NOTES ON WEAK UNITS OF PICARD 1- AND 2-STACKS

ETTORE ALDROVANDI AND AHMET EMIN TATAR

Abstract. The weak units of strict monoidal 1- and 2-categories are defined respectively in [10] and [9]. In this paper, we recall them for Picard 1- and 2-stacks. We show that they form a Picard 1- and 2-stack, respectively. We deduce by [13, Theorem 6.4] that there exists a length 2 (resp. 3) complex of abelian sheaves that represent the Picard stack (resp. Picard 2-stack) of the weak units. Lastly, we calculate such complexes.

1. Introduction

Saavedra in [12] gives an alternative way of defining units in monoidal categories. He observes that a unit e in monoidal category C can be defined as a cancellable idempotent object, an object e with the property that tensoring with e from both sides is an equivalence and that is equipped with isomorphism $\varphi : ee \to e$. In the traditional way, a unit is an object equipped with left and right constraints (i.e. the isomorphisms $\mathfrak{l}_X : eX \to X$ and $\mathfrak{r}_X : Xe \to X$) satisfying some compatibility conditions. In [10], Kock analyzes these two definitions of units in a monoidal category. He calls the units defined as cancellable idempotent objects *Saavedra units*, and the units extracted from the definition of bicategories with one object *classical units*. He shows that these two notions of units are equivalent and the category they form is contractible.

In a subsequent work [9], Joyal and Kock carry out the discussion for units of monoidal categories to units of monoidal 2-categories. They give an alternative definition to the notion of classical unit in monoidal 2-categories. In this classical notion a unit is an object equipped with left and right constraints which are weakly invertible 1-morphisms and with a 2-isomorphism between the left and the right constraints. These data are required to satisfy certain conditions (see [9, §6]). On the other hand, Joyal and Kock define a unit of a monoidal 2-category as an appropriate generalization of Saavedra unit. This is an object e with the property that tensoring with e from both sides is biequivalence and that is equipped with the weakly invertible morphism $\varphi : ee \to e$. Throughout this paper, we call this alternative definition of unit *Joyal-Kock unit*. In [9], Joyal and Kock show, as in the 1-categorical case, that these two notions of units are equivalent and their 2-category is contractible.

The language of Saavedra units and Joyal-Kock units in the context of Picard 1- and 2-stacks is very helpful. Their capability of expressing units without referring to left and right constraints is very beneficial if one considers the amount of data and coherence conditions required to define Picard 1- and 2-stacks. There is no need of mentioning units in their definitions because the fact *every object is cancellable* is part of the Picard data and these notions of units are equivalent to classical notions. The benefits of this economical way of defining Picard 1- and 2-stacks becomes more significant when it comes to define (2-)functors. With Saavedra units and Joyal-Kock units, we don't need to assume that a unit is transferred to a unit. It is enough to assure that the (2-)functor transfers the Picard structure to the Picard structure.

In this paper, we consider the Saavedra units (resp. Joyal-Kock units) in a Picard stack (resp. Picard 2-stacks). We prove they form Picard stacks (resp. 2-stack) of their own which we denote by $\mathscr{I}(\mathscr{C})$ (resp. $\mathbb{I}(\mathbb{C})$). Due to the contractibility results of Kock in [10] and Joyal-Kock in [9], we expect these (2-)stacks to be equivalence relations with contractible quotients, i.e, to be ultimately contractible spaces. We confirm this both by a direct geometric analysis and by explicitly computing complexes of abelian sheaves that represent them. The explicit computation of these complexes is interesting because we compare them directly to the complexes representing the homotopy fiber over 1 in the Postnikov exact sequence. The comparison allows us to characterize the Saavedra units as *rigid* model of the homotopy fibers over 1. We study this in more details in [3].

Organization of the paper. In section 2, we quickly recall Saavedra units and Joyal-Kock units of Picard 1- and 2- categories. In section 3, we remind basics of fibered 1- and 2-categories over a site. In section 4, we examine Saavedra units of Picard stacks. We show that Saavedra units of a Picard stack $\mathscr{C}$ form a Picard stack $\mathscr{I}(\mathscr{C})$. We calculate a complex of abelian sheaves such that the Picard stack associated to it is equivalent to $\mathscr{I}(\mathscr{C})$. In section 5, we follow the same plan as in section 4 for Joyal-Kock units.

Notation and Conventions. We work with strict 2-categories. A 2-groupoid is a 2-category whose 1-morphisms are weakly invertible and 2-morphisms are isomorphisms. A 2-functor is used in the sense of [8]. For compactness in the diagrams, we denote the tensoring operation in any category by juxtaposition. The usual notation $\otimes$ is used in the names of functors (i.e. $X \otimes -$ denotes the functor tensoring by X) and in cases to avoid ambiguities. We use capital roman letters for categories (C, D, ...), calligraphic letters for 2-categories ($\mathcal{C}$, $\mathcal{D}$, ...), script letters for stacks ($\mathscr{C}$, $\mathscr{D}$, ...) and double letters for 2-stacks ($\mathbb{C}$, $\mathbb{D}$, ...).

2. Quick Recall on Weak Units

In this section, we recall briefly the weak units of Picard 1- and 2- categories. The main references are [10] and [9] where these units are defined for strict monoidal 1- and 2- categories. For definitions of Picard 1- and 2-categories, we refer to [5].

2.1. **Saavedra Units.** Let C be a Picard category. A pair (e, φ) is called a unit element where e is an object and $\varphi : ee \to e$ is an isomorphism in C . A unit morphism $(e_1, \varphi_1) \to (e_2, \varphi_2)$ is given by an isomorphism $u : e_1 \to e_2$ in C such that the diagram

$$\begin{tikzcd} e_1 e_1 \arrow[r, "uu"] \arrow[d, "\varphi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & e_2 e_2 \arrow[d, "\varphi_2"] \\ e_1 \arrow[r, "u"'] & e_2 \end{tikzcd} \tag{2.1}$$

commutes. This defines the groupoid of Saavedra units I(C).

In [10] these units are called Saavedra units since they were first mentioned by Saavedra in [12]. I(C) is a Picard category since the classical notion of unit extracted from the definition of monoidal category is equivalent to the notion of Saavedra unit and I(C) is contractible ([10, Proposition 2.19]). Moreover due to the contractibility, I(C) is always a Picard category whether C is Picard or not. We define the tensor product of Saavedra units (e_1, φ_1) and (e_2, φ_2) as the Saavedra unit $(e_1 e_2, \varphi)$ where φ is the composition

$$(e_1 e_2)(e_1 e_2) \xrightarrow{\mathsf{a}^{-1}} ((e_1 e_2) e_1) e_2 \xrightarrow{\mathsf{a}^{-1}\mathsf{c}\mathsf{a}} ((e_1 e_1) e_2) e_2 \xrightarrow{\mathsf{a}} (e_1 e_1)(e_2 e_2) \xrightarrow{\varphi_1 \varphi_2} e_1 e_2 \, , \tag{2.2}$$

with $\mathsf{a}^{-1}\mathsf{c}\mathsf{a}$ given by

$$((e_1 e_2) e_1) e_2 \xrightarrow{\mathsf{a}^{-1}} (e_1 (e_2 e_1)) e_2 \xrightarrow{\mathsf{c}} (e_1 (e_1 e_2)) e_2 \xrightarrow{\mathsf{a}} ((e_1 e_1) e_2) e_2 \, . \tag{2.3}$$

The isomorphisms a and c represent the associativity and the braiding constraints, respectively. There is a choice involved in the definition of φ, but any two such choices are connected by a unique isomorphism. This unique isomorphism would be the pasting of the isomorphisms of the Picard data. We also note that our definition coincides with the one in [10, §2.21] if one assumes strict associativity and uses the compatibility between the braiding constraint and left and right unit constraints.

In [6], Deligne points out that C always has a Saavedra unit, since tensoring by an object X in C is an equivalence (see the proof of Proposition (2.1)).

2.2. **Joyal-Kock Units.** Let $\mathcal{C}$ be a Picard 2-category. A pair (e, φ) is called a unit element in $\mathcal{C}$ where e is an object and $\varphi : ee \to e$ is a 1-morphism in $\mathcal{C}$. A unit 1-morphism $(e_1, \varphi_1) \to (e_2, \varphi_2)$ is given by a pair (f, θ_f) where $f : e_1 \to e_2$ is a weakly invertible 1-morphism and θ_f is the 2-isomorphism

$$\begin{tikzcd} e_1e_1 \arrow[r, "ff"] \arrow[d, "\varphi_1"'] \arrow[dr, phantom, "\theta_f \nearrow"] & e_2e_2 \arrow[d, "\varphi_2"] \\ e_1 \arrow[r, "f"'] & e_2 \end{tikzcd} \tag{2.4}$$

A unit 2-morphism $(f, \theta_f) \Rightarrow (g, \theta_g)$ is given by a 2-isomorphism $\delta : f \Rightarrow g$ in $\mathcal{C}$ such that

$$\begin{tikzcd} e_1e_1 \arrow[r, "ff"'] \arrow[r, bend left=50, "gg"] \arrow[d, "\varphi_1"'] \arrow[dr, phantom, "\theta_f \nearrow"] & e_2e_2 \arrow[d, "\varphi_2"] \\ e_1 \arrow[r, "f"'] & e_2 \end{tikzcd} \quad = \quad \begin{tikzcd} e_1e_1 \arrow[r, "gg"] \arrow[d, "\varphi_1"'] \arrow[dr, phantom, "\theta_g \nearrow"] & e_2e_2 \arrow[d, "\varphi_2"] \\ e_1 \arrow[r, "g"] \arrow[r, bend right=50, "f"'] & e_2 \end{tikzcd} \tag{2.5}$$

Unit elements, unit 1-morphisms, and unit 2-morphisms of a Picard 2-category $\mathcal{C}$ form the 2-groupoid $\mathcal{I}(\mathcal{C})$ of Joyal-Kock units. We define the tensor product on $\mathcal{I}(\mathcal{C})$ in the same as the one on I(C). As in the case of Saavedra units, whether $\mathcal{C}$ is Picard or not, $\mathcal{I}(\mathcal{C})$ is always a Picard 2-category. However if $\mathcal{C}$ is Picard, then $\mathcal{I}(\mathcal{C})$ is not empty.

Proposition 2.1. *A Picard 2-category $\mathcal{C}$ always has a Joyal-Kock unit.*

This result is not surprising since Picard 2-categories have classical units that are equivalent to Joyal-Kock units [9, Thereom E]. We give a proof of this fact without referring to this equivalence. This Proposition and its proof generalize to group-like 2-categories.

Proof of Proposition 2.1. For any object X in $\mathcal{C}$, the 2-functor $-\otimes X$ from $\mathcal{C}$ to $\mathcal{C}$ is a biequivalence. Therefore, for any $X \in \mathcal{C}$ there exists $e_X \in \mathcal{C}$ with a 1-morphism $f : e_X X \to X$. $\mathrm{id}_{e_X} \otimes f$ is a 1-morphism in the category $\mathrm{Hom}_{\mathcal{C}}(e_X(e_X X), e_X X)$. $\mathsf{a}^{-1}_{e_X, e_X, X} \circ (\mathrm{id}_{e_X} \otimes f)$ is a 1-morphism in the category $\mathrm{Hom}_{\mathcal{C}}((e_X e_X)X, e_X X)$ which is equivalent to $\mathrm{Hom}_{\mathcal{C}}(e_X e_X, e_X)$, since tensoring is a biequivalence. We define $\varphi : e_X e_X \to e_X$ as the image of $\mathrm{id}_{e_X} \otimes f$ under this equivalence. $\square$

3. Quick Recall on Fibered 2-Categories

In this section, we briefly recall fibered 2-categories following [8, §1]. We also give a characterization of fibered 2-categories which generalizes [14, Proposition 3.22].

Consider a 2-category $\mathcal{C}$ associated with a 2-functor $p : \mathcal{C} \to \mathsf{S}$. We say $\mathcal{C}$ is a 2-category over S. For any $U \in \mathsf{S}$, we denote by $\mathcal{C}_U$ the fiber of $\mathcal{C}$ over U. It is a 2-category whose objects, 1-morphisms, and 2-morphisms respectively map to U, id_U, and $\mathrm{id}_{\mathrm{id}_U}$.

Let $\varphi : V \to U$ be in S, $X \in \mathcal{C}_V$, and $Y \in \mathcal{C}_U$. $\mathrm{Hom}_\varphi(X,Y)$ denotes the category of morphisms $f : X \to Y$ in $\mathcal{C}$ that are mapped to φ by p. If $U = V$ and $\varphi = \mathrm{id}_U$, then we denote the collection of these maps by $\mathrm{Hom}_U(X,Y)$.

For any $f \in \mathrm{Hom}_\varphi(X,Y)$, there exists a natural transformation

$$\tilde{f} : \mathrm{Hom}_U(-,X) \longrightarrow \mathrm{Hom}_\varphi(-,Y),$$

defined by post composing with f. We say that f is *cartesian* if $\tilde{f}$ is an equivalence, that is, if there exists a natural transformation

$$G : \mathrm{Hom}_\varphi(-,Y) \longrightarrow \mathrm{Hom}_U(-,X)$$

such that $\tilde{f} \circ G \simeq \mathrm{id}$ and $G \circ \tilde{f} \simeq \mathrm{id}$. More precisely, for any $X' \in \mathcal{C}_U$, the functor

$$\tilde{f}_{X'} : \mathrm{Hom}_U(X',X) \longrightarrow \mathrm{Hom}_\varphi(X',Y),$$

is an equivalence.

We call a 2-category $\mathcal{C}$ over S *fibered* if:

(1) for any morphism $\varphi : V \to U$ in S and any object $Y \in \mathcal{C}_U$, there exists a cartesian morphism $f : X \to Y$ over φ,
(2) composition of cartesian morphisms is cartesian.

We say $\mathcal{C}$ is *fibered in 2-groupoids* if for any $U \in \mathsf{S}$, $\mathcal{C}_U$ is a 2-groupoid.

In [14], Vistoli gives a characterization for fibered categories in groupoids (see [14, Proposition 3.22]). We generalize this to fibered 2-categories in 2-groupoids.

Proposition 3.1. *Let $p : \mathcal{C} \to \mathsf{S}$ be a 2-category over the site S. Then $\mathcal{C}$ is fibered in 2-groupoids if and only if*

(i) every 1-morphism is cartesian,
(ii) given $Y \in \mathcal{C}_U$ and $\varphi : V \to U$ in S, there exists $f : X \to Y$ with $p(f) = \varphi$.

The following auxiliary result is needed in the proof of the Proposition 3.1:

Lemma 3.2. *Let $p : \mathcal{C} \to \mathsf{S}$ be a 2-category over the site S satisfying (i) and (ii) of the Proposition 3.1. Then for any $X, Y \in \mathcal{C}_U$, $\mathrm{Hom}_U(X,Y)$ is fibered over S/U.*

Proof. Given a morphism $\varphi : i_2 \Rightarrow i_1$ in S/U and $g \in \mathrm{Hom}_U(X,Y)$ over i_1, we want to find a cartesian morphism $\alpha : f \Rightarrow g$ in $\mathrm{Hom}_U(X,Y)$ over φ. By (ii), we can pull back the objects

$X_{|V_1}$ and $Y_{|V_1}$ along φ as shown in the diagram.

$$\begin{tikzcd}
X_{|V_1|V_2} \arrow[r, "h_\varphi^X"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & X_{|V_1} \arrow[r, "h_1^X"] \arrow[d, "g"] & X \\
Y_{|V_1|V_2} \arrow[r, "h_\varphi^Y"'] & Y_{|V_1} \arrow[r, "h_1^Y"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
V_2 \arrow[r, "\varphi"] \arrow[rr, bend right, "i_2"'] & V_1 \arrow[r, "i_1"] & U
\end{tikzcd}$$

By (i), g, h_φ^X, and h_φ^Y are cartesian. Therefore there exists $f : X_{|V_1|V_2} \to Y_{|V_1|V_2}$ over i_2 and a 2-isomorphism $\alpha : f \Rightarrow g$. α is a cartesian since it is an isomorphism. $\square$

Proof of the Proposition 3.1. First we assume that $\mathcal{C}$ satisfies (i) and (ii). Let $\varphi : V \to U$ be in S and $Y \in \mathcal{C}_U$. By (ii), there exists $f : X \to Y$ over φ which is cartesian by (i). Also by (i), composition of cartesian is cartesian. Hence, all we need to show is that fibers are 2-groupoids.

Let $f : X \to Y$ be an object in $\mathrm{Hom}_U(X, Y)$. Since f is cartesian,

$$\tilde{f}_Y : \mathrm{Hom}_U(Y, X) \longrightarrow \mathrm{Hom}_U(Y, Y),$$

is an equivalence. There exists $h \in \mathrm{Hom}_U(Y, X)$ such that $f \circ h \simeq \mathrm{id}_Y$. On the other hand, h is also cartesian. Therefore

$$\tilde{h}_X : \mathrm{Hom}_U(X, Y) \longrightarrow \mathrm{Hom}_U(X, X),$$

is an equivalence, as well. Using essential surjectivity of $\tilde{h}_X$, we find $f' \in \mathrm{Hom}_U(X, Y)$ with $h \circ f' \simeq \mathrm{id}_X$ from which we deduce by composing both sides by f that $f \simeq f'$. This shows f is weakly invertible.

Let $\alpha : f \Rightarrow g$ be a 1-morphism in $\mathrm{Hom}_U(X, Y)$. By Lemma 3.2, $\mathrm{Hom}_U(X, Y)$ is fibered over S/U. Assume that α is over id_U. Then there exists $\beta : g \Rightarrow f$ such that $\alpha \circ \beta = \mathrm{id}_g$. By repeating the same argument for β, we show $\beta \circ \alpha = \mathrm{id}_f$.

Conversely, we assume that $\mathcal{C}$ is a fibered 2-category in 2-groupoids. (ii) follows immediately from the definition. To verify that $f : X \to Y$ a morphism of $\mathcal{C}$ over $\varphi : U \to V$ is cartesian, we have to show that

$$\tilde{f} : \mathrm{Hom}_U(-, X) \longrightarrow \mathrm{Hom}_\varphi(-, Y),$$

is an equivalence. Let $X' \in \mathcal{C}_U$ and let $g \in \mathrm{Hom}_\varphi(X', Y)$ be cartesian. Then $\tilde{g}$ is an equivalence and there exists $h \in \mathrm{Hom}_U(X', X)$ such that $g \circ h \simeq f$. Since fibers of $\mathcal{C}$ are 2-groupoids, h is weakly invertible. So $\tilde{h}$ is an equivalence. It follows that so does $\tilde{g} \circ \tilde{h} = \widetilde{g \circ h} \simeq \tilde{f}$. This finishes the proof. $\square$

4. Weak Units of Picard Stacks

We define weak units of a Picard stack $\mathscr{C}$. We call these units Saavedra units of $\mathscr{C}$. We show that such units form a Picard stack $\mathscr{I}(\mathscr{C})$. We deduce by [6, Lemme 1.4.13] that there exists a complex of abelian sheaves that represents $\mathscr{I}(\mathscr{C})$. We calculate such a complex. We end this section by extending the discussion to non Picard case using [2, Theorem 5.3.6].

4.1. **Saavedra Units of a Picard Stack.** We consider a Picard stack $\mathscr{C}$ represented by an abelian complex $\lambda : A \to B$. $\mathscr{C}$ can be modeled by $\mathrm{TORS}(A,B)$, the Picard stack of (A,B)-torsors. For details of $\mathrm{TORS}(A,B)$, we refer to [4] and [7]. Here, we give a brief reminder. An (A,B)-torsor is a pair (L,x), where L is an A-torsor and $x : L \to B$ is an A-equivariant morphism of sheaves. A morphism between two pairs (L,x) and (K,y) is an A -equivariant morphism of sheaves $\psi : L \to K$ such that the diagram

$$\begin{tikzcd} L \arrow[rr, "\psi"] \arrow[dr, "x"'] & \circlearrowleft & K \arrow[dl, "y"] \\ & B & \end{tikzcd} \tag{4.1}$$

commutes. The tensor product on $\mathrm{TORS}(A,B)$ is

$$(L,x) \otimes (K,y) := (L \wedge^A K, x \wedge y), \tag{4.2}$$

where $L \wedge^A K$ is the contracted product and $x \wedge y$ is the A-equivariant morphism from $L \wedge^A K$ to B given by $x(l) + y(k)$ with (l,k) in $L \wedge^A K$.

A *Saavedra unit* in $\mathrm{TORS}(A,B)$ is an idempotent (A,B)-torsor, that is, an (A,B)-torsor (L,x) with an (A,B)-torsor morphism

$$\varphi : (L,x) \otimes (L,x) \longrightarrow (L,x). \tag{4.3}$$

In other words, we have an (A,B)-torsor morphism

$$\varphi : (L \wedge^A L, 2x) \longrightarrow (L,x). \tag{4.4}$$

We denote these units by $((L,x),\varphi)$.

A *morphism of Saavedra units* in $\mathrm{TORS}(A,B)$

$$((L,x),\varphi) \longrightarrow ((K,y),\sigma),$$

is given by an (A,B)-torsor morphism $\psi : (L,x) \to (K,y)$ satisfying the commutative diagram

$$\begin{tikzcd} (L \wedge^A L, 2x) \arrow[r, "\psi\wedge\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & (K \wedge^A K, 2y) \arrow[d, "\sigma"] \\ (L,x) \arrow[r, "\psi"'] & (K,y) \end{tikzcd} \tag{4.5}$$

This defines the groupoid of Saavedra units of a Picard stack $\mathscr{C}$. We denote it by $\mathscr{I}(\mathscr{C})$.

Example 4.1. We consider the trivial (A,B)-torsor A with the A-equivariant map $\zeta : A \to B$. We note immediately that ζ is nothing but the morphism λ since $\zeta(0_A) = 0_B$ and from the A-equivariance $\zeta(a) = \zeta(0_A + a) = \zeta(0_A) + \lambda(a) = \lambda(a)$. We denote this trivial (A,B)-torsor by $(A,0)$. We equip $(A,0)$ with the composition

$$(A \wedge^A A, 0) \xrightarrow{\Phi} (A,0) \xrightarrow{\alpha} (A,0). \tag{4.6}$$

$\alpha \in \ker(\lambda)$ represents the A-equivariant map that sends the global section 0_A to α and Φ is the canonical morphism

$$\Phi : (A \wedge^A A, 0) \longrightarrow (A,0) \tag{4.7}$$

that sends the global section $(0_A, 0_A)$ to the global section 0_A. The composition (4.1) is an (A, B)-torsor morphism, since

$$\lambda \circ \alpha \circ \Phi(0_A, 0_A) = 0_A = \lambda(0_A) + \lambda(0_A).$$

In other words, since the diagram

$$\begin{tikzcd} (A \wedge^A A, 0) \arrow[rr, "\alpha\circ\Phi"] \arrow[dr, "\lambda+\lambda"'] & \circlearrowleft & (A, 0) \arrow[dl, "\lambda"] \\ & B & \end{tikzcd} \tag{4.8}$$

commutes. Therefore, $(A, 0)$ equipped with (4.6) is a Saavedra unit. We denote it by $((A, 0), \alpha)$. In case $\alpha = 0_A$, we obtain the special Saavedra unit $((A, 0), 0)$.

4.2. **Contractibility of Saavedra Units.** In [10, Proposition 2.19], Kock proves that Saavedra units of a monoidal category form a contractible category. His proof is based on the following. First, he proves that the category of classical units of a monoidal category is contractible. Second, he shows that the category of Saavedra units is equivalent to the category of classical units. In this section, we prove the same result for Picard stacks by directly constructing the unique isomorphism between any two Saavedra units.

Proposition 4.2. *All Saavedra units of $\mathscr{C}$ are uniquely isomorphic to each other. That is, $\mathscr{I}(\mathscr{C})$ is a contractible groupoid over the site S.*

Proof. Let $((L, x), \varphi)$ be a Saavedra unit of $\mathscr{C}$ and u be a local section of L. Since L is locally isomorphic to A, there exists a unique a_φ in A such that

$$\varphi(u, u) = u + a_\varphi. \tag{4.9}$$

From the commutativity of the diagram (4.1), we deduce the relation $2x(u) = x(u) + \lambda(a_\varphi)$ which simplifies to

$$x(u) = \lambda(a_\varphi). \tag{4.10}$$

If we choose another section u' with $u' = u + \alpha$ for some α in A, then there exists unique a'_φ in A such that u' satisfies relations

$$\varphi(u', u') = u' + a'_\varphi, \quad x(u') = \lambda(a'_\varphi) \tag{4.11}$$

similar to (4.9) and (4.10). On the other hand,

$$\varphi(u', u') = \varphi(u + \alpha, u + \alpha) = \varphi(u, u + 2\alpha) = \varphi(u, u) + 2\alpha. \tag{4.12}$$

Putting the relations (4.9) and (4.11) in (4.12) we find

$$a'_\varphi = a_\varphi + \alpha. \tag{4.13}$$

We consider the section $s = u - a_\varphi$. From (4.13), $s = u' - a'_\varphi$. This shows that s is a global section of the Saavedra unit $((L, x), \varphi)$. We also note that

$$x(s) = 0_B \quad \text{and} \quad \varphi(s, s) = s. \tag{4.14}$$

Let $((L, x), \varphi)$ and $((K, y), \sigma)$ be two Saavedra units of $\mathscr{C}$ with global sections s and t, respectively. We construct an isomorphism

$$\psi : ((L, x), \varphi) \longrightarrow ((K, y), \sigma) \tag{4.15}$$

by sending s to t. From (4.14), it follows that ψ is a Saavedra unit morphism. ψ is unique because s and t are uniquely determined by φ and σ, respectively. $\square$

4.3. **Picard Stack of Saavedra Units.** The category of Saavedra units $\mathscr{I}(\mathscr{C})$ of a Picard stack $\mathscr{C}$ is defined in section 4.1. In this section, we show that $\mathscr{I}(\mathscr{C})$ is in fact a Picard stack. We still assume that $\mathscr{C}$ is modeled by $\mathrm{T\scriptstyle{ORS}}(A, B)$.

By Proposition 4.2, all morphisms in $\mathscr{I}(\mathscr{C})$ are cartesian. Let $f : U \to V$ be a morphism in S and let $((L, s), \varphi)$ be a Saavedra unit over V. Since $\mathrm{T\scriptstyle{ORS}}(A, B)$ is a fibered category in groupoids, we can pull back φ along f to an (A, B)-torsor morphism over U. From the additivity of the pull back functor $f^* : \mathscr{C}_V \to \mathscr{C}_U$, the restriction of (L, x) over U $f^*(L, x) := (L_{|U}, x_{|U})$ equipped with $f^*(\varphi) : (L_{|U} \wedge^A L_{|U}, 2x_{|U}) \to (L_{|U}, x_{|U})$ is a Saavedra unit over U. The commutativity of the diagram

$$\begin{tikzcd}
(L_{|U} \wedge^A L_{|U}, 2x_{|U}) \arrow[r, "i_U \wedge i_U"] \arrow[d, "f^*(\varphi)"'] \arrow[dr, phantom, "\circlearrowleft"] & (L \wedge^A L, 2x) \arrow[d, "\varphi"] \\
(L_{|U}, x_{|U}) \arrow[r, "i_U"'] & (L, x)
\end{tikzcd}$$

implies that the restriction function on the underlying (A, B)-torsor $i_U : (L_{|U}, x_{|U}) \to (L, x)$ is a Saavedra unit morphism. From [14, Proposition 3.22], this shows that $\mathscr{I}(\mathscr{C})$ is a fibered category in groupoids over S. The facts that morphisms of Saavedra units from a sheaf and every descent datum is effective follow immediately from the Proposition 4.2 showing that $\mathscr{I}(\mathscr{C})$ is a stack.

The group-like structure on $\mathscr{I}(\mathscr{C})$ is defined by the contracted product

$$((L, s), \varphi) \otimes ((K, y), \sigma) := ((L \wedge^A K, x \wedge y), \varphi \wedge \sigma),$$

where $\varphi \wedge \sigma$ is defined by the relations (2.2) and (2.3). By Proposition 4.2, this structure is braided and satisfies Picard axioms. This shows,

Proposition 4.3. *$\mathscr{I}(\mathscr{C})$ is a Picard stack.*

4.4. **The Cocyclic Description of a Saavedra Unit.** In this section, we give the cocyclic description of Saavedra units which helps us find a complex representing the Picard stack of Saavedra units. These calculations are similar to the ones in the proof of the Proposition 4.2.

Let $((L, x), \varphi)$ be a Saavedra unit of $\mathscr{C}$ over U and $V_\bullet \to U$ a hypercover. We assume $\mathscr{C}$ is represented by the complex of abelian sheaves $\lambda : A \to B$. Chosen a local section $u \in L_{V_0}$, since L is locally isomorphic to A, there exists a unique $a \in A(V_1)$ satisfying

$$d_0^*(u) = d_1^*(u) + a. \tag{4.16}$$

By pulling back (4.16) to V_2, we find the relation

$$d_0^*(a) + d_2^*(a) = d_1^*(a). \tag{4.17}$$

Applying the A-equivariant map $x : L \to B$ to (4.16), we obtain

$$d_0^*(b) = d_1^*(b) + \lambda(a), \tag{4.18}$$

where $b = x(u) \in B(V_0)$. The pair (a, b) with relations (4.17) and (4.18) is a cocycle that represents the (A, B)-torsor (L, x). Next, we remember the (A, B)-torsor morphism φ that equips (L, x) with a Saavedra unit structure. Since both $\varphi(u, u)$ and u are in L_{V_0} and L is locally isomorphic to A, there exists unique $a_\varphi \in A(V_0)$ such that

$$\varphi(u, u) = u + a_\varphi. \tag{4.19}$$

By pulling back (4.19) to V_1 along d_i^* for $i=0,1$, we have

$$\varphi(d_i^*(u), d_i^*(u)) = d_i^*(u) + d_i^*(a_\varphi). \tag{4.20}$$

On the other hand

$$\varphi(d_0^*(u), d_0^*(u)) = \varphi(d_1^*(u)+a, d_1^*(u)+a) = \varphi(d_1^*(u), d_1^*(u)+2a) = \varphi(d_1^*(u), d_1^*(u))+2a. \tag{4.21}$$

Putting together (4.20) and (4.21), we find

$$a = d_0^*(a_\varphi) - d_1^*(a_\varphi). \tag{4.22}$$

From the commutativity of the diagram (4.1), we deduce the relation $2b = b + \lambda(a_\varphi)$ which implies

$$\lambda(a_\varphi) = b. \tag{4.23}$$

These calculations show that the collection (a, a_φ, b) where $a \in A(V_1)$, $a_\varphi \in A(V_0)$, and $b \in B(V_0)$ satisfying the relations (4.17), (4.18), (4.22), and (4.23) represent the Saavedra unit $((L,x),\varphi)$. The collection (a, a_φ, b) is a 1-cocycle with values in the complex

$$A \xrightarrow{(\mathrm{id}_A,\lambda)} \ker(\lambda - \mathrm{id}_B)\,. \tag{4.24}$$

If we chose another local section $u' \in L_{V_0}$, we find another 1-cocycle cohomologous to (a, a_φ, b). Therefore the set of equivalence classes of 1-cocycles with values in the morphism (4.24) classify Saavedra units. In fact, the equivalence classes form an abelian group which we denote by $\mathrm{H}^0(*, A \to \ker(\lambda - \mathrm{id}_B))$. Here $*$ represents the final object of the topos of sheaves on S, i.e. the sheaf whose value is the point at each object of S.

4.5. **A Complex of Abelian Sheaves defining the Stack of Saavedra Units.** [6, Lemme 1.4.13] tells us that any Picard stack can be represented by a length 2 complex of abelian sheaves. Let $\mathscr{C}$ be a Picard stack represented by $\lambda : A \to B$. In this section, we find a complex of abelian sheaves that represents $\mathscr{I}(\mathscr{C})$ the Picard stack of Saavedra units in terms of $\lambda : A \to B$.

From the cocyclic description of Saavedra units, we know that $\mathrm{H}^0(*, A \to \ker(\lambda - \mathrm{id}_B))$ classify Saavedra units of $\mathscr{C}$. Hence,

Proposition 4.4. *The Picard stack associated to the morphism (4.24) is equivalent to the Picard stack of Saavedra units $\mathscr{I}(\mathscr{C})$ where $\mathscr{C}$ is the Picard stack represented by $\lambda : A \to B$.*

Remark 4.5. [6, Lemme 1.4.13] also tells us that two quasi-isomorphic length 2 complexes of abelian sheaves represent equivalent Picard stacks. This helps us to find other representations of $\mathscr{I}(\mathscr{C})$.

(1) Since the morphism $\mathrm{id}_A : A \to A$ is quasi-isomorphic to (4.24), id_A provides another representation of $\mathscr{I}(\mathscr{C})$.
(2) From Proposition 4.2 and Example 4.1, we deduce that $\ker(\lambda)$ parametrizes Saavedra units of $\mathscr{C}$. As for morphisms of Saavedra units, it is enough to look at the morphisms between two Saavedra units of the form $((A,0),\alpha)$ where α is in $\ker(\lambda)$. Let ψ be the unique isomorphism between $((A,0),\alpha)$ and $((A,0),\beta)$. If we chase the global section $(0_A, 0_A)$ of $(A \wedge^A A)$ in a diagram similar to (4.5), we find that $\psi(0_A) = \beta - \alpha$. Since ψ is A-equivariant, it is defined by the image of 0. Hence, morphisms of Saavedra units are also parametrized by $\ker(\lambda)$. These calculations show that the morphism $\mathrm{id}_{\ker(\lambda)} : \ker(\lambda) \to \ker(\lambda)$ arises naturally in the realm of Saavedra units. Moreover, $\mathrm{id}_{\ker(\lambda)}$ is quasi isomorphic to (4.24) which therefore gives another representation of $\mathscr{I}(\mathscr{C})$.

Remark 4.6. We observe that the complex (4.24) is $\tau_{\leq -1}(C^\bullet)[-1]$ where $C^\bullet$ is the cone of the identity morphism $\mathscr{A}\to\mathscr{A}$ and $\tau_{\leq -1}(C^\bullet)[-1]$ is the soft truncation of $C^\bullet$ with entries shifted to the right by 1 unit. $C^\bullet$ determines a Picard 2-stack $\mathbb{C}_{\mathscr{A}}$ which satisfies the homotopy fiber sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathscr{A}) \arrow[r, "\mathrm{id}"] & \pi_1(\mathscr{A}) \arrow[r] & \mathscr{K} \arrow[r] & \mathscr{A} \arrow[r, "\mathrm{id}"] & \mathscr{A} \arrow[dllll, "\iota"] \\
& \mathbb{C}_{\mathscr{A}} \arrow[r, "\Delta"] & \mathrm{T{\scriptsize ORS}}(\mathscr{A}) \arrow[r, "\mathrm{id}_*"] & \mathrm{T{\scriptsize ORS}}(\mathscr{A}),
\end{tikzcd}$$

where Δ is as defined in [2, 6.3.10] and $\mathscr{K}$ is the homotopy kernel of $\mathrm{id}_{\mathscr{A}}$ which is equivalent to a point. The 0 in the above sequence is in fact $\pi_1(\mathscr{K})$ which is equal to $\pi_2(\mathbb{C}_{\mathscr{A}})$. From this sequence, we also observe that $\mathbb{C}_{\mathscr{A}}$ is the homotopy fiber of id_*. On the other hand, $\tau_{\leq -1}C^\bullet[-1]$ represents $\mathscr{A}\mathrm{ut}(I_{\mathbb{C}_{\mathscr{A}}})$ the Picard stack of automorphisms of the unit object $I_{\mathbb{C}_{\mathscr{A}}}$ in $\mathbb{C}_{\mathscr{A}}$. Therefore, $\mathscr{I}(\mathscr{C})$ can be characterized as the Picard stack of automorphisms of the unit object of the homotopy fiber of id_*.

Remark 4.7. We also note that there exists $\mathscr{I}(\mathscr{C})\to\mathscr{C}$ a morphism of Picard stacks defined by forgetting the morphism (4.4). This corresponds to the strict morphism of complexes

$$\begin{tikzcd}
A \arrow[r, "(\mathrm{id}_A{,}\lambda)"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowright"] & \ker(\lambda - \mathrm{id}_B) \arrow[d, "\mathrm{pr}_B"] \\
A \arrow[r, "\lambda"'] & B
\end{tikzcd}$$

under Deligne's equivalence ([6, Lemme 1.4.13]).

4.6. **Non Picard Case.** [2, Theorem 5.3.6] implies that a group-like stack $\mathscr{G}$ can be represented by a crossed module $\lambda : G\to H$ of sheaves. This allows us to represent the Saavedra units of a group-like stack by a length 2 complex of sheaves.

The category $\mathscr{I}(\mathscr{G})$ of Saavedra units of a group-like stack $\mathscr{G}$ is defined in the same way as the category of Saavedra units in the Picard case. The examples given for Picard case (see Example 4.1) extends to group-like case as follows. We note by $(G, 1)$ the trivial (G, H)-torsor whose G-equivariant map is λ. $(G, 1)$ equipped with the composition

$$(G \wedge^G G, 1) \xrightarrow{\Phi} (G, 1) \xrightarrow{\alpha} (G, 1)\,, \tag{4.25}$$

is a Saavedra unit of $\mathscr{G}$ where Φ is the canonical morphism that sends the global section $(1_G, 1_G)$ to 1_G and $\alpha \in \ker(\lambda)$ represents the G-equivariant morphism that maps 1_G to α. We denote these Saavedra units by $((G, 1), \alpha)$. In particular, if $\alpha = 1_G$, we have a Saavedra unit denoted by $((G, 1), 1)$.

The proof of the Proposition 4.2 generalizes without difficulty to non abelian case.

Proposition 4.8. *$\mathscr{I}(\mathscr{G})$ is a contractible groupoid over* S.

By calculations similar to section 4.4, we show that the elements of the set $\mathrm{H}^0(*, G\to \ker(\lambda\mathrm{inv}_H))$ where $\mathrm{inv}_H : h\to h^{-1}$, classify the Saavedra units of $\mathscr{G}$. The morphism

$$(\mathrm{id}_G, \lambda) : G \longrightarrow \ker(\lambda\mathrm{inv}_H) \tag{4.26}$$

is a crossed module of sheaves where the action of $\ker(\lambda \mathrm{inv}_H)$ on G is $g^{(g',h')} := g^{h'}$ for any $g \in G$ and $(g', h') \in \ker(\lambda \mathrm{inv}_H)$. Therefore the crossed module (4.26) represents $\mathscr{I}(\mathscr{G})$. We observe that (4.26) is the soft truncation of

$$G \xrightarrow{(\mathrm{id}_G,\lambda)} G \ltimes H \xrightarrow{\lambda \mathrm{inv}_H} H\,, \tag{4.27}$$

which is the cone of the morphism $\mathrm{id} : \mathscr{G} \to \mathscr{G}$.

By arguments similar to Remark 4.5, other representations of $\mathscr{I}(\mathscr{G})$ are the complexes $\mathrm{id}_G : G \to G$ and $\mathrm{id}_{\ker(\lambda)} : \ker(\lambda) \to \ker(\lambda)$. We can consider $1 \to 1$ as an abelian representation.

Remark 4.9. The crossed module structure on (4.26) equips $\mathrm{H}^0(*, G \to \ker(\lambda \mathrm{inv}_H))$ with a group structure defined by $(g_1, g_1', h_1)(g_2, g_2', h_2) = (g_1^{d_0^* h_2} g_2, g_1'^{h_2} g_2', h_1 h_2)$.

Remark 4.10. We could have first presented the Saavedra units of a group-like stack and deduced the Picard case. However, taken into account the fact that in the 2-dimensional case we only know only how to represent stacks with Picard structure by complexes and later in the paper we talk about 2-stacks, we prefer to focus on the Picard case even in 1-dimension.

Remark 4.11. We would like to point out the relation between $\mathscr{I}_{\mathscr{C}}$ the Saavedra units of a (Picard) stack $\mathscr{C}$ and $\mathscr{C}_1$ the connected components of the identity in $\mathscr{C}$. There exists a functor from $\mathscr{I}_{\mathscr{C}}$ to $\mathscr{C}_1$ defined by forgetting the morphism $\varphi : XX \to X$. $\mathscr{C}_1$ has a richer structure which makes it more interesting than $\mathscr{I}_{\mathscr{C}}$. More details about $\mathscr{C}_1$ can be found in [3].

5. Weak Units of Picard 2-Stacks

In this section, we follow the same plan as in section 4. We define weak units of a Picard 2-stack $\mathbb{C}$. We call them Joyal-Kock units of $\mathbb{C}$. We show that such units form a Picard 2-stack. Therefore by [13, Theorem 6.4], there exists a length 3 complex of abelian sheaves where the Picard 2-stack associated to it is equivalent to the Picard 2-stack of Joyal-Kock units. We conclude this section by computing such a complex.

5.1. **Joyal-Kock Units of a Picard 2-Stack.** Let $\mathbb{C}$ be a Picard 2-stack represented by the abelian complex

$$A \xrightarrow{\delta} B \xrightarrow{\lambda} C\,. \tag{5.1}$$

That is, $\mathbb{C}$ can be modeled by the Picard 2-stack $\mathrm{T{\small ORS}}(\mathscr{A}, C)$ of $\mathscr{A}$-torsors that become trivial over C and where $\mathscr{A} \simeq \mathrm{T{\small ORS}}(A, B)$. Let us remind $\mathrm{T{\small ORS}}(\mathscr{A}, C)$.

We refer to [4, §6.1] for the definition of a torsor over a group-like stack. For the notion of an $(\mathscr{G}, \mathscr{H})$-torsor with a stack morphism $\Lambda : \mathscr{G} \to \mathscr{H}$, we refer to [2, §6.3.4]. In this paper, we work with $(\mathscr{G}, \mathscr{H})$-torsors where $\mathscr{G}$ is the Picard stack $\mathscr{A}$ and $\mathscr{H}$ is the discrete Picard stack C. The Picard stack morphism $\Lambda : \mathscr{A} \to C$ associates to an (A, B)-torsor (L, x) a point $\lambda(x)$ in C. An object of $\mathrm{T{\small ORS}}(\mathscr{A}, C)$ consists of a pair $(\mathscr{L}, x)$, where $\mathscr{L}$ is an $\mathscr{A}$-torsor and $x : \mathscr{L} \to C$ is an $\mathscr{A}$-equivariant map with respect to Λ. A morphism between any two pairs in $\mathrm{T{\small ORS}}(\mathscr{A}, C)$ is given by the pair (F, γ_F)

$$(F, \gamma_F) : (\mathscr{L}, x) \longrightarrow (\mathscr{K}, y)\,,$$

where $F : \mathscr{L} \to \mathscr{K}$ is an $\mathscr{A}$-torsor morphism satisfying

$$\begin{tikzcd} \mathscr{L} \arrow[rr, "F"] \arrow[dr, "x"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathscr{K} \arrow[dl, "y"] \\ & C & \end{tikzcd} \tag{5.2}$$

and γ_F is a 2-morphism

$$
\begin{tikzcd}
\mathscr{L} \times \mathscr{A} \arrow[r, "F\times \mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\swarrow_{\gamma_F}"] & \mathscr{K} \times \mathscr{A} \arrow[d] \\
\mathscr{L} \arrow[r, "F"'] & \mathscr{K}
\end{tikzcd}
\tag{5.3}
$$

expressing the compatibility of the $\mathscr{A}$-torsor structures of $\mathscr{L}$ and $\mathscr{K}$.

A 2-morphism

$$
\begin{tikzcd}
(\mathscr{L}, x) \arrow[r, bend left, "{(F,\gamma_F)}"] \arrow[r, bend right, "{(G,\gamma_G)}"'] \arrow[r, phantom, "\Downarrow\Gamma"] & (\mathscr{K}, y)
\end{tikzcd},
$$

is given by a natural transformation $\Gamma : F \Rightarrow G$ satisfying the equation of natural transformations

$$
\begin{tikzcd}
\mathscr{L} \times \mathscr{A} \arrow[r, bend left=40, "F\times \mathrm{id}"] \arrow[r, "G\times \mathrm{id}"'] \arrow[r, phantom, bend left=20, "\Downarrow\Gamma\times \mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\swarrow_{\gamma_F}"] & \mathscr{K} \times \mathscr{A} \arrow[d] \\
\mathscr{L} \arrow[r, "G"'] & \mathscr{K}
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{L} \times \mathscr{A} \arrow[r, "F\times \mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\swarrow_{\gamma_G}"] & \mathscr{K} \times \mathscr{A} \arrow[d] \\
\mathscr{L} \arrow[r, "F"] \arrow[r, bend right=40, "G"'] \arrow[r, phantom, bend right=20, "\Downarrow\Gamma"] & \mathscr{K}
\end{tikzcd}
\tag{5.4}
$$

The tensor product on $\textsc{Tors}(\mathscr{A}, C)$ is similar to the tensor product in the stack case. For the definition of the contracted product of two $\mathscr{A}$-torsors, the reader can refer to [4, §6.7].

A *Joyal-Kock unit* in $\textsc{Tors}(\mathscr{A}, C)$ is an idempotent $(\mathscr{A}, C)$-torsor. That is, an $(\mathscr{A}, C)$-torsor $(\mathscr{L}, x)$ with an $(\mathscr{A}, C)$-torsor morphism

$$
(\varphi, \gamma_\varphi) : (\mathscr{L}, x) \otimes (\mathscr{L}, x) \longrightarrow (\mathscr{L}, x), \tag{5.5}
$$

where γ_φ is a 2-morphism of the form (5.3). In other words, with an $(\mathscr{A}, C)$-torsor morphism

$$
(\varphi, \gamma_\varphi) : (\mathscr{L} \wedge^{\mathscr{A}} \mathscr{L}, 2x) \longrightarrow (\mathscr{L}, x). \tag{5.6}
$$

We denote these units in short by $((\mathscr{L}, x), \varphi)$.

A *morphism of Joyal-Kock units* in $\textsc{Tors}(\mathscr{A}, C)$

$$
((\mathscr{L}, x), \varphi) \longrightarrow ((\mathscr{K}, y), \sigma),
$$

is given by an $(\mathscr{A}, C)$-torsor morphism

$$
(\psi, \gamma_\psi) : (\mathscr{L}, x) \longrightarrow (\mathscr{K}, y),
$$

and a 2-morphism of $(\mathscr{A}, C)$-torsors

$$\begin{tikzcd}
(\mathscr{L} \wedge^{\mathscr{A}} \mathscr{L}, 2x) \arrow[r, "\psi\wedge\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\theta_\psi \Uparrow"] & (\mathscr{K} \wedge^{\mathscr{A}} \mathscr{K}, 2y) \arrow[d, "\sigma"] \\
(\mathscr{L}, x) \arrow[r, "\psi"'] & (\mathscr{K}, y)
\end{tikzcd}$$

We denote these morphisms by the pair (ψ, θ_ψ).

A *2-morphism of Joyal-Kock units* in $\textsc{Tors}(\mathscr{A}, C)$

$$\begin{tikzcd}
((\mathscr{L}, x), \varphi) \arrow[r, bend left, "(\psi,\theta_\psi)"] \arrow[r, bend right, "(\phi,\theta_\phi)"'] \arrow[r, phantom, "\Downarrow\Gamma"] & ((\mathscr{K}, y), \sigma)
\end{tikzcd},$$

is given by a 2-morphism of $(\mathscr{A}, C)$-torsors $\Gamma : \psi \Rightarrow \phi$ satisfying the equation of 2-morphisms (5.7)

$$\begin{tikzcd}
(\mathscr{L} \wedge^{\mathscr{A}} \mathscr{L}, 2x) \arrow[r, bend left, "\psi\wedge\psi"] \arrow[r, "\phi\wedge\phi"'] \arrow[r, phantom, bend left=20, "\Downarrow\Gamma\wedge\Gamma"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\theta_\phi \Uparrow"] & (\mathscr{K} \wedge^{\mathscr{A}} \mathscr{K}, 2y) \arrow[d, "\sigma"] \\
(\mathscr{L}, x) \arrow[r, "\phi"'] & (\mathscr{K}, y)
\end{tikzcd}
=
\begin{tikzcd}
(\mathscr{L} \wedge^{\mathscr{A}} \mathscr{L}, 2x) \arrow[r, "\psi\wedge\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\theta_\psi \Uparrow"] & (\mathscr{K} \wedge^{\mathscr{A}} \mathscr{K}, 2y) \arrow[d, "\sigma"] \\
(\mathscr{L}, x) \arrow[r, "\psi"] \arrow[r, bend right, "\phi"'] \arrow[r, phantom, bend right=20, "\Downarrow\Gamma"] & (\mathscr{K}, y)
\end{tikzcd}$$

This defines the 2-groupoid of Joyal-Kock units of a Picard 2-stack $\mathbb{C}$. We denote it by $\mathbb{I}(\mathbb{C})$.

Example 5.1. We consider the trivial $(\mathscr{A}, C)$-torsor $\mathscr{A}$ with the $\mathscr{A}$-equivariant map $\zeta : \mathscr{A} \to C$ where ζ is the 2-stack morphism Λ. We denote this trivial $(\mathscr{A}, C)$-torsor by $(\mathscr{A}, 0)$. We equip $(\mathscr{A}, 0)$ with the composition

$$(\mathscr{A} \wedge^{\mathscr{A}} \mathscr{A}, 0) \xrightarrow{\Phi} (\mathscr{A}, 0) \xrightarrow{(L,s)} (\mathscr{A}, 0). \tag{5.8}$$

Φ is the canonical morphism

$$\Phi : (\mathscr{A} \wedge^{\mathscr{A}} \mathscr{A}) \longrightarrow (\mathscr{A}, 0), \tag{5.9}$$

that sends the global section $((A, 0), (A, 0))$ of $\mathscr{A} \wedge^{\mathscr{A}} \mathscr{A}$ to the global section $(A, 0)$. (L, s) is an $\mathscr{A}$-torsor such that $\Lambda(L, s) := \lambda(s) = 0$. That is, (L, s) is in the homotopy kernel of the stack morphism Λ. We recall that $\ker(\Lambda)$ is a stack represented by the complex $A \to \ker(\lambda)$. For details about the homotopy kernel of a stack morphism, we refer to [2, 6.1]. The commutativity of the diagram

$$\begin{tikzcd}
(\mathscr{A} \wedge^{\mathscr{A}} \mathscr{A}, 0) \arrow[rr, "(L,s)\circ\Phi"] \arrow[dr, "\Lambda+\Lambda"'] & \circlearrowleft & (\mathscr{A}, 0) \arrow[dl, "\Lambda"] \\
& C &
\end{tikzcd} \tag{5.10}$$

shows that the composition (5.8) is an $(\mathscr{A}, C)$-torsor morphism. Therefore, $(\mathscr{A}, 0)$ equipped with (5.8) is a Joyal-Kock unit. We denote it by $((\mathscr{A}, 0), (L, s))$. In case $(L, s) = (A, 0)$, we obtain the special Joyal-Kock unit which we denote by abuse of notation by $((\mathscr{A}, 0), 0)$.

5.2. **Contractibility of Joyal-Kock Units.** In this section, we show that there exists a Joyal-Kock unit morphism between any two Joyal-Kock units by giving a direct construction of the morphism. This result has been proven over a point in [9, Theorem C].

Proposition 5.2. *All Joyal-Kock units of* $\mathbb{C}$ *are equivalent up to a unique 2-isomorphism. That is,* $\mathbb{I}(\mathbb{C})$ *is a contractible 2-groupoid over* S.

Proof. Let $((\mathscr{L}, x), \varphi)$ be a Joyal-Kock unit of $\mathbb{C}$ and ℓ be a local section of $\mathscr{L}$. There exists an $\mathscr{A}$-torsor $(P, s)_\varphi$ unique up to a unique isomorphism such that

$$\varphi(\ell, \ell) \simeq \ell + (P, s)_\varphi. \tag{5.11}$$

From the commutativity of the diagram (5.2), we find

$$x(\ell) = \Lambda((P, s)_\varphi) = \lambda(s). \tag{5.12}$$

Choosing another local section ℓ' in $\mathscr{L}$ results in another set of relations

$$\varphi(\ell', \ell') \simeq \ell' + (P', s')_\varphi \quad \text{and} \quad x(\ell') = \lambda(s'), \tag{5.13}$$

Since $\mathscr{L}$ is locally isomorphic to $\mathscr{A}$, there exists an $\mathscr{A}$-torsor (Q, t) with

$$\ell' \simeq \ell + (Q, t). \tag{5.14}$$

By arguments similar to the proof of the Proposition 4.2, the relation (5.14) implies the unique isomorphism

$$(P', s')_\varphi \simeq (P, s)_\varphi + (Q, t). \tag{5.15}$$

From (5.15), the section $s = \ell - (P, s)_\varphi$ is uniquely isomorphic to $\ell' - (P', s')_\varphi$. That is, s is a global section of the Joyal-Kock unit $((\mathscr{L}, x), \varphi)$ unique up to a unique isomorphism and it satisfies

$$x(s) = 0_C \quad \text{and} \quad \varphi(s, s) \simeq s. \tag{5.16}$$

As in section 4.2, we define the morphism between two Joyal-Kock units $((\mathscr{L}, x), \varphi)$ and $((\mathscr{K}, y), \sigma)$ by mapping their global sections to each other. The relations (5.16) guarantee that this defines a Joyal-Kock unit morphism. This morphism is unique up to the choice of a global section. □

5.3. **Picard 2-Stack of Joyal-Kock Units.** We show that Joyal-Kock units of a Picard 2-stack $\mathbb{C}$ form a Picard 2-stack denoted by $\mathbb{I}(\mathbb{C})$. By repeating the arguments in section 4.3, we show that $\mathbb{I}(\mathbb{C})$ satisfies the conditions of the Proposition 3.1. So $\mathbb{I}(\mathbb{C})$ is a fibered 2-category in 2-groupoids. By Proposition 5.2, we deduce that $\mathbb{I}(\mathbb{C})$ is in fact a 2-stack over S.

We define a group-like structure on $\mathbb{I}(\mathbb{C})$ by

$$((\mathscr{L}, x), \varphi) \otimes ((\mathscr{K}, y), \sigma) := ((\mathscr{L} \wedge^{\mathscr{A}} \mathscr{K}, x + y), \varphi \wedge \sigma),$$

where $\varphi \wedge \sigma$ is of the form (2.2). One more time by Proposition 5.2, one can verify that this group-like structure satisfies the Picard axioms. Hence,

Proposition 5.3. $\mathbb{I}(\mathbb{C})$ *is a Picard 2-stack.*

5.4. **The Cocyclic Description of a Joyal-Kock Unit.** In this section, we give a cocyclic description of Joyal-Kock units of a Picard 2-stack $\mathbb{C}$ modeled by $\mathrm{Tors}(\mathscr{A}, C)$. This description help us to find a complex of abelian sheaves that represent $\mathbb{I}(\mathbb{C})$. The arguments presented here are the categorification of the ones in section 4.4.

Let $((\mathscr{L}, x), \varphi)$ be a Joyal-Kock unit of $\mathrm{Tors}(\mathscr{A}, C)$ over U and $V_\bullet \to U$ be a hypercover. Since $\mathscr{L}$ is locally not empty and isomorphic to $\mathscr{A}$, upon choosing a local section $\ell \in \mathscr{L}_{V_0}$, we find (P, s) an $\mathscr{A}$-torsor over V_1 such that

$$d_0^*(\ell) \simeq d_1^*(\ell) + (P, s). \tag{5.17}$$

By pulling back (5.17) to V_2, we find an isomorphism of $\mathscr{A}$-torsors over V_2

$$f : d_0^*(P, s) + d_2^*(P, s) \simeq d_1^*(P, s), \tag{5.18}$$

which satisfies a cocycle condition when pulled back to V_3. By applying the $\mathscr{A}$-equivariant morphism x to (5.17), we obtain the equation,

$$d_0^*(c) = d_1^*(c) + \lambda(s), \tag{5.19}$$

where $x(\ell) = c \in C(V_0)$ and $\lambda(s)$ is a point in $C(V_1)$. The collection $(f, (P, s))$ with the cocycle condition satisfied by f and relation (5.19) represents the $(\mathscr{A}, C)$-torsor $(\mathscr{L}, x)$.

Next, we remember the morphism φ that gives $(\mathscr{L}, x)$ a Joyal-Kock unit structure. Since ℓ and $\varphi(\ell, \ell)$ are in $\mathscr{L}_{V_0}$ and $\mathscr{L}$ is locally trivial, there exists an $\mathscr{A}$-torsor (Q, t) over V_0 such that

$$\varphi(\ell, \ell) \simeq \ell + (Q, t). \tag{5.20}$$

Applying d_i^* for $i = 1, 2$ to (5.20), we obtain

$$\varphi(d_i^*(\ell), d_i^*(\ell)) \simeq d_i^*(\ell) + d_i^*(Q, t). \tag{5.21}$$

From (5.17) and $\mathscr{A}$-equivariance of φ

$$\varphi(d_0^*(\ell), d_0^*(\ell)) \simeq d_1^*(\ell) + d_1^*(Q, t) + (P \wedge^A P, 2s). \tag{5.22}$$

After substituting (5.21) in (5.22), we find

$$d_0^*(\ell) + d_0^*(Q, t) \simeq d_1^*(\ell) + d_1^*(Q, t) + (P \wedge^A P, 2s), \tag{5.23}$$

which simplifies to

$$g : d_0^*(Q, t) - d_1^*(Q, t) \simeq (P, s), \tag{5.24}$$

where $-d_1^*(Q, t)$ denotes the inverse of the $d_1^*(Q, t)$. By pulling back (5.24), we find

$$d_0^*(g) + d_2^*(g) = d_1^*(g) + f. \tag{5.25}$$

Moreover, from the commutativity of the diagram (5.2), we find

$$\lambda(t) = c. \tag{5.26}$$

Hence, the collection $(f, (P, s), g, (Q, t), c)$ where $f \in \mathscr{A}_{V_2}$, $(P, s) \in \mathscr{A}_{V_1}$, $g \in \mathscr{A}_{V_1}$, $(Q, t) \in \mathscr{A}_{V_0}$, and $c \in C(V_0)$ satisfying the relations (5.19), (5.24), (5.25), and (5.26) plus the coherence condition on f when it is pulled back to V_3 describes a Joyal-Kock unit in $\mathrm{Tors}(\mathscr{A}, C)$. We note that this collection is a 1-cocycle with coefficients in the morphism of Picard stacks

$$(\mathrm{id}_{\mathscr{A}}, \Lambda) : \mathscr{A} \longrightarrow \ker(\Lambda - \mathrm{id}_C). \tag{5.27}$$

We refer to [1, §6.1] for a detailed treatment of cocycles with coefficients in a stack morphism.

If we choose another local section $\ell' \in \mathscr{L}_{V_0}$, we find a cohomologous 1-cocycle. Therefore the classes of 1-cocycles with values in the morphism (5.27) classify up to equivalence Joyal-Kock units. We denote this set of classes of cocycles by $\mathrm{H}^0(*, \mathscr{A} \to \ker(\Lambda - \mathrm{id}_C))$.

Remark 5.4. We obtain the morphism (5.27) by truncating the homotopy exact sequence of Picard stacks

$$\mathscr{A} \xrightarrow{(\mathrm{id}_{\mathscr{A}},\Lambda)} \mathscr{A} \oplus C \xrightarrow{\Lambda - \mathrm{id}_C} C,$$

where $\mathscr{A} \oplus C$ is a Picard stack represented by the complex $(\lambda, \mathrm{id}_C) : A \oplus C \to B \oplus C$. For details about exact sequences of group-like stacks, we refer to [2].

5.5. **A Complex of Abelian Sheaves defining the 2-Stack of Joyal-Kock Units.** The characterization theorem for Picard 2-stacks [13, Theorem 6.4] implies that all Picard 2-stacks can be represented by a length 3 complex of abelian sheaves. Let $\mathbb{C}$ be a Picard 2-stack represented by the complex (5.1). In this section, we compute a length 3 complex that represents $\mathbb{I}(\mathbb{C})$ in terms of (5.1).

Since the elements of the set $\mathrm{H}^0(*, \mathscr{A} \to \ker(\Lambda - \mathrm{id}_C))$ classify up to equivalence the Joyal-Kock units of $\mathrm{T{\scriptsize ORS}}(\mathscr{A}, C)$, the stack morphism (5.27) which corresponds to the morphism of complexes

$$\begin{tikzcd} A \arrow[r, "(\mathrm{id}_A{,}0)"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A \oplus C \arrow[d, "(\delta{,}\mathrm{id}_C)"] \\ B \arrow[r, "(\mathrm{id}_B{,}\lambda)"'] & \ker(\lambda - \mathrm{id}_C) \end{tikzcd}$$

represents $\mathbb{I}(\mathbb{C})$. From [1, Proposition 6.1.6] the 1-cocycles with coefficients in

$$A \xrightarrow{(\delta,(\mathrm{id}_A,0))} B \oplus (A \oplus C) \xrightarrow{(\mathrm{id}_B,\lambda)+(-\delta,0)} \ker(\lambda - \mathrm{id}_C). \tag{5.28}$$

the cone of (5.27) classify up to equivalence the Joyal-Kock units of $\mathrm{T{\scriptsize ORS}}(\mathscr{A}, C)$. Hence,

Proposition 5.5. *The Picard 2-stack associated to (5.28) is equivalent to the Picard 2-stack of Joyal-Kock units $\mathbb{I}(\mathbb{C})$ where $\mathbb{C}$ is the Picard 2-stack represented by (5.1).*

Remark 5.6. By [13, Theorem 6.4], length 3 complexes of abelian sheaves quasi-isomorphic to (5.28) provide other representations of $\mathbb{I}(\mathbb{C})$. For instance:

(1) The stack morphism $\mathrm{id}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ is quasi-isomorphic to (5.27). Therefore its cone

$$A \xrightarrow{(\mathrm{id}_A,\delta)} A \oplus B \xrightarrow{\delta - \mathrm{id}_B} B,$$

represents $\mathbb{I}(\mathbb{C})$.

(2) From Proposition 5.2 and Example 5.1, we deduce that $\mathrm{id}_{\ker(\Lambda)}$ the identity of $\ker(\Lambda)$ the homotopy kernel of Λ represents $\mathbb{I}(\mathbb{C})$. Since $\mathrm{id}_{\ker(\Lambda)}$ is quasi-isomorphic to (5.27), its cone

$$A \xrightarrow{(\mathrm{id}_A,\delta)} A \oplus \ker(\lambda) \xrightarrow{\delta - \mathrm{id}_{\ker(\lambda)}} \ker(\lambda),$$

represents $\mathbb{I}(\mathbb{C})$, as well.

Remark 5.7. We remark that the morphism of Picard 2-stacks $\mathbb{I}(\mathbb{C}) \to \mathbb{C}$ defined by forgetting the morphism (5.6) corresponds to the morphism of complexes

$$\begin{tikzcd}
A \arrow[r, "{(\delta,(\mathrm{id}_A,0))}"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowright"] & B \oplus (A \oplus C) \arrow[r, "{(\mathrm{id}_B,\lambda)+(-\delta,0)}"] \arrow[d, "\mathrm{pr}_B"] \arrow[dr, phantom, "\circlearrowright"] & \ker(\lambda - \mathrm{id}_C) \arrow[d, "\mathrm{pr}_C"] \\
A \arrow[r, "\delta"'] & B \arrow[r, "\lambda"'] & C
\end{tikzcd}$$

under the equivalence [13, Theorem 6.4].

Remark 5.8. In non-abelian setting, we know how to associate a group-like 2-stack to a 2-crossed module, say $(\delta, \lambda) : G \to H \to K$(see [11]). We model this 2-stack by $\mathrm{T{\small ORS}}(\mathscr{G}, K)$ where $\mathscr{G}$ is the group-like stack associated to $G \to H$. By adapting the ideas of section 5 to the non-abelian case we find, analogously to section 4.6 that Joyal-Kock units of $\mathrm{T{\small ORS}}(\mathscr{G}, K)$ form a Picard 2-stack represented by the complex

$$G \xrightarrow{(\delta,(\mathrm{id}_G,1))} H \oplus (G \oplus K) \xrightarrow{(\mathrm{id}_H,\lambda)(\delta^{-1},1)} \ker(\lambda \mathrm{inv}_K)\,.$$

We also remark that it is natural to expect that these results extend in a similar way to n-stacks that are at least group-like.

References

[1] Ettore Aldrovandi. 2-gerbes bound by complexes of *gr*-stacks, and cohomology. *J. Pure Appl. Algebra*, 212(5):994–1038, 2008.

[2] Ettore Aldrovandi and Behrang Noohi. Butterflies I: Morphisms of 2-group stacks. *Advances in Mathematics*, 221(3):687 – 773, 2009.

[3] Ettore Aldrovandi and Ahmet Emin Tatar. Cohomology and coherence. *In preparation*.

[4] Lawrence Breen. Bitorseurs et cohomologie non abélienne. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 401–476. Birkhäuser Boston, Boston, MA, 1990.

[5] Lawrence Breen. On the classification of 2-gerbes and 2-stacks. *Astérisque*, (225):160, 1994.

[6] Pierre Deligne. La formule de dualité globale, 1973. SGA 4 III, Exposé XVIII.

[7] Pierre Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.

[8] Monique Hakim. *Topos annelés et schémas relatifs*. Springer-Verlag, Berlin, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 64.

[9] Andre Joyal and Joachim Kock. Coherence for weak units, 2009.

[10] Joachim Kock. Elementary remarks on units in monoidal categories. *Math. Proc. Cambridge Philos. Soc.*, 144(1):53–76, 2008.

[11] Behrang Noohi. On weak maps between 2-groups, 2005.

[12] Neantro Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin, 1972.

[13] A. Emin Tatar. Length 3 complexes of abelian sheaves and Picard 2-stacks. *Adv. Math.*, 226(1):62–110, 2011.

[14] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.

Department of Mathematics, FSU, Tallahassee, USA
E-mail address: `aldrovandi@math.fsu.edu`

Department of Mathematics and Statistics, KFUPM, Dhahran, KSA
E-mail address: `atatar@kfupm.edu.sa`