb{R}$ and such that $\forall m \in \mathbb{N}$, $\lim_{y \to \pm\infty} |y|^m \rho_n(y) = 0$. We also assume that $\dfrac{\partial_y \rho_n(y)}{\rho_n(y)}$ has at most polynomial growth. For $m \geq 1$, we denote by $f \in \mathcal{C}^m_\uparrow(\mathbb{R}^n)$ the space of the functions $f : \mathbb{R}^n \to \mathbb{R}$ which are m times differentiable and such that f and its derivatives up to order m have at most polynomial growth.

A random variable F is called a simple functional if there exists some $n \in \mathbb{N}$ and some measurable function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$F = f(V_1, \ldots, V_n).$$

We denote by $S_{(n,m)}$ the space of the simple functionals such that $f \in \mathcal{C}_{\uparrow}^{m}(\mathbb{R}^n)$. A simple process of length k is a sequence of random variables $U = (U_i)_{i\leq k}$, $k \leq n$ such that

$$U_i = u_i(V_1, \dots, V_n).$$

We will also denote by $P_{n,m}^{k}$ the space of k-length simple processes.

DEFINITION 3.1 (Inner Product). *Let $U = (U_i)_{i\leq k}$ and $\tilde{U} = (\tilde{U}_j)_{j\leq k}$ be two k-length simple processes in $P_{(n,1)}^{k}$ then*

$$\langle U, \tilde{U} \rangle = \sum_{p=1}^{k} U_p \tilde{U}_p$$

is called the inner product of U and $\tilde{U}$.

DEFINITION 3.2 (Malliavin Derivative). *The k-length simple process $D^k : S_{(n,1)} \to P_{(n,0)}^{k}$, $k \leq n$ is called the Malliavin derivative operator and it is defined by $D^k F = (D_i F)_{i\leq k}$, where $F = f(V_1, \dots, V_n) \in S_{(n,1)}$ and $D_i^k F = \partial_i f(V_1, \dots, V_n)$, $i \leq k$.*

DEFINITION 3.3 (Skorohod Integral). *$\delta^k : P_{(n,0)}^{k} \to S_{(n,1)}$, $k \leq n$ is called the Skorohod integral operator and is defined for any k-length simple process $U \in P_{(n,0)}^{k}$ such that*

$$\delta^k(U) = -\sum_{i=1}^{k} [D_i U_i + \theta_i(V_i) U_i],$$

where

$$\theta_i(y) = \partial_y \ln[\rho_i(y)] = \begin{cases} \frac{\rho_i'(y)}{\rho_i(y)}, & \text{if } \rho_i(y) \rangle 0 \\ 0, & \text{if } \rho_i(y) = 0 \end{cases}$$

PROPOSITION 3.4 (Duality Formula). *Let $F \in S_{(n,1)}$ and $U \in P_{(n,0)}^{k}$, then*

$$E[\langle D^k F, U \rangle] = E[F \delta^k(U)].$$

Proof.

$$\begin{aligned} E[\langle D^k F, U \rangle] &= E[\sum_{i=1}^{k} D_i F\, U_i] \\ &= E[\sum_{i=1}^{k} (u_i \partial_i f)(V_1, \dots, V_n)] \\ &= \int_{\mathbb{R}^n} \left[\sum_{i=1}^{k} (u_i \partial_i f)(y_1, \dots, y_n) \right] \rho_1(y_1) \dots \rho_n(y_n) dy_1 \dots dy_n \end{aligned}$$

Now, we apply the regular integration by parts formula of Calculus for each i.
Let

$$\begin{aligned} \alpha_i &= u_i(y_1, \dots, y_n)\rho_i(y_i) & d\beta_i &= \partial_i f(y_1, \dots, y_n) dy_i \\ d\alpha_i &= \left[\partial_i u_i(y_1, \dots, y_n) + u_i(y_1, \dots, y_n) \frac{\rho_i'(y_i)}{\rho_i(y_i)} \right] \rho_i(y_i) dy_i & \beta_i &= f(y_1, \dots, y_n) \end{aligned}$$

and we continue from above,

$$
\begin{aligned}
&= \sum_{i=1}^{k} \int_{\mathbb{R}^{n-1}} [f(y_1,\dots,y_n)u_i(y_1,\dots,y_n)\rho_i(y_i)|_{\mathbb{R}}] \prod_{j=1,j\neq i}^{n} \rho_j(y_j)dy_j \\
&- \int_{\mathbb{R}^n} f(y_1,\dots,y_n) \sum_{i=1}^{k} \left[\partial_i u_i(y_1,\dots,y_n) + u_i(y_1,\dots,y_n)\frac{\rho_i'(y_i)}{\rho_i(y_i)}\right] \prod_{j=1}^{n} \rho_j(y_j)dy_j \\
&= \int_{\mathbb{R}^n} f(y_1,\dots,y_n)\delta^k(U(y_1,\dots,y_n)) \prod_{j=1}^{n} \rho_j(y_j)dy_j \\
&= E[F\delta^k(U)].
\end{aligned}
$$

□

DEFINITION 3.5 (Ornstein-Uhlenbeck Operator). *$L^k : S_{(n,2)} \to S_{(n,0)}$ is called the Ornstein-Uhlenbeck operator and is defined by*

$$
\begin{aligned}
L^k F &= -\sum_{i=1}^{k} L_i F = -\sum_{i=1}^{k} D_i D_i F + \theta_i D_i F \\
&= -\sum_{i=1}^{k} (\partial^2_{ii} f)(V_1,\dots,V_n) + \theta_i(V_i)(\partial_i f)(V_1,\dots,V_n)
\end{aligned}
$$

REMARK 3.1.

i. *Orntein-Uhlenbeck operator can be considered as the Skorohod integral operator applied to Malliavin derivative operator.*
ii. $E[F\,L^k G] = E[G\,L^k F]$
Proof. $E[F\,L^k G] = E[F\,\delta^k(D^k G)] = E[\langle D^k F, D^K G\rangle] = E[\delta^k(D^k F)\,G] = E[G\,L^k F]$, *where we used the duality formula in the third equality.* □
iii. $L^k(F\,G) = FL^k(G) + GL^k(F) - 2\langle D^k F, D^k G\rangle$
Proof. *We use the definition of Ornstein-Uhlenbeck operator.*

$$
\begin{aligned}
L^k(F\,G) &= -\sum_{i=1}^{k} L_i(F\,G) = -\sum_{i=1}^{k} D_i D_i(F\,G) + \theta_i(V_i) D_i(F\,G) \\
&= -\sum_{i=1}^{k} \{D_i[F\,D_iG + G\,D_iF] + \theta_i(V_i)[F\,D_iG + G\,D_iF]\} \\
&= -\sum_{i=1}^{k} \{D_iFD_iG + FD_iD_iG + D_iGD_iF + GD_iD_iF + \\
&+ \theta_i(V_i)F\,D_iG + \theta_i(V_i)G\,D_iF\} \\
&= -\sum_{i=1}^{k} \{F(D_iD_iG + \theta_i(V_i)\,D_iG) + G(D_iD_iF \\
&+ \theta_i(V_i)\,D_iF) + 2D_iFD_iG\} \\
&= FL^k(G) + GL^k(F) - 2\langle D^k F, D^k G\rangle \qquad \square
\end{aligned}
$$

DEFINITION 3.6 (Malliavin Covariance Matrix). *Let $F = (F_1, F_2, \dots, F_d)$ be an d-dimensional vector of simple functionals such that $F_i \in S_{(n,1)}$. The matrix $M^k_\sigma(F)$*

is called the Malliavin covariance matrix of F whose entries are given by

$$M_\sigma^k(F)_{ij} = \langle DF_i, DF_j\rangle = \sum_{l=1}^{k} \partial_l f_i \partial_l f_j(V_1,\ldots,V_n),$$

where $F_i = f_i(V_1,\ldots,V_n)$.

THEOREM 3.7 (Integration by Parts). *Let $F = (F_1, F_2, \ldots, F_d) \in S^d_{(n,2)}$ and $G \in S_{(n,1)}$. We assume that the $M_\sigma^k(F)$ is invertible and denote $M_\gamma^k(F) = \left[M_\sigma^k(F)\right]^{-1}$. We also assume that $E[detM_\gamma^k(F)]^4 < \infty$. Then for every smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$*

$$E[\partial_i\phi(F)G] = E[\phi(F)H_i^k(F,G)], \tag{3.1}$$

where

$$H_i^k(F,G) = \sum_{i=1}^{d} G\, M^k_{\gamma_{ji}}(F)\, L^k F - M^k_{\gamma_{ji}}(F)\langle D^k F, D^k G\rangle - G\langle D^k F, D^k M^k_{\gamma_{ji}}(F)\rangle \tag{3.2}$$

with $E[H^i(F,G)] < \infty$. *Proof.* Using the chain rule we write

$$\begin{aligned}
\langle D^k\phi(F), D^k F_j\rangle &= \sum_{p=1}^{k} D_p\phi(F)\, D_p F_j \\
&= \sum_{p=1}^{k}\sum_{i=1}^{d} \partial_i\phi(F) D_p F_i\, D_p F_j \\
&= \sum_{i=1}^{d} \partial_i\phi(F) \sum_{p=1}^{k} D_p F_i\, D_p F_j \\
&= \sum_{i=1}^{d} \partial_i\phi(F) M^k_{\sigma_{ij}}(F)
\end{aligned}$$

Thus, we obtain $\partial_i\phi(F) = \sum_{j=1}^{d} \langle D^k\phi(F), D^k F_j\rangle M^k_{\gamma_{ji}}(F)$. By Remark 1.6.(iii) we have,

$$\langle D^k\phi(F), D^k F_j\rangle = \frac{1}{2}\left[-L^k(\phi(F)F_j) + \phi(F)L^k F_j + F_j L^k(\phi(F))\right].$$

Therefore,

$$E[\partial_i\phi(F)G] = \frac{1}{2}E\left[\left(\sum_{j=1}^{d}\left[-L^k(\phi(F)F_j) + \phi(F)L^k F_j + F_j L^k(\phi(F))\right] M^k_{\gamma_{ji}}(F)\right) G\right]$$

When we apply Remark 1.6(ii) to the first and third terms in the summation by considering G and $M_\gamma^k(F)_{ji}$ in the multiplication, on the right hand side we get

$$= \frac{1}{2}E\left[\sum_{j=1}^{d}\left[-\phi(F)F_j L^k(GM^k_{\gamma_{ji}}(F)) + \phi(F)GM^k_{\gamma_{ji}}(F)L^k F_j + \phi(F)L^k(F_j M^k_{\gamma_{ji}}(F)G)\right]\right].$$

Again, by applying Remark 1.6.(iii) to the last term in the above sum, we obtain

$$= \frac{1}{2}E\left[\sum_{j=1}^{d}\phi(F)\Big[GM^k_{\gamma_{ji}}(F)L^kF_j - F_jL^k(GM^k_{\gamma_{ji}}(F)) + F_jL^k(M^k_{\gamma_{ji}}(F)G)\right.$$

$$\left. +\ M^k_{\gamma_{ji}}(F)GL^k(F_j) - 2\langle D^kF_j, D^k(M^k_{\gamma_{ji}}(F)G)\rangle\Big]\right]$$

$$= \frac{1}{2}E\left[\phi(F)\sum_{j=1}^{d}\Big[2GM^k_{\gamma_{ji}}(F)L^kF_j - 2\langle D^kF_j, D^k(M^k_{\gamma_{ji}}(F)G)\rangle\Big]\right]$$

By applying Remark 1.6(iv) to the last term of the above sum we conclude as

$$= E\left[\phi(F)\sum_{j=1}^{d}GM^k_{\gamma_{ji}}(F)L^kF_j - M^k_{\gamma_{ji}}(F)\langle D^kF_j, D^k(G)\rangle - G\langle D^kF_j, D^k(M^k_{\gamma_{ji}}(F))\rangle\right].$$

□

4. Applications of Malliavin Calculus. Throughout this section we will use a discretization scheme of the process. As a result, we use the following time grid

$$0 = t_0 \le t_1 \le \ldots \le t_{n-1} \le t_n = T.$$

Therefore, the continuous underlying process takes the following structure

$$S_T = S_0 e^{(r-q+w)T + \sum_{i=1}^{n} A\sqrt{\mathcal{S}_{t_i} - \mathcal{S}_{t_{i-1}}}Z_i + B\sum_{i=1}^{n}\mathcal{S}_{t_i} - \mathcal{S}_{t_{i-1}}}, \tag{4.1}$$

where $\mathcal{S}_t$ is the subordinating process. Since, digital and call options' payoff functions depend on the final value of the underlying, the Malliavin weights for them are the same. In the next sections we use the integration by parts formula that we obtain in Theorem (3.7). Thus, the Malliavin weight for 1-dimension ($d = 1$) is

$$H^k(F,G) = G\,M^k_\gamma(F)\,L^kF - M^k_\gamma(F) < D^kF, D^kG > -G < D^kF, D^kM^k_\gamma(F) >, \tag{4.2}$$

for $1 \le k \le n$.

The arbitrage free option price is given by $u(x) = e^{-rT} = e^{-rT}E[\phi(S_T)]$ with respect to the sensitivity parameter x. In the next sections we give the details of the calculations of the Malliavin weights for the Greeks when $x = S_0$, $x = r$ or $x = \sigma$, etc. Also, in the following subsections,

$$F = f(x_1, \ldots, x_n, y_1, \ldots, y_n) = S_0 \exp\left((r - q + w)T + \sum_{i=1}^{n} A\sqrt{y_i}x_i + B\sum_{i=1}^{n} y_i\right),$$

and $G_x = \frac{\partial S_T}{\partial x}$.

4.1. VG Model. In VG model S_T is given by

$$S_T = S_0 e^{(r-q+w)T + \sum_{i=1}^{n}\sigma\sqrt{\Delta G_i}Z_i + \theta\sum_{i=1}^{n}\Delta G_i}, \tag{4.3}$$

where G_t is the gamma process used to subordinate the Wiener process and $\Delta G_i = G_{t_i} - G_{t_{i-1}}$. Also, we define the following random set $RS = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) : (x_1, \ldots, x_n, y_1, \ldots, y_n) = (Z_1, \ldots, Z_n, \Delta G_1, \ldots, \Delta G_n)\}$.

4.1.1. Greeks: Δ.

$$\Delta = e^{-rT}\frac{\partial}{\partial S_0}E[\phi(S_T)] = e^{-rT}E[\phi'(S_T)\frac{S_T}{S_0}] = E[\phi(S_T)H^k_\Delta(S_T,\frac{S_T}{S_0})].$$

In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $S_T = F = f(Z_1,\dots,Z_n,\Delta G_1,\dots,\Delta G_n)$.
2. $D_jF = \frac{\partial f}{\partial x_j}|_{RS} = \sigma\sqrt{\Delta G_j}S_T$, $\forall j$ such that $1\le j\le k$.
3. $D_{jj}F = \frac{\partial^2 f}{\partial x_j^2}|_{RS} = \sigma^2\Delta G_j S_T$, $\forall j$ such that $1\le j\le k$.
4. $G_{S_0} = \frac{\partial S_T}{\partial S_0} = \frac{\partial f}{\partial S_0}|_{RS} = \frac{S_T}{S_0}$.
5. $D_jG_{S_0} = \sigma\sqrt{\Delta G_j}\frac{S_T}{S_0}$, $\forall j$ such that $1\le j\le k$.
6. $\Theta_j(x_j) = \frac{d\log(p_j(x_j))}{dx_j} = -x_j$, where $p_j(x_j) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x_j^2}{2}}$, $\forall j$ such that $1\le j\le k$.
7. $L_jF = D_{jj}F + \Theta_j(Z_j)D_jF = \sigma^2\Delta G_jS_T - Z_j\sigma\sqrt{\Delta G_j}S_T$, $\forall j$ such that $1\le j\le k$.
8. $L^k = -\sum_{i=1}^k D_{jj}F + \Theta_j(Z_j)D_jF = -S_T\sum_{i=1}^k\sigma^2\Delta G_j - Z_j\sigma\sqrt{\Delta G_j}$, for $1\le k\le n$.
9. $M^k_\sigma(F) = \sum_{i=1}^k\sigma^2\Delta G_iS_T^2 > 0$, for $1\le k\le n$.
10. $D_jM^k_\sigma(F) = \sigma\sqrt{\Delta G_j}S_T2S_T\sum_{i=1}^k\sigma^2\Delta G_i = 2\sigma\sqrt{\Delta G_j}\sum_{i=1}^k\sigma^2\Delta G_iS_T^2$
$= 2\sigma\sqrt{\Delta G_j}M^k_\sigma(F)$, $\forall j$ such that $1\le j\le k$.
11. $M^k_\gamma(F) = \dfrac{1}{M^k_\sigma(F)}$, for $1\le k\le n$.
12. $D_jM^k_\gamma(F) = -\dfrac{D_jM^k_\sigma(F)}{(M^k_\sigma(F))^2} = -\dfrac{2\sigma\sqrt{\Delta G_j}M^k_\sigma(F)}{(M^k_\sigma(F))^2} = -\dfrac{2\sigma\sqrt{\Delta G_j}}{M^k_\sigma(F)}$

Malliavin weight follows from the above calculations:

$$\begin{aligned}
H^k_\Delta(F,G_{S_0}) &= \frac{S_T}{S_0}M^k_\gamma(F)\left(-S_T\sum_{i=1}^k\left(\sigma^2\Delta G_j - Z_j\sigma\sqrt{\Delta G_j}\right)\right)\\
&- M^k_\gamma(F)\sum_{j=1}^k\sigma\sqrt{\Delta G_j}S_T\sigma\sqrt{\Delta G_j}\frac{S_T}{S_0} - \frac{S_T}{S_0}\sum_{j=1}^k\sigma\sqrt{\Delta G_j}S_T\frac{(-2\sigma\sqrt{\Delta G_j})}{M^k_\sigma(F)}\\
&= \frac{-1}{S_0}M^k_\gamma(F)\left(\sum_{i=1}^k\sigma^2\Delta G_jS_T^2 - S_T^2\sum_{j=1}^kZ_j\sigma\sqrt{\Delta G_j}\right)\\
&- \frac{1}{S_0}M^k_\gamma(F)\sum_{j=1}^k\sigma^2\Delta G_jS_T^2 + \frac{2}{S_0}M^k_\gamma(F)\sum_{j=1}^k\sigma^2\Delta G_jS_T^2\\
&= \frac{-1}{S_0}M^k_\gamma(F)\left(M^k_\sigma(F) - S_T^2\sum_{j=1}^kZ_j\sigma\sqrt{\Delta G_j}\right)\\
&- \frac{1}{S_0}M^k_\gamma(F)M^k_\sigma(F) + \frac{2}{S_0}M^k_\gamma(F)M^k_\sigma(F)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{S_0} + \frac{M_\gamma^k(F)}{S_0} S_T^2 \sum_{j=1}^{k} Z_j \sigma \sqrt{\Delta G_j} - \frac{1}{S_0} + \frac{2}{S_0} \\
&= \frac{M_\gamma^k(F)}{S_0} S_T^2 \sum_{j=1}^{k} Z_j \sigma \sqrt{\Delta G_j} \\
&= \frac{S_T^2 \sum_{j=1}^{k} Z_j \sigma \sqrt{\Delta G_j}}{S_0 \sum_{i=1}^{k} \sigma^2 \Delta G_i S_T^2} \\
&= \frac{\sum_{j=1}^{k} Z_j \sqrt{\Delta G_j}}{S_0 \sum_{i=1}^{k} \sigma \Delta G_i}. \qquad (4.4)
\end{aligned}
$$

Therefore, delta of a call or digital option is written via the Malliavin weight as

$$
\Delta = e^{-rT} E[\phi(S_T) \frac{\sum_{j=1}^{k} Z_j \sqrt{\Delta G_j}}{S_0 \sum_{i=1}^{k} \sigma \Delta G_i}]. \qquad (4.5)
$$

4.1.2. Greeks: Γ. We express the weight for gamma in terms of the weight of delta.

$$
\begin{aligned}
\Gamma &= e^{-rT} \frac{\partial^2}{\partial S_0^2} E[\phi(S_T)] = e^{-rT} \frac{\partial}{\partial S_0} E[\phi'(S_T) \frac{S_T}{S_0}] = e^{-rT} \frac{\partial}{\partial S_0} E[\phi(S_T) H_\Delta^k(S_T, \frac{S_T}{S_0})] \\
&= e^{-rT} E[\phi'(S_T) \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] + E[\phi(S_T) \frac{\partial}{\partial S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] \qquad (4.6) \\
&= e^{-rT} E[\phi(S_T) H^k(S_T, \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0}))] + e^{-rT} E[\phi(S_T) \frac{\partial}{\partial S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] \qquad (4.7)
\end{aligned}
$$

After taking the derivative in the last term, gamma is given as follows.

$$
\begin{aligned}
\Gamma &= e^{-rT} E[\phi(S_T) H^k(S_T, \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0}))] - e^{-rT} E[\phi(S_T) \frac{1}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] \\
&= e^{-rT} E[\phi(S_T) \left(H^k(S_T, \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})) - \frac{1}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0}) \right)]. \qquad (4.8)
\end{aligned}
$$

We use the integration by parts formula in order to remove the derivative in the first term of Equation (4.6). Since $F = S_T$, we use the F-related calculations that are done for delta. Thus, we deal with a shorter list of calculations for gamma. In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $G_\Gamma = \frac{S_T}{S_0} H_\Delta^k(F, G_{S_0})$.
2. $D_j G_\Gamma = \frac{\sigma\sqrt{\Delta G_j} S_T}{S_0} H_\Delta^k(F, G_{S_0}) + \frac{S_T}{S_0^2} \frac{\sqrt{\Delta G_j}}{\sum_{i=1}^{k} \sigma \Delta G_i}$, $\forall j$ such that $1 \le j \le k$.
3.

$$
\begin{aligned}
&G_\Gamma M_\gamma^k(F) L^k F \\
&\qquad = \frac{S_T}{S_0} H_\Delta^k(F, G_{S_0}) M_\gamma^k(F) \left(-S_T \sum_{j=1}^{k} \left(\sigma^2 \Delta G_j - Z_j \sigma \sqrt{\Delta G_j} \right) \right) \\
&\qquad = H_\Delta^k(F, G_{S_0}) \left(-M_\gamma^k(F) \sum_{j=1}^{k} \frac{S_T^2}{S_0} \sigma^2 \Delta G_j + \frac{\sum_{j=1}^{k} \frac{S_T^2}{S_0} Z_j \sigma \sqrt{\Delta G_j}}{\sum_{i=1}^{k} \sigma^2 \Delta G_i S_T^2} \right)
\end{aligned}
$$

$$= H^k_\Delta(F,G_{S_0})\left(-M^k_\gamma(F)M^k_\sigma(F) + \frac{\sum_{j=1}^k \frac{S_T^2}{S_0} Z_j\sigma\sqrt{\Delta G_j}}{\sum_{i=1}^k \sigma^2\Delta G_i S_T^2}\right)$$

$$= H^k_\Delta(F,G_{S_0})\left(-\frac{1}{S_0} + H^k_\Delta(F,G_{S_0})\right) \tag{4.9}$$

4. Recall that $M^k_\gamma(F)M^k_\sigma(F) = 1$.

$$\begin{aligned} &M^k_\gamma(F)\langle DF, DG_\Gamma\rangle_k \\ &= M^k_\gamma(F)\sum_{j=1}^k \sigma\sqrt{\Delta G_j}S_T\left[\frac{\sigma\sqrt{\Delta G_j}S_T}{S_0}H^k_\Delta(F,G_{S_0}) + \frac{S_T}{S_0^2}\frac{\sqrt{\Delta G_j}}{\sum_{i=1}^k\sigma\Delta G_i}\right] \\ &= \frac{H^k_\Delta(F,G_{S_0})M^k_\gamma(F)}{S_0}\sum_{j=1}^k\sigma^2\Delta G_j S_T^2 + \frac{M^k_\gamma(F)S_T^2}{S_0^2}\frac{\sum_{j=1}^k\sigma\Delta G_j}{\sum_{i=1}^k\sigma\Delta G_i} \\ &= \frac{H^k_\Delta(F,G_{S_0})M^k_\gamma(F)}{S_0}M^k_\sigma(F) + \frac{M^k_\gamma(F)S_T^2}{S_0^2}\frac{\sum_{j=1}^k\sigma\Delta G_j}{\sum_{i=1}^k\sigma\Delta G_i} \\ &= \frac{H^k_\Delta(F,G_{S_0})}{S_0} + \frac{M^k_\gamma(F)S_T^2}{S_0^2}. \end{aligned} \tag{4.10}$$

5.

$$\begin{aligned} &G_\Gamma\langle DF, DM^k_\gamma(F)\rangle_k \\ &= -\frac{S_T}{S_0}H^k_\Delta(F,G_{S_0})\sum_{j=1}^k\sigma\sqrt{\Delta G_j}S_T\frac{2\sigma\sqrt{\Delta G_j}}{M^k_\sigma(F)} \\ &= -\frac{2}{S_0}H^k_\Delta(F,G_{S_0})\sum_{j=1}^k\sigma^2\Delta G_j S_T^2 M^k_\gamma(F) \\ &= -\frac{2}{S_0}H^k_\Delta(F,G_{S_0})M^k_\sigma(F)M^k_\gamma(F) \\ &= -\frac{2}{S_0}H^k_\Delta(F,G_{S_0}). \end{aligned} \tag{4.11}$$

Thus,

$$\begin{aligned} H^k(F,G_\Gamma) &= (4.9) - (4.10) - (4.11) \\ &= H^k_\Delta(F,G_{S_0})\left(-\frac{1}{S_0} + H^k_\Delta(F,G_{S_0})\right) \\ &\quad -\frac{H^k_\Delta(F,G_{S_0})}{S_0} - \frac{M^k_\gamma(F)S_T^2}{S_0^2} + \frac{2}{S_0}H^k_\Delta(F,G_{S_0}) \\ &= (H^k_\Delta(F,G_{S_0}))^2 - \frac{M^k_\gamma(F)S_T^2}{S_0^2}. \end{aligned} \tag{4.12}$$

This implies

$$H^k_\Gamma(F,G_\Gamma) = (H^k_\Delta(F,G_{S_0}))^2 - \frac{M^k_\gamma(F)S_T^2}{S_0^2} - \frac{1}{S_0}H^k_\Delta(S_T,\frac{S_T}{S_0}). \tag{4.13}$$

Therefore,

$$\Gamma = e^{-rT}E[\phi(S_T)\left((H^k_\Delta(S_T,\frac{S_T}{S_0}))^2 - \frac{M^k_\gamma(F)S_T^2}{S_0^2} - \frac{1}{S_0}H^k_\Delta(S_T,\frac{S_T}{S_0})\right)]. \tag{4.14}$$

4.1.3. Greeks: ρ. We express the weight for rho in terms of the weight of delta.

$$\rho = \frac{\partial}{\partial r}\left(e^{-rT}E[\phi(S_T)]\right) = -Te^{-rT}E[\phi(S_T)] + e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial r}]. \tag{4.15}$$

We define $G_\rho = \dfrac{\partial S_T}{\partial r} = TS_T = TS_0\dfrac{S_T}{S_0} = TS_0 G_\Delta$. Thus, we rewrite ρ in terms of G_Δ as

$$\begin{aligned}\rho &= -Te^{-rT}E[\phi(S_T)] + e^{-rT}E[\phi'(S_T)TS_0G_\Delta]\\ &= -Te^{-rT}E[\phi(S_T)] + TS_0e^{-rT}E[\phi(S_T)H^k_\Delta(F,G_\Delta)]\\ &= e^{-rT}E[\phi(S_T)T(S_0H^k(F,G_\Delta)-1)]\end{aligned} \tag{4.16}$$

Equation (4.16) implies that

$$H^k_\rho(S_T, H^k(F,G_\Delta)) = T(S_0H^k(F,G_\Delta)-1), \tag{4.17}$$

where $F = S_T$.

4.1.4. Greeks: $\mathcal{D}$. We express the weight for drift in terms of the weight of delta.

$$\mathcal{D} = \frac{\partial}{\partial \theta}\left(e^{-rT}E[\phi(S_T)]\right) = e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial \theta}]. \tag{4.18}$$

Recall that $w = \dfrac{1}{\kappa}\ln(1-\theta\kappa-\sigma^2\dfrac{\kappa}{2})$. We define $G_{\mathcal{D}} = \dfrac{\partial S_T}{\partial \theta} = \left(\dfrac{\partial w}{\partial \theta}T + \displaystyle\sum_{i=1}^n \Delta G_i\right)S_T = S_0\left(\dfrac{\partial w}{\partial \theta}T + \displaystyle\sum_{i=1}^n \Delta G_i\right)\dfrac{\partial S_T}{\partial S_0} = S_0\left(\dfrac{\partial w}{\partial \theta}T + \displaystyle\sum_{i=1}^n \Delta G_i\right)G_\Delta$.

Lemma 4.1. *Let A be any random variable independent from Z_i for all $i = 1,\dots,k$. Then $H^k(F,AG) = AH^k(F,G)$ for all $i = 1,\dots,k$.*

Proof. Equation (4.2) implies that

$$\begin{aligned}H^k(F,AG) &= AG\,M^k_\gamma(F)\,L^kF\\ &\qquad -M^k_\gamma(F) < D^kF, D^k(AG) > -AG < D^kF, D^kM^k_\gamma(F) > .\end{aligned}$$

Since A is independent of Z_i for all $i = 1,\dots,k$, we write $M^k_\gamma(F) < D^kF, AD^kG >$ for the second term in the above equation. Also,

$$M^k_\gamma(F) < D^kF, AD^kG >= M^k_\gamma(F)A < D^kF, D^kG >$$

by the linearity of the inner product. This completes the proof. $\square$

We observe that $S_0\left(\frac{\partial w}{\partial\theta}T+\sum_{i=1}^n\Delta G_i\right)$ is independent of Zi's. Thus, Lemma (4.1) implies that

$$\begin{aligned}\mathcal{D}&=e^{-rT}E[\phi'(S_T)S_0\left(\frac{\partial w}{\partial\theta}T+\sum_{i=1}^n\Delta G_i\right)G_\Delta]\\&=e^{-rT}E[\phi(S_T)H^k_\Delta(S_T,S_0\left(\frac{\partial w}{\partial\theta}T+\sum_{i=1}^n\Delta G_i\right)G_\Delta)]\\&=e^{-rT}E[\phi(S_T)S_0\left(\frac{\partial w}{\partial\theta}T+\sum_{i=1}^n\Delta G_i\right)H^k_\Delta(S_T,G_\Delta)]\end{aligned}$$

Therefore,

$$\mathcal{D}=e^{-rT}E[\phi(S_T)S_0\left(\frac{-T}{1-\theta\kappa-\sigma^2\frac{\kappa}{2}}+\sum_{i=1}^n\Delta G_i\right)H^k_\Delta(S_T,G_\Delta)],\tag{4.19}$$

since $\dfrac{\partial w}{\partial\theta}=\dfrac{-1}{1-\theta\kappa-\sigma^2\frac{\kappa}{2}}$. Equation (4.19) implies

$$H^k_{\mathcal{D}}(S_T,G_\Delta)=S_0\left(\frac{-T}{1-\theta\kappa-\sigma^2\frac{\kappa}{2}}+\sum_{i=1}^n\Delta G_i\right)H^k_\Delta(S_T,G_\Delta).\tag{4.20}$$

4.1.5. Greeks: $\mathcal{V}$. We express the weight for vega in terms of the weight of delta.

$$\mathcal{V}=\frac{\partial}{\partial\sigma}\left(e^{-rT}E[\phi(S_T)]\right)=e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial\sigma}].\tag{4.21}$$

In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $G_\sigma=\dfrac{\partial S_T}{\partial\sigma}=\left(T\dfrac{\partial w}{\partial\sigma}+\sum_{i=1}^n\sqrt{\Delta G_i}Z_i\right)S_T$. For the further calculations D_jG_σ is needed explicitly.
2. $D_jG_\sigma=\sqrt{\Delta G_j}S_T+\left(T\dfrac{\partial w}{\partial\sigma}+\sum_{i=1}^n\sqrt{\Delta G_i}Z_i\right)\sigma\sqrt{\Delta G_j}S_T$.
3.

$$\begin{aligned}&G_\sigma M^k_\gamma(F)L^kF\\&=\left(T\frac{\partial w}{\partial\sigma}+\sum_{i=1}^n\sqrt{\Delta G_i}Z_i\right)S_TM^k_\gamma(F)\left(-S_T\sum_{j=1}^k\left(\sigma^2\Delta G_j-Z_j\sigma\sqrt{\Delta G_j}\right)\right)\\&=\left(T\frac{\partial w}{\partial\sigma}+\sum_{i=1}^n\sqrt{\Delta G_i}Z_i\right)\left(-1+M^k_\gamma(F)S_T^2\sum_{j=1}^kZ_j\sigma\sqrt{\Delta G_j}\right)\\&=-S_0\left(T\frac{\partial w}{\partial\sigma}+\sum_{i=1}^n\sqrt{\Delta G_i}Z_i\right)\left(\frac{1}{S_0}-\frac{\sum_{j=1}^kZ_j\sigma\sqrt{\Delta G_j}}{S_0\sum_{j=1}^k\sigma^2\Delta G_j}\right)\\&=-S_0\left(T\frac{\partial w}{\partial\sigma}+\sum_{i=1}^n\sqrt{\Delta G_i}Z_i\right)\left(\frac{1}{S_0}-H^k_\Delta(F,G_\Delta)\right)\end{aligned}\tag{4.22}$$

4.

$$
\begin{aligned}
&M_\gamma^k(F)\langle DF, DG_\sigma\rangle_k \\
&\quad = M_\gamma^k(F)\sum_{j=1}^k \sigma\sqrt{\Delta G_j}S_T\left(\sqrt{\Delta G_j}S_T\right. \\
&\quad \left. + \left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right)\sigma\sqrt{\Delta G_j}S_T\right) \\
&\quad = M_\gamma^k(F)\left\{\left(\frac{1}{\sigma}\sum_{j=1}^k \sigma^2\Delta G_j S_T^2\right)\right. \\
&\quad \left. + \left(T\frac{\partial w}{\partial \sigma}\sum_{j=1}^k \sigma^2\Delta G_j S_T^2\right) + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\left(\sum_{j=1}^k \sigma^2\Delta G_j S_T^2\right)\right\} \\
&\quad = \frac{1}{\sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i + T\frac{\partial w}{\partial \sigma}. \qquad (4.23)
\end{aligned}
$$

5.

$$
\begin{aligned}
&G_\sigma\langle DF, DM_\gamma^k(F)\rangle_k \\
&\quad = \left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right)S_T\left(\sum_{j=1}^k \sigma\sqrt{\Delta G_j}S_T\frac{(-2\sigma\sqrt{\Delta G_j})}{M_\sigma^k(F)}\right) \\
&\quad = \left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right)\left(-M_\gamma^k(F)\sum_{j=1}^k \sigma^2\Delta G_j S_T^2\right) \\
&\quad = -2\left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right). \qquad (4.24)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
H^k(F, G_\sigma) &= (4.22) - (4.23) - (4.24) \\
&= -S_0\left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right)\left(\frac{1}{S_0} - H_\Delta^k(F, G_\Delta)\right) \\
&\quad - \left\{\frac{1}{\sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i + T\frac{\partial w}{\partial \sigma}\right\} \\
&\quad +2\left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right) \\
&= (-1-1+2)T\frac{\partial w}{\partial \sigma} + (-1-1+2)\sum_{i=1}^n \sqrt{\Delta G_i}Z_i - \frac{1}{\sigma} \\
&\quad +S_0 H_\Delta^k(F, G_\Delta)\left(T\frac{\partial w}{\partial \sigma} + \sum_{i=1}^n \sqrt{\Delta G_i}Z_i\right) \\
&= S_0 H^k(F, G_\Delta)\sum_{i=1}^n \sqrt{\Delta G_i}Z_i + S_0 H^k(F, G_\Delta)T\frac{\partial w}{\partial \sigma} - \frac{1}{\sigma}. \qquad (4.25)
\end{aligned}
$$

As a result vega is given by

$$\mathcal{V} = e^{-rT}E[\phi(S_T)\left(S_0H^k(F,G_\Delta)\sum_{i=1}^{n}\sqrt{\Delta G_i}Z_i - \frac{S_0H^k(F,G_\Delta)T\sigma}{1-\theta\kappa-\frac{\sigma^2\kappa}{2}} - \frac{1}{\sigma}\right)] \quad (4.26)$$

since $\dfrac{\partial w}{\partial \sigma} = \dfrac{-\sigma}{1-\theta\kappa-\frac{\sigma^2\kappa}{2}}$.

4.1.6. Greeks: $\mathcal{V}_2$. We express the weight for vega2 in terms of the weight of delta.

$$\mathcal{V}_2 = \frac{\partial}{\partial\kappa}\left(e^{-rT}E[\phi(S_T)]\right) = e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial\kappa}] + E^{(\kappa)}[\phi(S_T)]. \quad (4.27)$$

where

$$E^{(\kappa)}[\phi(S_T)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(S_T)\prod_{i=1}^{n}f_{\mathcal{N}}(x_i;0,1) \times\frac{\partial}{\partial\kappa}\left(\prod_{i=1}^{n}f_G(y_i;a\Delta t_i,b)\right)dx_1\dots dx_n dy_1\dots dy_n, \quad (4.28)$$

where $a=b=1/\kappa$. We define

$$F = \prod_{i=1}^{n}f_G(y_i;\Delta t_i/\kappa,1/\kappa), \quad (4.29)$$

and we evaluate the partial derivative as follows:

$$\begin{aligned}
&\frac{\partial}{\partial\kappa}\left(\prod_{i=1}^{n}f_G(y_i;\Delta t_i/\kappa,1/\kappa)\right)\\
&=\frac{\partial}{\partial\kappa}e^{\log(F)}\\
&=F\frac{\partial}{\partial\kappa}\left(\sum_{i=1}^{n}\log(f_G(y_i;\Delta t_i/\kappa,1/\kappa))\right)\\
&=F\frac{\partial}{\partial\kappa}\left(\sum_{i=1}^{n}\frac{-\Delta t_i}{\kappa}\log(\kappa)-\log(\Gamma(\frac{\Delta t_i}{\kappa}))+(\frac{\Delta t_i}{\kappa}-1)y_i-\frac{y_i}{\kappa}\right)\\
&=F\sum_{i=1}^{n}\frac{\Delta t_i}{\kappa^2}\left\{\log(\kappa)-1+\psi(\frac{\Delta t_i}{\kappa})-\log(y_i)+\frac{y_i}{\Delta t_i}\right\}\\
&=\prod_{i=1}^{n}f_G(y_i;\Delta t_i/\kappa,1/\kappa)\left(\frac{T}{\kappa^2}(\log(\kappa)-1)\right.\\
&\left.+\sum_{i=1}^{n}\frac{\Delta t_i}{\kappa^2}\left\{\psi(\frac{\Delta t_i}{\kappa})-\log(y_i)+\frac{y_i}{\Delta t_i}\right\}\right),
\end{aligned} \quad (4.30)$$

where $\psi(x)$ is the digamma function as denoted in [2]. Therefore,

$$e^{-rT}E^{(\alpha)}[\phi(S_T)] = e^{-rT}E[\phi(S_T)\left(\frac{T}{\kappa^2}(\log(\kappa)-1) + \sum_{i=1}^{n}\frac{\Delta t_i}{\kappa^2}\left\{\psi(\frac{\Delta t_i}{\kappa})-\log(y_i)+\frac{y_i}{\Delta t_i}\right\}\right)]. \quad (4.31)$$

We define $G_{\mathcal{V}_2} = \dfrac{\partial S_T}{\partial \kappa} = \dfrac{\partial w}{\partial \kappa} T S_T = S_0 \dfrac{\partial w}{\partial \kappa} T \dfrac{\partial S_T}{\partial S_0} = S_0 \dfrac{\partial w}{\partial \kappa} T G_\Delta$. Since $S_0 \dfrac{\partial w}{\partial \kappa} T$ is independent of Zi's, Lemma (4.1) implies that

$$\begin{aligned}
\mathcal{V}_2 &= e^{-rT} E[\phi'(S_T) S_0 \frac{\partial w}{\partial \kappa} T G_\Delta] + \text{ Equation (4.31)} \\
&= e^{-rT} E[\phi(S_T) H^k_\Delta(S_T, S_0 \frac{\partial w}{\partial \kappa} T G_\Delta)] + \text{ Equation (4.31)} \\
&= e^{-rT} E[\phi(S_T) S_0 \frac{\partial w}{\partial \kappa} T H^k_\Delta(S_T, G_\Delta)] + \text{ Equation (4.31)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
H^k_{\mathcal{V}_2}(S_T, G_\Delta) = S_0 \frac{\partial w}{\partial \kappa} T H^k_\Delta(S_T, G_\Delta) &+ \frac{T}{\kappa^2}(\log(\kappa) - 1) \\
&+ \sum_{i=1}^n \frac{\Delta t_i}{\kappa^2} \left\{ \psi(\frac{\Delta t_i}{\kappa}) - \log(y_i) + \frac{y_i}{\Delta t_i} \right\} \quad (4.32)
\end{aligned}$$

with

$$\frac{\partial w}{\partial \kappa} = -\frac{(\theta + \sigma^2/2)\kappa + (1 - \theta\kappa - \sigma^2\kappa/2)\log(1 - \theta\kappa - \sigma^2\kappa/2)}{\kappa^2}. \tag{4.33}$$

The Malliavin weight given in Equation (4.32) has the term $\log(y_i)$ for $y_i > 0$. Thus, theoretically the Mallivin weight is well defined. However, in simulations when the weight is calculated for small values of y_i the calculation of the logarithm takes longer and results in very large negative numbers, as well. Thus, the variance of the weight gets larger. We observe that the Malliavin approach does not converge. In the next section, we introduce a localization method that improves the convergence .

4.2. NIG Model. In NIG model S_T is given by

$$S_T = S_0 e^{(r+w)T + \sum_{i=1}^n \delta\sqrt{\Delta IG_i} Z_i + \beta\delta^2 \sum_{i=1}^n \Delta IG_i}, \tag{4.34}$$

where IG_t is the inverse Gaussian process used to subordinate the Wiener process and $\Delta IG_i = IG_{t_i} - IG_{t_{i-1}}$. Also, we define the following random set $RS = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) : (x_1, \ldots, x_n, y_1, \ldots, y_n) = (Z_1, \ldots, Z_n, \Delta IG_1, \ldots, \Delta IG_n)\}$. Also, recall that $w = \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2})$.

4.2.1. Greeks: Δ.

$$\Delta = e^{-rT} \frac{\partial}{\partial S_0} E[\phi(S_T)] = e^{-rT} E[\phi'(S_T)\frac{S_T}{S_0}] = E[\phi(S_T) H^k_\Delta(S_T, \frac{S_T}{S_0})].$$

In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $S_T = F = f(Z_1, \ldots, Z_n, \Delta IG_1, \ldots, \Delta IG_n)$.
2. $D_j F = \frac{\partial f}{\partial x_j}|_{RS} = \delta\sqrt{\Delta IG_i} S_T$, $\forall j$ such that $1 \le j \le k$.
3. $D_{jj} F = \frac{\partial^2 f}{\partial x_j^2}|_{RS} = \delta^2 \Delta IG_i S_T$, $\forall j$ such that $1 \le j \le k$.
4. $G_{S_0} = \frac{\partial S_T}{\partial S_0} = \frac{\partial f}{\partial S_0}|_{RS} = \frac{S_T}{S_0}$.
5. $D_j G_{S_0} = \delta\sqrt{\Delta IG_i}\frac{S_T}{S_0}$, $\forall j$ such that $1 \le j \le k$.
6. $\Theta_j(x_j) = \frac{d \log(p_j(x_j))}{dx_j} = -x_j$, where $p_j(x_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}}$, $\forall j$ such that $1 \le j \le k$.

7. $L_jF = D_{jj}F + \Theta_j(Z_j)D_jF = \delta^2\Delta IG_i S_T - Z_j\delta\sqrt{\Delta IG_i}S_T$, $\forall j$ such that $1 \le j \le k$.
8. $L^k = -\sum_{i=1}^{k} D_{jj}F + \Theta_j(Z_j)D_jF = -S_T\sum_{i=1}^{k}\delta^2\Delta IG_i - Z_j\delta\sqrt{\Delta IG_i}$, for $1 \le k \le n$.
9. $M_\sigma^k(F) = \sum_{i=1}^{k}\delta^2\Delta IG_i S_T^2 > 0$, for $1 \le k \le n$.
10. $D_jM_\sigma^k(F) = \delta\sqrt{\Delta IG_i}S_T 2S_T\sum_{i=1}^{k}\delta^2\Delta IG_i$
$= 2\delta\sqrt{\Delta IG_i}\sum_{i=1}^{k}\delta^2\Delta IG_i S_T^2 = 2\delta\sqrt{\Delta IG_i}M_\sigma^k(F)$, $\forall j$ such that $1 \le j \le k$.
11. $M_\gamma^k(F) = \dfrac{1}{M_\sigma^k(F)}$, for $1 \le k \le n$.
12. $D_jM_\gamma^k(F) = -\dfrac{D_jM_\sigma^k(F)}{(M_\sigma^k(F))^2} = -\dfrac{2\delta\sqrt{\Delta IG_i}M_\sigma^k(F)}{(M_\sigma^k(F))^2} = -\dfrac{2\delta\sqrt{\Delta IG_i}}{M_\sigma^k(F)}$

The Malliavin weight follows from the above calculations:

$$
\begin{aligned}
H_\Delta^k(F, G_{S_0}) &= \frac{S_T}{S_0}M_\gamma^k(F)\left(-S_T\sum_{i=1}^{k}\left(\delta^2\Delta IG_i - Z_j\delta\sqrt{\Delta IG_i}\right)\right) \\
&\quad -M_\gamma^k(F)\sum_{j=1}^{k}\delta\sqrt{\Delta IG_i}S_T\delta\sqrt{\Delta IG_i}\frac{S_T}{S_0} \\
&\quad -\frac{S_T}{S_0}\sum_{j=1}^{k}\delta\sqrt{\Delta IG_i}S_T\frac{(-2\delta\sqrt{\Delta IG_i})}{M_\sigma^k(F)} \\
&= \frac{-1}{S_0}M_\gamma^k(F)\left(\sum_{i=1}^{k}\delta^2\Delta IG_i S_T^2 - S_T^2\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_i}\right) \\
&\quad -\frac{1}{S_0}M_\gamma^k(F)\sum_{j=1}^{k}\delta^2\Delta IG_i S_T^2 + \frac{2}{S_0}M_\gamma^k(F)\sum_{j=1}^{k}\delta^2\Delta IG_i S_T^2 \\
&= \frac{-1}{S_0}M_\gamma^k(F)\left(M_\sigma^k(F) - S_T^2\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_i}\right) \\
&\quad -\frac{1}{S_0}M_\gamma^k(F)M_\sigma^k(F) + \frac{2}{S_0}M_\gamma^k(F)M_\sigma^k(F) \\
&= \frac{-1}{S_0} + \frac{M_\gamma^k(F)}{S_0}S_T^2\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_i} - \frac{1}{S_0} + \frac{2}{S_0} \\
&= \frac{M_\gamma^k(F)}{S_0}S_T^2\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_i} = \frac{S_T^2\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_i}}{S_0\sum_{i=1}^{k}\delta^2\Delta IG_i S_T^2} \\
&= \frac{\sum_{j=1}^{k}Z_j\sqrt{\Delta IG_i}}{S_0\sum_{i=1}^{k}\delta\Delta IG_i}.
\end{aligned}
\tag{4.35}
$$

Therefore, the delta of a call or digital option via the Malliavin weight is

$$\Delta = e^{-rT} E[\phi(S_T) \frac{\sum_{j=1}^{k} Z_j \sqrt{\Delta IG_i}}{S_0 \sum_{i=1}^{k} \delta \Delta IG_i}]. \tag{4.36}$$

4.2.2. Greeks: Γ. We express the weight for gamma in terms of the weight of delta.

$$\begin{aligned}
\Gamma &= e^{-rT} \frac{\partial^2}{\partial S_0^2} E[\phi(S_T)] = e^{-rT} \frac{\partial}{\partial S_0} E[\phi'(S_T) \frac{S_T}{S_0}] = e^{-rT} \frac{\partial}{\partial S_0} E[\phi(S_T) H_\Delta^k(S_T, \frac{S_T}{S_0})] \\
&= e^{-rT} E[\phi'(S_T) \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] + E[\phi(S_T) \frac{\partial}{\partial S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] \qquad (4.37) \\
&= e^{-rT} E[\phi(S_T) H^k(S_T, \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0}))] + e^{-rT} E[\phi(S_T) \frac{\partial}{\partial S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] \qquad (4.38)
\end{aligned}$$

After taking the derivative in the last term, gamma is given as follows.

$$\begin{aligned}
\Gamma &= e^{-rT} E[\phi(S_T) H^k(S_T, \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0}))] - e^{-rT} E[\phi(S_T) \frac{1}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})] \\
&= e^{-rT} E[\phi(S_T) \left(H^k(S_T, \frac{S_T}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0})) - \frac{1}{S_0} H_\Delta^k(S_T, \frac{S_T}{S_0}) \right)]. \qquad (4.39)
\end{aligned}$$

We use the integration by parts formula in order to remove the derivative in the first term of Equation (4.37). Since $F = S_T$, we use the F-related calculations that are done for delta. Thus, we deal with a shorter list of calculations for gamma. In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $G_\Gamma = \frac{S_T}{S_0} H_\Delta^k(F, G_{S_0})$.
2. $D_j G_\Gamma = \frac{\delta \sqrt{\Delta IG_i} S_T}{S_0} H_\Delta^k(F, G_{S_0}) + \frac{S_T}{S_0^2} \frac{\sqrt{\Delta IG_i}}{\sum_{i=1}^{k} \delta \Delta IG_i}$, $\forall j$ such that $1 \le j \le k$.
3.

$$\begin{aligned}
&G_\Gamma M_\gamma^k(F) L^k F \\
&\quad = \frac{S_T}{S_0} H_\Delta^k(F, G_{S_0}) M_\gamma^k(F) \left(-S_T \sum_{j=1}^{k} \left(\delta^2 \Delta IG_i - Z_j \delta \sqrt{\Delta IG_i} \right) \right) \\
&\quad = H_\Delta^k(F, G_{S_0}) \left(-M_\gamma^k(F) \sum_{j=1}^{k} \frac{S_T^2}{S_0} \delta^2 \Delta IG_i + \frac{\sum_{j=1}^{k} \frac{S_T^2}{S_0} Z_j \delta \sqrt{\Delta IG_i}}{\sum_{i=1}^{k} \delta^2 \Delta IG_i S_T^2} \right) \\
&\quad = H_\Delta^k(F, G_{S_0}) \left(-M_\gamma^k(F) M_\sigma^k(F) + \frac{\sum_{j=1}^{k} \frac{S_T^2}{S_0} Z_j \delta \sqrt{\Delta IG_i}}{\sum_{i=1}^{k} \delta^2 \Delta IG_i S_T^2} \right) \\
&\quad = H_\Delta^k(F, G_{S_0}) \left(-\frac{1}{S_0} + H_\Delta^k(F, G_{S_0}) \right) \qquad (4.40)
\end{aligned}$$

since $M_\gamma^k(F) M_\sigma^k(F) = 1$.

4.

$$\begin{aligned}
&M^k_\gamma(F)\langle DF, DG_\Gamma\rangle_k \\
&\quad= M^k_\gamma(F)\sum_{j=1}^{k}\delta\sqrt{\Delta IG_i}S_T\left[\frac{\delta\sqrt{\Delta IG_i}S_T}{S_0}H^k_\Delta(F,G_{S_0})+\frac{S_T}{S_0^2}\frac{\sqrt{\Delta IG_i}}{\sum_{i=1}^k\delta\Delta IG_i}\right] \\
&\quad= \frac{H^k_\Delta(F,G_{S_0})M^k_\gamma(F)}{S_0}\sum_{j=1}^{k}\delta^2\Delta IG_i S_T^2+\frac{M^k_\gamma(F)S_T^2}{S_0^2}\frac{\sum_{j=1}^k\delta\Delta IG_i}{\sum_{i=1}^k\delta\Delta IG_i} \\
&\quad= \frac{H^k_\Delta(F,G_{S_0})M^k_\gamma(F)}{S_0}M^k_\sigma(F)+\frac{M^k_\gamma(F)S_T^2}{S_0^2}\frac{\sum_{j=1}^k\delta\Delta IG_i}{\sum_{i=1}^k\delta\Delta IG_i} \\
&\quad= \frac{H^k_\Delta(F,G_{S_0})}{S_0}+\frac{M^k_\gamma(F)S_T^2}{S_0^2}. \qquad (4.41)
\end{aligned}$$

5.

$$\begin{aligned}
&G_\Gamma\langle DF, DM^k_\gamma(F)\rangle_k \\
&\quad= -\frac{S_T}{S_0}H^k_\Delta(F,G_{S_0})\sum_{j=1}^{k}\delta\sqrt{\Delta IG_i}S_T\frac{2\delta\sqrt{\Delta IG_i}}{M^k_\sigma(F)} \\
&\quad= -\frac{2}{S_0}H^k_\Delta(F,G_{S_0})\sum_{j=1}^{k}\delta^2\Delta IG_i S_T^2 M^k_\gamma(F) \\
&\quad= -\frac{2}{S_0}H^k_\Delta(F,G_{S_0})M^k_\sigma(F)M^k_\gamma(F) \\
&\quad= -\frac{2}{S_0}H^k_\Delta(F,G_{S_0}). \qquad (4.42)
\end{aligned}$$

Thus,

$$\begin{aligned}
H^k(F,G_\Gamma) &= (4.40)-(4.41)-(4.42) \\
&= H^k_\Delta(F,G_{S_0})\left(-\frac{1}{S_0}+H^k_\Delta(F,G_{S_0})\right) \\
&\quad -\frac{H^k_\Delta(F,G_{S_0})}{S_0}-\frac{M^k_\gamma(F)S_T^2}{S_0^2}+\frac{2}{S_0}H^k_\Delta(F,G_{S_0}) \\
&= (H^k_\Delta(F,G_{S_0}))^2-\frac{M^k_\gamma(F)S_T^2}{S_0^2}. \qquad (4.43)
\end{aligned}$$

This implies

$$H^k_\Gamma(F,G_\Gamma) = (H^k_\Delta(F,G_{S_0}))^2-\frac{M^k_\gamma(F)S_T^2}{S_0^2}-\frac{1}{S_0}H^k_\Delta(S_T,\frac{S_T}{S_0}). \qquad (4.44)$$

Therefore,

$$\Gamma = e^{-rT}E[\phi(S_T)\left((H^k_\Delta(F,G_{S_0}))^2-\frac{M^k_\gamma(F)S_T^2}{S_0^2}-\frac{1}{S_0}H^k_\Delta(S_T,\frac{S_T}{S_0})\right)]. \qquad (4.45)$$

4.2.3. Greeks: ρ. We express the weight for rho in terms of the weight of delta.

$$\rho = \frac{\partial}{\partial r}\left(e^{-rT}E[\phi(S_T)]\right) = -Te^{-rT}E[\phi(S_T)] + e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial r}]. \tag{4.46}$$

We define $G_\rho = \dfrac{\partial S_T}{\partial r} = TS_T = TS_0\dfrac{S_T}{S_0} = TS_0G_\Delta$. Thus, we rewrite ρ in terms of G_Δ as

$$\begin{aligned}\rho &= -Te^{-rT}E[\phi(S_T)] + e^{-rT}E[\phi'(S_T)TS_0G_\Delta] \\ &= -Te^{-rT}E[\phi(S_T)] + TS_0e^{-rT}E[\phi(S_T)H^k_\Delta(F, G_\Delta)] \\ &= e^{-rT}E[\phi(S_T)T(S_0H^k(F, G_\Delta) - 1)] \end{aligned}\tag{4.47}$$

Equation (4.47) implies that

$$H^k_\rho(S_T, H^k(F, G_\Delta)) = T(S_0H^k(F, G_\Delta) - 1), \tag{4.48}$$

where $F = S_T$.

The sensitivities $\mathcal{D}$, $\mathcal{V}$, $\mathcal{V}2$ are given after calculating the sensitivities with respect to Barndoff-Nielsen parametrization. However, these parameters also determine the distribution of the inverse Gausssian increments. Therefore, differentiating under the expectation implies the differentiating the probability density of the inverse Gausssian increments, as well. Thus, the sensitivities with respect to these parameters are given as follows:

4.2.4. Sensitivity with respect to β.

$$\frac{\partial u(\beta)}{\partial \beta} = e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial \beta}] + e^{-rT}E^{(\beta)}[\phi(S_T)], \tag{4.49}$$

where

$$\begin{aligned}E^{(\beta)}[\phi(S_T)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(S_T)\prod_{i=1}^n f_{\mathcal{N}}(x_i; 0, 1)\frac{\partial}{\partial \beta} \\ \times\left(\prod_{i=1}^n f_{IG}(y_i; \Delta t_i, b)\right) dx_1 \ldots dx_n dy_1 \ldots dy_n,\end{aligned} \tag{4.50}$$

where $b = \delta\sqrt{\alpha^2 - \beta^2}$. We define

$$F = \prod_{i=1}^n f_{IG}(y_i; \Delta t_i, b), \tag{4.51}$$

and we evaluate the partial derivative as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\prod_{i=1}^{n} f_{IG}(y_i;\Delta t_i,b)\right)\\
&= \frac{\partial}{\partial\beta} e^{\log(F)}\\
&= F\frac{\partial}{\partial\beta}\left(\sum_{i=1}^{n}\log(f_{IG}(y_i;\Delta t_i,b))\right)\\
&= F\frac{\partial}{\partial\beta}\left(\sum_{i=1}^{n}\log\left(\frac{\Delta t_i}{\sqrt{2\pi}}\right)+\Delta t_i b-\frac{3}{2}\log(y_i)-\frac{(\Delta t_i)^2}{2y_i}-\frac{b^2y_i}{2}\right)\\
&= F\sum_{i=1}^{n}\Delta t_i\frac{\partial b}{\partial\beta}-\frac{\partial b}{\partial\beta}by_i\\
&= \prod_{i=1}^{n} f_{IG}(y_i;\Delta t_i,b)\sum_{i=1}^{n}(\Delta t_i-by_i)\frac{\partial b}{\partial\beta}.
\end{aligned}
\tag{4.52}
$$

Therefore,

$$
E^{(\beta)}[\phi(S_T)] = E[\phi(S_T)\sum_{i=1}^{n}(\Delta t_i - b\Delta IG_i)\frac{\partial b}{\partial\beta}]. \tag{4.53}
$$

The first term of Equation (4.49) involves the derivative operator. We apply Theorem (3.7) to remove the derivative in the expectation. We define $G_\beta = \dfrac{\partial S_T}{\partial\beta}$. In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $G_\beta = \left(T\dfrac{\partial w}{\partial\beta}+\delta^2\displaystyle\sum_{i=1}^{n}\Delta IG_i\right)S_T.$
2. $D_jG_\beta = \left(T\dfrac{\partial w}{\partial\beta}+\delta^2\displaystyle\sum_{i=1}^{n}\Delta IG_i\right)\delta\sqrt{\Delta IG_j}S_T$
3.

$$
\begin{aligned}
&G_\beta M_\gamma^k(F)L^kF\\
&= \left(T\frac{\partial w}{\partial\beta}+\delta^2\sum_{i=1}^{n}\Delta IG_i\right)S_TM_\gamma^k(F)\left(-S_T\sum_{j=1}^{k}\left(\delta^2\Delta IG_j - Z_j\delta\sqrt{\Delta IG_j}\right)\right)\\
&= \left(T\frac{\partial w}{\partial\beta}+\delta^2\sum_{i=1}^{n}\Delta IG_i\right)\left(-1+S_0\frac{\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_j}}{S_0\sum_{i=1}^{n}\delta^2\Delta IG_j}\right)\\
&= \left(T\frac{\partial w}{\partial\beta}+\delta^2\sum_{i=1}^{n}\Delta IG_i\right)(-1+S_0H_\Delta^k(F,G_\Delta)).
\end{aligned}
\tag{4.54}
$$

4.

$$
\begin{aligned}
&M_\gamma^k(F)\langle DF, DG_\beta\rangle_k\\
&\qquad = M_\gamma^k(F)\sum_{j=1}^{k}\left(T\frac{\partial w}{\partial\beta}+\delta^2\sum_{i=1}^{n}\Delta IG_i\right)\delta^2\Delta IG_jS_T^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) M_\gamma^k(F) \sum_{j=1}^{k} \delta^2 \Delta IG_j S_T^2 \\
&= \left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) \qquad (4.55)
\end{aligned}
$$

5.

$$
\begin{aligned}
&G_\beta \langle DF, DM_\gamma^k(F)\rangle_k \\
&= -\left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) S_T \sum_{j=1}^{k} \delta \sqrt{\Delta IG_j} S_T \frac{2\delta\sqrt{\Delta IG_j}}{M_\sigma^k(F)} \\
&= -2\left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) M_\gamma^k(F) \sum_{j=1}^{k} S_T^2 \delta^2 \Delta IG_j \\
&= -2\left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) \qquad (4.56)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
H^k(F, G_\beta) &= (4.54) - (4.55) - (4.56) \\
&= \left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) (S_0 H_\Delta^k(F, G_\Delta)). \qquad (4.57)
\end{aligned}
$$

Together with Equation (4.53) this implies

$$
\begin{aligned}
\frac{\partial u(\beta)}{\partial \beta} = e^{-rT} E[\phi(S_T) \Bigg(&\left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) S_0 H_\Delta^k(F, G_\Delta) \\
&+ \left(\sum_{i=1}^{n} (\Delta t_i - b\Delta IG_i) \frac{\partial b}{\partial \beta} \right) \Bigg)] \qquad (4.58a)
\end{aligned}
$$

Hence, after summing up Δt_i's over the time grid:

$$
\begin{aligned}
= e^{-rT} E[\phi(S_T) \Bigg(&\left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i \right) S_0 H_\Delta^k(F, G_\Delta) \\
&+ \left(T - \sum_{i=1}^{n} b\Delta IG_i \right) \frac{\partial b}{\partial \beta} \Bigg)] \qquad (4.58b)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\partial w}{\partial \beta} &= \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} - \frac{\delta(\beta+1)}{\sqrt{\alpha^2 - (\beta+1)^2}}, \qquad (4.59) \\
\frac{\partial b}{\partial \beta} &= -\frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}. \qquad (4.60)
\end{aligned}
$$

4.2.5. Sensitivity with respect to δ.

$$\frac{\partial u(\delta)}{\partial \delta} = e^{-rT} E[\phi'(S_T) \frac{\partial S_T}{\partial \delta}] + e^{-rT} E^{(\delta)}[\phi(S_T)], \tag{4.61}$$

where

$$E^{(\delta)}[\phi(S_T)] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(S_T) \prod_{i=1}^{n} f_{\mathcal{N}}(x_i; 0, 1) \frac{\partial}{\partial \delta} \times \left(\prod_{i=1}^{n} f_{IG}(y_i; \Delta t_i, b) \right) dx_1 \ldots dx_n dy_1 \ldots dy_n, \tag{4.62}$$

where $b = \delta\sqrt{\alpha^2 - \beta^2}$. We define

$$F = \prod_{i=1}^{n} f_{IG}(y_i; \Delta t_i, b), \tag{4.63}$$

and we evaluate the partial derivative as follows:

$$\begin{aligned}
&\frac{\partial}{\partial \delta}\left(\prod_{i=1}^{n} f_{IG}(y_i; \Delta t_i, b) \right) \\
&= \frac{\partial}{\partial \delta} e^{\log(F)} \\
&= F \frac{\partial}{\partial \delta} \left(\sum_{i=1}^{n} \log(f_{IG}(y_i; \Delta t_i, b)) \right) \\
&= F \frac{\partial}{\partial \delta} \left(\sum_{i=1}^{n} \log\left(\frac{\Delta t_i}{\sqrt{2\pi}} \right) + \Delta t_i b - \frac{3}{2}\log(y_i) - \frac{(\Delta t_i)^2}{2y_i} - \frac{b^2 y_i}{2} \right) \\
&= F \sum_{i=1}^{n} \Delta t_i \frac{\partial b}{\partial \delta} - \frac{\partial b}{\partial \delta} b y_i \\
&= \prod_{i=1}^{n} f_{IG}(y_i; \Delta t_i, b) \sum_{i=1}^{n} (\Delta t_i - b y_i) \frac{\partial b}{\partial \delta}.
\end{aligned} \tag{4.64}$$

Therefore,

$$E^{(\delta)}[\phi(S_T)] = E[\phi(S_T) \sum_{i=1}^{n} (\Delta t_i - b\Delta IG_i) \frac{\partial b}{\partial \delta}]. \tag{4.65}$$

The first term of Equation (4.61) involves the derivative operator. We apply Theorem (3.7) to remove the derivative in the expectation. We define $G_\delta = \dfrac{\partial S_T}{\partial \delta}$. In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $G_\delta = \left(T\dfrac{\partial w}{\partial \delta} + \displaystyle\sum_{i=1}^{n} \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^{n} \Delta IG_i \right) S_T.$
2. $D_j G_\delta = \sqrt{\Delta IG_j} S_T + \left(T\dfrac{\partial w}{\partial \delta} + \displaystyle\sum_{i=1}^{n} \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^{n} \Delta IG_i \right) \delta\sqrt{\Delta IG_j} S_T$

3.

$$
\begin{aligned}
& G_\delta M_\gamma^k(F) L^k F \\
& \quad = \left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) S_T M_\gamma^k(F) \\
& \quad \times \left(-S_T \sum_{j=1}^k \left(\delta^2 \Delta IG_j - Z_j \delta \sqrt{\Delta IG_j}\right)\right) \\
& \quad = \left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) \left(-1 + S_0 \frac{\sum_{j=1}^k Z_j \delta \sqrt{\Delta IG_j}}{S_0 \sum_{i=1}^n \delta^2 \Delta IG_j}\right) \\
& \quad = \left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) (-1 + S_0 H_\Delta^k(F, G_\Delta)). \qquad (4.66)
\end{aligned}
$$

4.

$$
\begin{aligned}
& M_\gamma^k(F) \langle DF, DG_\delta \rangle_k \\
& \qquad = M_\gamma^k(F) \sum_{j=1}^k \delta \Delta IG_j S_T^2 \\
& \qquad + \left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) \delta^2 \Delta IG_j S_T^2 \\
& \qquad = \frac{M_\gamma^k(F)}{\delta} \sum_{j=1}^k \delta^2 \Delta IG_j S_T^2 \\
& \qquad + M_\gamma^k(F) \sum_{j=1}^k \delta^2 \Delta IG_j S_T^2 \left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) \\
& \qquad = \frac{1}{\delta} + T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i \qquad (4.67)
\end{aligned}
$$

5.

$$
\begin{aligned}
& G_\delta \langle DF, DM_\gamma^k(F) \rangle_k \\
& \quad = -\left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) \\
& \quad \times S_T \sum_{j=1}^k \delta \sqrt{\Delta IG_j} S_T \frac{2\delta\sqrt{\Delta IG_j}}{M_\sigma^k(F)} \\
& \quad = -2\left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) M_\gamma^k(F) \sum_{j=1}^k S_T^2 \delta^2 \Delta IG_j \\
& \quad = -2\left(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^n \sqrt{\Delta IG_i} Z_i + 2\beta\delta \sum_{i=1}^n \Delta IG_i\right) \qquad (4.68)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
H^k(F,G_\delta) &= (4.66) - (4.67) - (4.68) \\
&= -\frac{1}{\delta} + \Bigg(T\frac{\partial w}{\partial \delta} \\
&\quad + \sum_{i=1}^{n}\sqrt{\Delta IG_i}Z_i + 2\beta\delta\sum_{i=1}^{n}\Delta IG_i\Bigg) S_0 H^k_\Delta(F,G_\Delta). \qquad (4.69)
\end{aligned}
$$

Together with Equation (4.65) this implies

$$
\begin{aligned}
\frac{\partial u(\delta)}{\partial \delta} &= e^{-rT}E[\phi(S_T)\Bigg(-\frac{1}{\delta} + \Bigg(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^{n}\sqrt{\Delta IG_i}Z_i + 2\beta\delta\sum_{i=1}^{n}\Delta IG_i\Bigg) S_0 H^k_\Delta(F,G_\Delta) \\
&\quad + \left(\sum_{i=1}^{n}(\Delta t_i - b\Delta IG_i)\frac{\partial b}{\partial \delta}\right)\Bigg)]
\end{aligned}
$$

Hence, after summing up Δt_i's over the time grid:

$$
\begin{aligned}
&= e^{-rT}E[\phi(S_T)\Bigg(-\frac{1}{\delta} + \Bigg(T\frac{\partial w}{\partial \delta} + \sum_{i=1}^{n}\sqrt{\Delta IG_i}Z_i + 2\beta\delta\sum_{i=1}^{n}\Delta IG_i\Bigg) S_0 H^k_\Delta(F,G_\Delta) \\
&+ \left(T - \sum_{i=1}^{n} b\Delta IG_i\right)\frac{\partial b}{\partial \delta}\Bigg)]
\end{aligned}
$$

with

$$
\frac{\partial w}{\partial \delta} = \sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2-\beta^2}, \qquad (4.71)
$$

$$
\frac{\partial b}{\partial \delta} = \sqrt{\alpha^2-\beta^2}. \qquad (4.72)
$$

4.2.6. Sensitivity with respect to α.

$$
\frac{\partial u(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(e^{-rT}E[\phi(S_T)]\right) = e^{-rT}E[\phi'(S_T)\frac{\partial S_T}{\partial \alpha}] + e^{-rT}E^{(\alpha)}[\phi(S_T)], \qquad (4.73)
$$

where

$$
\begin{aligned}
E^{(\alpha)}[\phi(S_T)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(S_T)\prod_{i=1}^{n} f_{\mathcal{N}}(x_i;0,1) \\
\times\frac{\partial}{\partial \alpha}\left(\prod_{i=1}^{n} f_{IG}(y_i;\Delta t_i, b)\right) dx_1\dots dx_n dy_1\dots dy_n, \qquad (4.74)
\end{aligned}
$$

where $b = \delta\sqrt{\alpha^2-\beta^2}$. We define

$$
F = \prod_{i=1}^{n} f_{IG}(y_i;\Delta t_i, b), \qquad (4.75)
$$

and we evaluate the partial derivative as follows:

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\left(\prod_{i=1}^{n} f_{IG}(y_i;\Delta t_i,b)\right)\\
&\quad= \frac{\partial}{\partial\alpha}e^{\log(F)}\\
&\quad= F\frac{\partial}{\partial\alpha}\left(\sum_{i=1}^{n}\log(f_{IG}(y_i;\Delta t_i,b))\right)\\
&\quad= F\frac{\partial}{\partial\alpha}\left(\sum_{i=1}^{n}\log\left(\frac{\Delta t_i}{\sqrt{2\pi}}\right)+\Delta t_i b-\frac{3}{2}\log(y_i)-\frac{(\Delta t_i)^2}{2y_i}-\frac{b^2y_i}{2}\right)\\
&\quad= F\sum_{i=1}^{n}\Delta t_i\frac{\partial b}{\partial\alpha}-\frac{\partial b}{\partial\alpha}by_i\\
&\quad= \prod_{i=1}^{n} f_{IG}(y_i;\Delta t_i,b)\sum_{i=1}^{n}(\Delta t_i-by_i)\frac{\partial b}{\partial\alpha}. \qquad (4.76)
\end{aligned}$$

Therefore,

$$E^{(\alpha)}[\phi(S_T)] = E[\phi(S_T)\sum_{i=1}^{n}(\Delta t_i - b\Delta IG_i)\frac{\partial b}{\partial\alpha}]. \qquad (4.77)$$

The first term of Equation (4.61) involves the derivative operator. We apply Theorem (3.7) to remove the derivative in the expectation. We define $G_\alpha = \dfrac{\partial S_T}{\partial\alpha}$. In the following list the detailed calculations of the terms in Equation (4.2) are given.

1. $G_\alpha = T\dfrac{\partial w}{\partial\alpha}S_T$.
2. $D_jG_\alpha = T\dfrac{\partial w}{\partial\alpha}\delta\sqrt{\Delta IG_j}S_T$
3.

$$\begin{aligned}
G_\alpha M^k_\gamma(F)L^kF &= T\frac{\partial w}{\partial\alpha}S_T M^k_\gamma(F)\left(-S_T\sum_{j=1}^{k}\left(\delta^2\Delta IG_j - Z_j\delta\sqrt{\Delta IG_j}\right)\right)\\
&= T\frac{\partial w}{\partial\alpha}\left(-1+S_0\frac{\sum_{j=1}^{k}Z_j\delta\sqrt{\Delta IG_j}}{S_0\sum_{i=1}^{n}\delta^2\Delta IG_j}\right)\\
&= T\frac{\partial w}{\partial\alpha}(-1+S_0H^k_\Delta(F,G_\Delta)). \qquad (4.78)
\end{aligned}$$

4.

$$\begin{aligned}
M^k_\gamma(F) < DF, DG_\alpha >_k &= M^k_\gamma(F)\sum_{j=1}^{k}T\frac{\partial w}{\partial\alpha}\delta^2\Delta IG_jS_T^2\\
&= T\frac{\partial w}{\partial\alpha} \qquad (4.79)
\end{aligned}$$

5.

$$
\begin{aligned}
G_\alpha < DF, DM_\gamma^k(F) >_k &= -T\frac{\partial w}{\partial \alpha}S_T\sum_{j=1}^{k}\delta\sqrt{\Delta IG_j}S_T\frac{2\delta\sqrt{\Delta IG_j}}{M_\sigma^k(F)} \\
&= -2T\frac{\partial w}{\partial \alpha}M_\gamma^k(F)\sum_{j=1}^{k}S_T^2\delta^2\Delta IG_j \\
&= -2T\frac{\partial w}{\partial \alpha} \qquad (4.80)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
H^k(F, G_\alpha) &= (4.78) - (4.79) - (4.80) \\
&= T\frac{\partial w}{\partial \alpha}S_0 H_\Delta^k(F, G_\Delta). \qquad (4.81)
\end{aligned}
$$

Together with Equation (4.77) this implies

$$
\begin{aligned}
\frac{\partial u(\alpha)}{\partial \alpha} &= e^{-rT}E[\phi(S_T)\left(T\frac{\partial w}{\partial \alpha}S_0H_\Delta^k(F, G_\Delta) + \left(\sum_{i=1}^{n}(\Delta t_i - b\Delta IG_i)\frac{\partial b}{\partial \alpha}\right)\right)] \\
&= e^{-rT}E[\phi(S_T)\left(T\frac{\partial w}{\partial \alpha}S_0H_\Delta^k(F, G_\Delta) + \left(T - \sum_{i=1}^{n}b\Delta IG_i\right)\frac{\partial b}{\partial \alpha}\right)],
\end{aligned}
$$

after summing up Δt_i's over the time grid, and with

$$
\begin{aligned}
\frac{\partial w}{\partial \alpha} &= \delta\alpha\left(\frac{1}{\sqrt{\alpha^2 - (\beta+1)^2}} - \frac{1}{\sqrt{\alpha^2 - \beta^2}}\right), \qquad (4.82) \\
\frac{\partial b}{\partial \alpha} &= \frac{\delta\alpha}{\sqrt{\alpha^2 - \beta^2}}. \qquad (4.83)
\end{aligned}
$$

5. Localization Functions. When the sensitivity is calculated by using the Malliavin calculus, the Malliavin weight might grow during the simulation causing a higher variance. Thus, the speed of convergence will be slower. In order to avoid this problem we use the variance reduction method that is introduced in [12]. We use a localization function which vanishes out of an interval $[K-\delta, K+\delta]$ for some $\delta > 0$. We first define functions B_δ and G_δ where B_δ is the derivative of G_δ.

$$
B_\delta(x) = \begin{cases} 0 & \text{if } x < K-\delta \\ \mathcal{L}'(x) & \text{if } x \in [K-\delta, K+\delta] \\ \mathcal{R}'(x) & \text{if } x > K+\delta \end{cases}, \qquad (5.1)
$$

and

$$
\begin{aligned}
G_\delta(x) &= \int_{-\infty}^{y} B_\delta(x)dx \\
&= \begin{cases} 0 & \text{if } y < K-\delta \\ \mathcal{L}(y) & \text{if } y \in [K-\delta, K+\delta] \\ \mathcal{R}(y) & \text{if } y > K+\delta, \end{cases} \qquad (5.2)
\end{aligned}
$$

where $\mathcal{R}(y) = y - K$ for a call option and $\mathcal{R}(y) = 1$ for a digital option. The smooth approximation of the pay-off function is $\mathcal{L}(y) = \frac{(y-(K-\delta))^2}{4\delta}$ for a call option.
We introduce $\mathcal{L}(y) = \exp\left(-\left(\frac{y-(K+a\delta)}{y-(K-a\delta)}\right)^b\right)$ as the smooth approximation of the digital option's pay-off function in the interval $[K-\delta, K+\delta]$, where a is a positive real number and b is an even integer.
Hence, we define the localization function

$$F_\delta(y) = max(R(y), 0) - G_\delta(y) \tag{5.3}$$

$$= \begin{cases} 0 & \text{if } y < K - \delta \\ -\mathcal{L}(y) & \text{if } y \in [K-\delta, K] \\ \mathcal{R}(y) - \mathcal{L}(y) & \text{if } y \in [K, K+\delta] \\ 0 & \text{if } y > K + \delta. \end{cases} \tag{5.4}$$

As it can be seen in Figure 5.1, G_δ becomes steeper for larger values of b enabling a better approximation of the digital option's pay-off function. However, the interval where G_δ is non-zero shrinks and the effect of localization vanishes.

By using Equation (5.3), pay-off function is given by $\phi(y) = F_\delta(y) + G_\delta(y)$ enabling us to write, for instance, the Greek, Δ as follows;

$$\begin{aligned}\frac{\partial E[\phi(S_T)]}{\partial S_0} &= \frac{\partial E[F_\delta(S_T)]}{\partial S_0} + \frac{\partial E[G_\delta(S_T)]}{\partial S_0} \\ &= E[F_\delta(S_T)H^k(S_T, \frac{\partial S_T}{\partial S_0})] + E[B_\delta(S_T)\frac{\partial S_T}{\partial S_0}]. \end{aligned} \tag{5.5}$$

We took $r = 0$ to simplify the expressions in above. Since F_δ vanishes out the interval $[K-\delta, K+\delta]$ the contribution of large values of S_T in the Malliavin weight will vanish, as well.

A third term needs to be added in Equation (5.5) when we are calculating the derivatives of an option with respect to the parameters in the Barndoff-Nielsen parametrization of NIG model. To be more specific, let us consider the partial derivative with respect to β, i.e.

$$\frac{\partial E[\phi(S_T)]}{\partial \beta} = \frac{\partial E[F_\delta(S_T)]}{\partial \beta} + \frac{\partial E[G_\delta(S_T)]}{\partial \beta} \tag{5.6}$$

$$= E[F_\delta(S_T)H(S_T, \frac{\partial S_T}{\partial \beta})] + E[B_\delta(S_T)\frac{\partial S_T}{\partial \beta}] + E^{(\beta)}[G_\delta(S_T)], \tag{5.7}$$

where $E^{(\beta)}[\cdot]$ is the expectation after the partial derivative of IG density is taken [3] and

$$H(S_T, \frac{\partial S_T}{\partial \beta}) = \left(T\frac{\partial w}{\partial \beta} + \delta^2 \sum_{i=1}^{n} \Delta IG_i\right) S_0 H_\Delta^k(F, G_\Delta) + \left(T - \sum_{i=1}^{n} b\Delta IG_i\right)\frac{\partial b}{\partial \beta} \tag{5.8}$$

is the weight in Equation (4.58b). We also introduce the variations of (5.6) by mixing it with the finite difference method when a function is smooth. We explain these variations and the notation that we use to denote them in the presence of above example for NIG model:

[3] The definition is given in Equation (4.50).

- **NIGM**: Global Malliavin sensitivity for NIG model,
$$\frac{\partial E[\phi(S_T)]}{\partial\beta} = E[\phi(S_T)H(S_T, \frac{\partial S_T}{\partial\beta})].$$
- **NIGML**: Local Malliavin sensitivity for NIG model,
$$\frac{\partial E[\phi(S_T)]}{\partial\beta} = E[F_\delta(S_T)H(S_T, \frac{\partial S_T}{\partial\beta})] + E[B_\delta(S_T)\frac{\partial S_T}{\partial\beta}] + E^{(\beta)}[G_\delta(S_T)].$$
- **NIGMLF**: Local Malliavin sensitivity for NIG model with finite difference method used in the second term of Equation (5.6),
$$\frac{\partial E[\phi(S_T)]}{\partial\beta} = E[F_\delta(S_T)H(S_T, \frac{\partial S_T}{\partial\beta})] + \frac{u(S_T, \beta + h) - u(S_T, \beta - h)}{2h},$$
where $u(S_T, \beta) = E[G_\delta(S_T)]$.
- **NIGMLFF**: Local Malliavin sensitivity for NIG model with finite difference method used in first and second terms of Equation (5.6). The weight $H(S_T, \frac{\partial S_T}{\partial\beta})$ given in Equation (5.8) has two terms. The second term comes from the partial derivative of the product density function, $\mathbf{F}(\beta)$, of IG distribution with respect to β. Since $\mathbf{F}$ is smooth in β, we use finite difference method to calculate the derivative. In order to accomplish this, we only shift β in the density. This is equivalent to simulating S_T from the following functional form, S_T^*
$$S_T^*(\beta \pm h) = S_0 e^{(r-q+w(\beta))T + \sum_{i=1}^n \delta\sqrt{\Delta IG_i(\beta\pm h)}Z_i + \beta\delta^2\sum_{i=1}^n \Delta IG_i(\beta\pm h)}. \quad (5.9)$$
Hence,
$$\begin{aligned}\frac{\partial E[\phi(S_T)]}{\partial\beta} &= E[F_\delta(S_T)H^k(S_T, \frac{\partial S_T}{\partial\beta})] \\ &+ \frac{E[F_\delta(S_T^*(\beta + h))] - E[F_\delta(S_T^*(\beta - h))]}{2h} \\ &+ \frac{u(S_T, \beta + h) - u(S_T, \beta - h)}{2h},\end{aligned}$$
where $H^k(S_T, \frac{\partial S_T}{\partial\beta})$ is the first term of the summation in (5.8).
- **VGMF**: Global Malliavin sensitivity for VG model with finite difference method used to calculate $E^{(\kappa)}[\phi(S_T)]$. When we are calculating Vega2 in the VG model the weight has a logarithmic term as functional of the Gamma random numbers. Thus, as smaller Gamma random numbers drawn during the Monte-Carlo simulations, logarithmic term gets larger values and this causes large variance in the simulations. In order to get rid of the large variance, we apply the finite difference method in order to take the derivative of the Gamma product density with respect to κ by following the method explained for **NIGMLFF** but without using any localization. Thus,
$$\begin{aligned}\mathcal{V}_2 &= \frac{\partial}{\partial\kappa}\left(E[\phi(S_T)]\right) \\ &= E[\phi(S_T)H^k(S_T, \frac{\partial S_T}{\partial\kappa})] + \frac{E[\phi(S_T^*(\kappa + h)] - E[\phi(S_T^*(\kappa - h)]}{2h}. \quad (5.10)\end{aligned}$$

where

$$H^k(S_T, \frac{\partial S_T}{\partial \kappa}) = S_0 \frac{\partial w}{\partial \kappa} T H^k_\Delta (S_T, G_\Delta)$$

and

$$S^*_T(\kappa \pm h) = S_0 e^{(r+w(\kappa))T + \sum_{i=1}^n \sigma \sqrt{\Delta G_i(\kappa \pm h)} Z_i + \theta \sum_{i=1}^n \Delta G_i(\kappa \pm h)}. \qquad (5.11)$$

6. Data and Numerical Results. We used S&P 500 index European option data for our numerical implementations, [1]. The figures that represent the numerical results are obtained for the average of the bid and ask prices in August 06, 2008. During the option's model calibration process we used the fast Fouriér transform method as it is discussed in [16]. As we implement the nonlinear least square method we used implied vega of the each option in order to weight the squared differences as suggested in [9]. Hence, we obtained the following values for the parameters of each model: $\sigma_{\text{BS}} = 0.2002$, $(\sigma_{\text{VG}}, \theta_{\text{ VG}}, \kappa_{\text{VG}}) = (0.0102, -1.4552, 0.0198)$, $(\alpha_{\text{NIG}}, \beta_{\text{ NIG}}, \delta_{\text{NIG}}) = (21.8248, -6.1882, 0.7934)$ and Cont-Tankov [9] parametrization values of NIG are $(\sigma_{\text{NIG}}, \theta_{\text{ NIG}}, \kappa_{\text{NIG}}) = (0.1947, -0.2346, 0.0602)$.

The comparison between different localizations of Malliavin and finite difference approaches depend on the benchmark line that is obtained by evaluating the integrals given in Section 2 and we call that line as VGFFT or NIGFFT. We also present $\pm 1\%$ of the FFT line. We used the centered finite difference (CF) method as it is given in Section 5. Thus, for gamma it turns out to be

$$\frac{\partial \Delta}{\partial S_0} = \frac{E[\phi(S_T(S_0+h))] - 2E[\phi(S_T(S_0))] + E[\phi(S_T(S_0-h))]}{h^2}.$$

The parameter values that are used during the simulations represent the market with $S_0 = 1289.19$, $r = 2.8152\%$ and $q = 2.04\%$. We run simulations for an ATM option with $T = 0.3726$. The choice of h depends on the number of paths, M such that $h = M^{-1/5}$ as discussed in [14]. The localization parameters are chosen as $(a, b, \delta) = (1, 2, 150)$. We generate inverse Gaussian random numbers by using the algorithm introduced in [18]. Since the European type call and digital options are path independent, we used one time step. This results in the parameter $a = T/\kappa$ of the gamma distribution to be larger than one. Hence, we use Fishman's generator, [11] for small number of simulations and for the large ones Marsaglia and Tsang's faster gamma random number generator, [17]. We also used the common random numbers in the path generation in order to reduce the variance in finite difference method.

In Figure 6.1, we present the simulation results of delta, gamma and rho under NIG and VG models for call options. We observe that NIGML has a remarkable improvement on NIGM when delta and rho are simulated. However, Malliavin approach does not bring improvement on CF. When gamma is simulated NIGM is converging in the $\pm 1\%$ band while CF does not. Due to the strong discontinuity of the digital option, the convergence of CF is very slow as it is seen in Figure 6.3. Improvement of localization in Malliavin approach is observed, as well.

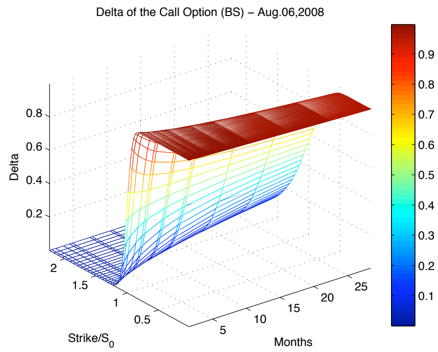
Figure 6.2 presents the alpha, beta and delta sensitivities of Barndoff-Nielsen parametrization of NIG model together with drift, vega and vega2 sensitivities of Cont-Tankov parametrization. NIGM and NIGML outperforms CF in Figures 6.2(a), 6.2(b). However, the simulations for delta sensitivity in Figure 6.2(c) shows that CF outperforms all of the variations of Malliavin approach except NIGMLFF. As we obtain the Figures 6.2(d), 6.2(e) and 6.2(f), we use Equation (2.10). The dotted lines of

$\pm 1\%$ of the FFT line are also obtained by mapping $\pm 1\%$ of the FFT lines of Figures 6.2(a), 6.2(b) and 6.2(c). NIGMBEST is the combination of the best of the variations of Malliavin approach under Barndoff-Nielsen parametrization. Thus, for this simulations we used $(\frac{\partial C}{\partial \alpha}, \frac{\partial C}{\partial \beta}, \frac{\partial C}{\partial \delta}) = (\mathrm{NIGM}, \mathrm{NIGML}, \mathrm{NIGMLFF})$. The digital option simulations are shown in Figure 6.4 with $(\frac{\partial C}{\partial \alpha}, \frac{\partial C}{\partial \beta}, \frac{\partial C}{\partial \delta}) = (\mathrm{NIGM}, \mathrm{NIGML}, \mathrm{NIGMLF})$. After a simulation of $M = 10^7$, it is clear that CF is outperformed by NIGMBEST and NIGM for all simulations for both call and digital options.

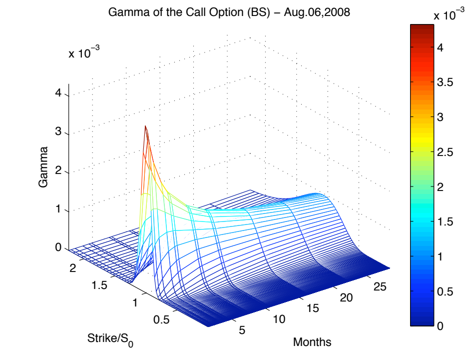
Finally, drift, vega and vega2 under VG model are given in Figure 6.5. For both digital and call options VGML and VGMLF improve the convergence of the Malliavin approach and outperforms the CF for digital option. However, none of the methods converge in the vega simulations of the digital option even after 10^8 simulations. On the other hand, VGML outperforms CF for call option after simulating vega. As we discussed in Section 4.1.6, even tough the Malliavin weight without localization is well-defined, it tends to explode during the Monte Carlo simulations as small gamma random numbers are generated. We used VGMF as a combined weight obtained by the Malliavin calculus and the finite difference method. The other method VGMLFF extends it by using a localization function. Simulations of vega2 of the digital option converge as we use these methods. Figure 6.5(f) shows that convergence in VGMLFF method is better. Figure 6.5(e) shows that in the call option case neither of the methods including the centered finite difference do not yield converging simulations.

Conclusion and Discussion. In this paper we introduce new sensitivities of the option contracts when the underlying is modelled by an exponential Variance Gamma or a Normal Inverse Gaussian process. We analyse new sensitivities, *drift*, *vega* and *vega2* together with the *Greeks* of the options that are written on S&P 500 index. We also introduced the use of the Malliavin calculus through Brownian subordination of the VG and NIG processes. We obtained discrete formulas that can be used in the Monte-Carlo simulations directly. In the digital option case we observed that Malliavin calculus outperforms the centered finite difference method. However, for call and digital options the use of the Malliavin calculus may not improve the convergence of the simulations when the index is modelled by Variance Gamma process. When we run the simulations for the sensitivities except for gamma the Malliavin approach and its variations that we introduced perform at most as good as the centered finite difference method. Exceptions that the variations of Malliavin approach performs slightly better are the *vega* of the call option and the drift of the digital option. On the other hand, when the index is modelled by the NIG process, Malliavin calculus outperforms centered finite difference method for all of the sensitivities except delta of the call option. However, even in that case a variation of the Malliavin approach is performing as good as centered finite difference method. We certainly suggest to use the Malliavin calculus for the sensitivity calculations of options when the index is modelled by a Normal Inverse Gaussian process and when it comes to model the index with the VG process we suggest to use the finite difference method.

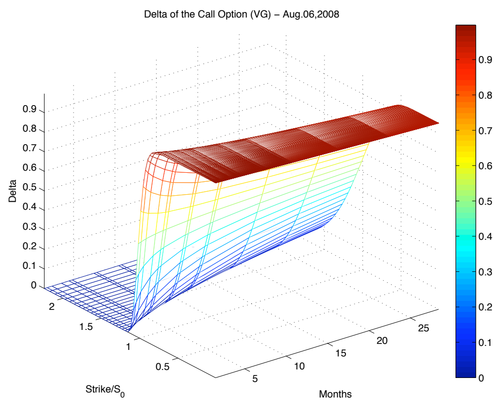
In Section 3 we give a generalized finite dimensional Malliavin calculus. Hence, as we calculate the Malliavin weight we only use the normal random variables that arise from the discretization scheme of the Brownian subordination of the VG and NIG processes. If gamma random variables are also introduced in the weight we obtained a better convergence in the VG case of the delta simulations. We will discuss the details of this approach and the applications to other sensitivities in our forthcoming paper.



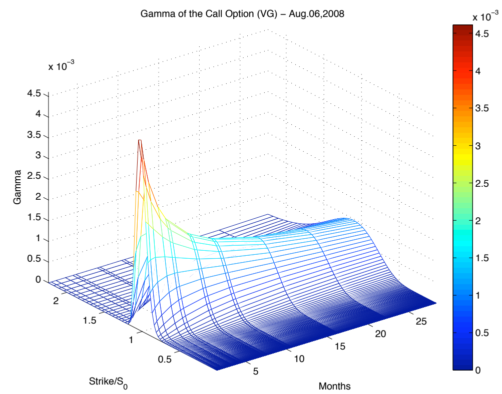
(a)



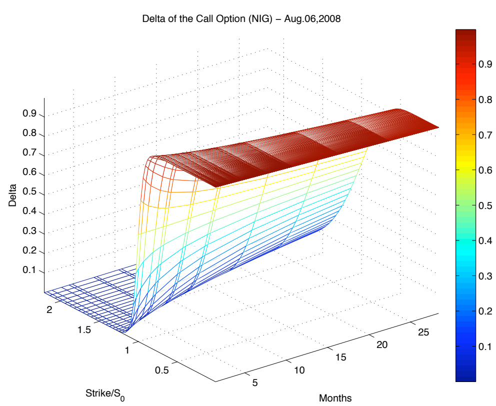
(b)



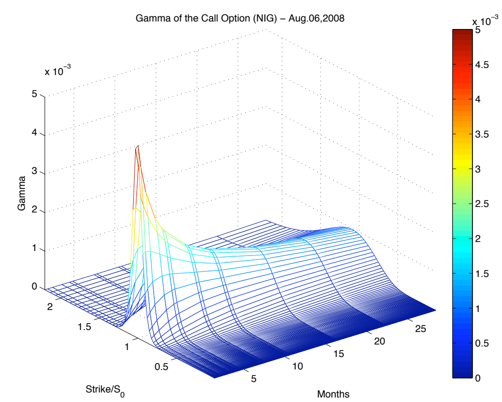
(c)



(d)



(e)



(f)

Fig. 2.1. *Call Option's Delta and Gamma under BS, VG and NIG models.*

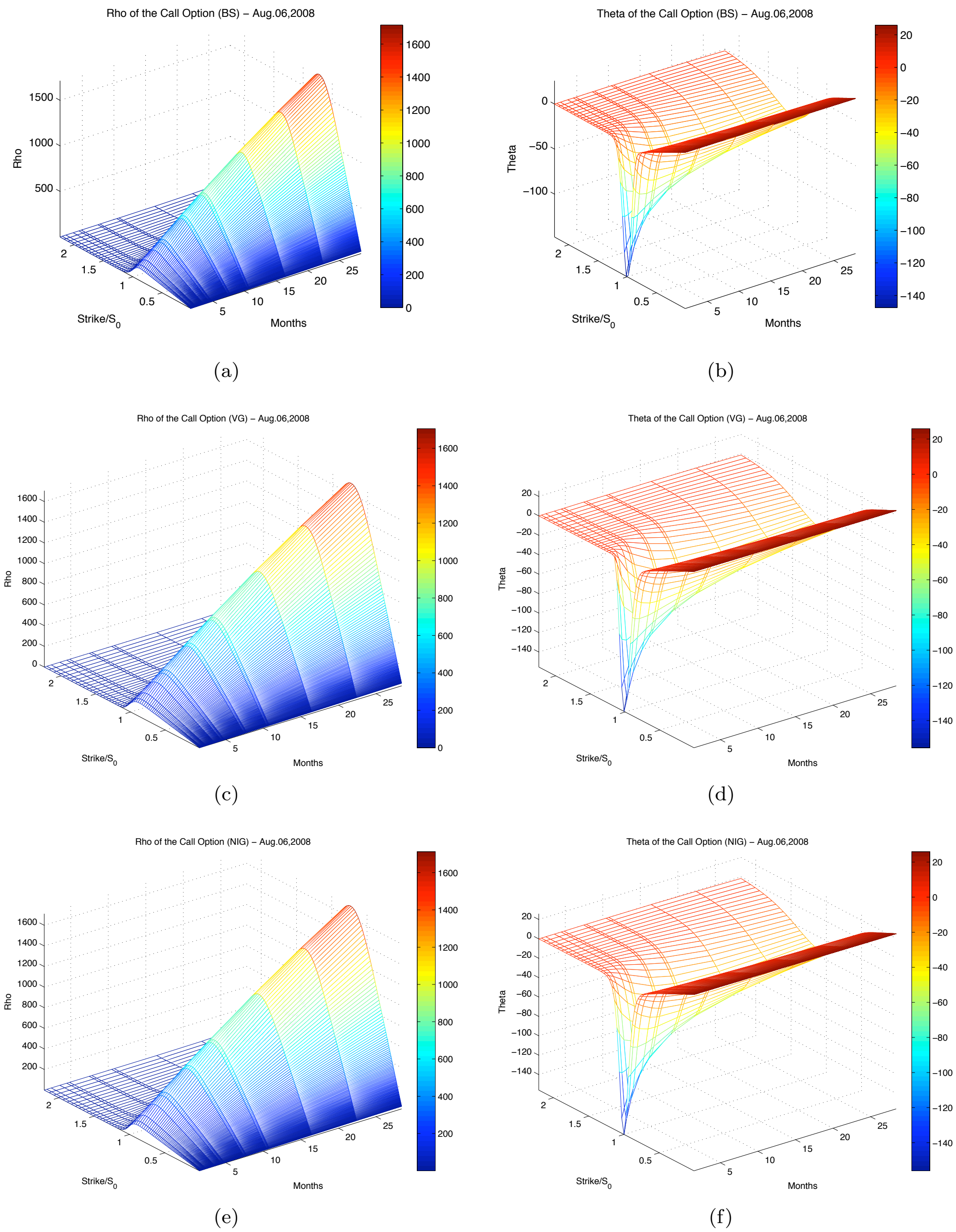


FIG. 2.2. *Call Option's Rho and Theta under BS, VG and NIG models.*

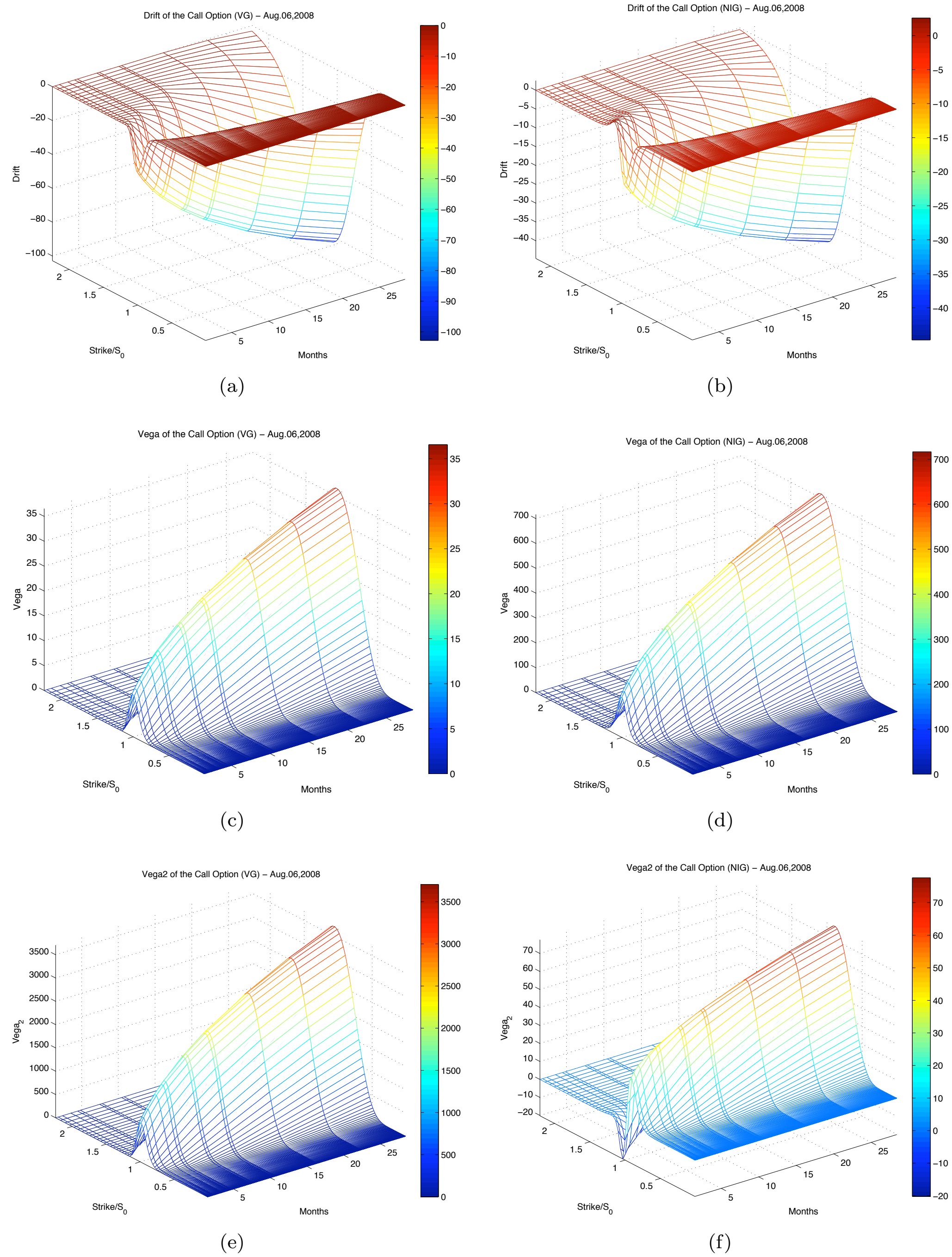


FIG. 2.3. *Call Option's Drift, Vega and Vega2 under VG and NIG models.*

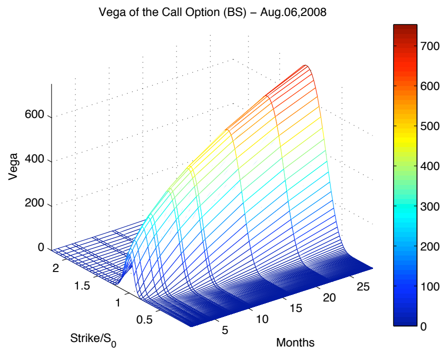


FIG. 2.4. *Call Option's Vega under BS model.*

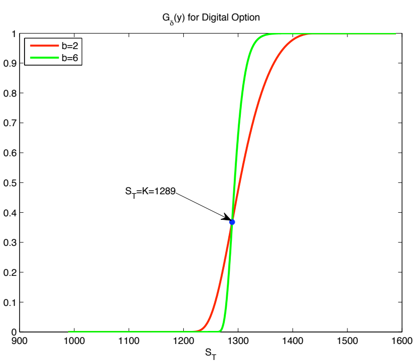


FIG. 5.1. G_δ *for digital option with* $a = 1$ *and* δ*=150.*

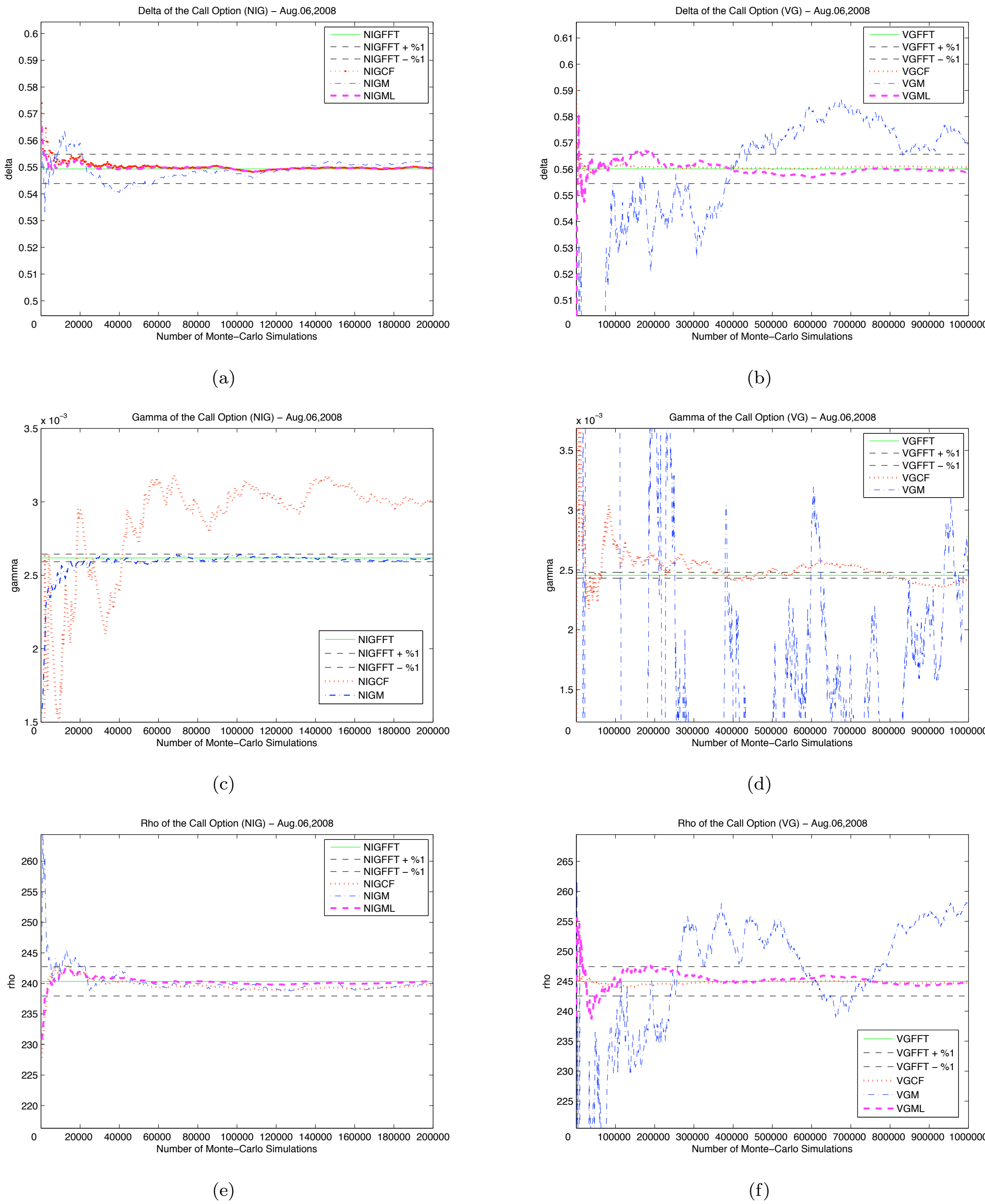


FIG. 6.1. *Call Option's Delta, Gamma, Rho under VG and NIG models.*

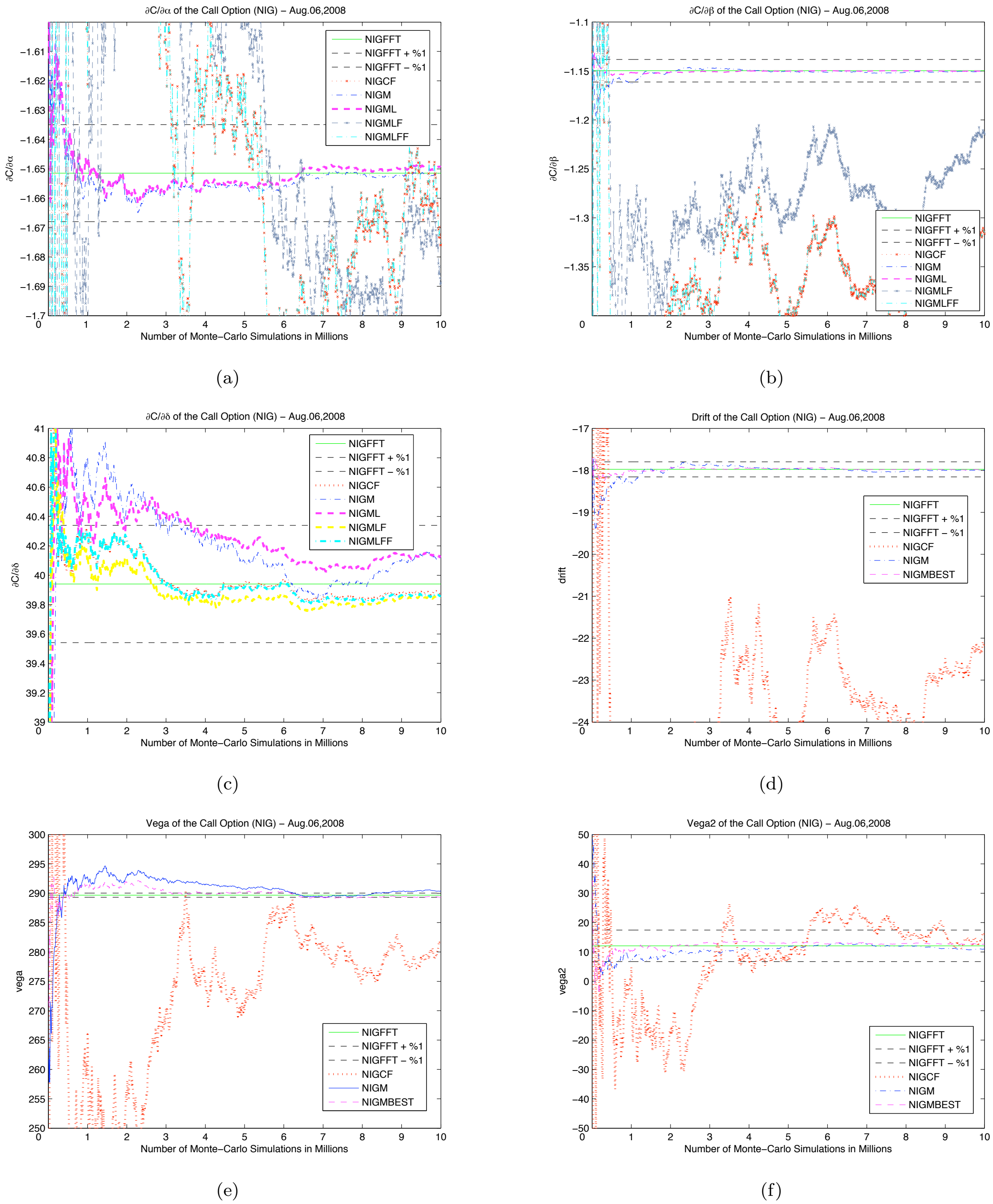
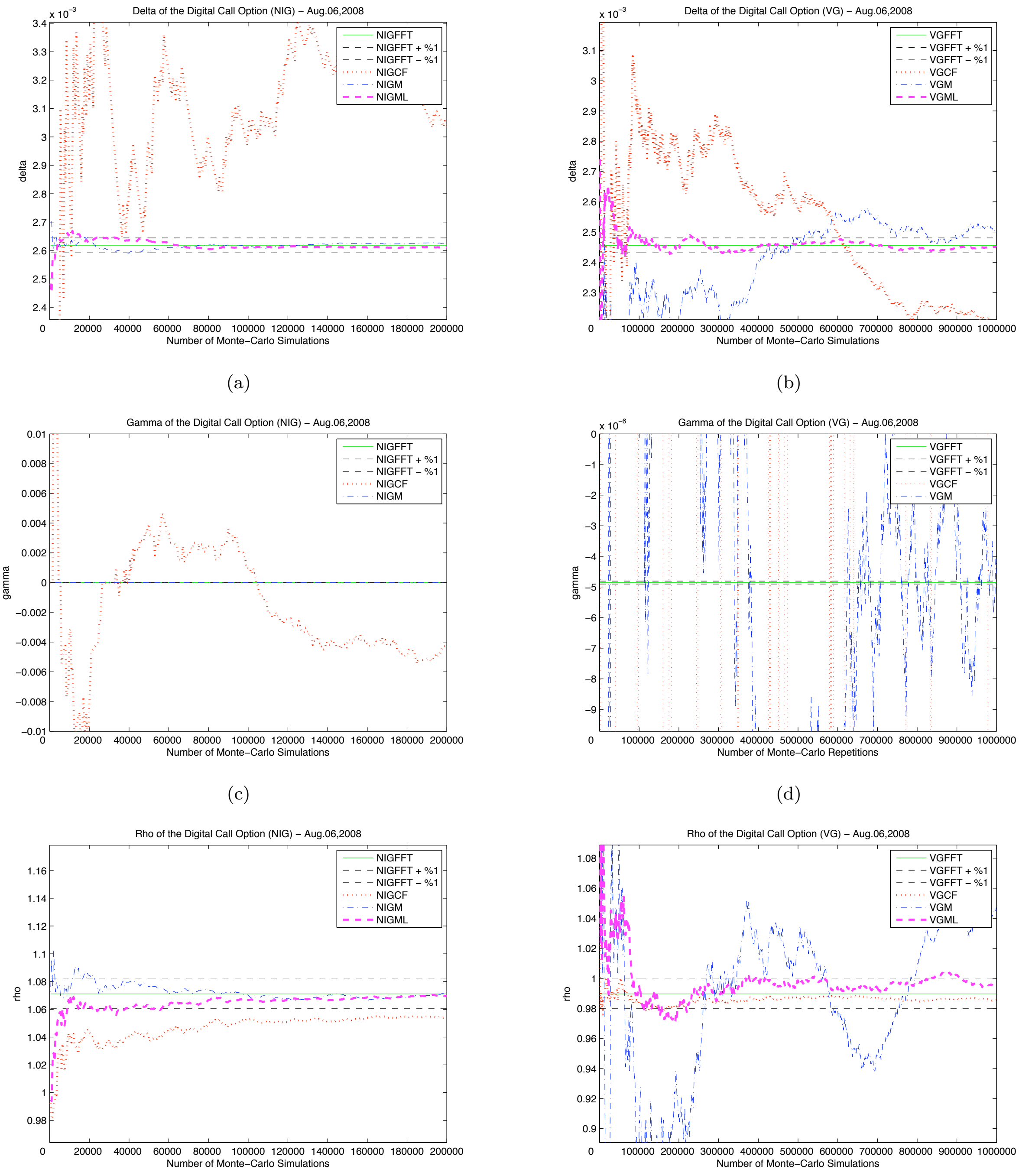


FIG. 6.2. *Call Option's Drift, Vega and Vega2 sensitivities under NIG model.*



(a) (b)

(c) (d)

(e) (f)

FIG. 6.3. *Digital Call Option's Delta, Gamma, Rho under VG and NIG models.*

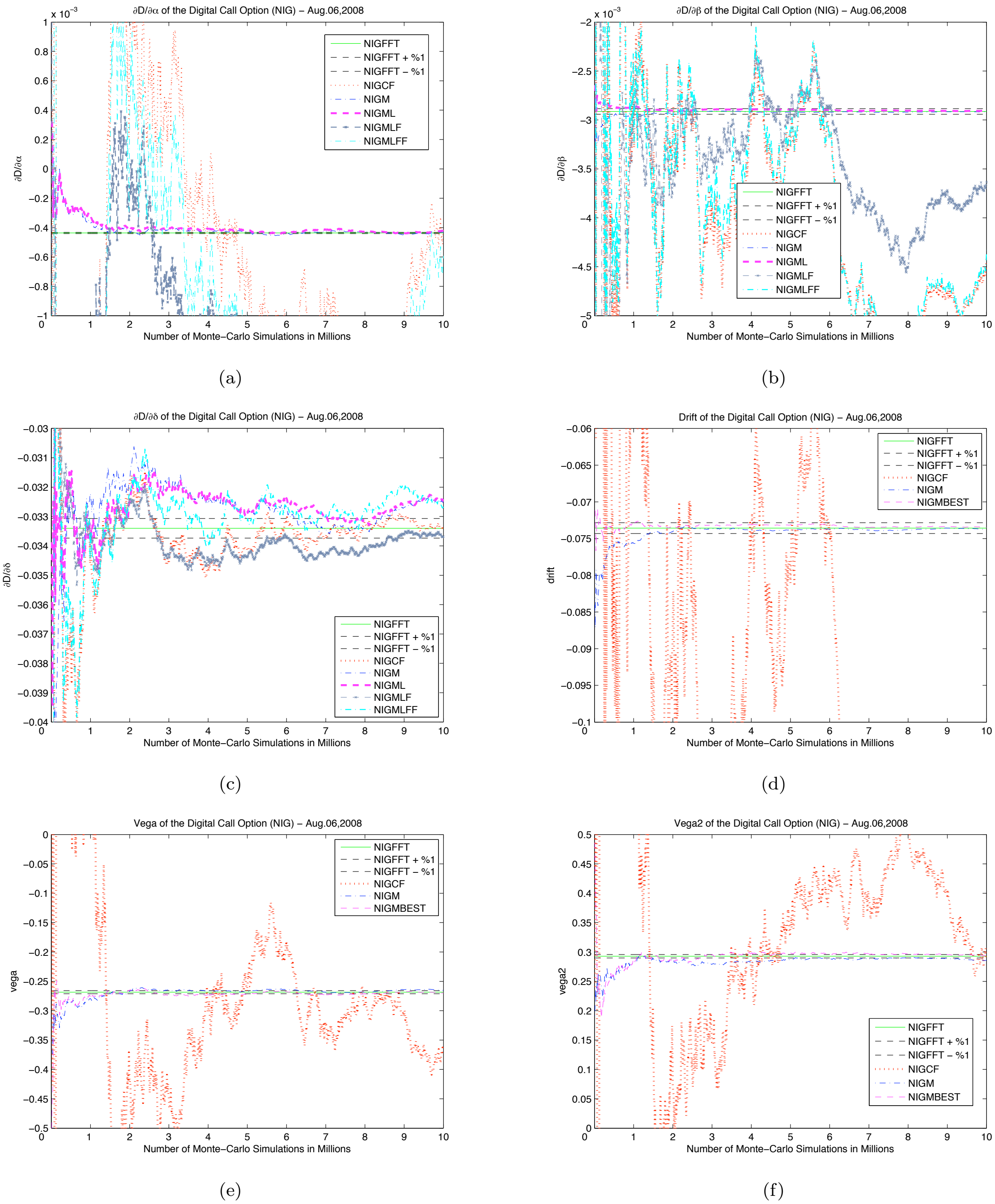


FIG. 6.4. *Digital Call Option's Drift, Vega and Vega2 sensitivities under NIG model.*

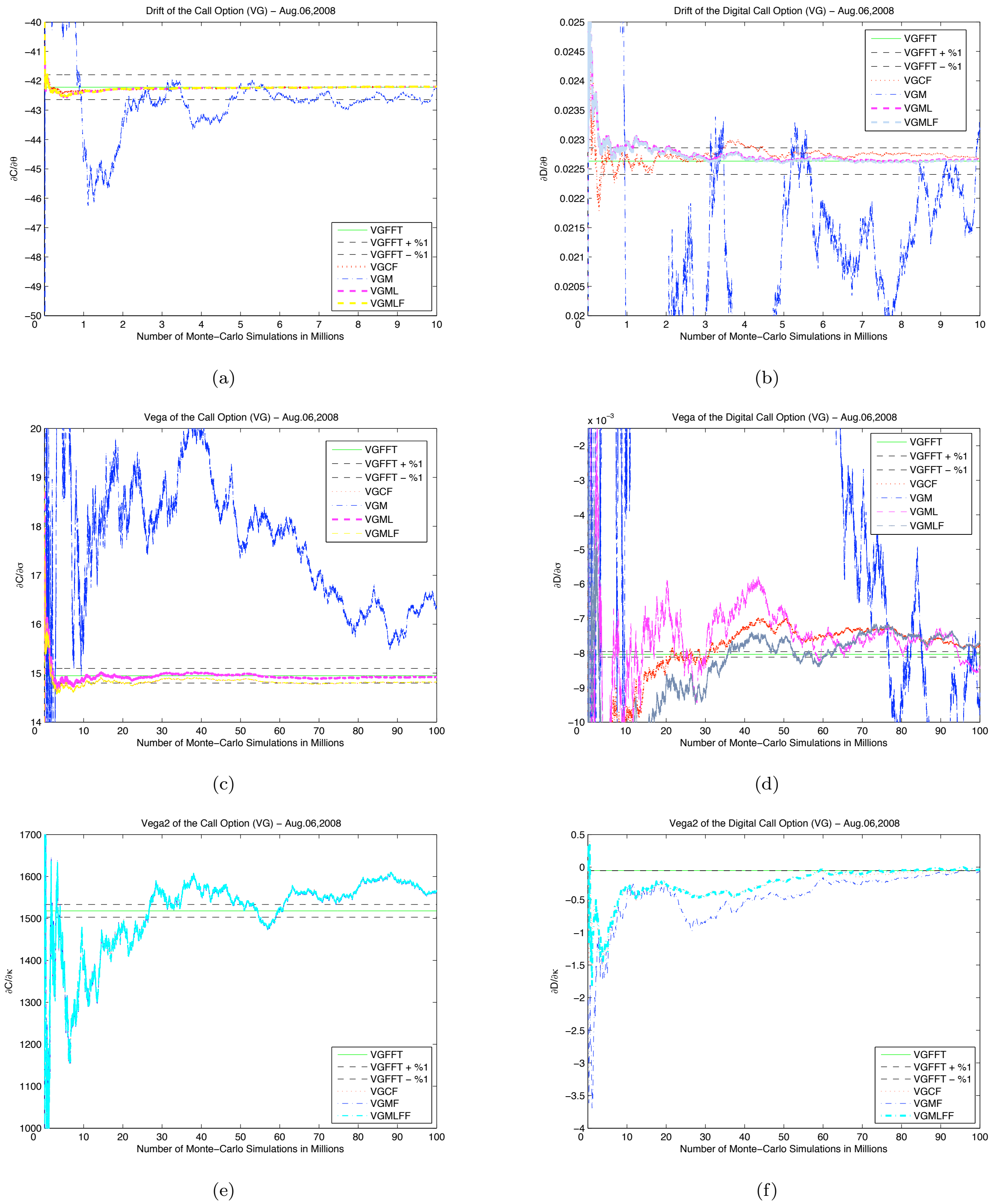


Fig. 6.5. *Digital Call and Call Options' Drift and Vega sensitivities under VG model.*

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