On the Curve $X(9)$

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Introduction

In this note we use the machinery of computing with theta functions [FKK] to describe a model for the modular curve $\mathbb{H}/\Gamma(9)$, also known as $X(9)$. We show that this curve of genus 10 is given as a complete intersection of 2 cubics. This description is an improvement over the one given in [FKK] where it was given as the intersection of 5 hypersurfaces. We construct differentials on the curve through the use of theta functions with characteristics. Finally we use this information to find the Weierstrass gap sequence for the Weierstrass points.

We use the following notation

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau(n+\varepsilon/2)^2+2\pi i(n+\varepsilon/2)(z+\varepsilon'/2)}$$
$$= \sum_{n=-\infty}^{\infty} q^{(n+\varepsilon/2)^2} e^{2\pi i(n+\varepsilon/2)(z+\varepsilon'/2)}$$
$$\theta(z,\tau) = \theta\begin{bmatrix}0\\ 0\end{bmatrix}(z,\tau)$$

where $q = e^{\pi i\tau}$.

Mappings of $X(k)$, k odd, into projective space are usually done using functions of the form

$$\theta\begin{bmatrix}x/k\\ 1\end{bmatrix}(0,k\tau)$$

for x odd. It is convenient to define a related q series. For fixed k odd define

$$\varphi_\ell(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{k(n+(2\ell-1)/2k)^2}$$
$$= e^{-\pi i(2\ell-1)/2k}\theta\begin{bmatrix}(2\ell-1)/k\\ 1\end{bmatrix}(0,k\tau).$$

The functions φ_ℓ were used by Klein [K1] to parametrize $X(5)$ and Klein and Hurwitz [K,H] used them to parametrize $X(7)$. For $k=7$ they showed that $\varphi_1^3\varphi_3+\varphi_3^3\varphi_2-\varphi_2^3\varphi_1=0$, which is the equation for $X(7)$. Hurwitz showed that the forms $\eta^3\varphi_\ell$, $\ell=1,2,3$ are the holomorphic differentials on $X(7)$, where

$$\begin{aligned}\eta(\tau) &= q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n})\\ &= q^{1/12}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+n}\\ &= \exp(-\pi i/6)\theta\begin{bmatrix}1/3\\1\end{bmatrix}(0,3\tau).\end{aligned}$$

We will obtain a similar result for $X(9)$. We use the methods developed in [FKK] using theta functions with characteristics and applying the residue theorem.

§1 The Geometry of the curve $X(9)$

First let us recall some basic facts about the curve $X(9)$. In general we will consider $X(k)$ as the compactification of $\mathbb{H}/\Gamma(k)$ obtained by filling in the cusps. The curve $X(9)$ is of genus 10 and has 36 cusps (For Algorithm to write the cusps see [Sh].) Inside the group $\Gamma/\Gamma(9)=\mathrm{PSL}_2(9)$, which is the automorphism group of $X(9)$, we find the translation $\tau\to\tau+3$. The mapping $\mathbb{H}/\Gamma(9)\to\mathbb{H}/\langle\Gamma(9),\tau\to\tau+3\rangle$ is $3:1$ mapping with fixed point in the following 9 cusps: $1/9$, $2/9$, $4/9$, $1/3$, $2/3$, $4/3$, $5/3$, $7/3$, $8/3$. The order of the ramification is 3 at each of the 9 points. Thus using Riemann-Hurwitz we conclude that $\mathbb{H}^2/\langle\Gamma(9),\tau\to\tau+3\rangle$ is a curve of genus 1.

In [FKK] a projective representation of $\mathrm{PSL}_2(k)$ was defined and an equivariant map given by theta constants $X(k)\to\mathbb{CP}^{(k-3)/2}$. In the case of $k=9$ we obtain a map $X(9)\to\mathbb{CP}^3$ induced by $\tau\to(\varphi_1(\tau),\varphi_2(\tau),\varphi_3(\tau),\varphi_4(\tau))$. The theta function transforms under $\tau\to\tau+1$ in the following manner: $\varphi_\ell(\tau+1)=\exp(\pi i/36)\exp(\pi i\ell(\ell-1)/9)\varphi_\ell(\tau)$. Since $\ell(\ell-1)$ is even we can write the transformation rule as $\varphi_\ell(\tau+1)=\exp(\pi i/4)\zeta^{-(2\ell-1)^2}\varphi_\ell(\tau)$ where $\zeta=\exp(2\pi i/9)$. This gives the representation on one generator of $\mathrm{PSL}_2(k)$.

In the following computations it will be useful to use the general transformation formula for theta functions with characteristic. Let

$$\chi = \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$$

be a characteristic and let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of $\mathrm{SL}_2(\mathbb{Z})$. Let γ act on characteristics by

$$\chi\gamma = \begin{bmatrix} a\varepsilon + c\varepsilon' - ac \\ b\varepsilon + d\varepsilon' + bd \end{bmatrix}.$$

We have the transformation rule for theta constants $\theta[\chi](\tau) = \theta[\chi](0, \tau)$

$$\theta[\chi]\left(\frac{a\tau + b}{c\tau + d}\right) = \kappa(\chi, \gamma)(c\tau + d)^{1/2}\theta[\chi\gamma](\tau),$$

where

$$\kappa(\chi, \gamma) = \exp\{2\pi i((-1/4)(a\varepsilon + c\varepsilon') - (1/8)(ab\varepsilon^2 + cd\varepsilon'^2 + 2bc\varepsilon\varepsilon'))\}\kappa(0, \tau)$$

See [BFK] for details.

Now we would like to prove the following theorem.

Theorem. *The map $\varphi : X(9) \to \mathbb{CP}^3$ is an embedding into the complete intersection of two cubic hypersurfaces C_1, C_2 given by the following equations respectively*

$$-x_1x_3^2 + x_1^2x_4 + x_3x_4^2 = 0$$
$$-x_2^3 + x_1^2x_3 - x_1x_4^2 + x_3^2x_4 = 0,$$

where $(x_1 : x_2 : x_3 : x_4)$ is a point in $\mathbb{CP}^3$.

Proof. Let $f(x_1, x_3, x_4)$ denote the left hand side of the first equation. In the variables $(x_1 : x_3 : x_4)$ the curve $f = 0$ is an elliptic curve in $\mathbb{CP}^2$. Let $g(x_1, x_3, x_4) = f(x_1, x_4, x_3)$. Then $g = 0$ is also an elliptic curve, and both curves are left invariant under the order three transformation $a : (x_1, x_3, x_4) \to (x_4, x_1, -x_3)$. The two curves intersect in the points

$(1:0:0)$, $(-1:\omega:\omega)$, and $(1:-1:-\omega)$ and these together with their images under a give nine intersection points. By Bézout's theorem, these are all the intersection points.

Now the two equations can be written $f = 0$ and $-x_2^3 + g = 0$. The projection $\psi(x_1 : x_2 : x_3 : x_4) \to (x_1 : x_3 : x_4)$ maps $C_1 \cap C_2$ onto $f = 0$. The map is 3 to 1 except at the 9 intersection points of $f = 0$ and $g = 0$ where there is just one preimage. Using the Riemann-Hurwitz formula as before gives the genus of $C_1 \cap C_2$ as 10.

Conversely, we are going to show that the mapping $\varphi : X(9) \to \mathbb{CP}^3$, is contained in $C_1 \cap C_2$.

Let us look at the function

$$\frac{\theta^2\begin{bmatrix}1/9\\1\end{bmatrix}\theta\begin{bmatrix}7/9\\1\end{bmatrix}}{\theta\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}3/9\\1\end{bmatrix}\theta\begin{bmatrix}15/9\\1\end{bmatrix}}.$$

of the variable (z, τ). This is a function on the torus $\mathbb{C}/\langle 1, \tau\rangle$ with three simple poles at the zeros of the denominator, i.e., at the points $z = 0$, $z = \tau/3$, $z = 2\tau/3$. Using the fact that the sum of the residues is 0 on any fundamental domain we compute the following identity

$$\theta^2\begin{bmatrix}1/9\\1\end{bmatrix}\theta\begin{bmatrix}7/9\\1\end{bmatrix} + \varepsilon^{13}\theta^2\begin{bmatrix}7/9\\1\end{bmatrix}\theta\begin{bmatrix}5/9\\1\end{bmatrix} + \varepsilon^{8}\theta^2\begin{bmatrix}5/9\\1\end{bmatrix}\theta\begin{bmatrix}1/9\\1\end{bmatrix}$$

where the theta functions are evaluated at $(0, \tau)$ and where $\varepsilon = \exp(\pi i/9)$. It follows that the equation $-x_1x_3^2 + x_1^2x_4 + x_3x_4^2 = 0$ holds for $x_\ell = \varphi_\ell$. We remark that in order to do the cancellation, were using the known transformation properties of theta functions with characteristics described earlier.

Let us now look at the second equation. This identity was pointed out to us by Professor F. Garvan who performed a computer search for such identities. Let us look at the function

$$\frac{\theta^2\begin{bmatrix}3/9\\1\end{bmatrix}\theta\begin{bmatrix}15/9\\1\end{bmatrix}\theta\begin{bmatrix}1/9\\1\end{bmatrix}}{\theta\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}7/9\\1\end{bmatrix}\theta\begin{bmatrix}13/9\\1\end{bmatrix}\theta\begin{bmatrix}11/9\\1\end{bmatrix}}$$

evaluated at (z, τ).

This is a function on the torus and the zeros are at $z=0$, $z=\tau/9$, $z=7\tau/9$, $z=8\tau/9$. The residue theorem will give just as previously that

$$C_1\frac{\theta\begin{bmatrix}3/9\\1\end{bmatrix}^2\theta\begin{bmatrix}1/9\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}7/9\\1\end{bmatrix}\theta\begin{bmatrix}13/9\\1\end{bmatrix}\theta\begin{bmatrix}11/9\\1\end{bmatrix}}+C_2\frac{\theta\begin{bmatrix}5/9\\1\end{bmatrix}^2\theta\begin{bmatrix}17/9\\1\end{bmatrix}\theta\begin{bmatrix}3/9\\1\end{bmatrix}}{\theta\begin{bmatrix}11/9\\1\end{bmatrix}\theta'\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}15/9\\1\end{bmatrix}\theta\begin{bmatrix}13/9\\1\end{bmatrix}}$$

$$+C_3\frac{\theta\begin{bmatrix}17/9\\1\end{bmatrix}^2\theta\begin{bmatrix}29/9\\1\end{bmatrix}\theta\begin{bmatrix}15/9\\1\end{bmatrix}}{\theta\begin{bmatrix}23/9\\1\end{bmatrix}\theta\begin{bmatrix}21/9\\1\end{bmatrix}\theta'\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}25/9\\1\end{bmatrix}}+C_4\frac{\theta\begin{bmatrix}19/9\\1\end{bmatrix}^2\theta\begin{bmatrix}31/9\\1\end{bmatrix}\theta\begin{bmatrix}17/9\\1\end{bmatrix}}{\theta\begin{bmatrix}25/9\\1\end{bmatrix}\theta\begin{bmatrix}23/9\\1\end{bmatrix}\theta\begin{bmatrix}29/9\\1\end{bmatrix}\theta'\begin{bmatrix}1\\1\end{bmatrix}}=0$$

where all functions are evaluated at $(0,\tau)$.

Using the fact that $\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)\neq 0$ and cancelling denominators, we obtain the following equation:

$$C_1\theta^3\begin{bmatrix}3/9\\1\end{bmatrix}+C_2\theta^2\begin{bmatrix}5/9\\1\end{bmatrix}\theta\begin{bmatrix}7/9\\1\end{bmatrix}+C_3\theta\begin{bmatrix}1/9\\1\end{bmatrix}\theta^2\begin{bmatrix}7/9\\1\end{bmatrix}+C_4\theta^2\begin{bmatrix}1/9\\1\end{bmatrix}\theta\begin{bmatrix}5/9\\1\end{bmatrix}=0$$

By computing the coefficients C_j, and using the modified coordinates, we obtain that the equation

$$-x_2^3+x_1^2x_3-x_1x_4^2+x_3^2x_4=0,$$

holds for $x_\ell=\varphi_\ell$ which is the second desired equation.

Hence we have a mapping $\varphi: X(9)\to \mathbb{CP}^3$ such that the image is also a curve of genus 10. An easy computation shows that $C_1\cap C_2$ is non singular and the smoothness implies irreducibility. Now the non-constant map φ must be 1–1 and an embedding. This completes the proof.

§2 Construction of differentials

Having obtained explicit model for the curve we would like to see whether we can construct the canonical embedding on differentials on $X(9)$. As was pointed out to us by

Professor B. van Geemen, the composition of the rational mappings. $X(9) \xrightarrow{\varphi} \mathbb{CP}^3 \xrightarrow{\mathrm{sq}} \mathbb{CP}^9$ where $\mathrm{sq}(x_1 : x_2 : x_3 : x_4) = (\cdots, x_i x_j, \cdots)$ in $\mathbb{CP}^9$ is the Veronese mapping, is the canonical embedding of the curve. In fact, the 9 points with $x_2 = 0$ are the ramification points so 2 times the sum of these nine points is a canonical divisor. The sum of the 9 points is cut out by the plane $x_2 = 0$, so it follows immediately that the surfaces of degree 2 intersect the curve in canonical divisors. Now there are 10 independent quadrics and these give 10 independent canonical divisors.

Hence, we conclude that

Corollary. *The map $X(9) \to \mathbb{CP}^9$ defined by $\tau \to (\cdots, \varphi_\ell(\tau)\varphi_m(\tau), \cdots)$, $1 \le \ell \le 4$, $1 \le m \le 4$ is the canonical embedding for $X(9)$.*

Furthermore the projective representation of $\mathrm{PSL}_2(9)$ mentioned above gives the action on differentials which is the natural action of $\mathrm{PSL}_2(9)$ on them It will become clear from the theorem below where we present the basis for differentials through our cusp forms. The $\varphi_i\varphi_j$ are forms of weight 1. We turn them into forms of weight 2 multiplying them by η^2.

Theorem. The set of forms $\eta^2\varphi_i\varphi_j d\tau$, $1 \le \ell \le 4$, $1 \le m \le 4$ *is a basis for holomorphic differentials for the curve $X(9)$.*

Proof.

We note that if one of $\varphi_i\varphi_j$ is a differential then all of them are since each function φ_i/φ_j is well defined on $X(9)$. We note that φ_2 is $\eta(3\tau)$ so it is enough to show that $\eta^2(\tau)\eta^2(3\tau)$ is a differential.

We would like to show that $\eta^2(\tau)\eta^2(3\tau)$ is a differential. The transformation formula for $\eta^2(\tau)$ is known given in [Kn, p. 51]: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\eta^2(\gamma\tau) = v_\eta(\gamma)(c\tau + d)\eta^2(\tau)$ where

$$v_\eta(\gamma) = \begin{cases} \exp \frac{\pi i}{6}[(a+d)c - bd(c^2-1) - 3c] & c \text{ odd} \\ \exp \frac{\pi i}{6}[(a+d)c - bd(c^2-1) + 3d - 3 - 3cd] & c \text{ even.} \end{cases}$$

Now if $\gamma \in \Gamma(9)$ we can write $\eta^2(3\gamma\tau) = \eta^2(\gamma'(3\tau))$ where $\gamma' = \begin{pmatrix} a & 3b \\ \frac{c}{3} & c \end{pmatrix}$. Thus if $\gamma \in \Gamma(9)$

then: $\eta^2(3\gamma\tau) = v'_\eta(\gamma)(c\tau + d)\eta^2(3\tau)$ where $v'_\eta = \{\exp \frac{\pi i}{6}[(a+d)\frac{c}{3} - 3bd(\frac{c^2}{9} - 1) - c]\}$ c odd. Hence we need to verify that $v_\eta(\gamma)v_\eta(\gamma') = 1$. But this is easy. Let us do it for c odd and c even follows likewise.

$$v_\eta(\gamma)v_\eta(\gamma') = \exp \frac{\pi i}{6}\left[\frac{4c}{3}(a+d) - bd\left(\frac{4c^2}{3} - 4\right) + b(d-1) - 4cd\right].$$

Again, using the fact that c is even and $c \equiv 0 \bmod 9$ we get that $v_\eta(\gamma)v_\eta(\gamma') \equiv 1$. (The only problematic term is $6(d-1)$. But since $d \not\equiv 0 \bmod 2$ we're done.)

Let us remark on how the formula proved in [Kn]. Since we know the transformation formula for $\tau \to \tau + 1$ and $-\frac{1}{\tau}$ we first check that $v_\eta(\tau+1) = \exp\frac{\pi i}{6}$ and $v_\eta(-\frac{1}{\tau}) = -i$. Then we show that v_η is a character $v_\eta : SL_2(\mathbb{Z}) \mapsto \mathbb{C}^*$ and since it coincides on the generators we're done. (The most work in [Kn] is essentially to show that $v_\eta(\gamma)$ is a character!)

Corollary. *The divisor of $\eta\varphi_\ell$ represents a theta characteristic on the surface.*

Proof. The squares are differentials.

A coordinate system for $X(9)$ at ∞ is given by $q^{2/9}$, so when we speak of the order of a q series we are thinking of it locally as a function on $X(9)$ and we mean its order in the variable $q^{2/9}$. The order of φ_j is $(2j-1)^2/8$ and the order at other cusps of $X(9)$ is computed by transforming the cusp to ∞ and using the transformation formulas for theta functions with characteristics and the projective representation on $\mathrm{PSL}_2(9)$. As in [FKK], table on page 245, we can compute the order of φ_j at each cusps. The order of $\eta(\tau)$ at each cusp is $3/8$. This allows us to compute the orders of the differentials at each cusp. For example, the divisor of $\eta^2\varphi_2^2$ is

$$\{(1/9)(2/9)(4/9)(1/3)(2/3)(4/3)(5/3)(7/3)(8/3)\}^2$$

and so the order is 2 at the 9 fixed points of $\tau \to \tau + 3$ on $X(9)$. If we choose σ to be a nonvanishing holomorphic differential on the quotient torus, then this is the divisor of the pullback differential. Thus $\eta^2\varphi_2^2$ is up to a constant multiple the pullback differential from the torus. Moreover we have that

$$\{(1/9)(2/9)(4/9)(1/3)(2/3)(4/3)(5/3)(7/3)(8/3)\}$$

is a theta characterestic on $X(9)$. Also we point out the fact that $\Gamma(9)/\Gamma(3)$ is the group $(Z/3Z)^3$. Hence X(9) can be also seen as a $(Z/3Z)^3$ covering of S^2 minus four points. The cusps are precisely the branch points of the covering. Moreover the nine cusps which are lying above ∞ correspond to the fixed points of $\tau \mapsto \tau + 3$. The others correspond to other unipotent subgroups in $PSL_2(3)$ and there are precisely four of them. We already saw that the set

$$\{(1/9),(2/9),(4/9),(1/3),(2/3),(4/3),(5/3),(7/3),(8/3)\},$$

which is the set of fixed points of $\tau \mapsto \tau+3$ is contained in the hyperplane given by $x_2 = 0$. Thus we conclude that there are 3 more hyperplanes each containing precisely 9 cusps.

We are also able to construct a basis of differentials adjust it to compute the Weierstrass gap sequence at the cusps.

Theorem. The forms

$$\varphi_1^2, \quad \varphi_1\varphi_2, \quad \varphi_2^2, \quad \varphi_1\varphi_3, \quad \varphi_2\varphi_3, \quad \varphi_1\varphi_4, \quad \varphi_2\varphi_4, \quad \varphi_3\varphi_4, \quad \varphi_4^2, \quad \varphi_3\varphi_4^2/\varphi_1$$

multiplied by η^2 are holomorphic differentials adjusted at 1/9 with the following orders

$$0, \ 1, \ 2, \ 3, \ 4, \ 6, \ 7, \ 9, \ 12, \ 15$$

and thus with the Weierstrass gap sequence

$$1, \ 2, \ 3, \ 4, \ 5, \ 7, \ 8, \ 10, \ 13, \ 16.$$

Proof. Straightforward computation.

We can also decompose the space of forms $V = \langle \varphi_\ell \varphi_m \rangle$, $\ell = 1, \ldots, 4$, $m = 1, \ldots, 4$ under the action of the map $\tau \to \tau + 1$. Let V_n be the eigenspace associated to the eigenvector ζ^n. From the formula for the action on the forms φ_ℓ, we find that

$$
\begin{aligned}
V_1 &= \langle \eta^2\varphi_1^2, \eta^2\varphi_3\varphi_4\rangle \\
V_2 &= \langle \eta^2\varphi_1\varphi_2\rangle \\
V_3 &= \langle \eta^2\varphi_2^2\rangle \\
V_4 &= \langle \eta^2\varphi_4^2, \eta^2\varphi_1\varphi_3\rangle. \\
V_5 &= \langle \eta^2\varphi_2\varphi_3\rangle \\
V_7 &= \langle \eta^2\varphi_3^2, \eta^2\varphi_1\varphi_4\rangle \\
V_8 &= \langle \eta^2\varphi_2\varphi_4\rangle
\end{aligned}
$$

This will be useful in subsequent studies of the representation of the group $\mathrm{PSL}_2(9)$ through its action on the holomorphic 1 forms.

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