On Series of Ordinals and Combinatorics

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Abstract. This paper deals mainly with generalizations of results in finitary combinatorics to infinite ordinals. It is well-known that for finite ordinals $\sum_{\beta<\alpha} \beta$ is the number of 2-element subsets of an α-element set. It is shown here that for any well-ordered set of arbitrary infinite order type α, $\sum_{\beta<\alpha} \beta$ is the ordinal of the set M of 2-element subsets where M is ordered in some natural way. This result is then extended to evaluating the ordinal of the set of all n-element subsets for each natural number $n \geq 2$

Moreover series $\sum_{\beta<\alpha} f(\beta)$ are investigated and evaluated, where α is a limit ordinal and the function f belongs to a certain class of functions containing polynomials with natural number coefficients. The tools developed for this result can be extended to cover all infinite α, but the case of finite α appears to be quite problematic.

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§0. Introduction

The research we report on here grew from an attempt to obtain a representation of the ordinal sum $\sum_{\beta<\alpha} \beta$ in terms of relatively simple finite-place functions on the ordinals. A very limited case of this problem is mentioned, by way of an exercise, in Sierpinski [S, p. 286, Exercise 6]. Specifically, Sierpinski asks the reader to show that $\sum_{\beta<\omega^n} \beta = \omega^{2n-1}$, where $n \geq 1$ is a natural number. We give here a complete solution to the wider problem.

Actually, we do much more. In the case that α is a limit ordinal, we obtain such a representation for the series $\sum_{\beta<\alpha} f(\beta)$, where the summand $f(\beta)$ comes from a class of functions broad enough to include polynomials in β with natural number coefficients. Our results could, using the tools given here, be extended to the case in which α is an infinite ordinal which is not a limit, but the finite case is much harder. For example, for finite α, the sum $\sum_{\beta<\alpha} \beta^n$ is intimately connected with the Bernoulli numbers. See Rademacher [R].

One of the tools developed here – the so-called "Gamma Lemma" – appears to be a contribution, in its own right, to the theory of infinite series of ordinals, and may well have value for summing series apart from its use here.

Further contributions of this paper pertain to ordinal combinatorics. We introduce a notion of "binomial expansion", whose coefficients have combinatorial significance analogous to the binomial coefficients in finite combinatorics. It is an elementary result of finite combinatorics that, for finite α, $\sum_{\beta<\alpha} \beta$ is the number of 2- element subsets of an α-element set. We show that a result of this kind obtains for transfinite α as well: $\sum_{\beta<\alpha} \beta$ is the ordinal of the family of 2- element subsets of any well ordered set whose ordinal is α, when those 2-element subsets are ordered in a certain natural way. We then solve the considerably more difficult problem of evaluating, for each natural number $n \geq 2$, the ordinal of the family of n-element subsets of a set whose ordinal is α.

Conventions:

We use a "bar" (e.g., $\bar{\alpha}$) to denote limit ordinals.

We say that an ordinal α is in Weak Cantor Normal Form (abbreviated WCNF) when it is expressed as $\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$. Here α_1 is called the degree of α. We assume familiarity with the arithmetical properties of the ordinals as set forth in Sierpinski [S]. For example, we use without comment the algorithm for multiplying normal forms given in [S, p. 320, Exercise 4].

An ordinal α which may be expressed in the form ω^x is referred to as a gamma number. If x is itself a gamma number, then α is called a delta number.

§1. Combinatorics and summation

Let α denote an ordinal. For each nonzero finite ordinal r, let $S(\alpha, r)$ denote the set of elements $(a_1, \ldots, a_r)$ of α^r such that $a_1 < a_2 < \ldots < a_r$. Order $S(\alpha, r)$ lexicographically from the right. Let $C(\alpha, r)$ denote the ordinal of $S(\alpha, r)$. By convention, set $C(\alpha, 0) = 1$.

Recall that for non-negative integers n, r, $C(n, r)$ is the number of r–combinations of a set with n elements. We have extended this notion to the situation in which n is any ordinal α, by ordering the r–combinations of elements of α and letting $C(\alpha, r)$ denote the resulting ordinal.

Proposition 1.1. *The ordinals $C(\alpha, r)$ satisfy, and are defined by, the following recursions:*

(0) $C(\alpha, 0) = 1$ *for all* α, *and* $C(0, r) = 0$ *for all* $r > 0$.

(1) $C(\alpha + 1, r) = C(\alpha, r) + C(\alpha, r - 1)$ *for all* α *and for all* $r \geq 1$.

(2) $C(\bar{\alpha}, r) = \sup\limits_{\alpha < \bar{\alpha}} C(\alpha, r)$ *for all* $\bar{\alpha}$ *and for all* r.

Proof. It is clear that the ordinals $C(\alpha, r)$ satisfy (0).

(1): Note that $C(\alpha, 1) = \alpha$, all α, so the result holds for $r = 1$. Assume $r > 1$. We are considering r–tuples $(a_1, \ldots, a_r)$ of elements of $\alpha + 1$, with $a_1 < a_2 < \ldots < a_r$. An r–tuple in which $a_r < \alpha$ is actually an r–tuple of elements of α. An r–tuple with $a_r = \alpha$ can be considered to be an $(r - 1)$–tuple of elements of α, by disregarding a_r. Since those of the former type precede those of the latter type in our ordering, the result follows.

(2): This follows from the fact that an r–tuple of elements of a limit ordinal $\bar{\alpha}$ is actually an r–tuple of elements of α for some $\alpha < \bar{\alpha}$.

It is clear that (0), (1), and (2) determine $C(\alpha, r)$ for all α, r. ■

We can view the numbers $C(\alpha, r)$ as "binomial coefficients," as follows.

Let A denote the class of countably infinite sequences of ordinals. We write $(\alpha_0, \alpha_1, \alpha_2, \ldots) \in A$ as $f = f(X) = \sum \alpha_i X^i$. If $\alpha_i = 0$ for $i > n$, we write

$f = \alpha_0 + \alpha_1 X + \ldots + \alpha_n X^n$. For $f = \sum \alpha_i X^i$, $g = \sum \beta_i X^i \in A$, we define $f + g = \sum(\alpha_i + \beta_i)X^i$, and $fg = \sum(\alpha_i\beta_0 + \alpha_{i-1}\beta_1 + \ldots + \alpha_0\beta_i)X^i$. Now define elements $(1+X)^\alpha$ of A recursively, as follows:

$$\begin{aligned}(1+X)^0 &= 1\\ (1+X)^{\alpha+1} &= (1+X)^\alpha(1+X)\\ (1+X)^{\bar\alpha} &= \lim_{\alpha<\bar\alpha}(1+X)^\alpha,\end{aligned}$$

where the limit is taken coefficientwise. If we write $(1+X)^\alpha = \sum C_{\alpha,r}X^r$, we find that the $C_{\alpha,r}$'s satisfy the recursion of Proposition 1.1, and thus $C_{\alpha,r} = C(\alpha, r)$ for all α, r. That is, $C(\alpha, r)$ is the coefficient of X^r in our "expansion" of $(1+X)^\alpha$.

Let us write $C_r(\alpha)$ in place of $C(\alpha, r)$. Thus, our functions $C_r(\alpha)$ are given recursively by

$C_0(\alpha) = 1$ for all α and $C_r(0) = 0$ for all $r > 0$

$C_r(\alpha+1) = C_r(\alpha) + C_{r-1}(\alpha)$ for all α and for all $r \geq 1$

$C_r(\bar\alpha) = \sup_{\alpha<\bar\alpha} C_r(\alpha)$.

For any function f from the ordinals to the ordinals, we define S_f, the "cumulative sum" of f, by: $S_f(\alpha) = \sum_{\beta<\alpha} f(\beta)$, where the sum is as defined in Sierpinski. By convention, we have $S_f(0) = 0$. Then S_f satisfies the following recursion:

$$\begin{aligned}S_f(0) &= 0\\ S_f(\alpha+1) &= S_f(\alpha) + f(\alpha)\\ S_f(\bar\alpha) &= \sup_{\alpha<\bar\alpha} S_f(\alpha).\end{aligned}$$

¿From the formal similarity of the above two sets of recursion relations, we see that, for $r \geq 1$, we have $C_r = S_{C_{r-1}}$, i.e., $C_r(\alpha) = \sum_{\beta<\alpha} C_{r-1}(\beta)$. This is a familiar relationship when α is finite. In the finite case, the numbers $C_r(\alpha)$ are "known", and the relationship is used to evaluate the sum. We shall reverse the process: after developing some summation tools, we shall use the relationship to recursively evaluate the numbers $C_r(\alpha)$.

§2. Summation tools.

Lemma 2.1. *Let f be nondecreasing. Then for all ordinals α, β we have*

$$S_f(\alpha) + f(\alpha)\beta \leq S_f(\alpha+\beta) \leq S_f(\alpha) + f(\alpha+\beta)\beta.$$

Proof. We have

$$\begin{aligned}
S_f(\alpha) + f(\alpha)\beta &= S_f(\alpha) + \sum_{\alpha<\beta} f(\alpha) \\
&\le S_f(\alpha) + \sum_{\gamma<\beta} f(\alpha+\gamma) = S_f(\alpha+\beta) \\
&\le S_f(\alpha) + \sum_{\gamma<\beta} f(\alpha+\beta) \\
&= S_f(\alpha) + f(\alpha+\beta)\beta. \blacksquare
\end{aligned}$$

Main Lemma 2.2. *Let f be nondecreasing. Let α, β be ordinals with $f(\alpha)\beta = f(\alpha+\beta)\beta$. Then $S_f(\alpha+\beta) = S_f(\alpha) + f(\alpha)\beta$.*

Proof. This is immediate from Lemma 2.1. $\blacksquare$

Degree Lemma 2.3. *Let f be nondecreasing. Let α, $\bar\beta$ be ordinals with* $\deg f(\alpha) = \deg f(\alpha+\bar\beta)$. *Then $S_f(\alpha+\bar\beta) = S_f(\alpha) + f(\alpha)\bar\beta$.*

Proof. If $\deg f(\alpha) = \deg f(\alpha+\bar\beta) = \gamma$, then $f(\alpha)\bar\beta = \omega^\gamma\bar\beta = f(\alpha+\bar\beta)\bar\beta$, so the result follows from the Main Lemma.

Gamma Lemma 2.4. *Let f be nondecreasing, and let ω^γ be an infinite gamma number. Then $S_f(\omega^\gamma) = \sup\limits_{\alpha<\omega^\gamma}(f(\alpha)\omega^\gamma)$.*

Proof. For any limit ordinal $\bar x$, we have

$$S_f(\bar x) = \sup_{x<\bar x}\left(\sum_{\beta<x} f(\beta)\right) \le \sup_{x<\bar x}\left(\sum_{\beta<x} f(x)\right) = \sup_{x<\bar x}(f(x)x) \le \sup_{x<\bar x}(f(x)\bar x).$$

Thus, with $\bar\alpha = \omega^\gamma$, we have $S_f(\bar\alpha) \le \sup\limits_{\alpha<\bar\alpha}(f(\alpha)\bar\alpha)$. Now for any $\alpha < \bar\alpha$ we have

$$\begin{aligned}
f(\alpha)\bar\alpha &\le \sum_{\beta<\alpha} f(\beta) + f(\alpha)\bar\alpha \\
&= \sum_{\beta<\alpha} f(\beta) + \sum_{\beta<\bar\alpha} f(\alpha) \\
&\le \sum_{\beta<\alpha} f(\beta) + \sum_{\beta<\bar\alpha} f(\alpha+\beta) \\
&= S_f(\alpha+\bar\alpha) \\
&= S_f(\bar\alpha)\,, \quad \text{since } \alpha+\bar\alpha = \bar\alpha.
\end{aligned}$$

Thus $\sup\limits_{\alpha<\bar{\alpha}} f(\alpha)\bar{\alpha} \leq S_f(\bar{\alpha})$ holds as well. ∎

Remark 2.5. Using the Gamma Lemma, it is very easy to do the exercise of Sierpinski referred to in our introduction.

Corollary 2.6. *Let f be nondecreasing, and let α, γ be ordinals with $\gamma > 0$. Then*

$$S_f(\alpha+\omega^\gamma) = S_f(\alpha) + \sup_{\beta<\omega^\gamma}(f(\alpha+\beta)\omega^\gamma).$$

Proof. Set $g(x) = f(\alpha+x)$. Then $S_f(\alpha+\omega^\gamma) = S_f(\alpha) + S_g(\omega^\gamma)$, and we evaluate $S_g(\omega^\gamma)$ using the Gamma Lemma. ∎

We introduce a function that will help us deal with the sups arising in Lemma 2.4 and Corollary 2.6, and will have another use later on. We define

$$\begin{aligned} m'(0,y) &= 0 \\ m'(x+1,y) &= xy+1 \\ m'(\bar{x},y) &= \sup_{x<\bar{x}}(xy) \end{aligned}$$

Lemma 2.7. *Let X be a set of ordinals which does not contain $\alpha = \sup\{x : x \in X\}$. Let y be any ordinal. Then $\sup\limits_{x\in X}(xy) = m'(\alpha,y)$.*

Proof. Note that, since $\alpha \notin X$, α must be a limit ordinal. Thus our result follows from the fact that multiplication is a nondecreasing function of the left argument. ∎

Combining the Gamma Lemma with Lemma 2.7 yields

Second Gamma Lemma 2.8. *Let f be nondecreasing, and let $\bar{\alpha} = \omega^\gamma$ be an infinite gamma number. Suppose that f is continuous at $\bar{\alpha}$, and that $f(\alpha) < f(\bar{\alpha})$ for $\alpha < \bar{\alpha}$. Then*

$$S_f(\omega^\gamma) = m'(f(\omega^\gamma),\omega^\gamma). \blacksquare$$

As another application of Lemma 2.7, we have the following observation:

$$m'(z\bar{x},y) = \sup_{x<\bar{x}}(zxy) = z\sup_{x<\bar{x}}(xy) = zm'(\bar{x},y).$$

WCNF Lemma 2.9. *Let f be nondecreasing, and let $\bar{\alpha} = \omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_k}$ in WCNF. Suppose that, for $1 \leq r \leq k$, f is continuous at $\omega^{\alpha_1} + \ldots + \omega^{\alpha_r}$ and $f(x) < f(\omega^{\alpha_1} + \ldots + \omega^{\alpha_r})$ for $x < \omega^{\alpha_1} + \ldots + \omega^{\alpha_r}$. Then*

$$\begin{aligned} S_f(\bar{\alpha}) &= S_f(\omega^{\alpha_1} + \ldots + \omega^{\alpha_{k-1}}) + m'(f(\bar{\alpha}), \omega^{\alpha_k}) \\ &= m'(f(\omega^{\alpha_1}), \omega^{\alpha_1}) + m'(f(\omega^{\alpha_1} + \omega^{\alpha_2}), \omega^{\alpha_2}) + \ldots + m'(f(\bar{\alpha}), \omega^{\alpha_k}). \end{aligned}$$

Proof. We obtain the first equality by arguing as in Corollary 2.6, and using the Second Gamma Lemma. The second equality then follows by induction on k. ■

We now head towards a formula for $m'(\bar{x}, y^\beta)$. We first need an auxiliary function.

Let a' be defined by:

$$\begin{aligned} a'(0, y) &= y \\ a'(x+1, y) &= x + y \\ a'(\bar{x}, y) &= \sup_{x<\bar{x}}(x + y). \end{aligned}$$

Proposition 2.10. *Let $x = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$, $y = \omega^{\beta_1} + \ldots + \omega^{\beta_\ell}$ in WCNF. Write $x_1 = \omega^{\alpha_1} + \ldots + \omega^{\alpha_{k-1}}$ ($x_1 = 0$ if $k = 1$). Then*

$$a'(x, y) = \begin{cases} x_1 + y = x + y, & \text{if } \alpha_k < \beta_1 \\ x_1 + y, & \text{if } \alpha_k = \beta_1 \\ x, & \text{if } \alpha_k > \beta_1. \end{cases}$$

Proof. If $\alpha_k = 0$, then $x = x_1 + 1$, and $a'(x, y) = a'(x_1 + 1, y) = x_1 + y$, as required. So suppose $\alpha_k \geq 1$. Then x is a limit ordinal, and we have

$$a'(x, y) = \sup_{z<x}(z + y) = x_1 + \sup_{z<\omega^{\alpha_k}}(z + y).$$

If $\alpha_k \leq \beta_1$, then $z+y = y$ for all $z < \omega^{\alpha_k}$, so $a'(x, y) = x_1 + y$. If $\alpha_k > \beta_1$, then $y < \omega^{\alpha_k}$, and $z+y < \omega^{\alpha_k}$ for all $z < \omega^{\alpha_k}$, so $\sup_{z<\omega^{\alpha_k}}(z+y) = \omega^{\alpha_k}$, and thus $a'(x, y) = x_1 + \omega^{\alpha_k} = x$. ■

Proposition 2.11. *Let $\bar{x} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ in WCNF, and let $y = \omega^\beta$ be an infinite gamma number. Then*

$$m'(\bar{x}, y) = \begin{cases} \bar{x}y, & \text{if } k > 1 \\ \omega^{a'(\alpha_1, \beta)}, & \text{if } k = 1. \end{cases}$$

Proof. If $k > 1$, then $\bar{x} = \omega^{\alpha_1} + \bar{z}$, where $\bar{z} < \omega^{\alpha_1} n$, some n. Then $m'(\bar{x}, y) = \sup_{x<\bar{x}}(xy) = \sup_{z<\bar{z}}((\omega^{\alpha_1} + z)y) = \sup_{z<\bar{z}}(\omega^{\alpha_1} y) = \omega^{\alpha_1} y = \bar{x}y$. So now we assume $k = 1$. If $\alpha_1 = \alpha_1' + 1$ is a successor, then $m'(\bar{x}, y) = \sup_{x<\bar{x}}(xy) = \sup_{n}(\omega^{\alpha_1'} ny) = \sup_{n}(\omega^{\alpha_1'}\omega^{\beta}) = \omega^{\alpha_1'+\beta} = \omega^{a'(\alpha_1,\beta)}$. Otherwise α_1 is a limit, and $m'(\bar{x}, y) = \sup_{x<\omega^{\alpha_1}}(xy) = \sup_{z<\alpha_1}(\omega^z\omega^{\beta}) = \omega^{\sup_{z<\alpha_1}(z+\beta)} = \omega^{a'(\alpha_1,\beta)}$. ∎

We shall need the following result later on.

Proposition 2.12. *Let $\Delta = \omega^{\omega^{\delta}}$ be an infinite delta number, and let m be a positive integer. If $\bar{x} < \Delta^{\omega}$, then $\bar{x}\Delta^{m-1} \leq m'(\bar{x}, \Delta^m)$.*

Proof. By Proposition 2.11, we see that we may assume that $\bar{x} = \omega^{\alpha}$, and that in that case we have $m'(\bar{x}, \Delta^m) = \omega^{a'(\alpha,\omega^{\delta}m)}$. We must show that $\alpha + \omega^{\delta}(m-1) \leq a'(\alpha, \omega^{\delta}m)$. Since $\bar{x} < \Delta^{\omega}$, we have $\bar{x} < \Delta^n$ for some n, and thus $\alpha < \omega^{\delta}n$ for some n. Thus if $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ in WCNF, we have $\alpha_k \leq \delta$. By Proposition 2.10, we have $a'(\alpha, \omega^{\delta}m) = \omega^{\alpha_1} + \ldots + \omega^{\alpha_{k-1}} + \omega^{\delta}m$, and this gives our result. ∎

Remark 2.13. In the above, $a'(\alpha, \omega^{\delta}m) = \alpha + \omega^{\delta}m$ unless $\alpha_k = \delta$. In this case, $\alpha_1 = \ldots = \alpha_k = \delta$, since $\bar{x} < \Delta^{\omega}$. So we have that $m'(\bar{x}, \Delta^m) = \bar{x}\Delta^m$ unless $\bar{x} = \Delta^k$, some $k \geq 1$, in which case $m'(\bar{x}, \Delta^m) = \bar{x}\Delta^{m-1}$.

§3. Functions of polynomial type.

Let $\mathcal{P}$ be the class of nondecreasing functions f from the ordinals to the ordinals which have the following properties: $f(\alpha) > 0$ for $\alpha > 0$, and for all ordinals α, β with $\alpha > 0$ and $\beta \leq \alpha n$ for some positive integer n, we have $\deg f(\alpha) = \deg f(\alpha + \beta)$. We call $\mathcal{P}$ the class of functions of polynomial type.

It is easy to see that a nondecreasing function f with $f(\alpha) > 0$ for $\alpha > 0$ lies in $\mathcal{P}$ iff for all ordinals $\alpha > 0$ we have $\deg f(\alpha) = \deg f(\omega^{\alpha_1})$, where $\alpha = \omega^{\alpha_1} + \ldots$ in WCNF.

The class $\mathcal{P}$ clearly contains the nonzero constant functions and the identity function. For $f, g \in \mathcal{P}$ and α in WCNF as above we have $\deg((f + g)(\alpha)) = \deg(f(\alpha) + g(\alpha)) = \max(\deg f(\alpha), \deg g(\alpha)) = \max(\deg f(\omega^{\alpha_1}), \deg g(\omega^{\alpha_1})) = \deg((f + g)(\omega^{\alpha_1}))$, so $f + g \in \mathcal{P}$. Similarly, $fg \in \mathcal{P}$. In particular, $\mathcal{P}$ contains the "polynomials" $x^r\alpha_r + x^{r-1}\alpha_{r-1} + \ldots + x\alpha_1 + \alpha_0$.

Proposition 3.1. *Let f be a function of polynomial type. Let α, $\bar{\beta}$ be ordinals with $\bar{\beta} \leq \alpha n$, some positive integer n. Then $S_f(\alpha + \bar{\beta}) = S_f(\alpha) + f(\alpha)\bar{\beta}$.*

Proof. Since $\deg f(\alpha) = \deg f(\alpha + \bar{\beta})$, this follows from the Degree Lemma. ∎

Remark 3.2. Using Proposition 3.1, the cumulative sum of a function f of polynomial type can be evaluated at any limit ordinal $\bar{\alpha}$ once we know how to calculate it at the gamma numbers. For we can write $\bar{\alpha} = \omega^{\alpha_1} + \bar{\beta}$, with $\bar{\beta} < \omega^{\alpha_1} n$ for some n, and then we have $S_f(\bar{\alpha}) = S_f(\omega^{\alpha_1}) + f(\omega^{\alpha_1})\bar{\beta}$.

We now investigate the evaluation, at a gamma number, of S_f where f is a function of polynomial type.

For any $f \in \mathcal{P}$ and any ordinal α, let is write $t_f(\alpha)$ for the degree of $f(\omega^\alpha)$. Note that we then have $f(\omega^\alpha)\bar{\beta} = \omega^{t_f(\alpha)}\bar{\beta}$ for any $\bar{\beta}$.

Lemma 3.3. *Let $f \in \mathcal{P}$. The for any $\gamma \geq 0$ we have*

$$S_f(\omega^{\gamma+1}) = \omega^{t_f(\gamma)+\gamma+1}.$$

Proof. We have

$$\begin{aligned} S_f(\omega^{\gamma+1}) &= \sup_{\beta < \omega^{\gamma+1}} (f(\beta)\omega^{\gamma+1}) \quad \text{by the Gamma Lemma} \\ &= \sup_n (f(\omega^\gamma n)\omega^{\gamma+1}) \\ &= f(\omega^\gamma)\omega^{\gamma+1} \quad \text{since } f(\omega^\gamma),\ f(\omega^\gamma n) \text{ have the same degree} \\ &= \omega^{t_f(\gamma)+\gamma+1} \blacksquare \end{aligned}$$

Let $\bar{\gamma} = \sup_{\gamma < \bar{\gamma}} \gamma$ be a limit ordinal. Then for $f \in \mathcal{P}$ we have

$$\begin{aligned} S_f(\omega^{\bar{\gamma}}) &= S_f(\sup_{\gamma < \bar{\gamma}}(\omega^{\gamma+1})) \\ &= \sup_{\gamma < \bar{\gamma}} S_f(\omega^{\gamma+1}) \\ &= \sup_{\gamma < \bar{\gamma}} \omega^{t_f(\gamma)+\gamma+1} \\ &= \omega^{\sup\limits_{\gamma < \bar{\gamma}}(t_f(\gamma)+\gamma+1)} \end{aligned}$$

Thus we have established the following:

Theorem 3.4. *Let f be a function of polynomial type. Then for all $\gamma > 0$ we have*

$$S_f(\omega^\gamma) = \omega^{d_f(\gamma)},$$

where $d_f(\gamma)$ is the function defined by

$$\begin{aligned} d_f(0) &= 0, \\ d_f(\gamma+1) &= t_f(\gamma)+\gamma+1, \\ d_f(\bar{\gamma}) &= \sup_{\gamma<\bar{\gamma}}[t_f(\gamma)+\gamma+1]. \blacksquare \end{aligned}$$

We now proceed to calculate d_f for f of the form $f(X) = X^r n_r + X^{r-1}n_{r-1} + \ldots + Xn_1 + n_0$, where $r, n_0, \ldots, n_r$ are finite ordinals and $n_r > 0$.

Clearly $t_f(\alpha) = \alpha r$, all α. Thus we have

$$\begin{aligned} d_f(0) &= 0, \\ d_f(\gamma+1) &= \gamma r + \gamma + 1 = \gamma(r+1)+1, \quad \text{and} \\ d_f(\bar{\gamma}) &= \sup_{\gamma<\bar{\gamma}}(\gamma(r+1)+1) \\ &= \sup_{\gamma<\bar{\gamma}}(\gamma(r+1)). \end{aligned}$$

We shall denote d_f as d_{r+1}. Thus, for $r \geq 1$, we have $d_r(0) = 0$, $d_r(\gamma+1) = \gamma r + 1$, and $d_r(\bar{\gamma}) = \sup_{\gamma<\bar{\gamma}}(\gamma r)$. In terms of our function m' introduced earlier, we have $d_r(\gamma) = m'(\gamma, r)$.

Proposition 3.5. *Let γ be an ordinal, and let $r \geq 1$. Then $d_r(\gamma) = \gamma r$, unless γ is of the form $\gamma = \omega^\alpha m$ ($\alpha \geq 0$, $m \geq 1$), in which case $d_r(\gamma) = d_r(\omega^\alpha m) = \omega^\alpha(rm-(r-1))$.*

Proof. The assertion is true for $\gamma = 0$. Suppose $\gamma = \gamma' + 1$ is a successor. Then $d_r(\gamma) = \gamma' r + 1$. If γ is finite, then $\gamma = \omega^\alpha m$ with $\alpha = 0$ and $m = \gamma' + 1$. We have $\omega^\alpha(rm-(r-1)) = r(\gamma'+1) - r + 1 = r\gamma' + 1 = d_r(\gamma)$. If γ is infinite, then γ is not of the indicated form. We have $\gamma r = (\gamma'+1)r = \gamma' r + 1 = d_r(\gamma)$. Now suppose that γ is not a successor. If $\gamma = \omega^\alpha m$ ($\alpha \geq 0$, $m \geq 1$), then $\gamma = \sup_{z<\omega^\alpha}[\omega^\alpha(m-1)+z]$, so $d_r(\gamma) = \sup_{z<\omega^\alpha}([\omega^\alpha(m-1)+z]r) = \sup_{z<\omega^\alpha}[\omega^\alpha(rm-r)+z] = \omega^\alpha(rm-r)+\omega^\alpha = \omega^\alpha(rm-(r-1))$. Otherwise, $\gamma = \omega^\alpha m + \bar{\beta}$ ($\alpha \geq 1$, $m \geq 1$, $\bar{\beta} < \omega^\alpha$), and $d_r(\gamma) = \sup_{z<\bar{\beta}}([\omega^\alpha m + z]r) = \sup_{z<\bar{\beta}}[\omega^\alpha rm + z] = \omega^\alpha rm + \bar{\beta} = \gamma r$. $\blacksquare$

Theorem 3.6. *Let $f(X) = X^r n_r + X^{r-1}n_{r-1} + \ldots + Xn_1 + n_0$, where $r \geq 0$, $n_0, \ldots, n_r$ are finite ordinals, and $n_r \neq 0$. Let $\bar{\alpha} = \omega^{\alpha_1} + \bar{\beta}$, where $\bar{\beta} \leq \bar{\alpha}n$ for some n. Then*

$$S_f(\bar{\alpha}) = \omega^{d_{r+1}(\alpha_1)} + \omega^{\alpha_1 r}\bar{\beta}.$$

Proof. We have

$$\begin{aligned}
S_f(\bar{\alpha}) &= S_f(\omega^{\alpha_1}) + f(\omega^{\alpha_1})\bar{\beta} \quad \text{by Remark 3.2} \\
&= \omega^{d_{r+1}(\alpha_1)} + [\omega^{\alpha_1 r} n_r + \ldots + n_0]\bar{\beta} \quad \text{by Theorem 3.4} \\
&= \omega^{d_{r+1}(\alpha_1)} + \omega^{\alpha_1 r}\bar{\beta}. \blacksquare
\end{aligned}$$

Remark 3.7. Let us say that an ordinal $\alpha > 0$ is *typical* if its degree α_1 is not of the form $\omega^\beta m$ ($\beta \geq 0$, $m \geq 1$). Now let $\bar{\alpha}$ be a typical limit ordinal. Then $d_{r+1}(\alpha_1) = \alpha_1(r+1)$, and we have

$$S_f(\bar{\alpha}) = \omega^{\alpha_1(r+1)} + \omega^{\alpha_1 r}\bar{\beta} = \bar{\alpha}^{r+1}.$$

It is easy to evaluate S_{X^r} on the infinite successor ordinals $\bar{\alpha}+n$, $n \geq 1$. We have $(\bar{\alpha}+i)^r = \bar{\alpha}^r + \bar{\alpha}^{r-1}i + \bar{\alpha}^{r-2}i + \ldots + \bar{\alpha}i + i$, and thus

$$\begin{aligned}
S_{X^r}(\bar{\alpha}+n) &= S_{X^r}(\bar{\alpha}) + \bar{\alpha}^r + (\bar{\alpha}+1)^r + \ldots + (\bar{\alpha}+n-1))^r \\
&= S_{X^r}(\bar{\alpha}) + \bar{\alpha}^r n + \bar{\alpha}^{r-1}(n-1) + \bar{\alpha}^{r-2}(n-1) + \ldots + \bar{\alpha}(n-1) + (n-1).
\end{aligned}$$

A similar formula can be obtained for $S_f(\bar{\alpha}+n)$, when f is as above. As we have observed in the introduction, the evaluation of S_{X^r} at the finite ordinals is much more difficult.

The methods of this section can be extended to a slightly wider class of functions, as follows. Let us say that a nonzero nondecreasing function f satisfies the *degree condition* if for all ordinals α, β with $f(\alpha) > 0$ and $\beta \leq \alpha n$ for some positive integer n, we have $\deg f(\alpha) = \deg f(\alpha+\beta)$. Suppose f satisfies the degree condition. Let γ be the smallest ordinal for which $f(\gamma) > 0$. Define $g(\alpha) = f(\gamma+\alpha)$, all α. Then $g(\alpha) > 0$ for $\alpha > 0$. Suppose $\alpha > 0$ and $\alpha \leq \beta n$, for some n. Then $\gamma+\alpha+\beta \leq \gamma+\alpha+\alpha n = \gamma+\alpha(n+1) \leq (\gamma+\alpha)(n+1)$ so $\deg g(\alpha) = \deg f(\gamma+\alpha) = \deg f(\gamma+\alpha+\beta) = \deg g(\alpha+\beta)$. Thus g is a function of polynomial type. For any $\alpha \geq \gamma$, there exists a unique β with $\gamma+\beta = \alpha$. Then $f(\alpha) = g(\beta)$, and so we can compute f if we can compute g.

§4. First results for C_r

We begin by evaluating C_r at the gamma numbers. This involves obtaining more information about the relationship between d_r and our earlier functions a' and m'.

Lemma 4.1. *Let α be an ordinal, and let $r \geq 1$. Then $a'(d_r(\alpha),\alpha) = d_{r+1}(\alpha)$.*

Proof. Write $\alpha = \omega^{\alpha_1}m_1 + \ldots + \omega^{\alpha_k}m_k$ in Cantor Normal Form. If $k = 1$, then $a'(d_r(\alpha),\alpha) = a'(d_r(\omega^{\alpha_1}m_1),\omega^{\alpha_1}m_1) = a'(\omega^{\alpha_1}(rm_1-(r-1)),\omega^{\alpha_1}m) = \omega^{\alpha_1}((r+1)m_1 - r) = d_{r+1}(\alpha)$. If $k > 1$, then $a'(d_r(\alpha),\alpha) = a'(\alpha r,\alpha) = \alpha r + \alpha = \alpha(r+1) = d_{r+1}(\alpha)$. $\blacksquare$

Lemma 4.2. *Let α be an ordinal, and let $r \geq 1$. Then $m'(\omega^{d_r(\alpha)}, \omega^\alpha) = \omega^{d_{r+1}(\alpha)}$.*

Proof. $m'(\omega^{d_r(\alpha)}, \omega^\alpha) = \omega^{a'(d_r(\alpha),\alpha)} = \omega^{d_{r+1}(\alpha)}$, by Lemma 4.1. ∎

Proposition 4.3. *Let ω^γ be an infinite gamma number, and let $r \geq 1$. Then $C_r(\omega^\gamma) = \omega^{d_r(\gamma)}$.*

Proof. By induction on r. The result holds for $r = 1$. Assume $r \geq 1$, and the result holds for r. Then

$$\begin{aligned} C_{r+1}(\omega^\gamma) &= m'(C_r(\omega^\gamma), \omega^\gamma) \quad \text{by the Second Gamma Lemma} \\ &= m'(\omega^{d_r(\gamma)}, \omega^\gamma) \\ &= \omega^{d_{r+1}(\gamma)} \quad \text{by Lemma 4.2. ∎.} \end{aligned}$$

Lemma 4.4. *For all α and all $r \geq 0$, $C_r(\alpha) \leq \alpha^r$.*

Proof. By induction on r. The result is clear for $r = 0$. Assume $r \geq 0$, and the result holds for r. Then $C_{r+1}(\alpha) = \sum\limits_{\beta<\alpha} C_r(\beta) \leq \sum\limits_{\beta<\alpha} \beta^r \leq \sum\limits_{\beta<\alpha} \alpha^r = \alpha^r\alpha = \alpha^{r+1}$. ∎

Proposition 4.5. *Let $\bar{\alpha}$ be a typical limit ordinal, and let $r \geq 0$. Then*

$$C_r(\bar{\alpha}) = \bar{\alpha}^r.$$

Proof. The result is true for $r = 0$, so we may assume $r \geq 1$. Write $\bar{\alpha} = \omega^{\alpha_1} + \bar{\beta}$, where $\bar{\beta} \leq \omega^{\alpha_1}n$ for some n. Since $\bar{\alpha}$ is typical, we have $C_s(\omega^{\alpha_1}) = \omega^{\alpha_1 s}$ for all s. Now $\bar{\alpha} \leq \omega^{\alpha_1}(n+1)$, so $C_{r-1}(\bar{\alpha}) \leq C_{r-1}(\omega^{\alpha_1}(n+1)) = (\omega^{\alpha_1}(n+1))^{r-1} = \omega^{\alpha_1(r-1)}(n+1)$. Thus the Degree Lemma applies, and we have

$$\begin{aligned} C_r(\bar{\alpha}) &= C_r(\omega^{\alpha_1} + \bar{\beta}) \\ &= C_r(\omega^{\alpha_1}) + C_{r-1}(\omega^{\alpha_1})\bar{\beta} \\ &= \omega^{\alpha_1} + \omega^{\alpha_1(r-1)}\bar{\beta} \\ &= \bar{\alpha}^r. \text{ ∎} \end{aligned}$$

§5. Propagation results.

In this section, we investigate cases in which, under certain hypotheses, properties of a function f lead to similar properties of S_f. If S_f satisfies the same hypotheses, then we can apply our results again: the properties "propagate."

We first investigate a property related to the Main Lemma.

Let f be a nondecreasing function, and let α, β be ordinals. We say that (α, β) is an *ML pair for* f if $S_f(\alpha+\beta) = S_f(\alpha) + f(\alpha)\beta$ – that is, if the conclusion of the Main Lemma holds. It turns out that (α, β) is often an ML pair for f even when the hypotheses of the Main Lemma are not satisfied.

When β is an infinite gamma number, we have a necessary and sufficient condition for (α, β) to be an ML pair for f.

Proposition 5.1. *Let f be nondecreasing, α an ordinal, and γ an infinite gamma number. Then (α, γ) is an ML pair for f iff $f(\alpha)\gamma = f(\alpha+\beta)\gamma$ for all $\beta < \gamma$.*

Proof. By Corollary 2.6, we have that $S_f(\alpha+\gamma) = S_f(\alpha) + \sup_{\beta<\gamma}(f(\alpha+\beta)\gamma)$. Since f is nondecreasing, $f(\alpha)\gamma$ is the smallest element of the set of ordinals of which we are taking the supremum. Thus the supremum equals $f(\alpha)\gamma$ iff the set has no other elements. ∎

We use the above result to obtain a second criterion.

Proposition 5.2. *Let f be nondecreasing, α an ordinal for which $f(\alpha) > 0$, and γ an infinite gamma number. Write $\gamma = \Delta_1 \cdots \Delta_k$, where $\Delta_1 \geq \ldots \geq \Delta_k$ are delta numbers. Then (α, γ) is an ML pair for f iff $f(\alpha+\beta) < f(\alpha)\Delta_1$ for all $\beta < \gamma$.*

Proof. Suppose that $f(\alpha+\beta) < f(\alpha)\Delta_1$ for all $\beta < \gamma$. Then for $\beta < \gamma$ we have $f(\alpha+\beta) < f(\alpha)\beta'$ for some $\beta' < \Delta_1$. Since $\beta'\Delta_1 = \Delta_1$, this yields $f(\alpha+\beta)\gamma \leq f(\alpha)\beta'\gamma = f(\alpha)\gamma$. As $f(\alpha)\gamma \leq f(\alpha+\beta)\gamma$ is clear, we have that (α, γ) is an ML pair for f by Proposition 5.1.

Conversely, if $f(\alpha+\beta) \geq f(\alpha)\Delta_1$ for some $\beta < \gamma$, then $f(\alpha+\beta)\gamma \geq f(\alpha)\Delta_1\gamma$. Thus $f(\alpha+\beta)\gamma > f(\alpha)\gamma$ and, again by Proposition 5.1, (α, γ) is not an ML pair for f. ∎

The previous result can be sharpened when f is strictly increasing and continuous at $\alpha + \gamma$.

Corollary 5.3. *Suppose, in addition, that $f(\alpha+\beta) < f(\alpha+\gamma)$ for $\beta < \gamma$, and f is continuous at $\alpha+\gamma$. Then (α, γ) is an ML pair for f iff $f(\alpha+\gamma) \leq f(\alpha)\Delta_1$.* ∎

We are interested in determining when the fact that (α, γ) is an ML pair for f implies that (α, γ) is an ML pair for S_f. In general, we have the following:

Theorem 5.4. *Let f be nondecreasing, α an ordinal such that $f(\beta) > 0$ for some $\beta < \alpha$, and γ an infinite gamma number. Write $\gamma = \Delta_1 \cdots \Delta_k$ where $\Delta_1 \geq \ldots \geq \Delta_k$ are delta numbers. Suppose that (α, γ) is an ML pair for f. Then (α, γ) is an ML pair for S_f iff $f(\alpha)\gamma \leq S_f(\alpha)\Delta_1$.*

Proof. We see that Corollary 5.3 applies to S_f. Thus (α, γ) is an ML pair for S_f iff $S_f(\alpha + \gamma) \leq S_f(\alpha)\Delta_1$. Since (α, γ) is an ML pair for f, we have $S_f(\alpha + \gamma) = S_f(\alpha) + f(\alpha)\gamma$. Thus the condition $f(\alpha)\gamma \leq S_f(\alpha)\Delta_1$ is clearly necessary for (α, γ) to be an ML pair for S_f. If the condition holds, then

$$S_f(\alpha + \gamma) = S_f(\alpha) + f(\alpha)\gamma \leq S_f(\alpha) + S_f(\alpha)\Delta_1 = S_f(\alpha)(1 + \Delta_1) = S_f(\alpha)\Delta_1,$$

so (α, γ) is an ML pair for S_f, as required. ∎

We use the above result for the following:

Theorem 5.5. *Let f be nondecreasing, $\bar{\alpha}$ an ordinal such that $f(\beta) > 0$ for some $\beta < \bar{\alpha}$, Δ an infinite delta number, and m a positive integer. Write $\bar{\alpha} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ in WCNF. Suppose that $f(\bar{\alpha}) < \Delta^\omega$, that $\Delta^\omega \leq \omega^{\alpha_k}$, and that f is continuous at $\bar{\alpha}$. Then: if $(\bar{\alpha}, \Delta^m)$ is an ML pair for f, $(\bar{\alpha}, \Delta^m)$ is also an ML pair for S_f.*

Proof. By Theorem 5.4, $(\bar{\alpha}, \Delta^m)$ is an ML pair for S_f iff $f(\bar{\alpha})\Delta^m \leq S_f(\bar{\alpha})\Delta$. We shall show $f(\bar{\alpha})\Delta^{m-1} \leq S_f(\bar{\alpha})$. Using Corollary 2.6 and the fact that $\omega^{\alpha_k} \geq \Delta^m$, we have

$$S_f(\bar{\alpha}) \geq \sup_{\beta < \bar{\alpha}} (f(\beta)\omega^{\alpha_k}) \geq \sup_{\beta < \bar{\alpha}} (f(\beta)\Delta^m).$$

If $f(\beta) = f(\bar{\alpha})$ for some $\beta < \bar{\alpha}$, then $\sup\limits_{\beta < \bar{\alpha}} (f(\beta)\Delta^m) = f(\bar{\alpha})\Delta^m$, and we have our result. Otherwise, $\sup\limits_{\beta < \bar{\alpha}} (f(\beta)\Delta^m) = m'(f(\bar{\alpha}), \Delta^m) \geq f(\bar{\alpha})\Delta^{m-1}$ by Proposition 2.12. ∎

The above theorem is not quite a propagation result, as the hypothesis "$f(\bar{\alpha}) < \Delta^\omega$" may not be preserved – i.e., we may not have $S_f(\bar{\alpha}) < \Delta^\omega$. However, if we assume in addition that $\bar{\alpha} < \Delta^\omega$ we obtain what we want:

Lemma 5.6. *Let f be nondecreasing, and let α, β be ordinals, $\beta \geq 2$. If $\alpha < \beta^\omega$ and $f(\alpha) < \beta^\omega$, then $S_f(\alpha) < \beta^\omega$.*

Proof. Say $\alpha < \beta^n$, $f(\alpha) < \beta^m$ where n, m are finite. Then $S_f(\alpha) = \sum\limits_{x < \alpha} f(x) \leq \sum\limits_{x < \beta^n} \beta^m = \beta^m \beta^n = \beta^{m+n} < \beta^\omega$. ∎

Our next propagation result will be applied initially with g being the identity function.

Proposition 5.7. *Let g be nondecreasing, continuous, and strictly increasing past some finite n. Let Δ be an infinite delta number with $g(\Delta) = \Delta$. Write $f = S_g$. Then $f(\Delta) = \Delta$. If $f(\Delta + \bar{\beta}) = \Delta g(\bar{\beta})$ for all $\bar{\beta} < \Delta\omega$, then $S_f(\Delta + \bar{\beta}) = \Delta f(\bar{\beta})$ for all $\bar{\beta} < \Delta\omega$.*

Proof. We have $f(\Delta) = m'(g(\Delta), \Delta) = m'(\Delta, \Delta) = \Delta$, using Lemma 2.8. Write $\bar{\beta} = \omega^{\alpha_2} + \ldots + \omega^{\alpha_k}$ in WCNF. Note $\omega^{\alpha_2} \leq \Delta$, as $\bar{\beta} < \Delta\omega$. By Lemma 2.9 we have

$$
\begin{aligned}
S_f(\Delta + \bar{\beta}) = m'(f(\Delta), \Delta) + m'(f(\Delta + \omega^{\alpha_2}), \omega^{\alpha_2}) + \ldots + \\
m'(f(\Delta + \omega^{\alpha_2} + \ldots + \omega^{\alpha_k}), \omega^{\alpha_k}) \\
&= \Delta + m'(\Delta g(\omega^{\alpha_2}), \omega^{\alpha_2}) + \ldots + m'(\Delta g(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k}), \omega^{\alpha_k}) \\
&\qquad \text{by the above, and the assumption} \\
&= \Delta + \Delta[m'(g(\omega^{\alpha_2}), \omega^{\alpha_2}) + \ldots + m'(g(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k}), \omega^{\alpha_k}] \\
&\qquad \text{by the observation after Lemma 2.8} \\
&= \Delta + \Delta f(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k}) \quad \text{by Lemma 2.9 again} \\
&= \Delta f(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k}) \quad \text{since } f(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k}) \text{ is infinite.} \quad \blacksquare
\end{aligned}
$$

The following will be applied initially with f being the identity function.

Proposition 5.8. *Let f be nondecreasing, continuous, and strictly increasing past some finite n. Let Δ be an infinite delta number. Let $s \geq 1$, $m \geq 2$. Assume $f(\Delta^m s) < \Delta^\omega$. Suppose $\bar{\alpha} = \Delta^m s + \omega^{\alpha_{s+1}} + \ldots + \omega^{\alpha_k}$ is WCNF, with $\Delta^m > \omega^{\alpha_{s+1}}$. Then:*

(1) $S_f(\Delta^m s) < \Delta^\omega$.

(2) If $f(\Delta^m s + \omega^{\alpha_{s+1}})\omega^{\alpha_{s+1}} = f(\Delta^m s)\omega^{\alpha_{s+1}}$ *then*

$$
\begin{aligned}
S_f(\Delta^m_s + \omega^{\alpha_{s+1}}) &= S_f(\Delta^m_s) + f(\Delta^m_s)\omega^{\alpha_{s+1}}, \quad \text{and} \\
S_f(\Delta^m s + \omega^{\alpha_{s+1}})\omega^{\alpha_{s+1}} &= S_f(\Delta^m s)\omega^{\alpha_{s+1}}.
\end{aligned}
$$

(3) Suppose $k > s + 1$. If, in addition, $\deg f(\Delta^m s + \omega^{\alpha_{s+1}}) = \deg f(\bar{\alpha})$, *then*

$$\deg(S_f(\Delta^m s + \omega^{\alpha_{s+1}})) = \deg S_f(\bar{\alpha}),$$

and we have:

$$S_f(\bar{\alpha}) = S_f(\Delta^m s) + f(\Delta^m s)\omega^{\alpha_{s+1}} + f(\Delta^m s + \omega^{\alpha_{s+1}})(\omega^{\alpha_{s+2}} + \ldots + \omega^{\alpha_k}).$$

Proof. (1) follows from Lemma 5.6.

(2) The first assertion follows by the Main Lemma 2.2. By Lemma 2.9 we have

$$S_f(\Delta^m s) = m'(f(\Delta^m), \Delta^m) + m'(f(\Delta^m 2), \Delta^m) + \ldots + m'(f(\Delta^m s), \Delta^m),$$

so $m'(f(\Delta^m s), \Delta^m) \le S_f(\Delta^m s) \le m'(f(\Delta^m s), \Delta^m)s$, and we have

$$\begin{aligned} m'(f(\Delta^m s, \Delta^m)\omega^{\alpha_{s+1}} &= S_f(\Delta^m s)\omega^{\alpha_{s+1}} \\ &\le S_f(\Delta^m s + \omega^{\alpha_{s+1}})\omega^{\alpha_{s+1}} \\ &\le [m'(f(\Delta^m s), \Delta^m)s + f(\Delta^m s)\omega^{\alpha_{s+1}}]\omega^{\alpha_{s+1}}. \end{aligned}$$

Writing $\bar{x} = f(\Delta^m s)$, we see that it is enough to show

$$m'(\bar{x}, \Delta^m)\omega^{\alpha_{s+1}} = [m'(\bar{x}, \Delta^m)s + \bar{x}\omega^{\alpha_{s+1}}]\omega^{\alpha_{s+1}}.$$

By assumption, we have $\bar{x} < \Delta^\omega$. By Remark 2.13, we have that either $m'(\bar{x}, \Delta^m) = \bar{x}\Delta^m$, or $\bar{x} = \Delta^k$ and $m'(\bar{x}, \Delta^m) = \bar{x}\Delta^{m-1}$. If $m'(\bar{x}, \Delta^m) = \bar{x}\Delta^m$, then $\deg(\bar{x}\omega^{\alpha_{s+1}}) \le \deg m'(\bar{x}, \Delta^m)$, and our result is clear. The same argument applies in the second case, if $\omega^{\alpha_{s+1}} \le \Delta^{m-1}$. Otherwise, $\omega^{\alpha_{s+1}} = \Delta^{m-1}\beta$, with $\beta < \Delta$, and we have

$$\begin{aligned} [m'(\bar{x}, \Delta^m)s + \bar{x}\omega^{\alpha_{s+1}}]\omega^{\alpha_{s+1}} &= [\bar{x}\Delta^{m-1} + \bar{x}\Delta^{m-1}\beta]\Delta^{m-1}\beta \\ &= \bar{x}\Delta^{m-1}\beta\Delta^{m-1}\beta \\ &= \bar{x}\Delta^{m-1}\Delta^{m-1}\beta \\ &= m'(\bar{x}, \Delta^m)\omega^{\alpha_{s+1}}, \end{aligned}$$

so our result holds in this case as well.

(3) From the Degree Lemma 2.3, we have

$$S_f(\bar{\alpha}) = S_f(\Delta^m s + \omega^{\alpha_{s+1}}) + f(\Delta^m s + \omega^{\alpha_{s+1}}) \cdot (\omega^{\alpha_{s+2}} + \ldots + \omega^{\alpha_k}).$$

¿From the above,

$$S_f(\Delta^m s + \omega^{\alpha_{s+1}}) = S_f(\Delta^m s) + f(\Delta^m s)\omega^{\alpha_{s+1}} = S_f(\Delta^m s) + f(\Delta^m s + \omega^{\alpha_{s+1}})\omega^{\alpha_{s+1}}.$$

The result is clear. ∎

§6. Main results for C_r.

Recall that we showed in §4 that $C_r(\bar{\alpha}) = \bar{\alpha}^r$ if $\bar{\alpha}$ is typical, and that $C_r(\omega^\gamma) = \omega^{d_r(\gamma)}$ for an infinite gamma number ω^γ. In this section, we use our propagation results to compute C_r at any limit ordinal $\bar{\alpha}$, in terms of the WCNF $\bar{\alpha} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ of $\bar{\alpha}$, $k \geq 2$. Note that $C_0(\bar{\alpha}) = 1$, $C_1(\bar{\alpha}) = \bar{\alpha}$, and $C_2(\bar{\alpha}) = S_X(\bar{\alpha}) = \omega^{d_2(\alpha_1)} + \omega^{\alpha_1}(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k})$ from §3.

We first consider the case in which ω^{α_1} is a delta number.

Proposition 6.1. *Suppose that ω^{α_1} is a delta number. Then for $r \geq 2$ we have $C_r(\bar{\alpha}) = \omega^{\alpha_1} C_{r-1}(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k})$.*

Proof. The result holds for $r = 2$, as the term $\omega^{d_2(\alpha_1)} = \omega^{\alpha_1}$ is absorbed. We apply Proposition 5.7, with $g = C_1 =$ the identity function, obtaining $C_3(\bar{\alpha}) = \omega^{\alpha_1} C_2(\omega^{\alpha_2} + \ldots + \omega^{\alpha_k})$. Continuing in the same manner, we obtain our result for all r. ∎

Example 6.2. Using Proposition 6.1, we obtain

$$\begin{aligned} C_3(\omega^\omega + \omega^2) &= \omega^\omega C_2(\omega^2) \\ &= \omega^\omega \omega^3 = \omega^{\omega+3} \end{aligned}$$

However, $C_3(\omega^\omega) + C_2(\omega^\omega)\omega^2 = \omega^\omega + \omega^\omega \omega^2 = \omega^{\omega+2}$. So $(\omega^\omega, \omega^2)$ is not an ML pair for C_2, even though it is for C_1. In fact, with $f = C_2$, $\alpha = \omega^\omega$, $\beta = \omega^2$, both of the inequalities of Lemma 2.1 are strict.

We next consider the case $\bar{\alpha} = \Delta^m s$, where Δ is an infinite delta number.

Proposition 6.3. *Let Δ be an infinite delta number, and let $m, s \geq 1$. Then for $r \geq 1$, we have*

$$C_r(\Delta^m s) = \begin{cases} \Delta^{rm-(r-s)}, & \text{if } 1 \leq s \leq r \\ \Delta^{rm}(s - r + 1), & \text{if } s \geq r. \end{cases}$$

Proof. As the case $s = 1$ is known, we consider the case in which $s > 1$. Using Lemma 4.4 and Theorem 5.5 (initially with $f = C_1$), we have that $(\Delta^m(s-1), \Delta^m)$ is an ML pair for C_r, all $r \geq 1$. We shall establish our result by induction on r. The case $r = 1$ is clear. Assume that $r \geq 1$, and that the result holds for r. We establish the result for $r + 1$ by a secondary induction on s. As noted above, the result holds for $s = 1$. Suppose that $s > 1$ and that the result holds for $s - 1$. We have

$$\begin{aligned} C_{r+1}(\Delta^m s) &= C_{r+1}(\Delta^m(s-1) + \Delta^m) \\ &= C_{r+1}(\Delta^m(s-1)) + C_r(\Delta^m(s-1))\Delta^m. \end{aligned}$$

We first suppose that $1 \leq s - 1 \leq r$. Then $1 \leq s - 1 \leq r + 1$ holds as well, we have

$$\begin{aligned} C_{r+1}(\Delta^m(s-1)) + C_r(\Delta^m(s-1))\Delta^m &= \Delta^{(r+1)m-((r+1)-(s-1))} + \Delta^{rm-(r-(s-1))}\Delta^m \\ &= \Delta^{(r+1)m-(r+2-s)} + \Delta^{(r+1)m-(r+1-s)} \\ &= \Delta^{(r+1)m-(r+1-s)}, \end{aligned}$$

as required. Otherwise, $s-1>r$, so $s-1\geq r+1$, and we have

$$\begin{aligned}C_{r+1}(\Delta^m(s-1))&+C_r(\Delta^m(s-1))\Delta^m\\&=\Delta^{(r+1)m}((s-1)-(r+1)+1)+\Delta^{rm}((s-1)-r+1)\Delta^m\\&=\Delta^{(r+1)m}(s-r-1)+\Delta^{(r+1)m}\\&=\Delta^{(r+1)m}(s-(r+1)+1),\end{aligned}$$

as required. ∎

We now add on additional terms.

Proposition 6.4. *Suppose $\bar{\alpha}=\Delta^m s+\omega^{\alpha_{s+1}}+\omega^{\alpha_{s+2}}+\ldots+\omega^{\alpha_k}$ in WCNF, where Δ is an infinite delta number, $m\geq 2$, $s\geq 1$, and $\Delta^m>\omega^{\alpha_{s+1}}$. Then $(\Delta^m s,\omega^{\alpha_{s+1}})$ is an ML pair for C_r, all $r\geq 1$. If $k>s+1$, then $(\Delta^m s+\omega^{\alpha_{s+1}},\omega^{\alpha_{s+2}}+\ldots+\omega^{\alpha_k})$ is an ML pair for C_r, all $r\geq 1$.*

Proof. When $r=1$, the first assertion follows by the Main Lemma, and the second follows by the Degree Lemma. We apply Proposition 5.8 recursively, beginning with $f=C_1$, to obtain that the hypotheses of those Lemmas are satisfied for each C_r, $r\geq 1$. This gives our result. ∎

We combine our results in the following theorem, which allows rapid evaluation of $C_r(\bar{\alpha})$ in terms of the WCNF of $\bar{\alpha}$.

Theorem 6.5. *Let $\bar{\alpha}=\omega^{\alpha_1}+\omega^{\alpha_2}+\ldots+\omega^{\alpha_k}$ in WCNF. Let $r\geq 2$.*

I. If ω^{α_1} is not a power of a delta number, then $C_r(\bar{\alpha})=\bar{\alpha}^r$.

II. Suppose $\omega^{\alpha_1},\ldots,\omega^{\alpha_k}$ are delta numbers $\Delta_1,\ldots,\Delta_k$. Then

$$C_r(\bar{\alpha})=\begin{cases}\Delta_1\cdots\Delta_{r-1}(\Delta_r+\ldots+\Delta_k), & \textit{if } r\leq k\\ \Delta_1\cdots\Delta_k, & \textit{if } r\geq k.\end{cases}$$

III. Suppose $\omega^{\alpha_1},\ldots,\omega^{\alpha_s}$ are delta numbers $\Delta_1,\ldots,\Delta_s$, but $\omega^{\alpha_{s+1}}$ is not a delta number. Then

$$C_r(\bar{\alpha})=\begin{cases}\Delta_1\Delta_2\cdots\Delta_{r-1}(\Delta_r+\ldots+\Delta_s+\omega^{\alpha_{s+1}}+\ldots+\omega^{\alpha_k}), & \textit{if } r\leq s\\ \Delta_1\cdots\Delta_s C_{r-s}(\omega^{\alpha_{s+1}}+\ldots+\omega^{\alpha_k}), & \textit{if } r>s.\end{cases}$$

IV. Suppose $\omega^{\alpha_1}=\Delta^m$, Δ a delta number, $m\geq 2$.

(a) If $\omega^{\alpha_1}=\omega^{\alpha_2}=\omega^{\alpha_k}=\Delta^m$, then $C_r(\bar{\alpha})=C_r(\Delta^m k)$ is as given in Proposition 6.3.

(b) *Say* $\omega^{\alpha_1} = \omega^{\alpha_2} = \omega^{\alpha_s} = \Delta^m$, $\Delta^m > \omega^{\alpha_{s+1}}$. *If* $k = s+1$, *then* $C_r(\bar\alpha) = C_r(\Delta^m s) + C_{r-1}(\Delta^m s)\omega^{\alpha_{s+1}}$. *If* $k > s+1$, *then we have*

$$C_r(\bar\alpha) = C_r(\Delta^m s + \omega^{\alpha_{s+1}}) + C_{r-1}(\Delta^m s + \omega^{\alpha_{s+1}})(\omega^{\alpha_{s+2}} + \ldots + \omega^{\alpha_k}).$$

Proof. I is Proposition 4.5. II and III follow from Proposition 6.1, and IV comes from Proposition 6.4. ■

Example 6.8. Usually, only a few applications of the Theorem are required to evaluate $C_r(\bar\alpha)$. We give a reasonably complicated example:

$$\begin{aligned}
C_8(\omega^\omega + \omega^{10}3 + \omega^5 + \omega^2) &= \omega^\omega C_7(\omega^{10}3 + \omega^5 + \omega^2) \quad \text{by (III)}\\
&= \omega^\omega[C_7(\omega^{10}3 + \omega^5) + C_6(\omega^{10}3 + \omega^5)\omega^2] \quad \text{by (IVb)}\\
&= \omega^\omega[C_7(\omega^{10}3) + C_6(\omega^{10}3)\omega^5 + [C_6(\omega^{10}3) + C_5(\omega^{10}3)\omega^5]\omega^2]\\
&\qquad \text{by (IVb)}\\
&= \omega^\omega[\omega^{66} + \omega^{57}\omega^5 + [\omega^{57} + \omega^{48}\omega^5]\omega^2] \quad \text{by (IVa)}\\
&= \omega^{\omega+66} + \omega^{\omega+62} + \omega^{\omega+59}. \quad ■
\end{aligned}$$

References

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