THE GROTHENDIECK ALGEBRA OF A HOPF ALGEBRA, I

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0 Introduction

We began in [4] a study of the Grothendieck group $\mathcal{G}(H)$ of the category of finite dimensional right comodules of a Hopf algebra H. We begin here the study of the representation theory of the algebras $\mathcal{G}(H)^K$ obtained from $\mathcal{G}(H)$ by extending the scalars. Our primary motivation has been to try to establish Kaplansky's conjecture that if a finite dimensional cosemisimple Hopf algebra H over an algebraically closed field contains a simple subcoalgebra of dimension n^2, then n divides the dimension of H. As a more direct attack on the problem has not been successful, we have been studying the problem from a general point of view. Our question can be viewed as a conjecture about a character of $\mathcal{G}(H)$, so we have been developing information about the representation theory of $\mathcal{G}(H)$ in order to answer it. Of course, this information may be useful in other contexts as well. Our methods here have a geometric flavor; further results will appear in a subsequent paper.

After considering some generalities in Section 1, we concentrate on the case in which H is cosemisimple and finite dimensional. In Section 2, we show that $\mathcal{G}(H)^{\mathbb{C}}$ is semisimple, and obtain invariants t_j from its irreducible characters μ_j. We obtain detailed information related to the principal representation in Section 3. In Section 4, we study an element z of $\mathcal{G}(H)$, the class of the right coadjoint corepresentation of H, which has connections with the orthogonality relations for the characters of $\mathcal{G}(H)^{\mathbb{C}}$ and with the invariants t_j. We show in Section 5 that, except in certain very special cases, the subalgebra of $\mathcal{G}(H)^{\mathbb{Q}}$ generated by the image of the class of a simple comodule must have dimension at least

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four. As a corollary, we obtain that a cosemisimple Hopf algebra of dimension at least seven over an algebraically closed field must have at least four simple subcoalgebras.

We assume general familiarity with Hopf algebras, as presented in [5]. We review terminology from [4] as needed.

1 Generalities

Let H be a coalgebra defined over a field k, and let $\mathcal{M}^H$ denote the category of finite dimensional right H-comodules. We let $\mathcal{G}(H)$ denote the Grothendieck group of $\mathcal{M}^H$. Roughly speaking, $\mathcal{G}(H)$ is an abelian group with generators corresponding to isomorphism classes of objects of $\mathcal{M}^H$, and relations equating the class of the middle term of a short exact sequence in $\mathcal{M}^H$ with the sum of the classes of the end terms. (A precise definition is given in [4].) The image of the class of $M \in \mathcal{M}^H$ in $\mathcal{G}(H)$ is denoted $[M]$. $\mathcal{G}(H)$ is a free abelian group, with *standard basis* $\mathcal{X}$ consisting of the images of the classes of the simple right H-comodules. The elements of the standard basis are also referred to as the *basic elements* of $\mathcal{G}(H)$. If the basic element x occurs with nonzero coefficient when $z \in \mathcal{G}(H)$ is expressed in terms of the standard basis, then x is called a *basic component* of z.

We first discuss the behavior of the Grothendieck group upon extension of the base field.

Let L/k be a field extension. We use an upper L (on the right) to denote the result of applying the functor $(\ \cdot\ ) \otimes L$. Recall that, for k-vector spaces V, W, we have $(V \otimes W)^L \simeq (V^L) \otimes_L (W^L)$. For any k-coalgebra C, C^L is an L-coalgebra. For any right C-comodule M, M^L is a right C^L-comodule.

Proposition 1 *Let H be a k-coalgebra, and let L/k be a field extension. There is a monomorphism of abelian groups $(\ \cdot\ )^L : \mathcal{G}(H) \longrightarrow \mathcal{G}(H^L)$ given by: for all $M \in \mathcal{M}^H$, $([M])^L = [M^L]$. If H is a bialgebra, then $(\ \cdot\ )^L$ is a ring homomorphism. If H is a Hopf algebra, then $(\ \cdot\ )^L$ preserves " $*$ ".*

Proof. Since $(\ \cdot\ ) \otimes L$ sends a short exact sequence of C-comodules to a short exact sequence of C^L-comodules, the stated assignment defines a homomorphism of abelian groups. We must show that it is a monomorphism.

Let $x \in \mathcal{G}(H)$ be basic. Then there is a unique simple subcoalgebra C of H such that $x = [V]$ for a simple right C-comodule V. We have $x^L = [V^L]$. Since V^L is a C^L-comodule, we have by Proposition 2 of [4] that each basic component of x^L is associated with a simple subcoalgebra of C^L. Thus if x, y are distinct basic elements of $\mathcal{G}(H)$, x^L and y^L have no common basic components. It now follows easily that $(\ \cdot\ )^L$ is a monomorphism.

Now suppose that H is a bialgebra. It is easy to see that for right H-comodules M, N, we have that $(M \otimes N)^L \simeq M^L \otimes_L N^L$ as right H-comodules. Thus $(\ \cdot\ )^L$ preserves the ring structure. Now suppose that H is a Hopf algebra. Let M be a finite dimensional right H-comodule. Then $M^* = Hom_k(M,k)$ is a right H-comodule in a natural way, as described in [4, §3.]. It is not hard to see that the vector space isomorphism $(M^*)^L = (Hom_k(M,k))^L \simeq Hom_L(M^L, L) = (M^L)^*$ preserves the H^L-comodule structure. It follows that $(x^*)^L = (x^L)^*$ for all $x \in \mathcal{G}(H)$, as required. □

Let H be a k-coalgebra. Recall from [4] that for $M, N \in \mathcal{M}^H$ we write $i(M,N) = dim_k \mathcal{M}^H(M,N)$. For basic $x, y \in \mathcal{G}(H)$, associated to simple H-comodules V_x, V_y respectively, we define $i(x,y) = i([V_x],[V_y]) = i(V_x, V_y)$. Then we extend i to a biadditive function from $\mathcal{G}(H) \times \mathcal{G}(H)$ to $\mathbb{Z}$.

Remark 2 *When k is algebraically closed, i is the same as the function m of [4]. In any case, if $x \in \mathcal{G}(H)$ is basic, then for all $z \in \mathcal{G}(H)$ we have $i(x,z) = i(x,x)m(x,z)$, so $i(x,z) > 0$ iff $m(x,z) > 0$.*

Proposition 3 *Let H be a k-coalgebra, and let $M, N \in \mathcal{M}^H$.*

1. *For every field extension L/k, we have $i(M,N) = i(M^L, N^L)$.*
2. *$i(M,N) \leq i([M],[N])$, with equality if M and N are sums of simple subcomodules.*

Proof.

1. The dual version of [1, 29.5] gives us that $\mathcal{M}^H(M,N) \otimes L \simeq \mathcal{M}^{H^L}(M^L, N^L)$ as L-vector spaces, and our assertion follows.
2. First suppose that M is simple. We prove our assertion by induction on the number of composition factors of N. If N is simple, we have equality by definition. Otherwise, we have a short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$, where N', N'' have fewer composition factors than does N. The kernel of the induced mapping $\mathcal{M}^H(M,N) \longrightarrow \mathcal{M}^H(M,N'')$ can be identified with a subspace of $\mathcal{M}^H(M,N')$, so $i(M,N) \leq i(M,N') + i(M,N'')$. Using the inductive hypothesis, we thus have that $i(M,N) \leq i([M],[N']) + i([M],[N'']) = i([M],[N])$. Moreover, it is clear that if $N = N' \oplus N''$ then $i(M,N) = i(M,N') + i(M,N'')$, and thus $i(M,N) = i([M],[N])$. Hence we are done for the case in which M is simple. A similar induction on the number of composition factors of M completes the proof. □

Corollary 4 *Let H be a cosemisimple k-coalgebra, and let L/k be a field extension such that H^L is cosemisimple. Then $(\ \cdot\ )^L : \mathcal{G}(H) \longrightarrow \mathcal{G}(H^L)$ preserves the form i.*

Proof. Using both parts of Proposition 3, we have that for $[M],[N] \in \mathcal{G}(H)$, $i([M],[N]) = i(M,N) = i(M^L,N^L) = i([M^L],[N^L]) = i(([M])^L, ([N])^L)$. It follows that $i(x,y) = i(x^L, y^L)$ for all $x, y \in \mathcal{G}(H)$. □

We shall later on need the following general result. Recall from [4] that the coalgebra associated to $M \in \mathcal{M}^H$ is the smallest subcoalgebra C_M of H such that $\rho(M) \subseteq M \otimes C_M$, where $\rho : M \longrightarrow M \otimes H$ is the comodule structure map. If $\{m_j\}$ is a basis of M, and $\rho(m_j) = \sum_i m_i \otimes c_{ij}$, then C_M is the span of the elements c_{ij}.

Proposition 5 *Let H be a bialgebra, let $M \in \mathcal{M}^H$, and let $C = C_M$ be the subcoalgebra of H associated to M. Write $x = [M] \in \mathcal{G}(H)$. Then:*

1. *For each $n \geq 1$, the simple subcoalgebras of C^n are the subcoalgebras corresponding to the basic components of x^n.*
2. *The simple subcoalgebras of the subbialgebra B of H generated by C are in $1-1$ correspondence with the basic components of the powers of x.*
3. *If B is a Hopf algebra, then 1 is a basic component of x^n for some $n \geq 1$.*

Proof.

1. Let $N \in \mathcal{M}^H$, and let D be the subcoalgebra of H associated to N. Looking at a basis for $M \otimes N$ obtained in the usual manner from bases of M and N, we observe that CD is the subcoalgebra associated to $M \otimes N$. It thus follows easily by induction that, for all $n \geq 1$, C^n is the subcoalgebra associated to $\otimes^n M$. Since $x^n = [\otimes^n M]$, our assertion follows by [4, Proposition 2].
2. The subbialgebra generated by C is $B = k1 + C + C^2 + C^3 + \cdots$. By convention, we are considering $1 = x^0$ to be a power of x. Since every simple subcoalgebra of B other than $k1$ is a subcoalgebra of C^n for some $n \geq 1$, we are done by (1).
3. Now suppose that B is a Hopf algebra, with antipode S. Since $k1 \subseteq S(C)C$, we have by [4, Proposition 5(2)] that 1 is a basic component of yx for some basic element y corresponding to a simple subcoalgebra of $S(C)$. Since $S(C) \subseteq B$, y is a basic component of some x^n by the above, and thus 1 is a basic component of x^{n+1}. □

We now begin the study of the algebras obtained from the Grothendieck group.

Let H be a coalgebra. For any commutative ring R, we let $\mathcal{G}(H)^R = \mathcal{G}(H) \otimes_{\mathbb{Z}} R$, the R-module obtained by extending the scalars. We shall be concerned here primarily with the case in which R is a subfield of the complex numbers—indeed, primarily with the case $R = \mathbb{C}$ itself.

We define an inner product on $\mathcal{G}(H)^{\mathbb{C}}$. For $u = \sum_{x \in \mathcal{X}} \alpha_x x$, $v = \sum_{y \in \mathcal{X}} \beta_y y \in \mathcal{G}(H)^{\mathbb{C}}$, we set $(u, v) = \sum_{x,y \in \mathcal{X}} \alpha_x \overline{\beta_y} i(x, y)$. Since $i(x, y) = i(y, x)$ for any basic elements x, y, and thus $(u, v) = \overline{(v, u)}$ for any $u, v \in \mathcal{G}(H)^{\mathbb{C}}$, it is easily verified that $(\ \cdot\ ,\ \cdot\ )$ is indeed an inner product on $\mathcal{G}(H)^{\mathbb{C}}$. Then the *length* of $u \in \mathcal{G}(H)^{\mathbb{C}}$ is $\|u\| = (u, u)^{\frac{1}{2}}$. Note that $(u, v) = i(u, v)$ for all $u, v \in \mathcal{G}(H)$.

Now suppose that H is a Hopf algebra. We extend the ring anti-homomorphism $* : \mathcal{G}(H) \longrightarrow \mathcal{G}(H)$ [4, Proposition 8] to a conjugate linear mapping $* : \mathcal{G}(H)^{\mathbb{C}} \longrightarrow \mathcal{G}(H)^{\mathbb{C}}$ via $(\sum_{x \in \mathcal{X}} \alpha_x x)^* = \sum_{x \in \mathcal{X}} \overline{\alpha_x} x^*$.

Two important situations in which the inner product is preserved are given by the following.

Proposition 6 1. *Let H be a cosemisimple k-coalgebra, and let L/k be a field extension such that H^L is cosemisimple. Then for all $u, v \in \mathcal{G}(H)$ we have $(u, v) = (u^L, v^L)$.*

2. *Let H be a Hopf algebra whose antipode is injective. Then for all $u, v \in \mathcal{G}(H)^{\mathbb{C}}$ we have $(u^*, v^*) = (v, u)$.*

Proof.

1. Since $(u, v) = i(u, v)$ for all $u, v \in \mathcal{G}(H)$, our assertion follows by Corollary 4.

2. It suffices to establish this for u, v basic. Since the antipode is injective, u^*, v^* are also basic, and thus [4, Proposition 7(2)] gives us that $i(u^*, v^*) = i(v, u)$, as required. □

We now begin our study of representations.

Let H be a coalgebra. The *degree map* is the unique group homomorphism $d : \mathcal{G}(H) \longrightarrow \mathbb{Z}$ such that $d([M]) = dim_k M$ for all $M \in \mathcal{M}^H$. (This map was denoted *deg* in [4].) For any commutative ring R, $d = d^R : \mathcal{G}(H)^R \longrightarrow R$ is also called the degree map. When H is a bialgebra, d is an R-algebra map, and we call it the *principal representation* of $\mathcal{G}(H)^R$.

When H is a finite dimensional Hopf algebra, we have by [3, Proposition 3] that for all $M \in \mathcal{M}^H$, $M \otimes H \simeq H^{dim M} \simeq H \otimes M$ as right H-comodules. Thus, in $\mathcal{G}(H)$ we have $[M][H] = d([M])[H] = [H][M]$. For any commutative ring R in which $(dim H)1$ is a unit, $e_1 = \frac{1}{dim H}[H]$ is thus a central idempotent of $\mathcal{G}(H)^R$, which we call the *principal idempotent*.

We shall make use of two norms N_1, N_2 in our study of the principal representation, playing off the ease of computation of N_2 against the better theoretical behavior of N_1.

For a finite dimensional Hopf algebra H, we define $N_1 : \mathcal{G}(H)^{\mathbb{C}} \longrightarrow \mathbb{R}$ by:

$$N_1(x) = \sup_{u \neq 0} \frac{||xu||}{||u||} = \sup_{||u||=1} ||xu||, \quad \text{all } x \in \mathcal{G}(H)^{\mathbb{C}}.$$

For any Hopf algebra H, we define $N_2 : \mathcal{G}(H)^{\mathbb{C}} \longrightarrow \mathbb{R}$ by:

$$N_2(\sum_{x \in \mathcal{X}} \alpha_x x) = \sum_{x \in \mathcal{X}} |\alpha_x| d(x), \quad \text{all } x \in \mathcal{G}(H)^{\mathbb{C}}.$$

It is easy to see that each of N_1, N_2 satisfies the requirements to be a norm N: $N(x) \geq 0$; $N(x) = 0$ iff $x = 0$; $N(\alpha x) = |\alpha| N(x)$; $N(x + y) \leq N(x) + N(y)$.

It is also easy to see that N_1, N_2 are compatible with multiplication: $N(xy) \leq N(x)N(y)$ for all $x, y \in \mathcal{G}(H)^{\mathbb{C}}$. This implies that the eigenvalues of left multiplication by x are bounded by $N(x)$: if $y \neq 0$ and $xy = \alpha y$ for some $\alpha \in \mathbb{C}$, then $|\alpha| \leq N(x)$.

Proposition 7 *Let H be a finite dimensional Hopf algebra.*

1. *For all $x \in \mathcal{G}(H)^{\mathbb{C}}$, $|d(x)| \leq N_1(x)$.*

2. *For all $x, y \in \mathcal{G}(H)^{\mathbb{C}}$, $|(xy, y)| \leq N_1(x)(y, y)$. Furthermore, $|(xy, y)| = N_1(x)(y, y)$ iff $xy = \alpha y$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = N_1(x)$.*

Proof.

1. Since $x[H] = d(x)[H]$, we must have $|d(x)| \leq N_1(x)$.

2. Using the Cauchy-Schwartz inequality and the definition of N_1, we have $|(xy, y)| \leq ||xy|| \, ||y|| \leq N_1(x)||y|| \, ||y|| = N_1(x)(y, y)$. For the second assertion, we may assume $y \neq 0$. By Cauchy-Schwartz, the first inequality is then an equality iff $xy = \alpha y$ for some $\alpha \in \mathbb{C}$. The second inequality is then an equality as well iff $|\alpha| = N_1(x)$. □

2 Basic Results

Our fundamental tool is the following theorem, which is an extension of Theorem 9 of [4].

Theorem 8 *Let H be a cosemisimple Hopf algebra. Then $(uv, w) = (v, u^*w) = (u, wv^*)$ for all $u, v, w \in \mathcal{G}(H)^{\mathbb{C}}$.*

Proof. It suffices to establish our assertion for u, v, w in $\mathcal{G}(H)$. By Propositions 1 and 6, we may assume that the base field is algebraically closed. This gives us that our inner product agrees on $\mathcal{G}(H)$ with the function m of [4]. Using Theorem 9 of [4], we have $(uv, w) = (w, uv) = (u, wv^*)$. Since " $*$ " is an involution [4, Proposition 8], we then calculate that $(uv, w) = (v^*u^*, w^*) = (v^*, w^*u) = (v, u^*w)$. □

We obtain from Theorem 8 some basic information about the structure of the algebras we are considering.

Theorem 9 *Let H be a cosemisimple Hopf algebra.*

1. *For $u \in \mathcal{G}(H)^{\mathbb{C}}$, $u = 0$ iff $uu^* = 0$.*

2. *$\mathcal{G}(H)^{\mathbb{C}}$ contains no nonzero nil right ideals.*

3. *If H is finite dimensional, then $\mathcal{G}(H)^{\mathbb{Q}}$ is separable.*

Proof.

1. From Theorem 8, $(1, uu^*) = (u, u) = ||u||^2$, so if $uu^* = 0$ then $u = 0$.

2. Let u be a nonzero element of a nil right ideal N of $\mathcal{G}(H)^{\mathbb{C}}$. Then $w = uu^*$ is also nonzero. Since $w \in N$, there exists a positive integer n such that $w^n \neq 0$ but $w^{n+1} = 0$. But then $0 \neq w^n(w^n)^* = w^n(w^*)^n = w^{2n} = 0$, a contradiction.

3. When H is finite dimensional, $\mathcal{G}(H)^{\mathbb{C}}$ is semisimple by (2), and thus is a direct sum of matrix algebras. $\square$

Proposition 10 *Let H be a finite dimensional cosemisimple Hopf algebra. Let $e_1, \ldots, e_m$ be the central primitive idempotents of $\mathcal{G}(H)^{\mathbb{C}}$, with $e_1 = \frac{1}{dim H}[H]$ being the principal idempotent. For each j, let μ_j be the irreducible character of $\mathcal{G}(H)^{\mathbb{C}}$ corresponding to e_j, and let $n_j = \mu_j(1)$ be the degree of μ_j. Then:*

1. *$e_j^* = e_j$ for each j.*
2. *$(\ \cdot\ , e_j) = t_j \mu_j$, where $t_j = \frac{\|e_j\|^2}{n_j}$ is a positive real number.*

Proof.

1. Since " $*$ " is an involution, e_j^* is again a central primitive idempotent. Since $e_j e_j^*$ is nonzero by Theorem 9(1), we must have $e_j^* = e_j$.
2. Note that μ_j is defined by the properties that $\mu_j(ab) = \mu_j(ba)$ for all $a, b \in \mathcal{G}(H)^{\mathbb{C}}$ and $\mu_j(e_i) = \delta_{ij} n_j$. We have $(ab, e_j) = (a, e_j b^*) = (be_j, a^*) = (e_j, a^* b^*) = (ba, e_j)$, using (1), Theorem 8 and Proposition 6. Since $(e_i, e_j) = \delta_{ij}\|e_j\|^2$, our result follows. $\square$

Remark 11 *It follows from Proposition 10 that for every character μ of $\mathcal{G}(H)^{\mathbb{C}}$ we have $\mu(x^*) = \overline{\mu(x)}$ for all $x \in \mathcal{G}(H)^{\mathbb{C}}$.*

Remark 12 *The invariants t_j have been shown [6] to be quite useful when the base field is of characteristic zero, in which case basic elements of $\mathcal{G}(H)$ can be identified with irreducible characters in H.*

3 The Principal Representation

Our first goal is to tighten the connection between d and our norms, thus greatly increasing the utility of Proposition 7.

Lemma 13 *Let H be a finite dimensional cosemisimple Hopf algebra. Suppose that $x \in \mathcal{G}(H)^{\mathbb{C}}$ commutes with x^*. Then for any norm N on $\mathcal{G}(H)^{\mathbb{C}}$ which is compatible with multiplication, $|(xy, y)| \leq N(x)(y, y)$ for all $y \in \mathcal{G}(H)^{\mathbb{C}}$.*

Proof. We may assume that y is nonzero. We have by Theorem 8 that the adjoint of left multiplication by x on the inner product space $\mathcal{G}(H)^{\mathbb{C}}$ is left multiplication by x^*. Since x commutes with x^*, left multiplication by x is a normal operator, so $\mathcal{G}(H)^{\mathbb{C}}$ has an orthonormal basis consisting of its eigenvectors. So we can write $y = \sum_i y_i$, where each y_i is nonzero, $(y_i, y_j) = 0$ if $i \neq j$, and $xy_i = \alpha_i y_i$ for some $\alpha_i \in \mathbb{C}$. Then $|(xy, y)| = |\sum_{i,j}(\alpha_i y_i, y_j)| = |\sum_i(\alpha_i y_i, y_i)| \leq \sum_i |\alpha_i|(y_i, y_i) \leq \sum_i N(x)(y_i, y_i) = N(x)(y, y)$. $\square$

We shall consider $\mathcal{G}(H)^{\mathbb{R}} \subseteq \mathcal{G}(H)^{\mathbb{C}}$. Write

$$\mathcal{G}(H)^{\mathbb{R}}_+ = \{\sum_{x \in \mathcal{X}} \alpha_x x : \quad \text{for all } x \in \mathcal{X}, \alpha_x \in \mathbb{R} \text{ and } \alpha_x \geq 0\}.$$

Any $u \in \mathcal{G}(H)^{\mathbb{R}}$ can be written uniquely as $u = u_+ - u_-$, where $u_+, u_- \in \mathcal{G}(H)^{\mathbb{R}}_+$ have no common basic components. We call u_+ the positive part of u, and $-u_-$ the negative part of u.

Theorem 14 *Let H be a finite dimensional cosemisimple Hopf algebra.*

1. *For all $x \in \mathcal{G}(H)^{\mathbb{C}}$, $|d(x)| \leq N_1(x) \leq N_2(x)$.*
2. *For all $x \in \mathcal{G}(H)^{\mathbb{R}}_+$, $d(x) = N_1(x) = N_2(x)$.*

Proof.

1. We have $|d(x)| \leq N_1(x)$ by Proposition 7. For all $u \in \mathcal{G}(H)^{\mathbb{C}}$ we have $(xu, xu) = (x^*xu, u) \leq N_2(x^*x)(u, u)$ by Lemma 13, since $(x^*x)^* = x^*x$. Thus $\|xu\|^2 \leq N_2(x^*x)\|u\|^2 \leq N_2(x^*)N_2(x)\|u\|^2 = N_2(x)^2\|u\|^2$. So $N_1(x) = \sup_{\|u\|=1} \|xu\| \leq N_2(x)$, as required.
2. Since $d(x) = N_2(x)$ for $x \in \mathcal{G}(H)^{\mathbb{R}}_+$, this follows immediately from (1). □

Proposition 15 *Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field k. Then for all basic $x, y \in \mathcal{G}(H)$, $\|xy\|^2 \leq d(xy)$.*

Proof. Applying " * " if necessary, we see that we may assume $d(x) \leq d(y)$. Since k is algebraically closed, we have $(y, y) = 1$. We compute, using Proposition 7(2) and Theorem 14(2), that $\|xy\|^2 = (xy, xy) = (x^*xy, y) \leq N_1(x^*x)(y, y) = d(x^*x) = d(x)^2 \leq d(x)d(y) = d(xy)$. □

Let H be a finite dimensional cosemisimple Hopf algebra. For $x \in \mathcal{G}(H)^{\mathbb{R}}_+$, we now study the eigenspace for the eigenvalue $d(x)$ for left multiplication by x on $\mathcal{G}(H)^{\mathbb{C}}$. By Theorem 14(2), $d(x)$ is the largest eigenvalue. We call the eigenspace the *principal eigenspace* for (left multiplication by) x, and denote it $E(x)$. Since $z \in \mathcal{G}(H)^{\mathbb{C}}$ lies in $E(x)$ iff its real and imaginary parts are in $E(x)$, we may restrict our attention to $\mathcal{G}(H)^{\mathbb{R}}$.

Proposition 16 *Let H be a finite dimensional cosemisimple Hopf algebra. Let $x \in \mathcal{G}(H)^{\mathbb{R}}_+$.*

1. *If $v \in \mathcal{G}(H)^{\mathbb{R}}$ lies in $E(x)$, then v_+ and v_- are also in $E(x)$.*
2. *For each basic component x_i of x we have $E(x) \subseteq E(x_i)$.*

Proof.

1. We have $d(x)v_+ - d(x)v_- = d(x)v = xv = xv_+ - xv_-$. Thus $d(x)v_+$ is the positive part of $xv_+ - xv_-$. Since positive contributions come exclusively from xv_+, and $d(d(x)v_+) = d(xv_+)$, we must have $d(x)v_+ = xv_+$. Thus $d(x)v_- = xv_-$ as well.

2. It follows from (1) that it suffices to show that if $w \in E(x)$ lies in $\mathcal{G}(H)^{\mathbb{R}}_+$, then $w \in E(x_i)$. Let $\alpha > 0$ be the coefficient of x_i in x, where x is expressed in terms of the standard basis. Write $x = u + v$, where $u = \alpha x_i, v = x - \alpha x_i$ are in $\mathcal{G}(H)^{\mathbb{R}}_+$. We have $(uw, w) + (vw, w) = ((u + v)w, w) = (d(u + v)w, w) = d(u + v)(w, w) = d(u)(w, w) + d(v)(w, w)$. Since $(uw, w) \le d(u)(w, w)$ and $(vw, w) \le d(v)(w, w)$ by Proposition 7(2) and Theorem 14(2), we must have equality in each case. Now Proposition 7(2) gives our desired conclusion. □

For any subbialgebra B of a finite dimensional Hopf algebra H, let us write $y_B = [B] \in \mathcal{G}(H)$. When H is cosemisimple, and $x_{i_1}, \dots, x_{i_r}$ are the standard basis elements of $\mathcal{G}(H)$ corresponding to the simple subcoalgebras $C_{i_1}, \dots, C_{i_r}$ such that $B = C_{i_1} \oplus \cdots \oplus C_{i_r}$, then $y_B = d'(x_{i_1})x_{i_1} + \cdots + d'(x_{i_r})x_{i_r}$, where, if x is the basic element corresponding to the simple subcoalgebra C, we write $[C] = d'(x)x$.

Proposition 17 *Let H be a finite dimensional cosemisimple Hopf algebra, and let $x \in \mathcal{G}(H)^{\mathbb{R}}_+$. Let B be the subbialgebra of H corresponding to the standard subring of $\mathcal{G}(H)$ generated by the basic components of x. Then $E(x) = y_B\mathcal{G}(H)^{\mathbb{C}}$.*

Proof. Since $x \in \mathcal{G}(B)^{\mathbb{R}}$ we have $xy_B = d(x)y_B$, so $y_B\mathcal{G}(H)^{\mathbb{C}} \subseteq E(x)$. For the reverse inclusion, it is enough to show that every $u \in \mathcal{G}(H)^{\mathbb{R}}$ such that $xu = d(x)u$ lies in $y_B\mathcal{G}(H)^{\mathbb{R}}$. By Proposition 16(1), u_+ and u_- are also in $E(x)$, so we might as well assume that $u \in \mathcal{G}(H)^{\mathbb{R}}_+$. By Proposition 16(2), $x_iu = d(x_i)u$ for each basic component x_i of x, so it is harmless to assume that x is the sum of its basic components, and thus that $x = [M]$, where the subcoalgebra associated to $M \in \mathcal{M}^H$ generates B. By Proposition 5, each basic element y corresponding to a simple subcoalgebra of B is a basic component of x^n for some n. Using Proposition 16(2) again, we obtain that $yu = d(y)u$ for each such y. Thus $y_Bu = d(y_B)u$, so $u = y_Bd(y_B)^{-1}u \in y_B\mathcal{G}(H)^{\mathbb{R}}$. □

We now obtain more detailed information about $y_B\mathcal{G}(H)$.

Proposition 18 *Let H be a finite dimensional cosemisimple Hopf algebra, and let B be a subbialgebra of H. Let $\{x_1, \dots, x_n\}$ be the standard basis of $\mathcal{G}(H)$ and let $C_1, \dots, C_n$ be the corresponding simple subcoalgebras.*

1. *There is an equivalence relation $\sim$ on the set of simple subcoalgebras of H, given by $C_i \sim C_j$ iff $C_i \subseteq B\,C_j$.*

2. *For each equivalence class $\mathcal{C} = \{C_{i_1}, \dots, C_{i_r}\}$, $l_{\mathcal{C}} = d'(x_{i_1})x_{i_1} + \cdots + d'(x_{i_r})x_{i_r}$ lies in $y_B\mathcal{G}(H)$. The elements $l_{\mathcal{C}}$ form a basis of $y_B\mathcal{G}(H)^{\mathbb{Q}}$.*

Proof.

1. The relation $\sim$ is clearly reflexive, and easily seen to be transitive. We show that it is symmetric. By Remark 2 and [4, Proposition 5(2)], we have that $C_i \subseteq BC_j$ iff $(x_i, y_B x_j) > 0$. But $y_B^* = y_B$, so $(x_i, y_B x_j) = (y_B x_i, x_j) = (x_j, y_B x_i)$, and our conclusion follows.

2. Our argument in (1) shows that for any j, $y_B x_{i_j} = \alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r}$ for some positive integers $\alpha_1, \ldots, \alpha_r$. Unless $y_B x_{i_j}$ is a scalar multiple of $l_{\mathcal{C}} = d'(x_{i_1})x_{i_1} + \cdots + d'(x_{i_r})x_{i_r}$, we can find $\alpha > 0$ such that $(y_B x_{i_j} - \alpha y_H)_+$ is a nonzero vector in which at least one of the possible basic components $x_{i_1}, \ldots, x_{i_r}$ does not appear. Now $y_B x_{i_j} - \alpha y_H$ is in $E(y_B)$, so, by Proposition 16(1), $(y_B x_{i_j} - \alpha y_H)_+$ is as well. But our first observation then gives us that each of $x_{i_1}, \ldots, x_{i_r}$ must be a basic component, a contradiction. We thus conclude that $y_B x_{i_j}$ must be a scalar multiple of $l_{\mathcal{C}}$, so $l_{\mathcal{C}} \in y_B \mathcal{G}(H)$. The $l_{\mathcal{C}}$'s are clearly linearly independent, and our argument shows that they span $y_B \mathcal{G}(H)^{\mathbb{Q}} = \sum_i \mathbb{Q} y_B x_i$, so they form a basis of $y_B \mathcal{G}(H)^{\mathbb{Q}}$. □

Corollary 19 *Let H be a finite dimensional cosemisimple Hopf algebra, and let $x \in \mathcal{G}(H)_+^{\mathbb{R}}$. Let K be the subfield of $\mathbb{R}$ generated by the coefficients of x with respect to the standard basis of $\mathcal{G}(H)$. Let B be the subbialgebra of H corresponding to the standard subring of $\mathcal{G}(H)$ generated by the basic components of x. Then $y_B \in K\langle x \rangle$.*

Proof. Let $f(t) \in K[t]$ be the minimal polynomial of x. Since $f(d(x)) = d(f(x)) = 0$, we have $f(t) = (t - d(x))g(t)$ in $K[t]$. Then we see that $K\langle x \rangle$ contains a nonzero element $u = g(x)$ with $xu = d(x)u$. Since $u \in \mathcal{G}(B)^K$, Propositions 17 and 18 give us that u is a scalar multiple of y_B. □

4 The Element z

Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field k. In this section, we study the element $z = \sum_i x_i^* x_i$, where $\{x_1, \ldots, x_n\}$ is the standard basis of $\mathcal{G}(H)$.

We first provide an interpretation of z. Any subcoalgebra C of a Hopf algebra H becomes a right H-comodule via $\triangleright$: $C \longrightarrow C \otimes H$, $\triangleright(c) = \sum c_2 \otimes S(c_1)c_3$, all $c \in C$. Suppose that C is a matrix coalgebra, with basis $\{c_{ij}\}$ such that $\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$, $\epsilon(c_{ij}) = \delta_{ij}1$. Let V be a simple right C-comodule, with basis $\{v_j\}$ such that $\rho(v_j) = \sum_i v_i \otimes c_{ij}$. The right comodule structure on V^* is given in terms of the dual basis $\{v_i^*\}$ by $\rho(v_i^*) = \sum_j v_j^* \otimes S(c_{ij})$. Then $\varphi : C \longrightarrow V^* \otimes V$ given by $\varphi(c_{ij}) = v_i^* \otimes v_j$ is an isomorphism from $(C, \triangleright)$ to $V^* \otimes V$. We thus see that when H is a finite dimensional cosemisimple Hopf algebra over an algebraically closed field, $z = [(H, \triangleright)]$.

Note that $\sum_i x_i^* x_i$ is a rearrangement of $\sum_i x_i x_i^*$, so we also have $z = \sum_i x_i x_i^*$.

Much of our next result could be obtained from [2, (9.17)] by introducing a different form, but it is clearer to just prove what we need in our context.

Theorem 20 *Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field. Let $z = \sum_i x_i x_i^*$, where $\{x_1, \dots x_n\}$ is the standard basis of $\mathcal{G}(H)$. Let $e_j, \mu_j, t_j,\ j = 1, \dots, m$, be as in Proposition 10. Then:*

1. *Let $d_j = \sum_i \mu_j(x_i) x_i^*$. Then $e_j = t_j d_j$.*
2. *$(\ \cdot\ , z) = \sum_j n_j \mu_j$.*
3. *$z = \sum_j \frac{n_j}{t_j} e_j = \sum_j n_j d_j$.*

Proof.

1. We have $(x_r, d_j) = \overline{\mu_j(x_r^*)} = \mu_j(x_r)$ for all r, so $(\ \cdot\ , d_j) = \mu_j$. Since $t_j \mu_j = (\ \cdot\ , e_j)$ by Proposition 10(2), $e_j = t_j d_j$, as required.
2. For any $x \in \mathcal{G}(H)^{\mathbb{C}}$ we have $(x, z) = \sum_i (x, x_i x_i^*) = \sum_i (x x_i, x_i)$. This is the trace of left multiplication by x, calculated using the standard basis. So $(\ \cdot\ , z)$ is the character of the left regular representation of $\mathcal{G}(H)^{\mathbb{C}}$, which is $\sum_j n_j \mu_j$.
3. Since $t_j \mu_j = (\ \cdot\ , e_j)$, we have $(\ \cdot\ , \sum_j \frac{n_j}{t_j} e_j) = \sum_j n_j \mu_j = (\ \cdot\ , z)$, so $\sum_j \frac{n_j}{t_j} e_j = z$. □

Remark 21 *We observe that z is central. More generally, if H is a Hopf algebra with a bijective antipode, and M is a right H-comodule, then the map $\varphi : (H, \triangleright) \otimes M \longrightarrow M \otimes (H, \triangleright)$ given by $\varphi(h \otimes m) = \sum m_0 \otimes h m_1$ is an H-comodule isomorphism.*

Corollary 22 *Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field. Write $z = \sum_{x \in \mathcal{X}} x x^*$. Then for irreducible characters μ, μ' of $\mathcal{G}(H)^{\mathbb{C}}$ we have*

$$\sum_{x \in \mathcal{X}} \mu(x) \mu'(x^*) = \begin{cases} \mu(1)^{-1} \mu(z) & \text{if } \mu = \mu' \\ 0 & \text{if } \mu \neq \mu'. \end{cases}$$

Proof. Let us use the notation of Theorem 20, with $\mu = \mu_j,\ \mu' = \mu_k$. We have $\sum_i \mu_j(x_i) \mu_k(x_i^*) = \mu_k(\sum_i \mu_j(x_i) x_i^*) = \mu_k(d_j)$. Since $d_j = t_j^{-1} e_j$ we get 0 if $k \neq j$, and we have $\mu_j(z) = n_j \mu_j(d_j) = \mu_j(1) \mu_j(d_j)$, so $\mu_j(d_j) = \mu_j(1)^{-1} \mu_j(z)$ as required. □

Further information about z and the elements t_j is provided by the following.

Proposition 23 *Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field. Let z, $\{x_1, \dots, x_n\}$, and e_j, μ_j, t_j, $j = 1, \dots, m$, be as in Theorem 20. Then:*

1. $t_1 = \frac{1}{dimH}$, *and* $d(z) = dim(H)$.

2. $\sum\limits_j t_j n_j = 1$.

3. $\|z\|^2 = \sum\limits_j \frac{n_j^3}{t_j}$, *and* $n_j \leq t_j dimH$.

4. $\|z\| \leq \sum\limits_i d(x_i)$.

Proof.

1. Since H is the direct sum of the matrix coalgebras corresponding to $x_1, \ldots, x_n$, we have $[H] = \sum\limits_i d(x_i)x_i$. Thus $d(z) = \sum\limits_i d(x_i x_i^*) = \sum\limits_i d(x_i)^2 = d([H]) = dim(H)$. We have $e_1 = \frac{1}{dimH}[H]$, so by Proposition 10(2), $t_1 = t_1\mu_1(1) = (1, e_1) = \frac{1}{dimH}$.

2. We have $1 = (1,1) = (1, \sum\limits_j e_j) = \sum\limits_j (1, e_j) = \sum\limits_j t_j \mu_j(1) = \sum\limits_j t_j n_j$.

3. Since $z = \sum\limits_j \frac{n_j}{t_j} e_j$, we have $\|z\|^2 = \sum\limits_j \frac{n_j^2}{t_j^2}(e_j, e_j) = \sum\limits_j \frac{n_j^2}{t_j^2} t_j n_j = \sum\limits_j \frac{n_j^3}{t_j}$. Moreover, each eigenvalue $\frac{n_j}{t_j}$ of left multiplication by z is less than or equal to $N_1(z) = d(z) = dimH$.

4. Using Proposition 15, we have $\|z\| = \|\sum\limits_i x_i x_i^*\| \leq \sum\limits_i \|x_i x_i^*\| \leq \sum\limits_i d(x_i x_i^*)^{\frac{1}{2}} = \sum\limits_i d(x_i)$. □

5 The Subalgebra Generated by a Basic Element

Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field. We now show that, except in very special circumstances, a basic element of $\mathcal{G}(H)$ generates a subalgebra of $\mathcal{G}(H)^{\mathbb{Q}}$ of dimension at least 4.

Theorem 24 *Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field. Let $x \in \mathcal{G}(H)$ be a basic element. Then $dim\,\mathbb{Q}\langle x\rangle \geq 4$ unless one of the following holds:*

1. $d(x) = 1$, *and* $x^n = 1$ *for* $n = 1, 2,$ *or* 3.

2. $x = x^*$, *and there exist distinct basic elements* $x_3, \ldots, x_m$ *and a nonnegative integer* r *such that* $x^2 = 1 + rx + d(x_3)x_3 + \cdots + d(x_m)x_m$. *In this case,* $d(x) > d(x_i)$ *for* $i = 3, \ldots, m$.

Proof. Our assertion is clear when $d(x) = 1$, so we may assume that $d(x) > 1$. Let C be the simple subcoalgebra of H corresponding to x, and let B be the subbialgebra of H generated by C.

Suppose $dim\,\mathbb{Q}\langle x\rangle \leq 3$. Then x^3 lies in the span of $1, x, x^2$, so by Proposition 5 we have $B = k1 + C + C^2$. Let us say that the simple subcoalgebras of B are $C_1 = k1$, $C_2 = C, C_3, \ldots, C_m$, with corresponding basic elements $x_1 = 1, x_2 = x, x_3, \ldots, x_m$, of degrees $d_1 = 1, d_2, d_3, \ldots, d_m$ respectively. We have $x^2 = s_1 1 + s_2 x + s_3 x_3 + \cdots + s_m x_m$ for nonnegative integers $s_1, s_2, \ldots, s_m$.

Now B is a Hopf subalgebra of H, so $x^* \in \{x, x_3, \ldots, x_m\}$. We have $(xx^*, 1) = 1$, so $xx^* = 1 + r_2 x + r_3 x_3 + \cdots + r_m x_m$ for nonnegative integers $r_2, \ldots, r_m$. By applying d, we learn that $gcd(d_2, \ldots, d_m) = 1$.

If $dim\,\mathbb{Q}\langle x\rangle = 2$, then $B = k1 + C$. In this case, $C_3, \ldots, C_m$ do not exist, and the above argument shows $d_2 = 1$, a contradiction. So we may suppose that $dim\,\mathbb{Q}\langle x\rangle = 3$. For $i \geq 3$, C_i is a subcoalgebra of C^2, so $s_i > 0$.

We have $y_B = 1 + d_2 x + d_3 x_3 + \cdots + d_m x_m \in \mathbb{Q}\langle x\rangle$ by Corollary 19, so $\{1, x, y_B\}$ is a basis of $\mathbb{Q}\langle x\rangle$. Write $x^2 = \alpha 1 + \beta x + \gamma y_B$ for $\alpha, \beta, \gamma \in \mathbb{Q}$. We have $\beta + \gamma d_2 = s_2 \in \mathbb{Z}$, and $\gamma d_i = s_i \in \mathbb{Z}$ for $i \geq 3$. We shall show that $\beta \in \mathbb{Z}$, giving that $\gamma d_2 \in \mathbb{Z}$. Since $gcd(d_2, \ldots, d_m) = 1$, this will give us that $\gamma \in \mathbb{Z}$.

Now $xx_i = \sum_j m_{ij} x_j$ for certain integers m_{ij}, so x satisfies the monic polynomial $det(tI - [m_{ij}]) \in \mathbb{Z}[t]$. Thus the minimal polynomial for x in $\mathbb{Q}[t]$ is a monic polynomial with integer coefficients. So the trace of left multiplication by x on $\mathbb{Q}\langle x\rangle$ is an integer. Using the basis $\{1, x, y_B\}$, we obtain that $\beta + d_2$ is an integer. This gives us that β is an integer, and thus that γ is an integer, as desired.

For any $i \geq 3$, we have $0 < \gamma d_i = s_i = (x^2, x_i) = (x, x^* x_i) \leq d_i$. Thus $\gamma = 1$.

If $x^* \neq x$, then $(x^2, x^*) = (y_B, x^*) = d_2$, so we must have $x^2 = d_2 x^*$. But this gives $m = 3, d_3 = d_2$, and contradicts our gcd condition. So $x^* = x$, and we have $x^2 = 1 + s_2 x + d_3 x_3 + \cdots + d_m x_m$. Applying d yields $d_2^2 > d_i^2$ for $i \geq 3$, and thus that $d_2 > d_i$. □

Corollary 25 *A cosemisimple Hopf algebra H of dimension at least 7 over an algebraically closed field has at least 4 simple subcoalgebras.*

Proof. If H is a group algebra we are done, so we may assume that $\mathcal{G}(H)$ has basic elements of degree larger than 1. Let x be one of minimal degree. Suppose that the number of simple subcoalgebras of H (and thus $dim\,\mathcal{G}(H)^{\mathbb{Q}}$) is at most 3. Then by Theorem 24 $x^2 = 1 + rx + d_3 x_3$, where $d(x) > d(x_3)$. By minimality, $d(x_3) = 1$. Applying d, we obtain that $d(x)$ divides 2, and thus that $d(x) = 2$. Thus H has dimension 6, a contradiction. □

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