REDUCTION NUMBER BOUNDS ON ANALYTIC DEVIATION 2 IDEALS AND COHEN-MACAULAYNESS OF ASSOCIATED GRADED RINGS

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1. Introduction

Our purposes in this article are to generalize recent Cohen-Macaulay graded algebra results of Tang [18] and Simis-Ulrich-Vasconcelos [16] in the case of analytic deviation 2 ideals, and to extend to analytic deviation 2 some reduction number bounds that were established in [1]. The content of our graded algebra generalizations is the weakening of the assumption on the base ring from Gorenstein to Cohen-Macaulay. This requires a minimal reduction-generating set selection analogous to that constructed in [11, Proposition 2.3], but valid in the Cohen-Macaulay case. We are able to achieve this by making use of ideas developed in [3]. After producing the graded algebra results, we then apply them to help in identifying reduction number and relation type bounds on analytic deviation 2 ideals. The reduction number bounds are

[1]partially supported by the NSF
[2]partially supported by the NSA

expressed (as they were in [1]) in terms of properties of a presentation matrix of the ideal. Let us first recall some definitions.

Suppose (R, m) is a Noetherian local ring and I is an ideal of R. The *analytic spread* of I is defined by $\ell(I) = dim(\underset{n \geq 0}{\oplus} I^n/mI^n)$ (see [15]) and the *analytic deviation* of I is defined by $ad(I) = \ell(I) - ht(I)$ (see [10]). An ideal $J \subseteq I$ is called a reduction of I if $JI^n = I^{n+1}$ for $n \gg 0$, and is called a *minimal reduction* of I if J is minimal with respect to being a reduction of I. If R/m is assumed to be infinite then $\ell(I) = \mu(J)$ for *any* minimal reduction J of I (see [15]). By $\mu(J)$ we mean the size of a minimal generating set of J. If J is a reduction of I then the reduction number of I with respect to J is the integer $r_J(I) = min\{n \geq 0 | JI^n = I^{n+1}\}$. The *reduction number* of I is defined by $r(I) = min\{r_J(I) | J$ is a minimal reduction of $I\}$. For I an ideal of R let $R[It] = \underset{n \geq 0}{\oplus} I^n t^n$ denote the Rees algebra of I, and let $G(I) = \underset{n \geq 0}{\oplus} (I^n/I^{n+1})$ denote the associated graded ring of R with respect to I. If I is minimally generated by $a_1, ..., a_d$ then there is a canonical homomorphism $R[T_1, ..., T_d] \to R[It]$. Let Q denote the kernel of this homomorphism and let Q_i be the ideal generated by $\{F \in Q | F$ is homogeneous and $deg(F) \leq i\}$. The *relation type* of I, denoted by $N(I)$, is the smallest integer i such that $Q_i = Q$. A Notherian graded ring $S = R_0 + R_1 + R_2 + ...$ where (R_0, m) is a local ring, is known to be Cohen-Macaulay if and only if S_N is Cohen-Macaulay where $N = m_0 + R_1 + R_2 + ...$ is the homogeneous maximal ideal of S (see [14] pr [9]).

In [8, Theorem 1.4] Goto and Nakamura, assuming that R is Gorenstein, $ad(I) = 2$, $r(I) \leq 2$, R/I is Cohen-Macaulay, and a local condition on $\mu(I)$, showed that $G(I)$ is Cohen-Macaulay if and only if $depth(R/I^2) \geq dim(R/I) - 2$. Tang proved the following analogous result, which is valid for ideals of arbitrary analytic deviation.

Theorem 1.1. [Ta, Theorem 5.5]*Let (R, m) be a Gorenstein local ring and I an ideal of R. Assume that $ad(I) \geq 2, \mu(I_P) = ht(I)$ for all primes $P \supseteq I$ with $ht(P/I) \leq ad(I) - 1, depth(R/I^i) \geq dim(R/I) - i + 1$ for $i = 1, ..., ad(I)$, and $r(I) \leq ad(I) + 1$. Then $G(I)$ is Cohen-Macaulay if and only if $depth(R/I^{ad(I)+1}) \geq dim(R/I) - ad(I)$.*

In particular, (1.1) shows that in the case $ad(I) = 2$ (with the other assumptions intact), if $depth(R/I^2) \geq dim(R/I) - 1$ and $r(I)$ is at most *three* then $G(I)$ is Cohen-Macaulay if and only if $depth(R/I^3) \geq dim(R/I) - 2$. Our goal in section 2 is to improve Tang's result in the analytic deviation 2 case by weakening the assumption on R from Gorenstein to Cohen-Macaulay, and relaxing the local condition on $\mu(I)$. Here is the statement (see section 2 for the definition of n_0 and n_1).

Theorem 2.10. *Let (R, m) be a Cohen-Macaulay local ring with R/m infinite, and let I be an ideal of R. Assume that $ad(I) = 2, s = ht(I), n_0 = 0, n_1 \leq 1$, R/I is Cohen-Macaulay, and $depth(R/I^j) \geq dim(R/I) - j + 1$ for $j = 2, 3$. If $r(I) \leq 3$ then $G(I)$ is Cohen-Macaulay.*

Remark. If $G(I)$ is Cohen-Macaulay then it follows from [7, Proposition 3.3] that $depth(R/I^j) \geq dim(R/I) - ad(I)$ for *all* $j \geq 1$, hence for $j = ad(I) + 1$ in particular.

The weakening of the Gorenstein assumption to the Cohen-Macaulay assumption rests on Proposition 2.2 and our use of it throughout section 2.

In [1, Theorem 2.2] the authors, along with C. Huneke, established reduction number bounds for certain analytic deviation one ideals of a Cohen-Macaulay local ring. The bounds were dependent upon behavior of the ideal of entries of a presentation matrix of the ideal. Our main result in section 3 is the following, an analytic deviation 2 version of that result.

Theorem 3.4. *Let (R, m) be a local ring with R/m infinite and let I be an ideal of R. Assume that $ad(I) = 2, ht(I) \geq 1, R/I$ is Cohen-Macaulay, $G(I)$ is Cohen-Macaulay, and $\mu(I) = \ell(I) + 1$. Let J be a minimal reduction of I and let j be a positive integer such that $I \subseteq I_1(\phi)^j$.*

(a) If R is Cohen-Macaulay, $n_0 = 0, n_1 \leq 1$, and I is syzygetic, then $r_J(I) \leq \lceil \frac{r_J(I)-1}{j} \rceil + 2$.

(b) If R is Gorenstein and $\mu(I_P) = ht(I)$ for all primes $P \supseteq I$, then $r_J(I) \leq \lceil \frac{r_J(I)}{j} \rceil + 2$.

We are able to use this result, along with Theorem 2.10, to generalize the analytic deviation 2 case of a result due to Simis, Ulrich, and Vasconcelos (see [16, Theorem 4.10]). Our contributions are to weaken from Gorenstein to Cohen-Macaulay, in the case of analytic deviation 2, their assumption on the base ring R (we also use different techniques), and to include the statement that $G(I)$ is Cohen-Macaulay among the list of equivalences. This result (stated below) is in section 4. A matrix satisfies the row condition if, after invertible operations, some row of the matrix generates the ideal of entries of the matrix.

Theorem 4.3. *Let (R, m) be a 4-dimensional Cohen-Macaulay local ring with R/m infinite, and let I be a perfect height 2 ideal of R. Assume that $\mu(I_P) \leq ht(P)$ for all $P \supseteq I$ with $ht(P) \leq 3$ and $\mu(I) = 5$. Then $ad(I) = 2$ and the following are equivalent:*

(1) Every presentation matrix ϕ of I satisfies the row condition;
(2) $G(I)$ is Cohen-Macaulay;
(3) $r(I) \leq 3$;
(4) $N(I) \leq 4$;
(5) $R[It]$ is Cohen-Macaulay.

For ease of reference we include the following well known result.

Depth Lemma Let R be a Noetherian local ring and $0 \to A \to B \to C \to 0$ an exact sequence of finitely generated R-modules. Then either

(a) $depth(A) \geq depth(B) = depth(C)$, or
(b) $depth(B) \geq depth(A) = depth(C) + 1$, or
(c) $depth(C) > depth(A) = depth(B)$.

Acknowledgement We are grateful to Craig Huneke for conversations about this paper.

2. Cohen-Macaulay graded rings for ideals of analytic deviation 2

In this section we prove that in a Cohen-Macaulay local ring certain ideals of analytic deviation two and reduction number ≤ 3 have Cohen-Macaulay associated graded rings (in which case $R[It]$ is usually also Cohen-Macaulay).

The conclusions are analogous to certain results on analytic deviation 1 ideals from [2], [1], and analytic deviation 2 ideals from [8]. The main result is Theorem 2.10, which is the Cohen-Macaulay case of a theorem of Tang's for Gorenstein rings [Ta, Theorem 5.5].

We start with a result whose origins are in [11] and [3] and which provides the technical framework for our arguments.

Definition 2.1. Let (R, m) be a Cohen-Macaulay local ring and $I \subseteq R$. For $i \geq 0$ let $n_i(I) = \max\{r_J(I_P) | J$ is a reduction of I_P, $ht(P) = i + ht(I)$, and $\mu(J) = \min\{ht(P), \mu(I_P)\}\}$. In most cases the ideal I is clear from context, so we will abbreviate $n_i(I)$ by n_i.

We will use the following notation: Let $\mathcal{P} = \{P \supseteq I | ht(P/I) \leq 1\}$, and if $\{a_1, ..., a_m\}$ is a minimal generating set for a minimal reduction J of I then set $J_i = (a_1, ..., a_i)$.

Proposition 2.2. ([11, Proposition 3.3] and [3, Lemma 7.2]) *Let (R, m) be a Cohen-Macaulay local ring with R/m infinite and let I be an ideal of R of analytic deviation 2. Let $s = ht(I) \geq 1$. Let J be a minimal reduction of I. Then there exists a set of generators $\{a_1, \dots, a_{s+2}\}$ for J such that*

(1) a_i is R-regular, and $\{a_1, ..., a_s\}$ is an R-regular sequence that generates a reduction of I generically;

(2) $ht((a_1, \dots, a_s)I^{n_0} : I^{n_0+1}) \geq s$ and $ht((a_1, ..., a_{s+1})I^{n_1} : I^{n_1+1}) \geq s + 1$;

(3) $ht(((a_1, \dots, a_s)I^{n_0} : I^{n_0+1}) + I) \geq s + 1$ and $ht(((a_1, ..., a_{s+1}I^{n_1}) : I^{n_1+1}) + I) \geq s + 2$;

(4) If $n_0 = 0$ (i.e., I is generically a complete intersection) then $((a_1, ..., a_s) : a_{s+1}) = ((a_1, ..., a_s) : a_{s+2}) = ((a_1, ..., a_s) : I)$;

(5) $((a_1, ..., a_s) : a_{s+1}) \cap I^{n_0+1} \subseteq (a_1, ..., a_s)$;

(6) If $n_0 = 0$ and $n_1 \leq 1$ and all the primes in $Ass(R/I)$ have height at most s then $((a_1, ..., a_{s+1})I : a_{s+2}) \cap I^2 = (a_1, ..., a_{s+1})I$.

(7) If R is Gorenstein, R/I is Cohen-Macaulay and $\mu(I_P) = s$ for $P \in \mathcal{P}$ then $((a_1, \dots, a_{s+1}) : a_{s+2}) \cap I = (a_1, \dots, a_{s+1})$.

Proof. Let $a_1, \dots, a_{s+2}$ be a basic generating set for J as provided by [3, Lemma 7.2]. By using the argument of [1, Proposition 3.1] we may assume that $a_1, ..., a_s$ form an R-regular sequence, but we include here a slightly different argument for achieving this. We may adjust each a_i by an element of I^2 without affecting its image a_i' in $G(I)$, and thus retain that the $a_i's$ form a basic generating set for J. By using McAdam's avoidance lemma (see [11, Remark 1.10]) it is possible to make this adjustment one by one so that at the i^{th} stage ($i < s$) we choose $y \in I^2$ such that $a_{i+1} + y$ misses the associated primes of $R/(a_1, ..., a_i)$; the idea is to observe that $I^2 + a_{i+1}$ is not contained in any associated prime P of $R/(a_1, ..., a_i)$, then use McAdam's lemma. (1) is now immediate from [3, Lemma 7.2].

The elements a_{s+1} and a_{s+2} may be chosen to avoid the minimal primes of $(a_1, \ldots, a_s)$ which do not contain I. Thus (2) and (3) follow from [3, Lemma 7.2].

We can check (4) locally at minimal primes of J_s. If P is a minimal prime of J_s containing I then each colon ideal is R_P. If P does not contain I then P does not contain a_{s+1} or a_{s+2} either so all colon ideals are $(J_s)_P$.

To check (5), again we may localize at a minimal prime P of J_s. If $I \subseteq P$ then $I_P^{n_0+1} = (J_s I^{n_0})_P \subseteq (J_s)_P$. If $I \not\subseteq P$ then $(J_s : a_{s+1})_P = (J_s)_P$.

To see (6) we first show that the hypotheses guarantee that primes in $Ass(R/J_{s+1}I)$ have height at most $s+1$. Then the equality can be checked locally. We first note that $J_s I \cap a_{s+1} I \subseteq a_{s+1}((J_s : a_{s+1}) \cap I) = a_{s+1} J_s$ (using (5) for the second equality). We have the short exact sequence

$$0 \to R/J_s I \cap a_{s+1} I = R/a_{s+1} J_s \to R/J_s I \oplus R/a_{s+1} I \to R/J_{s+1} I \to 0.$$

Localize at a prime P in $Ass(R/J_{s+1}I)$. If P is associated to either $R/J_s I$ or $R/a_{s+1}I$ then the exact sequences

$$0 \to (R/I)^s \to R/J_s I \to R/J_s \to 0$$

and

$$0 \to (a_{s+1})/a_{s+1} I \to R/a_{s+1} I \to R/(a_{s+1}) \to 0$$

show that $ht(P) \leq s$ (the first of these sequences follows because, using that J_s is generated by a regular sequence, $J_s/IJ_s = (J_s \cap I)/IJ_s \cong$

$\mathrm{Tor}_1(R/J_s, R/I) \cong (R/I)^s$). Thus we can assume that $depth(R/a_{s+1}J_s) = 1$. Then the exact sequence

$$0 \to (a_{s+1})/a_{s+1}J_s \cong R/J_s \to R/a_{s+1}J_s \to R/(a_{s+1}) \to 0$$

shows that $ht(P) \leq s+1$. Therefore we can adjust our choice of a_{s+2} to avoid any associated primes of $J_{s+1}I$ not containing I. Now let P be any associated prime of $J_{s+1}I$. If $I \subseteq P$ then $I_P^2 = I_P(J_{s+1})_P$ since $n_1 \leq 1$. If $I \nsubseteq P$ then $((J_{s+1}I) : a_{s+2})_P = (J_{s+1}I)_P$.

(7) is [11, Proposition 3.3(vi)]. The proof for this does not rely on the hypothesis in Propostion 3.3 that I have reduction number ≤ 1. ∎

The next five lemmas are analogous to results from [8] and [18], but with the Gorenstein assumption from those papers weakened to the Cohen-Macaulay assumption.

Lemma 2.3. *Let (R, m) be a local Cohen-Macaulay ring with infinite residue field, let I be an ideal of height $s \geq 1$ and analytic deviation 2. Assume as well that every associated prime of R/I has height s. Let J be a reduction chosen as in Proposition 2.2.*

(1) If $n_0 \leq 1$ then $J_s \cap I^2 = J_sI$.

(2) If $n_0 = 0$ and $n_1 \leq 1$ then $J_{s+1} \cap I^2 = J_{s+1}I$.

Proof. This follows quickly by checking locally at the appropriate associated primes, using that if $P \in Ass(R/J_sI)$ then $ht(P) \leq s$ and if $P \in Ass(R/J_{s+1}I)$ then $ht(P) \leq s+1$, which were established in the proof of 2.2(6) above. ∎

Lemma 2.4. *Let (R, m) be a Cohen-Macaulay ring and let I be an ideal of height $s \geq 1$ and $ad(I) = 2$. Let $J = (a_1, \ldots, a_{s+2})$ be chosen as in Proposition 2.2. Assume that $n_0 \leq 1$. Suppose $depth(R/I) \geq \dim(R/I)$ and $depth(R/I^2) \geq \dim(R/I) - 1$. Then for $1 \leq t \leq s$, (a) $depth(R/J_tI) \geq \dim(R/I)$ and (b) $depth(R/J_tI^2) \geq \dim(R/I) - 1$.*

Proof. We first prove (a) by induction on t. If $t = 1$ then the exact sequence

$$0 \to (a_1)/a_1 I \to R/a_1 I \to R/(a_1) \to 0$$

and the isomorphism $(a_1)/a_1 I \cong R/I$ show that $depth(R/a_1 I) \geq \dim(R/I)$. Assume now that $s \geq t > 1$. The sequence

$$0 \to J_t I/J_{t-1} I \to R/J_{t-1} I \to R/J_t I \to 0$$

is exact and

$$J_t I/J_{t-1} I \cong a_t I/(a_t I \cap J_{t-1} I) = a_t I/a_t J_{t-1} \cong I/J_{t-1}$$

where the equality can be checked generically. Since $depth(I/J_{t-1}) \geq \dim((R/I) + 1$ we can apply the Depth Lemma to see that $depth(R/J_t I) \geq \dim(R/I)$.

We now prove (b) by induction on t. If $t = 1$ then the exact sequence

$$0 \to (a_1)/a_1 I^2 \to R/a_1 I^2 \to R/(a_1) \to 0$$

and the isomorphism $(a_1)/a_1 I^2 \cong R/I^2$ show that $depth(R/a_1 I^2) \geq \dim(R/I) - 1$.

Assume now that $t > 1$ and the claim is true for $t-1$. We have the exact sequence

$$0 \to J_t I^2/J_{t-1} I^2 \to R/J_{t-1} I^2 \to R/J_t I^2 \to 0$$

and the isomorphisms $J_t I^2/J_{t-1} I^2 \cong a_t I^2/(a_t I^2 \cap J_{t-1} I^2) = a_t I^2/a_t J_{t-1} I \cong I^2/J_{t-1} I$. The Depth Lemma shows that $depth(R/J_t I^2) \geq \dim(R/I) - 1$ as long as $depth(I^2/J_{t-1} I) \geq \dim(R/I)$. This is clear from the exact sequence $0 \to I^2/J_{t-1} I \to R/J_{t-1} I \to R/I^2 \to 0$. ∎

Lemma 2.5. *Let (R, m) and I be as in Proposition 2.2, and assume additionally that R/I is Cohen-Macaulay, $n_0 = 0$, $n_1 \leq 1$, $r(I) \leq 3$ and $depth(R/I^2) \geq dim(R/I) - 1$. Let J be a minimal reduction of I satisfying $JI^3 = I^4$ and let $\{a_1, ..., a_{s+2}\}$ be a generating set of J satisfying (2.2), (1)-(6). Then $J_{s+1} \cap I^n = J_{s+1} I^{n-1}$ for all $n \geq 1$.*

Proof. This is clear for $n = 1$ and follows from Proposition 2.3 for $n = 2$.

Suppose $n = 3$. It suffices to show the equality locally at primes in $Ass(R/J_{s+1}I^2)$. We first note that

$$J_sI^2 \cap a_{s+1}I^2 = a_{s+1}J_sI.$$

Let $ra_{s+1} \in J_sI^2 \cap a_{s+1}I^2$ where $r \in I^2$. Then $r \in ((J_s) : a_{s+1}) \cap I^2$ which is contained in J_s by (2.2)(5). Thus $r \in J_s \cap I^2 = J_sI$. It follows that $ra_{s+1} \in a_{s+1}J_sI$. The reverse inclusion is clear.

Now consider the exact sequence

$$0 \to R/a_{s+1}J_sI \to R/J_sI^2 \oplus R/a_{s+1}I^2 \to R/J_{s+1}I^2 \to 0$$

of R-modules. If $P \in Ass(R/J_{s+1}I^2)$, then by the Depth Lemma either $P \in Ass(R/J_sI^2) \cup Ass(R/a_{s+1}I^2)$, or $depth(R/a_{s+1}J_sI)_P = 1$. We claim that these conditions force $ht(P) \leq s+1$. If the former condition holds then Lemma 2.4(b) shows $ht(P) \leq s+1$. Now suppose that $depth(R/a_{s+1}J_sI)_P = 1$ and consider the exact sequence

$$0 \to (a_{s+1})/a_{s+1}J_sI \to R/a_{s+1}J_sI \to R/(a_{s+1}) \to 0.$$

The Depth Lemma yields that $depth(R/(a_{s+1}))_P = 1$, or $depth((a_{s+1})/a_{s+1}J_sI)_P = 1$. The isomorphism $(a_{s+1})/a_{s+1}J_sI \cong R/J_sI$ and Lemma 2.4(a) show that $ht(P) \leq s+1$. We have therefore established that it suffices to show $J_{s+1} \cap I^3 = J_{s+1}I^2$ locally at prime ideals of height at most $s+1$.

If $I \not\subseteq P$ then the equality holds locally at P (regardless of the value of $ht(P)$) so assume that $I \subseteq P$. If $ht(P) \leq s+1$ then the equality holds locally because $(J_{s+1})_P$ is a reduction of I_P with reduction number less than 2.

We now proceed by induction on n. Let $n \geq 4$. By the reduction number assumption we have that $JI^{n-1} = I^n$, thus $J_{s+1} \cap I^n = J_{s+1} \cap I^2 \cap I^n = J_{s+1}I \cap I^n$ (from the $n = 2$ case) $= J_{s+1}I^{n-1} + J_{s+1}I \cap a_{s+1}I^{n-1}$. Hence it suffices to show that $J_{s+1}I \cap a_{s+2}I^{n-1} \subseteq J_{s+1}I^{n-1}$. If $ra_{s+2} \in J_{s+1}I \cap a_{s+2}I^{n-1}$ where $r \in I^{n-1}$ then $r \in (J_{s+1}I : a_{s+2}) \cap I^3 \subseteq J_{s+1}$ by 2.2(6). Thus $r \in J_{s+1} \cap I^{n-1} = J_{s+1}I^{n-2}$ by the induction hypothesis. Therefore $ra_{s+2} \in J_{s+1}I^{n-1}$, proving Lemma 2.5. ∎

Lemma 2.6. *Let (R, m), I, and J be as in Lemma 2.5. Then $J_s \cap I^n = J_sI^{n-1}$ for all $n \geq 1$.*

Proof. The proof is by induction on n. The case $n = 1$ is clear and $n = 2$ is given by Lemma 2.3. Writing $J_s \cap I^n = J_s \cap J_{s+1} \cap I^n$ we see that $J_s \cap I^n = J_s \cap J_{s+1} I^{n-1}$ by Lemma 2.5, thus $J_s \cap I^n = J_s I^{n-1} + J_s \cap a_{s+1} I^{n-1}$. Let $r a_{s+1} \in J_s \cap a_{s+1} I^{n-1}$ where $r \in I^{n-1}$. Then (assuming $n > 2$) $r \in (J_s : a_{s+1}) \cap I^2 \subseteq J_s$ by condition (5) of (2.2). Thus $r \in J_s \cap I^{n-1} = J_s I^{n-2}$ by the induction hypothesis. Therefore $r a_{s+1} \in J_s I^{n-1}$ completing the proof. ∎

Lemma 2.7. *Let (R, m) be a Cohen-Macaulay local ring with R/m infinite and let I be an ideal of R of positive height. Assume that $n_0(I) = 0$, $n_1(I) \leq 1$, R/I is Cohen-Macaulay, and $depth(R/I^2) \geq \dim(R/I) - 1$. Let $J = (a_1, ..., a_{s+2})$ be a minimal reduction of I with $a_1, ..., a_{s+2}$ chosen to satisfy (1)-(6) of (2.2). Then*
(1) $depth(R/(J_s + I^2)) \geq \dim(R/I) - 1$, and
(2) if $depth(R/I^3) \geq \dim(R/I) - 2$ then $depth(R/(J_s + I^3)) \geq \dim(R/I) - 2$.

Proof. First note that $J_s \cap I^2 = J_s I$ by Proposition 2.3(a). Also $J_s \cap I^3 = J_s I^2$ because by the proof of Lemma 2.5 we have $J_{s+1} \cap I^3 = J_{s+1} I^2$ (reduction number three was not used for that part of the proof), thus we may apply the argument given in the proof of Lemma 2.6.

(1) This follows by applying the Depth Lemma to the exact sequence

$$0 \to R/J_s I \to R/J_s \oplus R/I^2 \to R/(J_s + I^2) \to 0.$$

($depth(R/J_s I \geq \dim(R/I)$ by Lemma 2.4(a).)

(2) By applying the Depth Lemma to

$$0 \to R/J_s I^2 \to R/J_s \oplus R/I^3 \to R/(J_s + I^3) \to 0$$

we obtain that either $depth(R/(J_s + I^3)) \geq \dim(R/I) - 2$, or $depth(R/(J_s + I^3)) = depth(R/J_s I^2) - 1 \geq \dim(R/I) - 2$ by Lemma 2.4(a). ∎

Before continuing we recall the definition of local reduction numbers from [3] and an important fact concerning them.

Definition 2.8. Let (R, m) be any local ring and let $I \subseteq R$ be an ideal of height s and analytic spread ℓ. The i^{th} local reduction number of I is

$$r_i(I) = \begin{cases} i-s, & i<s \\ \max\{r(I_P)-ht(P) \mid ht(P)=\ell(I_P) \text{ and } ht(P)\le i\}+i, & i=s,\dots,\ell \end{cases}$$

Theorem 2.9. [3, Theorem 4.4] *Assume $G(I)$ is CM. Let $\ell=\ell(I)$ and let J be any minimal reduction of I. Then* $\max\{r_{\ell-1}(I)+1, r_J(I)\}=r_\ell(I)$.

We can now state the main theorem of this section, which generalizes [Ta, Theorem 5.5] for $ad(I)=2$. We expect that the appropriate generalization to Cohen-Macaulay rings for all analytic deviations remains true.

Theorem 2.10. *Let (R,m) be a Cohen-Macaulay local ring with R/m infinite, and let I be an ideal of R. Assume that $ad(I)=2$, $s=ht(I)$, $n_0=0$, $n_1\le 1$, R/I is Cohen-Macaulay, and $depth(R/I^j)\ge dim(R/I)-j+1$ for $j=2,3$. If $r(I)\le 3$ then $G(I)$ is Cohen-Macaulay.*

Proof. If $s=ht(I)=0$ then following [8] we may adjoin an analytic variable z to R and replace I by (I,z) and J by (J,z). By [8, Lemma 2.2] all hypotheses except for the depth conditions are left undisturbed and the Cohen-Macaulayness of the associated graded ring is unaffected. To see that the depth conditions hold we use the exact sequence

$$0\to zS/(I,z)^n\to S/(I,z)^n\to R/I^n\to 0$$

and the isomorphism $zS/(I,z)^n\cong S/((I,z)^n:z)=S/(I,z)^{n-1}$, where $S=R[[z]]$. Thus we may assume that $s\ge 1$.

Let J be a minimal reduction of I satisfying $JI^3=I^4$ and by Proposition 2.2 choose a generating set $\{a_1,\dots,a_{s+2}\}$ for J fulfilling conditions (1)-(6) of (2.2). By Lemma 2.6 it holds that $\{a_1',\dots,a_s'\}$ is a $G(I)$-regular sequence. Thus $G(I/(a_1,\dots,a_s))\cong G(I)/(a_1',\dots,a_s')$, so that $G(I)$ is Cohen-Macaulay if and only if $G(I/(a_1,\dots,a_s))$ is Cohen-Macaulay. Further, if we set $R_1=R/(a_1,\dots,a_s)$ and $I_1=I/(a_1,\dots,a_s)$ then $n_0(I_1)=0$ and $n_1(I_1)\le 1$, R_1/I_1 is Cohen-Macaulay, $depth(R_1/I_1^j)\ge dim(R/I)-j+1$ for $j=2,3$ (by Lemma 2.7), $\ell(I_1)=2$ (by [11, Lemma 2.12]), and $r(I_1)\le r(I)\le 3$. Therefore by converting to R_1, I_1, and $J_1 = JR_1$ we may assume that $ht(I) = 0$, $\ell(I) = 2$, and $(b,c)I^3 = I^4$ where b, c is a minimal reduction of I satisfying the conditions that $((b) : c) \cap I^2 \subseteq (b)$, $((0) : b) \cap I = 0$, $((0) : c) \cap I = 0$, and by Lemma 2.5, $(b) \cap I^n = bI^{n-1}$.

The remainder of the proof is by induction on $dim(R)$, starting with the case $dim(R) = 2$. By using that $ht((0) : I + I) = 1$ and $depth(R/I^2) \geq 1$ we may choose elements $y \in ((0) : I)$ and z avoiding the associated primes of R/I^2 such that their images form a system of parameters (hence a regular sequence) for R/I. We claim that $\{y' + b', z' + c'\}$ is a $G(I)$-regular sequence. Suppose there is an equation

$$(y' + b')(u_0' + ..., u_n') = 0 \tag{2.11}$$

in $G(I)$, where u_i' is homogeneous of degree i. Assume that $n \geq 1$ and let u_i be a lifting of u_i'. Then $bu_n \in (b) \cap I^{n+2} = bI^{n+1}$ so we may write $bu_n = bu$ for some $u \in I^{n+1}$. Hence $u_n - u \in ((0) : b) \cap I = 0$, therefore $u_n' = 0$. This shows that we may assume $n = 0$ in (2.11). We have $yu_0 \in I$, therefore $u_0 \in I$ so $u_0' = 0$. This shows that $y' + b'$ is a $G(I)$-regular element.

Suppose there is an equation

$$(z' + c')(u_0' + ... + u_n') = (y' + b')(v_0' + ... + v_n') \tag{2.12}$$

in $G(I)$ where u_i', v_i' are homogeneous of degree i. We again will reduce n and use induction. Suppose first that $n \geq 2$. Then for liftings u_n and v_n we have $cu_n - bv_n \in I^{n+2} = (b,c)I^{n+1}$, hence we may write $cu_n - bv_n = cr + bs$ for some $r, s \in I^{n+1}$. Thus $u_n - r \in ((b) : c) \cap I^n = (b) \cap I^n = bI^{n-1}$, so we may write $u_n = r + bt$ for some $t \in I^{n-1}$. Therefore $u_n' = (y' + b')t'$ (recall that $y \in ((0) : I)$). This shows that we may assume $n \leq 1$ in (2.12).

If $n \leq 1$ in (2.12) then we have the inclusions $cu_0 + zu_1 - bv_0 - yv_1 = cu_0 + zu_1 - bv_0 \in I^2$ and $zu_0 - yv_0 \in I$. By using that $\{y, z\}$ is an R/I-regular sequence, we get that $u_0 - ry, v_0 - rz \in I$ for some $r \in R$. Writing $u_0 = ry + g$ for some $g \in I$ we see that

$$u_0' = r'y'.$$

We will be done if we can show $u_1 - rb \in I^2$ for then $u_1' = r'b'$. Since $z(u_1 - rb) = -cu_0 + i$ where $i \in I^2$, we have $z^2(u_1 - rb) = zcu_0 + zi = zcry + zcg + zi = zcg + zi \in I^2$. Thus $u_1 - rb \in (I^2 : z^2) = I^2$.

This proves that $G(I)$ is Cohen-Macaulay in the case $dim(R) = 2$. We now proceed by induction. Let $dim(R) = d > 2$. We first show by induction on n that $depth(R/I^n) > 0$ for all $n \geq 1$. By assumption this is true for $n = 1, 2, 3$. If $n \geq 4$ then $I^n = (b, c)^{n-3}I^3 = bI^{n-1} + c^{n-3}I^3$. Considering the exact sequence

$$0 \to R/(bI^{n-1} \cap c^{n-3}I^3) \to R/bI^{n-1} \oplus R/c^{n-3}I^3 \to R/I^n \to 0,$$

and the Depth Lemma, it suffices to show that $depth(R/bI^{n-1}) > 0$, $depth(R/c^{n-3}I^3) > 0$, and $depth(R/(bI^{n-1} \cap c^{n-3}I^3)) > 1$. Note that $b^iI^k \cong I^k$ and $c^iI^k \cong I^k$ for all i, k, because $((0) : b) \cap I = ((0) : c) \cap I = 0$. Thus, the induction hypothesis along with the exact sequences

$$0 \to I^k \to R \to R/I^k \to 0, \tag{2.13}$$

$$0 \to I^{n-1} \to R \to R/bI^{n-1} \to 0,$$

and

$$0 \to I^3 \to R \to R/c^{n-3}I^3 \to 0$$

imply that $depth(R/bI^{n-1}) > 0$ and $depth(R/c^{n-3}I^3) > 0$.

To show that $depth(R/(bI^{n-1} \cap c^{n-3}I^3)) > 1$, we first observe that $bI^{n-1} \cap c^{n-3}I^3 = bc^{n-3}I^2$. The inclusion $\supseteq$ is clear and $bI^{n-1} \cap c^{n-3}I^3 = c^{n-3}((b : c^{n-3}) \cap I^3) \subseteq c^{n-3}((b) \cap I^3) = c^{n-3}bI^2$.

Note that $bc^{n-3}I^2 \cong I^2$, again because $((0) : b) \cap I = ((0) : c) \cap I = 0$. Hence the exact sequence

$$0 \to bc^{n-3}I^2 \to R \to R/bc^{n-3}I^2 \to 0,$$

the hypothesis $depth(R/I^2) \geq d-1$, and (2.13), imply that $depth(R/bc^{n-3}I^2) \geq d - 1 > 1$. We conclude that $depth(R/I^n) > 0$ for all $n \geq 1$. Hence $depth(G(I)) > 0$ since $\bigcup_n \mathrm{Ass}(R/I^n)$ is a finite set [5].

The set $\mathcal{A}= \{P \in Spec(R) | (I_P^2) \neq (b)I_P$ and $ht(P/I) = 2\} \cup \{P \in Spec(R) | I_p \neq 0$ and $ht(P/I) = 1\}$ is finite ($I_P^2 = (b)I_P$ for all primes $P \supseteq I$ such that $ht(P/I) \leq 1$ because $\{b, c\}$ is a basic generating set for (b, c)), hence we may choose $x \in m \backslash I$ such that x' is $G(I)$-regular and $x \notin P$ if $P \in \mathcal{A}$. Setting $R_1 = R/(x)$ and $I_1 = (I, x)/(x)$ it follows that $G(I)$ is

Cohen-Macaulay if and only if $G(I_1)$ is Cohen-Macaulay. By our choice of x, $n_0(I_1) = 0$. Checking that $n_1(I_1) \leq 1$ is more involved. Let $P \supseteq (I, x)$ be a prime of height 2. Then $I_P^2 = bI_P$. If $\ell(I_P) = 0$ then I_P is nilpotent so that $I_P = 0$ (check generically). Otherwise $\ell(I_P) = 1$ and $G(I_P)$ is Cohen-Macaulay by [11, Theorem 2.1] (it extends to $ht(I) = 0$ as described above). Then $r_1(I_P) = 1$, so by Theorem 2.9, $r_K(I_P) \leq 1$ for any minimal reduction $K \subseteq I_P$. Hence $n_1(I_1) \leq 1$.

From the exact sequence

$$0 \to R/xI^k \to R/(x) \oplus R/I^k \to R/((x) + I^k) \to 0$$

for $k = 2, 3$ it follows that $depth(R_1/I_1^j) \geq dim(R_1) - j + 1$ for $j = 2, 3$. Finally we have that $ht(I_1) = ht(I)$, $\ell(I_1) = \ell(I)$ ([11, Lemma 2.12]), and $r(I_1) \leq 3$. Therefore the induction hypothesis on $d = dim(R)$ implies that $G(I_1)$ is Cohen-Macaulay, proving Theorem 2.10.

■

Remark 2.14. If $dim(R/I) = 2$ in Theorem 2.10 then (of course) the condition $depth(R/I^3) \geq dim(R/I) - 2$ is satisfied.

Corollary 2.15. *Let R and I be as in Theorem 2.10. If $s \geq 2$ then $R[It]$ is Cohen-Macaulay.*

Proof. If $s \geq 2$ then $\ell(I) \geq 4$, hence $r(I) = 3 \leq \ell(I) - 1$. Let $P \supseteq I$ be a prime with $\ell(I_P) = ht(P) < \ell(I)$. Then either P is minimal over I and $r(I_P) = 0 \leq ht(P) - 1 = s - 1$ or $ht(P) = s + 1$ and $r(I_P) \leq 1 \leq ht(P) - 1 = s$. Thus $R[It]$ is Cohen-Macaulay by [3, Theorem 5.6]. ■

3. Bounds on relation type and reduction number for ideals with Cohen-Macaulay associated graded ring

In order to analyze the relation type of an analytic deviation 2 ideal I when $G(I)$ is Cohen-Macaulay we must establish the conclusion of Lemma 2.5 without the reduction number 3 assumption, but instead with the assumption that $G(I)$ is Cohen-Macaulay. The following lemma does that.

Lemma 3.1. *Let (R, m) be Cohen-Macaulay with R/m infinite and let I be an ideal of R. Assume that $ad(I) = 2$, $s = ht(I) \geq 1$, $n_0 = 0$, $n_1 \leq 1$, R/I is Cohen-Macaulay, and $G(I)$ is Cohen-Macaulay. Let $J = (a_1, ..., a_{s+2})$ be a minimal reduction of I with generators chosen to satisfy (2.2), (1)-(6). Then $J_{s+1} \cap I^n = J_{s+1}I^{n-1}$ for all $n \geq 1$.*

Proof. As $G(I)$ is Cohen-Macaulay $\{a_1', ..., a_s'\}$ is a $G(I)$-regular sequence, hence $J_s \cap I^n = J_sI^{n-1}$ for all $n \geq 1$. We show by induction on n that $J_{s+1} \cap I^n = J_{s+1}I^{n-1}$ for $n \geq 1$. If $n = 1$ then the equality is clear and if $n = 2$ it follows from Propostion 2.3(2). Let $n \geq 3$ and write $J_{s+1} \cap I^n = J_{s+1} \cap I^{n-1} \cap I^n = J_{s+1}I^{n-2} \cap I^n$ by using the induction hypothesis. Let $r \in J_{s+1} \cap I^n$ and write $r = r_1a_1 + ... + r_{s+1}a_{s+1}$ where $r_j \in I^{n-2}$. Since J is generated by a basic generating set and $G(I)$ is Cohen-Macaulay , the sequence $a_1', \ldots, a_{s+1}'$ is $[0, \ldots, 0, n_0 + 1]$-regular by [3, Theorem 3.2]. Thus $r_{s+1}' \in ((a_1', ..., a_s') : a_{s+1}')_{n-2} = (a_1', ..., a_s')_{n-2}$ This forces $r_{s+1} \in J_sI^{n-3} + I^{n-1}$. Writing $r_{s+1} = t_1a_1 + ... + t_sa_s + t$ where $t_j \in I^{n-3}$ and $t \in I^{n-1}$, and substituting above, yields that $r = (r_1 + t_1a_{s+1})a_1 + ... + (r_s + t_sa_{s+1})a_s + ta_{s+1}$. Therefore $(r - ta_{s+1}) \in J_s \cap I^n = J_sI^{n-1}$. It follows that $r \in J_{s+1}I^{n-1}$. ∎

The next two lemmas are analytic deviation 2 analogues of [2, Proposition 2.5] and [1, Lemma 2.1]. Let $I = (a_1, ..., a_d)$ be an ideal of R and let Q denote the kernel of the homorphism $R[T_1, ..., T_d] \to R[It]$. Let Q_n be the ideal generated by the homogeneous polynomials of Q having degree at most n.

Lemma 3.2. *Let (R, m) be a Cohen-Macaulay local ring with R/m infinite and let I be an ideal of R. Assume that $ad(I) = 2$, $s = ht(I) \geq 1$, $n_0 = 0$, $n_1 \leq 1$, R/I is Cohen-Macaulay, and $G(I)$ is Cohen-Macaulay. Let $J = (a_1, ..., a_{s+2})$ be a minimal reduction of I with $a_1, ..., a_{s+2}$ chosen to satisfy the conclusions of (2.2). Then $(T_1, ..., T_{s+2}) \cap Q_n \subseteq Q_{n-1}$ for all $n \geq 3$.*

If R is Gorenstein and I is as above, with the additional restriction that I_Q is a complete intersection for $Q \in \mathcal{P}$ then the desired conclusion holds for $n \geq 2$.

Proof. The approach comes from the proof found in [2, Proposition 2.5]. Write $I=(a_1,...,a_{s+2},i_1,...,i_m)$ minimally and let $F\in(T_1,...,T_{s+2})\cap Q_n$ be homogeneous of degree n. Then $F=T_1G_1+...+T_{s+2}G_{s+2}$ where either $G_j=0$ or $deg(G_j)=n-1$. Evaluating implies $r=G_{s+2}(a_1,...,a_{s+2},i_1,...,i_m)\in((J_{s+1}):a_{s+2})\cap I^{n-1}\subseteq J_{s+1}\cap I^{n-1}$ by (2.2)(6). By Lemma 3.1 we get that $r\in J_{s+1}I^{n-2}$. Thus, modulo Q_{n-1}, we may assume that $F\in(T_1,...,T_{s+1})\cap Q_n$. Inductively we obtain that $F\in Q_{n-1}$.

When R is Gorenstein and I_Q is a complete intersection for all primes $Q\in\mathcal{P}$ Proposition 2.2(7) allows us to make the argument above in the case that $n-1=1$. ∎

Lemma 3.3. *Let (R,m) be a Cohen-Macaulay local ring and let I be an ideal of R. Assume that $ad(I)=2$, $ht(I)\geq 1$, $n_0=0$, $n_1\leq 1$, R/I is Cohen-Macaulay, $G(I)$ is Cohen-Macaulay, and $\mu(I)=\ell(I)+1$. Let ϕ be a presentation matrix of I. Then $I_1(\phi)Q_n\subseteq Q_{n-1}$ for all $n\geq 3$.*

If R is Gorenstein, I_Q is a complete intersection for all primes $Q\in\mathcal{P}$ and all other hypotheses are the same then the same conclusion holds for $n\geq 2$.

Proof. By passing to $R(X)=R[X]_{m[X]}$ we may assume that R/m is infinite. Let $s=ht(I)$ and choose a generating set $a_1,...,a_{s+3}$ of I such that every subset of $s+2$ elements is a minimal reduction of I. Let $F\in Q_n$ be homogeneous and let $b\in I_1(\phi)$. Because ϕ is a presentation matrix for I we may assume that $b\in((a_1,...,a_{s+2}):a_{s+3})$. Then since $J=(a_1,\ldots,a_{s+2})$ is a minimal reduction of I we can rechoose the generating set according to Proposition 2.2 and relabel. Write $F=rT_{s+3}^n+G$ where $G\in(T_1,...,T_{s+2})R[T_1,...,T_{s+3}]$ and $deg(G)=n$. There exists a linear polynomial $bT_{s+3}-r_1T_1-...-r_{s+2}T_{s+2}\in Q$ by our assumption on b, thus $bF\in Q_1+(T_1,...,T_{s+2})\cap Q_n$. By Lemma 3.2 it follows that $bF\in Q_{n-1}$, completing the proof. ∎

The next result parallels [1, Theorem 2.2] and gives bounds for the reduction number of certain analytic deviation 2 ideals. The argument is very similar to the proof of that theorem, with adjustments made for the increase in analytic deviation. We recall that an ideal I is said to by *syzygetic* if

$Q_1 = Q_2$.

Theorem 3.4. *Let (R, m) be a local ring with R/m infinite and let I be an ideal of R. Assume that $ad(I) = 2$, $ht(I) \geq 1$, R/I is Cohen-Macaulay, $G(I)$ is Cohen-Macaulay, and $\mu(I) = \ell(I) + 1$. Let ϕ be a presentation matrix of I. Let J be a minimal reduction of I and let j be a positive integer such that $I \subseteq I_1(\phi)^j$.*

(a) If R Cohen-Macaulay, $n_0 = 0$, $n_1 \leq 1$, and I syzygetic, then $r_J(I) \leq \lceil \frac{r_J(I)-1}{j} \rceil + 2$.

(b) If R Gorenstein and I_P is a complete intersection for $P \in \mathcal{P}$ then $r_J(I) \leq \lceil \frac{r_J(I)}{j} \rceil + 2$.

Proof. Let $s = ht(I)$. Then we may write $I = (a_1, ..., a_{s+3})$ where $J = (a_1, ..., a_{s+2})$ and $a_1, ..., a_{s+2}$ are chosen to satisfy Proposition 2.2 (1)-(6). Let $n = r_J(I)$. Then there exists a homogeneous polynomial $F \in Q$ of the form $F = T_{s+3}^{n+1} + G$ where G contains no term having T_{s+3}^{n+1} as a factor. By Lemma 3.3 we obtain that $I_1(\phi)^n F \subseteq Q_1$ if we are in case (b) and $I_1(\phi)^{n-1} F \subseteq Q_1$ if we are in case (a). If C denotes the ideal generated by last row of ϕ then $C = (J : a_{s+3})$, and by arguing as in the proof of [1, Theorem 2.2] we have that $I_1(\phi)^n \subseteq C$ in case (b) and $I_1(\phi)^{n-1} \subseteq C$ in case (a). By assumption $a_{s+3} \in I_1(\phi)^j$. Hence in case (b) we obtain $a_{s+3}^{\lceil \frac{n}{j} \rceil + 1} \in J$, and in case (a) we get $a_{s+3}^{\lceil \frac{n-1}{j} \rceil + 1} \in J$. Set $k = \lceil \frac{n}{j} \rceil + 2$ and consider case (b). We have shown that $I^{k-1} \subseteq J$, thus $I^{k+1} \subseteq JI^2$. By (2.2)(3) we have that $ht((J_{s+1} I^{n_1} : I^{n_1+1}) + I) \geq s + 2$, and by (2.2)(2) we have that $ht((J_s : I) + I) \geq s + 1$. We may now apply similar arguments as in the proof of [2, Theorem 5.2] (see also [1, proof of Theoerem 2.2]) to conclude that $I^{k+1} = JI^k$. Note that because of the analytic deviation 2 assumption, in order to apply [2, Proposition 5.10] as in the argument from [2, Theorem 5.2] we need to know that $I^{k+1} \subseteq JI^2$. This proves (b). The proof for (a) is obtained similarly, by starting with $k = \lceil \frac{n-1}{j} \rceil + 2$ instead. ■

As in [1] this theorem leads to explicit bounds on the reduction number. The following corollaries are analogous to [1, Corollaries 2.3 and 2.4]. Their proofs are similar to the argument of [1, Corollary 2.3].

Corollary 3.5. *Let R, I, and J be as in Theorem 3.4(a).*
(a) If $j > 1$ then $r_J(I) \leq 4$.
(b) If $j > 2$ then $r_J(I) \leq 3$.

Corollary 3.6. *Let R, I, and J be as in Theorem 3.4(b).*
(a) If $j > 2$ then $r_J(I) \leq 4$.
(b) If $j > 3$ then $r_J(I) \leq 3$.

We can use Corollary 3.6 to give another instance in which the Cohen-Macaulay property of $R[It]$ is equivalent to the Cohen-Macaulay property of $G(I)$.

Corollary 3.7. *Let (R, m) be a Cohen-Macaulay local ring and let I be an ideal of R. Assume that $ht(I) = 2$, $ad(I) = 2$, $\mu(I) = 5$, $n_0 = 0$, $n_1 \leq 1$, R/I is Cohen-Macaulay, I has finite projective dimension, and $depth(R/I^i) \geq dim(R/I) - i + 1$ for $i = 2, 3$. Then $R[It]$ is Cohen-Macaulay if and only if $G(I)$ is Cohen-Macaulay.*

Proof. Note that since $n_0 = 0$ and I has finite projective dimension, I is syzygetic [17]. If $R[It]$ is Cohen-Macaulay then $G(I)$ is Cohen-Macaulay by [12, Proposition 1.1]. Assume that $G(I)$ is Cohen-Macaulay. By the Hilbert-Burch Theorem [6] I is generated by the 4×4 minors of a 5×4 presentation matrix ϕ. Thus $I \subseteq I_1(\phi)^4$, therefore $r(I) \leq 3$ by Corollary 3.5(b). By [3, Theorem 5.1] $R[It]$ is Cohen-Macaulay. ∎

4. The row condition for height 2, analytic deviation 2 ideals

In this section we prove a row condition theorem for height 2, analytic deviation 2 ideals of a 4-dimensional Cohen-Macaulay local ring R. We recall a definition from [2].

Definition 4.1. Let I be an ideal in a local ring R and let ϕ be a presentation matrix of I. We say that ϕ satisfies the *row condition* if by performing invertible row and column operations we can obtain a matrix ψ such that the ideal of entries $I_1(\psi)$ is generated by some row of ψ.

In the papers [20], [2], and [1], results were proved that flushed the Cohen-Macaulay property of $R[It]$ out of the row condition on an ideal I, and conversely. These results all dealt with analytic deviation 1 ideals. More recently the paper [16] has extended this idea to certain strongly Cohen-Macaulay ideals in Gorenstein rings. In particular they prove the equivalence of (1), (3), and (4) of our Theorem 4.3 below (and that each implies (5)), not just for analytic deviation 2 but for analytic deviation $dim(R) - 2$, $dim(R)$ arbitrary. We make two improvements on their result, at the cost of restricting to analytic deviation 2. First we weaken the assumption on R from Gorenstein to Cohen-Macaulay. In addition we add the statements '$G(I)$ is Cohen-Macaulay' and '$R[It]$ is Cohen-Macaulay' to the list of equivalences. We first recall an exact sequence, discovered in [17], involving the 1^{st} Koszul homology of I. If H_1 denotes the first homology module of the Koszul complex built on a generating set $x_1, ..., x_n$ of I, then there is an exact sequence

$$0 \to \delta(I) \to H_1 \to [R/I]^n \to I/I^2 \to 0$$

of R-modules where $\delta(I) = ker(Sym_2(I) \to I^2)$. Recall that I is syzygetic if and only if $\delta(I) = 0$, therefore if I is syzygetic then there is an exact sequence

$$0 \to H_1 \to [R/I]^n \to I/I^2 \to 0. \tag{4.2}$$

The next result is an analytic deviation 2 version of [2, Theorem 8.2] (analytic deviation 2 is not included in the hypotheses, but is in fact forced by the other assumptions). Its proof contains arguments that are close, if not identical, to some of those in [2, Theorem 8.2] and [1, Theorem 3.1], although there are a few differences that compensate for the larger analytic deviation. For convenience we repeat most of the details.

Theorem 4.3. *Let (R, m) be a 4-dimensional Cohen-Macaulay local ring with R/m infinite, and let I be a perfect height 2 ideal of R. Assume that*

$\mu(I_P) \leq ht(P)$ for all $P \in \mathcal{P}$ and $\mu(I) = 5$. Then $ad(I) = 2$ and the following are equivalent:

(1) Every presentation matrix ϕ of I satisfies the row condition;

(2) $G(I)$ is Cohen-Macaulay;

(3) $r(I) \leq 3$;

(4) $N(I) \leq 4$;

(5) $R[It]$ is Cohen-Macaulay.

Remark 4.4. An ideal satisfying the hypotheses of Theorem 4.3 is generated be the 4×4 minors of any of the 5×4 matrices that present it by the Hilbert-Burch Theorem [6].

Proof. First of all we note that I is syzygetic [17, remark following Theorem 2.2] and the Koszul homology of I is Cohen-Macaulay [4, Theorem 2.1]. Thus $ad(I) = 2$ by [19, Proposition 2.6]. Also, the condition that $\mu(I_P) \leq ht(P)$ for $P \neq m$ shows that $n_0 = n_1 = 0$. By [12, Proposition 1.1] (5) implies (2), and by [8, Theorem 1.1] (5) implies (3), and (2), (3) imply (5). By Corollary 3.5(b), (2) implies (3). By the exact sequence (4.2) we obtain that $depth(I/I^2) \geq 1$. This forces $depth(R/I^2) \geq 1$, therefore (3) implies (2) by Theorem 2.9. We have thus established the equivalence of (2), (3), and (5).

Assume that (4) holds and let J be a minimal reduction of I. We may write $I = (a_1, ..., a_5)$ with $J = (a_1, ..., a_4)$. As usual let $Q = ker(R[T_1, ..., T_5] \to R[a_1t, ..., a_5t])$. Set $n = r_J(I)$. Then $a_5^{n+1} \in JI^n$, thus there exists a homogeneous polynomial $F \in Q$ of the form $F = T_5^{n+1} + G$ where $deg(G) = n + 1$ and F is a minimal generator of Q. By (4) we must have that $n+1 \leq 4$, hence $n \leq 3$. Therefore (3) holds. Assume that (3) (and hence (2) and (5)) holds. Then [3, Corollary 6.3] shows that $N(I) \leq \ell(I) = 4$. That $N(I) \leq 4$ can also be shown by elementary methods as follows. Let J be a minimal reduction of I satisfying $JI^3 = I^4$. Write $I = (a_1, ..., a_5)$ and $J = (a_1, ..., a_4)$ and assume that $a_1, ..., a_4$ satisfy Proposition 2.2 (1)-(6). Let $F \in Q_n$ be homogeneous with $n > 4$. If $F \in (T_1, ..., T_4)$ then $F \in Q_4$ by repeated (if necessary) use of Lemma 3.2. If $F \notin (T_1, ..., T_4)$ then write $F = rT_5^n + G$ where $G \in (T_1, ..., T_4)$ is homogeneous of degree n. By using that $a_5^4 \in JI^3$, F is congruent modulo Q_4 to an element in

$(T_1, ..., T_4) \cap Q_n$. By applying Lemma 3.2 again we obtain that $F \in Q_4$, establishing the equivalence of (3) and (4).

Assume that (1) holds and and let J be a minimal reduction of I. Write $I = (a_1, ..., a_5)$ with $J = (a_1, ..., a_4)$. Let ϕ be a presentation matrix of I built on $a_1, ..., a_5$. We may assume that $I_1(\phi) = (x_1, ..., x_4)$ is generated by the 5^{th} row of I. Then $(x_1, ..., x_4) = (J : a_5) = (J : I)$. By using that ϕ is a presentation matrix of I it follows that $(x_1, ..., x_4)I \subseteq (x_1, ..., x_4)J$. The determinant trick then yields that $a_5^4 \in JI^3$, therefore $JI^3 = I^4$ showing (3). Assume that (3) holds and let J be a minimal reduction of I satisfying $JI^3 = I^4$. Write $J = (a_1, ..., a_4)$ and $I = (a_1, ..., a_5)$ and let ϕ be a presentation matrix of I. By condition (4) we have that $Q = Q_4$, thus condition (2) allows us to apply Lemma 3.3 concluding that $I_1(\phi)Q \subseteq Q_3$. As the Koszul homology modules of the ideal I are Cohen-Macaulay, $Q_3 \subseteq Q_1$ by the result of Ulrich and Vasconcelos [19, Theorem 2.7] (as stated this theorem needs R to be Gorenstein, but the part of the proof giving $Q_3 \subseteq Q_1$ requires only that R be Cohen-Macaulay; see also [19, Theorem 4.1]). Thus $I_1(\phi)Q \subseteq Q_1$. As $a_5^4 \in JI^3$ there exists $F = T_5^4 + G \in Q$ homogeneous where $G \in (T_1, ..., T_4)R[T_1, ..., T_5]$ and $deg(G) = 4$. Let $y \in I_1(\phi)$ be arbitrary. Then $yF \in Q_1$, say $yF = L_1G_1 + ... + L_5G_5$ where $Q_1 = (L_1, ..., L_5)$. If the last row of ϕ is $[\,x_1 \quad x_2 \quad x_3 \quad x_4\,]$ then $(x_1, ..., x_4) = (J : a_5)$. We claim that $y \in (J : a_5)$. From the equation $yF = L_1G_1 + ... + L_5G_5$ we see that the coefficient of T_5^4, which is y, must be in the ideal generated by the coefficients of T_5 in $L_1, ..., L_5$. But this ideal is contained in $(J : I)$ because each L_i is in Q. This shows that $I_1(\phi) \subseteq (x_1, ..., x_4)$, and therefore $I_1(\phi) = (x_1, ..., x_4)$ establishing (1). The proof of Theorem 4.3 is now complete. ∎

We record a corollary that follows [2, Corollary 8.3].

Corollary 4.5. *Let (R,m) and I satisfy the conditions of Theorem 4.3. Let* ϕ *be a* 5×4 *matrix presenting I. If* $I_1(\phi)$ *is generated by some column of* ϕ*, and the other columns have entries in* $mI_1(\phi)$*, then* $R[It]$ *is not Cohen-Macaulay.*

Proof. See the proof of [2, Corollary 8.3]. ∎

Example 4.6. Let $T = k[u,v,w,x]$, let $N = (u,v,w,x)T$, let $R = T_N$, let $m = NT_N$, and let I be the ideal of R generated by the 4×4 minors of

$$\phi = \begin{bmatrix} u & v & w & x \\ 0 & w & x & u \\ w & x & 0 & v \\ x & u & v & 0 \\ u^2 & uv & vx & w^2 \end{bmatrix}.$$

Then $G(I)$ is Cohen-Macaulay.

Proof. I is perfect and is not contained in any principal ideal so has height two. Theorem 4.3 applies as long as we know that $\mu(I_P) \leq ht(P)$ for $P \neq m$. This can be seen in the following way. Let $P \neq m$ be a prime in R containing I. Then P does not contain all of the variables. One may invert one of the variables and do row and column operations to get a matrix of the form

$$\phi' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

If $ht(P) = 3$ then this shows that $\mu(I_P) \leq 3$. If one of the entries $*$ of the matrix ϕ' is a unit then further operations show that I_P is two generated. Using the computer program MACAULAY one can show that the ideal A generated by the entries $*$ has height 3. Thus if $ht(P/I) = 0$ then I_P is two generated. ∎

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