Fibered faces, Penner sequences, and handlebody mapping classes

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Abstract

In this paper we describe special loci on the fibered face of a 3-manifold that correspond to generalized Penner sequences of mapping classes and to handlebody mapping classes. As an application, we show that the logarithm of the minimum dilatation of handlebody mapping classes on a closed genus g surface behaves asymptotically like the inverse of the genus. We also show that the minimum dilatation of mapping classes with homological dilatation equal to one shares this asymptotic behavior.

1 Introduction

In [Pen], R. Penner established a technique for constructing a sequence of pseudo-Anosov mapping classes $\phi_g : S_g \to S_g$, where S_g is a closed oriented surface of genus g, whose dilatations satisfy

$$\lambda(\phi_g)^g \leq C \tag{1}$$

for some constant C. These mapping classes have the property that up to composition by a periodic map, ϕ_g is supported on a subsurface of fixed type, and its restriction to this subsurface is a fixed mapping class.

Thurston showed that the set of pseudo-Anosov mapping classes on orientable surfaces of finite type can be partitioned into subsets corresponding to rational points on special regions called fibered faces in the first cohomology of hyperbolic 3-manifolds. Convergent sequences of rational points on fibered faces with increasing denominators correspond to mapping classes with increasing topological Euler characteristic and converging normalized dilatation. For such sequences the logarithm of the dilatation behaves asymptotically like the inverse of the Euler characteristic. Thus, fibered faces are a natural source of sequences of mapping classes that satisfy inequalities of the type given in (1). This leads to the question:

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Question 1.1 *Let $\phi_n : S_n \to S_n$ be a sequence of pseudo-Anoosv mapping classes satisfying*

$$\lambda(\phi_n)^{|\chi(S_n)|} < C$$

for some constant C. What further conditions are necessary on the sequence to ensure that they correspond to a sequence on a fibered face?

In this paper, we define a generalized Penner sequence, and show that each Penner sequence corresponds to a convergent sequence on a fibered face. A version of Penner sequences was also studied by M. Bauer in [Bau]. In [Val] Valdivia defined a more restrictive version of Penner sequences than the one given in this paper, and shows that they arise from a single 3-manifold by analyzing the fundamental group and applying the Mostow-Prasad rigidity theorem. Our definition of Penner sequence does not require the mapping classes to be of Penner-type (i.e., a product of a pair of Dehn twists on multicurves), and our proof is constructive.

We next apply Penner sequences to the minimum dilatation problem for pseudo-Anosov handlebody mapping classes. A mapping class ϕ on a surface S is a handlebody mapping class if there is an identification of S with the boundary of a handlebody H so that ϕ extends to H. Let δ_g^H be the minimum dilatation of a pseudo-Anosov handlebody mapping class of genus g. Our first application is the following.

Theorem 1.2 *The minimum dilatation of pseudo-Anosov handlebody mapping classes satisfies*

$$\log(\delta_g^H) \asymp \frac{1}{g}.$$

In [FLM], Farb Leininger and Margalit proved that the dilatations of pseudo-Anosov mapping classes in the Torelli subgroup of $\mathrm{Mod}(S_g)$ is bounded from below by a constant $c > 1$. Thus, mapping classes that act trivially on first homology cannot have small dilatation. If we look, however, at mapping classes whose action on first homology has spectral radius equal to one, we get a different story.

Theorem 1.3 *The minimum dilatation δ_g^Δ of pseudo-Anosov mapping classes whose homological dilatation is one satisfies*

$$\log(\delta_g^\Delta) \asymp \frac{1}{g}.$$

This paper is organized as follows. Section 2 contains background on the minimum dilatation problem. In Section 3 we recall properties of fibered cones and fibered faces. We study the locus of points on a fibered face corresponding to handlebody mapping classes in Section 4. In Section 5, we describe a procedure for constructing generalized Penner sequences of mapping classes on fibered faces, and prove some properties. Section 6 contains two examples of Penner sequences, one is the Penner's original example, and the other is a sequence of handlebody mapping classes that has the properties required to prove Theorem 1.2 and Theorem 1.3.

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2 Small dilatation problem

Let S_g be a closed oriented surface of genus g, and let $\mathrm{Mod}(S_g)$ be the group of isotopy classes of self-homeomorphisms of S_g, called the *mapping class group of* S_g. By the Nielsen-Thurston classification, elements of $\mathrm{Mod}(S_g)$ are either periodic, reducible, or pseudo-Anosov. Each pseudo-Anosov mapping class ϕ has a *dilatation* $\lambda(\phi) > 1$ which measures the uniform rate of stretching and contraction of ϕ along an associated pair of ϕ-invariant transverse measured singular foliations. The set of logarithms of dilatations for a fixed S_g is the length spectrum for closed Teichmüller geodesics on the moduli space of S_g, and is a discrete subset of the real numbers strictly greater than one. The dilatations are also algebraic Perron units of degree bounded by $6g - 6$. These and further properties of pseudo-Anosov mapping classes are developed in [Thu], [FS] and [CB]. In [Pen] R. Penner studied the minimum dilatations $\delta(\mathrm{Mod}(S_g))$ as a function of genus and showed

$$\log(\delta(\mathrm{Mod}(S_g))) \asymp \frac{1}{g}.$$

Penner's result has inspired many further questions about minimum dilatations, including the following.

Question 2.1 *For which sequences of subgroups $G_g \subset Mod(S_g)$ does*

$$\log(\delta(G_g)) \asymp \frac{1}{g}$$

hold?

We say that a sequence of subgroups $G_g \subset \mathrm{Mod}(S_g)$ *supports small dilatations* if the answer to Question 2.1 is affirmative.

Examples of G_g that do not support small dilatation can be found in [FLM], [BL]. Mapping class groups that commute with a hyperbolic involution do support small dilatations (see [HK] and [Tsa]). In this paper, we show that the handlebody subgroup of $\mathrm{Mod}(S_g)$ (Theorem 1.2) and the subgroup of $\mathrm{Mod}(S_g)$ consisting of mapping classes with homological dilatation equal to 1 (Theorem 1.3) both support small dilatations.

Let S_g be a genus g surface, H_g a handlebody of genus g, and $p : S_g \to H_g$ an identification of S_g with the boundary of H_g. Let $\mathrm{Mod}_H(S_g, p) \subset \mathrm{Mod}(S_g)$ be the subgroup consisting of mapping classes $\phi : S_g \to S_g$ so that for some mapping class $\phi_h : H_g \to H_g$, the diagram

$$\begin{tikzcd} S_g \arrow[r, "\phi"] \arrow[d, "p"] & S_g \arrow[d, "p"] \\ H_g \arrow[r, "\phi_h"] & H_g \end{tikzcd}$$

commutes.

The subgroup $\mathrm{Mod}_H(S_g, p) \subset \mathrm{Mod}(S_g)$ is called a *handlebody subgroup* of S_g, and has been studied by several authors. For example, B. Wajnryb found a finite presentation of the handlebody subgroup in [Waj]. Although $\mathrm{Mod}_H(S_g, p)$ depends on p, any two such subgroups can be obtained one from the other by conjugating by an element of $\mathrm{Mod}(S_g)$. Hence $\delta(\mathrm{Mod}_H(S_g, p))$ does not depend on the choice of p. H. Masur showed that the limit set of the handlebody subgroup has measure zero in Thurston's sphere of measured foliations [Mas]. Thus, while the handlebody subgroups are small in a measure theoretic sense, Theorem 1.2 shows that they are general enough to support small dilatations.

3 Mapping classes and fibered cones.

Thurston's theory of fibered faces provides a way of associating mapping classes to primitive integral elements in conical regions, called fibered cones, inside $H^1(M;\mathbb{Z})$, where M is a 3-manifold [Thu]. McMullen introduced a method for analyzing the mapping classes defined by a fibered cone using multicyclic coverings of M [McM1].

Let $\mathcal{P}$ be the collection of all pseudo-Anosov mapping classes on an oriented surfaces of finite type. Given a mapping class $\phi \in \mathrm{Mod}(S)$, let $M(\phi)$ be the *mapping torus* of ϕ, defined by

$$M = S \times [0,1]/(x,1) \sim (\phi(x), 0).$$

Given a hyperbolic 3-manifold M, let

$$\Phi(M) = \{\phi \in \mathcal{P} \ : \ M = M(\phi)\},$$

and let

$$\Psi(M) = \{\psi : M \to S^1 \ : \ \psi \text{ is a fibration}\}.$$

Since the mapping tori of pseudo-Anosov maps are always hyperbolic, for each M there is a canonical one to one correspondence between $\Psi(M)$ and $\Phi(M)$, and $\mathcal{P}$ partitions into a disjoint union of $\Phi(M)$. The following well-known proposition gives a way to identify fibrations of M and their monodromy using coverings.

Proposition 3.1 *The mapping class ϕ is the monodromy of a fibration $\psi : M \to S^1$ if and only if the $\mathbb{Z}$-covering of M corresponding to the map $\pi_1(M) \to \mathbb{Z}$ obtained by composing ψ with the Hurewicz map is homeomorphic to $S \times \mathbb{R}$, and the covering automorphism group is generated by T_ϕ, where*

$$\begin{aligned} T_\phi : S \times \mathbb{R} &\to S \times \mathbb{R} \\ (s, u) &\mapsto (\phi(s), u - 1). \end{aligned}$$

Thurston showed how to partition $\Psi(M)$ further in a natural way. He defines a norm, known as the *Thurston norm* $|| \ ||_T$ on $H^1(M;\mathbb{R})$ considered as a vector space over $\mathbb{R}$. The norm of a general integral element ψ is the minimum absolute value of the topological Euler characteristic of a surface representing the dual of ψ after throwing away components of non-negative curvature. When $\psi \in H^1(M;\mathbb{Z})$ is fibered with fiber S, then

$$||\psi||_T = |\chi(S)|.$$

The *Thurston norm ball* is the set of elements of $H^1(M;\mathbb{Z})$ with Thurston norm less than or equal to one.

Theorem 3.2 (Thurston) *The Thurston norm ball is the convex hull of a finite set of integral points. For every top dimensional face F of the Thurston norm ball, let $C_F = F \cdot \mathbb{R}^+$ be the corresponding cone. Then*

$$\Psi(M, F) = C_F \cap \Psi(M)$$

is empty, or it equals $C_F \cap H^1(M;\mathbb{Z})$.

If $\Psi(M,F)$ is nonempty, we say F is a *fibered face*, and C_F is a *fibered cone.* Let $\Phi(M,F) \subset \mathcal{P}$ be the collection of monodromies of $\Psi(M,F)$. When F is a fibered face, there is a natural way to associate $\Phi(M,F)$ with rational points on F. To each rational point on a fibered face there is a corresponding primitive integral element (whose coordinates are relatively prime) on the ray through the rational point. This determines a fibration of M over S^1 with connected fibers S and a monodromy in $\Phi(M,F) \cap \mathrm{Mod}(S)$.

Given a fibered face F, define the *normalized dilatation* of $\psi : M \to S^1$ with monodromy ϕ to be

$$\overline{\lambda}(\psi) = \lambda(\phi)^{||\psi||_T}.$$

We will sometimes also write $\overline{\lambda}(\phi) = \overline{\lambda}(\psi)$.

Theorem 3.3 (Fried [Fri]) *The normalized dilatation extends to a continuous map on the C_F which is constant on rays through the origin, has a unique minimum in the interior of F, and goes to infinity toward the boundary of F.*

See also McMullen [McM1] for an alternate proof.

Corollary 3.4 *If $G_g \subset Mod(S_g)$ is a sequence of subsets, and*

$$G_g \cap (F_c \cdot \mathbb{R})$$

is nonempty for all g, where F_c is a compact subset of F, then G_g supports small dilatations.

Corollary 3.5 *Let $\psi \in \Psi(M,F)$, and let ψ_n be a sequence elements of $\Psi(M,F)$ whose projections in F converge to the projection of ψ. Let ϕ_n be the monodromy of ψ_n, and ϕ the monodromy of ψ. Then*

$$\lim_{n\to\infty} \overline{\lambda}(\phi_n) = \overline{\lambda}(\phi).$$

The individual fibrations of a 3-manifold M can be studied by associating them to cyclic coverings M (cf. [McM1], §10). Fix a mapping class $\phi : S \to S$, let M be the mapping torus of ϕ, and let $\psi : M \to S^1$ be the corresponding fibration of M over the circle. Let $\beta_1, \dots, \beta_k$ be elements of $H^1(S;\mathbb{Z})^{\phi-\mathrm{inv}}$ that form a basis for $H^1(S;\mathbb{R})^{\phi-\mathrm{inv}}$. Since M is a quotient space of $S \times I$, $H^1(S;\mathbb{R})^{\phi-\mathrm{inv}}$ determines a linear subspace of $H^1(M;\mathbb{R})$. Furthermore, $\{\beta_1, \ldots, \beta_k, \psi\}$ forms a basis of $H^1(M;\mathbb{R})$.

Let

$$\rho_{\underline{\beta}} : \widetilde{S} \to S$$

be the unbranched multi-cyclic covering defined by

$$\underline{\beta} = (\beta_1, \ldots, \beta_k) : \pi_1(S) \to \mathbb{Z}^k.$$

Then the maximal abelian covering $\rho : \widetilde{M} \to M$ satisfies a commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r] \arrow[ddr, "\rho"'] & \widetilde{S} \times \mathbb{R} \arrow[d, "\rho_{\underline{\beta}}"] \\ & S \times \mathbb{R} \arrow[d] \\ & M, \end{tikzcd}$$

where $\widetilde{M} \to \widetilde{S} \times \mathbb{R}$ is a homeomorphism.

Let $\alpha : M \to S^1$ be an element of $\Psi(M, F)$, and let K_α be the kernel of

$$\alpha_* : H_1(M : \mathbb{Z}) \to \mathbb{Z}.$$

Let $M_\alpha = M/K_\alpha$ be the quotient space.

Since α is a fibration, M_α is homeomorphic to $S_\alpha \times \mathbb{R}$ for some surface S_α.

Lemma 3.6 *The monodromy ϕ_α is the mapping class induced by any element $f_\alpha \in \alpha_*^{-1}(1)$. That is, the mapping class ϕ_α is the unique mapping class (up to choice of identification of M_α with $S_\alpha \times \mathbb{R}$) that satisfies the commutative diagram*

$$\begin{tikzcd} \widetilde{M} \arrow[r, "f_\alpha"] \arrow[d] & \widetilde{M} \arrow[d] \\ S_\alpha \times \mathbb{R} \arrow[r, "T_{\phi_\alpha}"] & S_\alpha \times \mathbb{R}. \end{tikzcd}$$

4 Handlebody locus

In this section, we identify linear slices of a fibered cone of a 3-manifold, so that the monodromy of the integral points are handlebody mapping classes. Using the notation of the previous section, we find sufficient conditions on α so that for some realization $p_\alpha : S_\alpha \to H_\alpha$ of S_α as the boundary of a handlebody H_α we have $\phi_\alpha \in \mathrm{Mod}_H(S_\alpha, p_\alpha)$.

Define $W_{\phi,p} \subset H^1(S;\mathbb{Z})$ to be the subspace consisting of $\beta : H_1(S) \to \mathbb{Z}$, where

(1) $\beta \in H^1(S;\mathbb{Z})^{\phi-\mathrm{inv}}$, and

(2) $\beta(K_p) = 0$, where K_p is the kernel of $p_* : H_1(S) \to H_1(H)$, or equivalently β factors through a map $\beta^h : H_1(H) \to \mathbb{Z}$, making the diagram

$$\begin{tikzcd} H_1(S) \arrow[r, "p_*"] \arrow[dr, "\beta"'] & H_1(H) \arrow[d, "\beta^h"] \\ & \mathbb{Z} \end{tikzcd}$$

commute.

Since p_* is surjective β^h is determined by β.

Theorem 4.1 *Let $\phi : S \to S$ be a handlebody mapping class, let $M = M(\phi)$ be the mapping torus of ϕ, let $\psi : M \to S^1$ be the corresponding fibration, and let $F \subset H^1(M;\mathbb{R})$ be the fibered face of $H^1(M:\mathbb{R})$ containing ψ. Let $V_{\phi,p} \subset H^1(M;\mathbb{R})$ be the subspace generated by $W_{\phi,p}$ and ψ. Then the monodromy of elements of $V_{\phi,p} \cap \Psi(M,F)$ are handlebody mapping classes.*

Proof. Let $\alpha \in V_{\phi,p}$. Write $\alpha = \beta + m\psi$, where $\beta \in W_{\phi,p}$. Since $\beta \in H^1(S;\mathbb{Z})$, it defines a cyclic covering $\rho_\beta : \widetilde{S}_\beta \to S$. Let ζ be a generator of the group of covering automorphisms. We have intermediate coverings of the maximal abelian covering $\widetilde{\rho} : \widetilde{M} \to M$ as shown in the diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\simeq"] \arrow[dr] \arrow[ddr] \arrow[dddr, "\widetilde{\rho}"'] & \widetilde{S} \times \mathbb{R} \arrow[d] \\ & \widetilde{S}_\beta \times \mathbb{R} \arrow[d] \\ & S \times \mathbb{R} \arrow[d] \\ & M. \end{tikzcd}$$

By property (1) of $W_{\phi,p}$, ϕ lifts to $\widetilde{\phi} : \widetilde{S}_\beta \to \widetilde{S}_\beta$, and induces a mapping class $\phi^h : H \to H$. Furthermore, we have a commutative diagram:

$$\begin{tikzcd} \widetilde{S}_\beta \arrow[loop left, "\widetilde{\phi}"] \arrow[r, "\widetilde{p}_\beta"] \arrow[d, "\rho_\beta"] & \widetilde{H}_\beta \arrow[loop right, "\widetilde{\phi}^h"] \arrow[d, "\rho^h_\beta"] \\ S \arrow[loop left, "\phi"] \arrow[r, "p"] & H \arrow[loop right, "\phi^h"] \end{tikzcd}$$

The map

$$\begin{array}{rcl} T_{\widetilde{\phi}} : \widetilde{S}_\beta \times \mathbb{R} & \to & \widetilde{S}_\beta \times \mathbb{R} \\ (s,t) & \mapsto & (\widetilde{\phi}(s), t-1). \end{array}$$

defines a covering automorphism of $\widetilde{S}_\beta \times \mathbb{R} \to M$, and

$$\begin{aligned} T_{\widetilde{\phi^h}} : \widetilde{H}_\beta \times \mathbb{R} &\to \widetilde{H}_\beta \times \mathbb{R} \\ (s,t) &\mapsto (\widetilde{\phi}^h(s), t-1). \end{aligned}$$

defines a covering automorphism of $\widetilde{H}_\beta \times \mathbb{R} \to X$, where X is the mapping torus of ϕ^h.

The covering automorphism ζ satisfies the commutative diagram

$$\begin{tikzcd} \widetilde{S}_\beta \arrow[loop left, "\zeta"] \arrow[r, "\widetilde{p}_\beta"] \arrow[d, "\rho_\beta"] & \widetilde{H}_\beta \arrow[loop right, "\zeta^h"] \arrow[d, "\rho^h_\beta"] \\ S \arrow[r, "p"] & H. \end{tikzcd}$$

Let K_α be the kernel of $\alpha_* : H_1(M;\mathbb{Z}) \to \mathbb{Z}$. Then $\widetilde{M}/K_\alpha$ is homeomorphic to the quotient of $\widetilde{S}_\beta \times \mathbb{R}$ by a cyclic subgroup $\langle \kappa \rangle \subset \langle \eta, T_{\widetilde{\phi}} \rangle$. Here κ generates $K_\alpha \cap \langle \zeta, T_{\widetilde{\phi}} \rangle$. Write κ as

$$\kappa = u\zeta + vT_{\widetilde{\phi}},$$

and let

$$\kappa^h = u\zeta^h + vT_{\widetilde{\phi^h}}.$$

Then κ and κ^h define covering automorphisms on $\widetilde{S}_\beta \times \mathbb{R}$ and $\widetilde{H}_\beta \times \mathbb{R}$, respectively.

Let

$$S_\alpha \times \mathbb{R} = (\widetilde{S}_\beta \times \mathbb{R})/\langle \kappa \rangle = \widetilde{M}/K_\alpha,$$

and

$$H_\alpha \times \mathbb{R} = (\widetilde{H}_\beta \times \mathbb{R})/\langle \kappa^h \rangle.$$

The various covering maps are shown in the following diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\simeq"] \arrow[dr] \arrow[ddr, "\widetilde{\rho}_\alpha"'] \arrow[dddr, "\widetilde{\rho}"'] & \widetilde{S} \times \mathbb{R} \arrow[d] & \\ & \widetilde{S}_\beta \times \mathbb{R} \arrow[r, "\widetilde{p}_\beta \times \mathrm{id}"] \arrow[d, "/\kappa"] & \widetilde{H}_\beta \times \mathbb{R} \arrow[d, "/\kappa^h"] \\ & S_\alpha \times \mathbb{R} \arrow[r, "f"] \arrow[d] & H_\alpha \times \mathbb{R} \\ & M. & \end{tikzcd}$$

Since

$$\widetilde{p}_\beta \times \mathrm{id} : \widetilde{S}_\alpha \times \mathbb{R} \to \widetilde{H}_\alpha \times \mathbb{R}$$

is an identification of $\widetilde{S}_\alpha \times \mathbb{R}$ with the boundary of $\widetilde{H}_\alpha \times \mathbb{R}$, it follows that since unbranched coverings are surjective local homeomorphisms, the map defined on the quotient

$$f : S_\alpha \times \mathbb{R} \to H_\alpha \times \mathbb{R}$$

is also an identification with the boundary. Thus the monodromy of α is a handlebody mapping class. $\square$

Remark. The inclusion

$$M \to X$$

gives rise to a map

$$H^1(X; \mathbb{Z}) \to H^1(M; \mathbb{Z}),$$

and the elements in $V_{\phi,p}$ lie in the image of this map.

The relation between M and X can be thought of as a 1-dimension higher version of surfaces S to handlebodies H, and there is a natural homomorphism

$$\mathrm{Mod}(X) \to \mathrm{Mod}(M).$$

Theorem 4.1 shows that given a handlebody monodromy ϕ for a fibration of M, there are linear sections of the fibered cone determined by ϕ in $H^1(M; \mathbb{Z})$ that lie in the image of $H^1(X; \mathbb{Z})$ and correspond to handlebody mapping classes.

Question 4.2 *What are the distinguishing properties of the image of the map Mod$(X) \to$ Mod(M)?*

5 Penner sequences and fibered faces

In this section, we define a (generalized) Penner sequence of mapping classes based on the main example found in [Pen]. Then we show how these Penner sequences are realized as sequences of monodromies associated to a convergent sequence of rational points on a single fibered face (Theorem 5.3).

By *Penner sequence* we mean a sequence of mapping classes (S_n, ϕ_n) which can be written as

$$\phi_n = r_n \circ \widehat{\delta}_n \circ \widehat{\eta}_n, \qquad n \geq \ell$$

where

(i) r_n is periodic of period n,

(ii) there is a fundamental domain $\Sigma_n \subset S_n$ of r_n on which $\widehat{\eta}_n$ is supported,

(iii) $\widehat{\delta}_n$ is a right Dehn twist centered on a simple closed curve γ_n, where for

$$\gamma_n \subset \Sigma_n^\ell := \bigcup_{k=1}^{\ell} (r_n)^k \Sigma_n,$$

(v) $(\Sigma_n, \widehat{\eta}_n|_{\Sigma_n}) = (\Sigma, \widehat{\eta})$, for some $(\Sigma, \widehat{\eta})$ independent of n, and

(vi) $(\Sigma_n^\ell, \gamma_n) = (\Sigma^\ell, \gamma)$ as pairs of topological spaces, where Σ^ℓ is a union of ℓ copies of Σ attached along boundary segments or components of the boundary, and (Σ^ℓ, γ) independent of n.

The arguments in this section work if $\widehat{\delta}_n$ is a left Dehn twist for all n. Penner sequences of the form $\phi_n = r_n \circ \eta_n \circ \delta_n$ may be analyzed in a similar way (see the Remark at the end of this section).

Next we encapsulate the essential data that describe a Penner sequence. A *relatively simple closed curve* on a surface S is a curve that is either a simple closed curve in the interior of S or is the homeomorphic image of a closed interval in S whose interior lies in the interior of S, and whose boundary lies in the boundary of S. A *Penner triple* (S, ϕ, τ) consists of a surface S of finite type, possible with boundary and punctures, a mapping class ϕ in $\mathrm{Mod}(S)$, and a finite union of relatively simple closed curves τ such that $\phi = \eta \circ \delta$ where η has a representative that fixes τ, and δ is a Dehn twist along a simple closed curve γ.

Given a Penner triple there is an associated 3-manifold M, the mapping torus of (S, ϕ), a fibration $\psi : M \to S^1$, and an associated element $\psi \in H^1(M;\mathbb{Z})$. There is also an element μ of $H^1(M;\mathbb{Z})$ defined by

$$\begin{array}{rcl} \pi_1(S) & \to & \mathbb{Z} \\ g & \mapsto & \iota_{\mathrm{alg}}(g, \tau). \end{array}$$

We will prove the following.

Lemma 5.1 *Given a Penner triple (S, ϕ, τ), the mapping classes (S_n, ϕ_n) determined by the sequence $\psi_n = \mu - n\psi$ on the fibered face of the mapping torus of ϕ is a Penner sequence.*

Let $\widetilde{S} \to S$ be the $\mathbb{Z}$ covering determined by the map

$$\begin{array}{rcl} \pi_1(S) & \to & \mathbb{Z} \\ g & \mapsto & \iota_{\mathrm{alg}}(g, \tau). \end{array}$$

Identify ζ with a generator for the cyclic group of covering automorphisms for this covering.

Let Σ_τ be the closure of $S \setminus \tau$ so that the boundary of Σ_τ contains the union of two copies of τ, $\tau^+ \cup \tau^-$, where either two connected components of the boundary if τ was a closed curve, or one connected component of the boundary otherwise. Then we can decompose $\widetilde{S}$ as

$$\widetilde{S} = \bigcup_{n \in \mathbb{Z}} \zeta^n \Sigma_\tau$$

where $\zeta^n(\tau^+)$ is identified with $\zeta^{n+1}(\tau^-)$ for all $n \in \mathbb{Z}$.

Let $\widetilde{\gamma}$ be the lift of γ such that

(i) $\widetilde{\gamma} \cap \Sigma_\tau \neq \emptyset$, and

(ii) $\widetilde{\gamma} \subset \bigcup_{n \geq 0} \zeta^n \Sigma_\tau$.

Let $\ell > 0$ be the smallest integer so that $\widetilde{\gamma} \cap \zeta^\ell \Sigma = \emptyset$.

The mapping classes δ and η lift to mapping classes $\widetilde{\delta}$ and $\widetilde{\eta}$ on $\widetilde{S}$, since their induced maps on $\pi_1(S)$ commute with the defining map of the covering space.

For $a \in \mathbb{Z}$, let $\widehat{\delta}_a$ be the Dehn twist on $\widetilde{S}$ centered at $\zeta^a \widetilde{\gamma}$. Then we have

(A1) $\widehat{\delta}_a \circ \widehat{\delta}_b = \widehat{\delta}_b \circ \widehat{\delta}_a$ for all $a, b \in \mathbb{Z}$,

(A2) $\widetilde{\delta} = \circ_{a \in \mathbb{Z}} \widehat{\delta}_a$, and

(A3) $\widehat{\delta}_a = \zeta^{-a} \circ \widehat{\delta}_0 \circ \zeta^a$.

Let $\widehat{\eta}_a$ be the restriction of $\widetilde{\eta}$ to $\zeta^a \Sigma$. Since η is the identity on τ, $\widehat{\eta}_a$ extends by the identity to all of $\widetilde{S}$. By abuse of notation we will also denote by $\widehat{\eta}_a$ the mapping class defined by this extension. Then we have

(B1) $\widehat{\eta}_a \circ \widehat{\eta}_b = \widehat{\eta}_b \circ \widehat{\eta}_a$ for all $a, b \in \mathbb{Z}$,

(B2) $\widetilde{\eta} = \circ_{a \in \mathbb{Z}} \widehat{\eta}_a$, and

(B3) $\widehat{\eta}_a = \zeta^a \circ \widehat{\eta}_0 \circ \zeta^{-a}$ for all $a \in \mathbb{Z}$.

Because δ_a is the identity map on $\zeta^b \Sigma$ for all $b < a$, we have

(C1) $\eta_b \circ \delta_a = \delta_a \circ \eta_b$

Let $\widetilde{\phi}$ be the lift of $\phi = \eta \circ \delta$ to $\widetilde{S}$. Then by (A2) and (B2), we have

$$\widetilde{\phi} = \widetilde{\delta} \circ \widetilde{\eta} = (\circ_{b \in \mathbb{Z}} \widehat{\delta}_b) \circ (\circ_{a \in \mathbb{Z}} \widehat{\eta}_a)$$

For any $x \in \widetilde{S}$, let $a \in \mathbb{Z}$ be the maximum element such that $x \in \zeta^a \Sigma$. Then for any $N > \ell$,

$$\begin{aligned} \widetilde{\phi}(x) &= (\circ_{b \in \mathbb{Z}} \widehat{\delta}_b) \circ \widehat{\eta}_a \\ &= \widehat{\delta}_{a-N} \circ \widehat{\delta}_{a-N+1} \circ \cdots \circ \widehat{\delta}_a \circ \widehat{\eta}_a \\ &= \widehat{\delta}_{a-N} \circ \eta_{a-N} \circ \cdots \circ \widehat{\delta}_a \circ \widehat{\eta}_a \\ &= \zeta^{a-N} (\zeta \circ \widehat{\phi})^N \zeta^{-a}, \end{aligned}$$

where $\widehat{\phi} = \widehat{\eta}_0 \circ \widehat{\delta}_0$.

Let $x \in \zeta^a \Sigma$, and let $N > \ell$. Then

$$\zeta^N (T_{\widetilde{\phi}})(x,t) = \zeta^a \left((\zeta \circ \widehat{\phi})^N (x), t+1 \right) \zeta^{-a}.$$

Define

$$\begin{aligned} R_N : \widetilde{S} \times \mathbb{R} &\rightarrow \widetilde{S} \times \mathbb{R} \\ (x,t) &\mapsto (\zeta \circ \widehat{\phi}(x), t + \frac{1}{N}) \end{aligned}$$

We thus have the following.

Lemma 5.2 *The mapping class $(R_N)^N$ is conjugate to $\zeta^N T_{\widetilde{\phi}}$ for all large enough N.*

The covering automorphism group of $\widetilde{S} \times \mathbb{R} \to M$ is generated by $Z = \zeta \times \mathrm{id}$ and

$$\begin{aligned} T = T_{\widetilde{\phi}} : \widetilde{S} \times \mathbb{R} &\rightarrow \widetilde{S} \times \mathbb{R} \\ (s,t) &\mapsto (\widetilde{\phi}(s), t-1). \end{aligned}$$

Let $K_n \subset \langle T, Z \rangle$ be the subgroup generated by $T^{-1} Z^n$. If $\psi : M \to S^1$ is the fibration corresponding to ϕ, and $\mu \in H^1(M;\mathbb{Z})$ is dual to Z, under the isomorphism

$$H^1(M;\mathbb{Z}) \simeq \mathrm{Hom}(H_1(M;\mathbb{Z}), \mathbb{Z}),$$

then K_n is the kernel of the map

$$\psi_n = n\psi - \mu.$$

Let F be the fibered face in $H^1(M;\mathbb{Z})$ that contains the projection of ψ to the boundary of the Thurston norm ball. Let $\Pi \subset H^1(M;\mathbb{Z})$ be the plane spanned by ψ and μ. Then the projections of the sequence ψ_n on the boundary of the Thurston norm ball are eventually in $F \cap \Pi$ and converge to the projection of ψ.

Let X_n be the quotient of $\widetilde{S} \times \mathbb{R}$ by TZ^n. Then since the normalization of ψ_n is eventually in F, X_n is isomorphic to $S_n \times \mathbb{R}$ for some surface S_n, and Z induces a mapping class ϕ_n on S_n so that M is the quotient of $S_n \times \mathbb{R}$ by T_{ϕ_n}.

By Lemma 3.6, the mapping class (S_n, ϕ_n) induced by Z is the monodromy of ψ_n. Lemma 5.2 implies

$$\phi_n = r_n \circ \widehat{\delta}_n \circ \widehat{\eta}_n,$$

where r_n, $\widehat{\eta}_n$, and $\widehat{\delta}_n$ are the maps induced on X_n by R, $\widehat{\eta}_0$, and $\widehat{\delta}_0$. This completes the proof of Lemma 5.1.

We have shown the following.

Theorem 5.3 *Let (S_n, ϕ_n), $n \geq \ell$ be a Penner sequence. For any fixed $n > \ell$, let S be the quotient of S_n by r_n, and let $q_n : S_n \to S$ be the quotient map. Let $\tau = q_n(\Sigma_n \cap \zeta\Sigma_n)$, and let $\gamma = q(\gamma_n)$.*

Let $\phi = \delta \circ \eta$, where η is induced by $(S_n, \widehat{\eta}_n)$, and δ is the right Dehn twist centered at γ. Then (S, ϕ, τ) is a Penner triple independent of n, and (S_n, ϕ_n) is its associated Penner sequence. Thus, the mapping torus for the mapping classes in a Penner sequence is a 3-manifold that is independent of n.

Proposition 5.4 *Let (S_n, ϕ_n) be a Penner sequence, and (S, ϕ, τ) the associated Penner triple. Then ϕ_n is pseudo-Anosov for all n if and only if ϕ is pseudo-Anosov.*

Proof. The maps ϕ and ϕ_n are the monodromy of fibrations of $M_\phi = M_{\phi_n}$. Thus if one of the mapping classes is pseudo-Anosov, then they all are. $\square$

Proposition 5.5 *If (S_n, ϕ_n) is a pseudo-Anosov Penner sequence, with associated triple (S, ϕ, τ), then*

$$\lim_{n \to \infty} \overline{\lambda}(\phi_n) = \overline{\lambda}(\phi).$$

From the proof of Theorem 5.3 we obtain the following lemma, which will be useful in Section 6.

Lemma 5.6 *The topological Euler characteristic of S_n satisfies*

$$\chi(S_n) = n\chi(\Sigma_\tau) - s,$$

where s is the number of components of τ which intersect that boundary of S_n.

Proof. The surface S_n can be identified with the union of n-copies of Σ_τ glued together along τ^+ and τ^-, where the nth copy of Σ_τ is identified using $\eta_n \circ \delta_n$ with the first ℓ copies of Σ_τ. $\square$

Remark. Mapping classes of the form $\phi_n = r_n \circ \widehat{\eta}_n \circ \widehat{\delta}_n$ (where the order of η_n and δ_n is reversed) may be studied in a the same way as above. In this case the Penner sequence of a Penner triple will be the monodromies associated to

$$\psi_n = n\psi + \mu.$$

6 Example

In this section, we compute the Alexander and Teichmüller polynomials for two sequences constructed from Penner triples: the first is Penner's original example, and the second is a Penner sequence of handlebody mapping classes. This allows us to find explicit equations for the homological and geometric dilatations of the corresponding Penner sequences. The Teichmüller polynomial is defined in [McM1]. Other computations of Teichmüller polynomials can be found in, for example, in [AD], [KT] and [Hir].

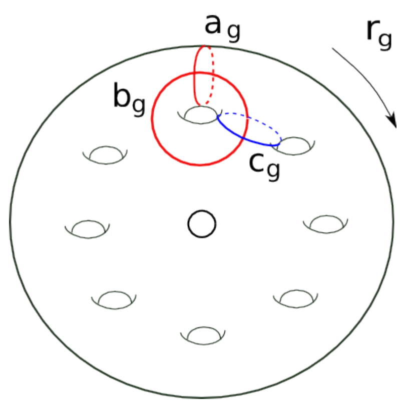

Figure 1: Penner triple.

6.1 Penner's original example

Penner's original sequence (S_g, ϕ_g) of mapping classes in [Pen] is shown in Figure 1. The surface S_g has genus g and two boundary components. The mapping class ϕ_g is the composition of Dehn twists along the curves a_g, b_g and c_g with a rotation r_g of period g. This Penner sequence is formed from the Penner triple (S, ϕ, τ) illustrated in Figure 2, where ϕ is the mapping class on the torus with two boundary components given by the product of Dehn twists $\delta_c \circ \delta_b^{-1} \circ \delta_a$ centered at the curves a, b and c, and $\tau = d$ is the path d connecting the two boundary components.

Proposition 6.1 *Penner's original sequence of mapping classes ϕ_g satisfies*

$$\lim_{g\to\infty} \overline{\lambda}(\phi_g) = \overline{\lambda}(\phi) \approx 46.9787.$$

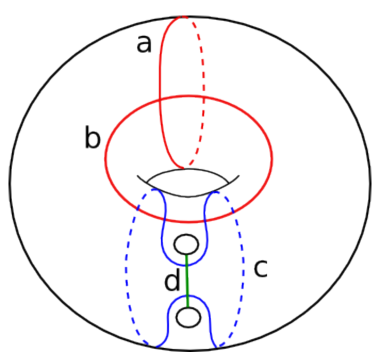

Figure 2: Penner's original example.

The action of ϕ on the first homology $H^1(S, \mathbb{Z})$ is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence has a 1-dimensional invariant subspace. Thus, the mapping torus M has $b_1(M) = 2$. The cyclic covering $\widetilde{S} \to S$ defined by τ is drawn in Figure 3. Let ζ generate the group of covering automorphisms. Then $\zeta \times \{\mathrm{id}\}$ and $T_{\widetilde{\phi}}$ define generators for $H_1(M;\mathbb{Z})$. Let μ be the dual of $\zeta \times \mathrm{id}$, that is, the extension of the map $\pi_1(S) \to \mathbb{Z}$ defined by τ, and let ψ be the fibration map dual to ϕ.

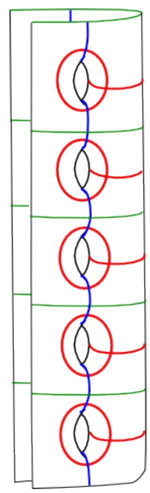

Figure 3: The simultaneous cyclic covering of Penner's examples.

Let $t, u \in H_1(M;\mathbb{Z})$ be duals to μ and $T_{\widetilde{\phi}}$ respectively. Let $\mathfrak{t}$ be the train track for ϕ given by smoothing the union of a, b and c at the intersections (see [Pen]). The Teichmüller polynomial is the characteristic polynomial for the action of the lift of ϕ on the cyclic covering of S defined by τ on the lift $\widetilde{\mathfrak{t}}$ of $\mathfrak{t}$, or more precisely on the space of allowable measure on $\widetilde{\mathfrak{t}}$. Using the switch conditions, we can replace the space of allowable measures with the space of labels on the lifts of the curves a, b and c. Then the Teichmüller polynomial of the fibered face defined by ϕ is a factor of the characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1+t \\ 1+t^{-1} & 2(1+t^{-1}) & 1+(1+t)(1+t^{-1}). \end{bmatrix},$$

and is given by

$$\Theta(u,t) = u^2 - u(5 + t + t^{-1}) + 1. \tag{2}$$

The Alexander polynomial Δ is the characteristic polynomial of the action of the lift $\widetilde{\phi}$ of ϕ on the first homology of $\widetilde{S}$. The lifts of a, b and c generate $H_1(\widetilde{S};\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$ module, and the action of $\widetilde{\phi}$ on these generators is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1-t \\ 1-t^{-1} & 2(1-t^{-1}) & 1+(1-t)(1-t^{-1}) \end{bmatrix}.$$

We thus have

$$\Delta(u,t) = \Theta(u,-t) = u^2 - u(5 - t - t^{-1}) + 1. \tag{3}$$

By the relation between the Alexander and Thurston nomrs [McM2], it follows that the fibered cone C_ψ in $H^1(M;\mathbb{R})$ containing ψ is given by elements $\alpha\psi + \beta\mu$, where

$$\alpha > |\beta|,$$

and the Thurston norm is given by

$$||(a,b)||_T = \max\{2|a|, 2|b|\}.$$

The dilatation $\lambda(\phi_{(\alpha,\beta)})$ corresponding to primitive integral points (α,β) in C_ψ is the largest solution of the polynomial equation

$$\Theta(x^\alpha, x^\beta) = 0.$$

In particular, Penner's examples (S_g, ϕ_g) correspond to the points $(g,1) \in C_\psi$, and we have the following.

Proposition 6.2 *The dilatation of ϕ_g is given by the largest root of the polynomial*

$$\Theta(x^g, x) = x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1.$$

The symmetry of Θ with respect to $t \mapsto -t$ and Theorem 3.3 implies that the minimum normalized dilatation realized in $\Phi(M,F)$ must occur at $(\alpha,\beta) = (1,0)$. Thus, we have the following.

Proposition 6.3 *The minimum normalized dilatation for the monodromies in C_ψ is given by $\overline{\lambda}(\phi) \approx 46.9787$.*

A pseudo-Anosov mapping class is *orientable* if it has orientable invariant foliations, or equivalently the geometric and homological dilatations are the same, and the spectral radius of the homological action is realized by a real (possibly negative) eigenvalue (see, for example, [LT] p. 5). Given a polynomial f, the largest complex norm amongst its roots is called the *house of* f, denoted $h(f)$. Thus, ϕ_g is orientable if and only if

$$h(\Delta(x^g, x)) = h(\Theta(x^g, x)). \tag{4}$$

Proposition 6.4 *The mapping classes (S_g, ϕ_g) are orientable if and only if g is even.*

Proof. By Equation (3), the homological dilatation of ϕ_g is the largest complex norm amongst roots of

$$\Delta(x^g, x) = x^{2g} + x^{g+1} - 5x^g + x^{g-1} + 1.$$

Let λ be the real root of $\Delta(x^g, x)$ with largest absolute value. Plugging λ into $\Theta(x^g, x)$ gives

$$\Theta(\lambda^g, \lambda) = -2\lambda^{g+1} - 2\lambda^{g-1} \neq 0.$$

while for $-\lambda$ we have

$$\Theta(-\lambda^g, -\lambda) = (-\lambda)^{g+1} - (\lambda)^{g+1} + (-\lambda)^{g-1} - (\lambda^{g-1}).$$

It follows that $h\Delta(x^g, x) = \lambda = h\Theta(x^g, x)$ if and only if g is even. $\square$

6.2 Handlebody example

In this section, we find a Penner triple (S, ϕ, τ) that produces a Penner sequence of handlebody mapping classes.

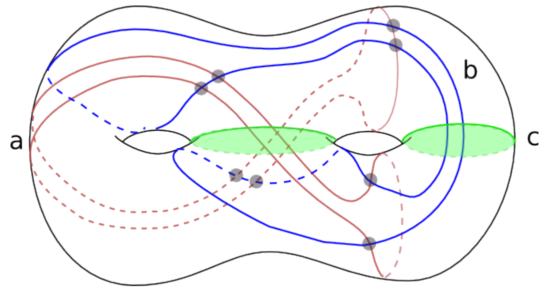

Figure 4: Two simple closed curves on the surface S bounding disks in the handlebody H.

Consider the two simple closed curves a (in red/light gray) and b (in blue/dark gray) on the genus 2 surface S pictured in Figure 4. Let $p : S \to H$ be the inclusion of S as the boundary of the genus 2 handlebody shown in the drawing. Let $\eta = \delta_a$ be the right Dehn twists centered at a and let $\delta = \delta_b^{-1}$ be the left Dehn twist centered at b. Let $\phi = \delta \circ \eta$.

Since a is homologically trivial in S, the Dehn twist η is an element of the Torelli subgroup of $\mathrm{Mod}(S)$, that is, it acts trivially on $H_1(S; \mathbb{Z})$. Thus, the induced action of ϕ on $H_1(S; \mathbb{Z})$ is that of the Dehn twist δ, and $\lambda_{\mathrm{hom}}(\phi) = 1$.

Proposition 6.5 *The space $W_{\phi,p} \subset H^1(S, \mathbb{R})$ equals to the kernel K of the map*

$$p_* : H_1(S; \mathbb{Z}) \to H_1(H; \mathbb{Z})$$

Proof. Both a and b have trivial algebraic intersection with elements in K. $\square$

It follows that $W_{\phi,p}$ is freely generated by the boundaries of the two shaded (green) disk in Figure 4.

Corollary 6.6 *The fibered cone for the fibration of M_ϕ associated to ϕ has a 2-dimensional subcone for which the monodromy of integral elements are handlebody mapping classes.*

Corollary 6.7 *For any τ in the kernel p_*, (S, ϕ, τ) is a Penner triple.*

Let τ be the boundary of the disk labeled with a c in Figure 4.

Figure 5 illustrates the cyclic covering $\widetilde{S} \to S$ defined by the element $\mu \in H^1(M; \mathbb{Z})$ corresponding to τ.

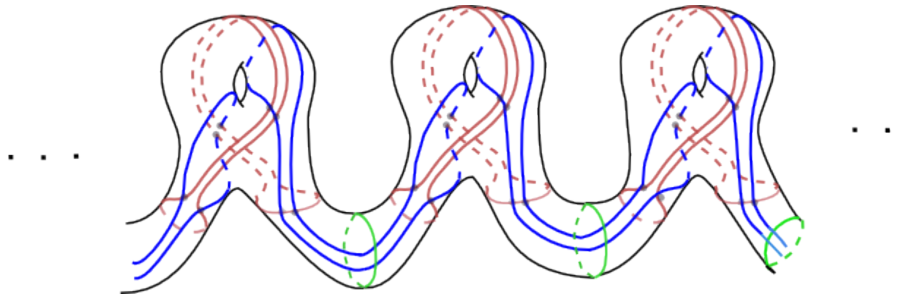

Figure 5: Cyclic covering corresponding to an element of $W_{\phi,p}$

Let $\Pi \subset H^1(M : \mathbb{R})$ be the planar section spanned by ψ and μ. Consider the sequence of elements $\psi_n = n\psi - \mu \in H^1(M; \mathbb{Z})$. We know that for large n, ψ_n belongs to the fibered face of ψ, and by Theorem 4.1, the monodromy ϕ_n of ψ_n is a handlebody mapping class for all n. Corollary 3.5 implies the following.

Proposition 6.8 *The normalized dilatations $\lambda(\phi_n)$ satisfy*

$$\lim_{n\to\infty} \overline{\lambda}(\phi_n) = \overline{\lambda}(\phi)$$

As in our first example, the dilatation $\lambda(\phi)$ and its normalization can be computed by considering the action of ϕ on the space of allowable measure on the train track that are generated by the unoriented curves a and b. The transition matrix for ϕ restricted to this subspace is given by

$$\begin{bmatrix} 1 & 8 \\ 8 & 65 \end{bmatrix}.$$

It follows that $\lambda(\phi)$ is the largest root of the polynomial equation

$$x^2 - 66x + 1 = 0.$$

Thus we have $\lambda(\phi) \approx 65.9848$, and $\overline{\lambda}(\phi) \approx (65.98)^2 \approx 4353.99$.

The transition matrix for the lifting $\widetilde{\phi}$ on the lifted train track is given by

$$\begin{bmatrix} 1 & 4+4t \\ 4+4t^{-1} & 16(1+t)(1+t^{-1})+1 \end{bmatrix},$$

and the Teichmüller polynomial for the fibered face F of M corresponding to ϕ specialized to the planar section $F \cap \Pi$ is thus

$$\Theta(u,t) = u^2 - u(34 + 16(t + t^{-1})) + 1.$$

This implies the following.

Proposition 6.9 *For each $n \geq 1$, the dilatations of ϕ_n is the largest root of*

$$x^{2n} - 16x^{n+1} - 34x^n - 16x^{n-1} + 1 = 0,$$

and the normalized dilatations $\overline{\lambda}(\phi_n)$ converge to $\overline{\lambda}(\phi) \approx 4353.99$.

The following proposition completes our proof of Theorem 1.2.

Proposition 6.10 *The genus of S_n is $g = n + 1$.*

Proof. The Euler characteristic of S_n is $-2n$ by Lemma 5.6. $\square$

The surface S_n is a union of n copies of Σ, which has genus $g = 1$ and 2 boundary components labeled τ^+ and τ^- (see Figure 6). These are glued together in a cyclic manner yielding the desired genus, and

$$\phi_n = r_n \circ \delta_n \circ \eta_n,$$

where a_n is the red/light grey curve, and b_n is the blue/dark grey curve.

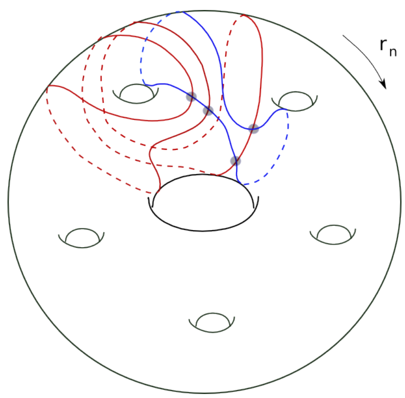

Figure 6: The mapping class (S_n, ϕ_n).

The homological action of ϕ_n is that of a Dehn twist on a multicurve, and hence the homological dilatation is equal to one. This proves Theorem 1.3.

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