# Boundary Layer for a Class of Nonlinear Pipe Flow

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**Abstract**

We establish the mathematical validity of the Prandtl boundary layer theory for a family of (nonlinear) parallel pipe flow. The convergence is verified under various Sobolev norms, including the physically important space-time uniform norm, as well as the $L^\infty(H^1)$ norm. Higher order asymptotics is also studied.

*Keywords:* Navier-Stokes system, parallel pipe flow, boundary layer, Prandtl theory, no-slip boundary condition

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## 1. Introduction

Boundary layers associated with slightly viscous incompressible fluid flow equipped with the physical **no-slip no-penetration** boundary condition are of great importance. From the physical point of view, in the absence of body force, it is the vorticity generated by the boundary layer and later advected into the main

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[1]Supported in part by National Science Foundation grant DMS-1009713 and DMS-1009714.
[2]Supported in part by National Youth grant, China (No. 11001184).
[3]Supported in part by the National Science Foundation, a COFRA award from FSU, and a 111 project from the Chinese Ministry of Education at Fudan University.

stream that drive the flow (see for instance the classical treatise by Schlichting [23] and the references therein). Indeed, many physical phenomena cannot be explained in a satisfactory fashion without accounting for boundary layer effects (D'Alembert's paradox is one). From the mathematical point of view, the boundary layer problem is a serious challenge since the slightly viscous fluid equation, the Navier-Stokes system at small viscosity, can be viewed as a singular perturbation of the Euler system that governs the flow of inviscid fluids (see for instance the book by Oleinik [20] and the review paper by E [3]).

Moreover, the leading order singular behavior governed by the so-called Prandtl equation [21, 20] may be ill posed (see the recent work by Guo and Nguyen [8], Gerard-Varet and Dormy [5], Grenier [7], and E and Engquist [4]). Even if the Prandtl boundary-layer system is well posed, one still needs to verify a spectral constraint on the Prandtl solution to ensure the convergence as was pointed out in [31]. The verification of such kind of spectral constraint may not be straightforward and it is still unknown if the classical Oleinik profile (as presented in her classical treatise [20], see also Xin and Zhang [35]) that leads to a well-posed Prandtl system satisfies the spectral constraint.

The well-posedness of the Prandtl system is already a challenge (see the works cited above). Our knowledge on the validity on the Prandtl boundary layer theory under *Dirichlet boundary condition* is also very limited and the validity itself remains a conudrum. Besides various cases where the Navier-Stokes system reduces to the trivial linear heat equation (either in half-space, or in a channel, or in a disk), the only known results on the validity of Prandtl theory are either for analytical data in half-space due to Sammartino and Caflisch [22], or channel flow with uniform injection and suction at the boundary by Temam and Wang [27, 28], or a special class of plane parallel flow introduced in [30] with the boundary layer behavior carefully investigated by Mazzucato, Niu and Wang [19]. Therefore, it is worthwhile to identify special type of flows for which the Prandtl theory may be rigorously validated.

In this work, we investigate the validity of Prandtl boundary layer theory associated with a special type of parallel pipe flow introduced in [30]. In this case we assume that the fluids occupy an infinitely long pipe with circular cross-section of radius 1, and with the $x-$ axis being the axis of the pipe. We impose that the flow is parallel to the axis of the pipe all the time (therefore no component of the velocity in the radial direction), and the flow is periodic in $x$ with period $L$ for simplicity. The classical Poiseuille flow is a special case of our ansatz provided we identify the mean pressure gradient as part of the (periodic in $x$) body force. Hence the spatial domain is $Q = \Omega \times [0, L]$, where $\Omega = \{(r, \phi)\big| 0 \leq r \leq 1, \phi \in [0, 2\pi]\}$

is the unit disk and $L$ is the horizontal period in the cylindrical coordinates with $\phi$ being the azimuthal angle and $r$ being the distance to the axis of the pipe (see figure 1 below).

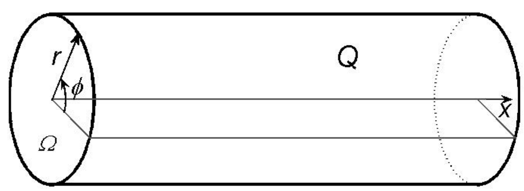


Figure 1: Cross section on the circular pipe.

Throughout the paper, we will denote the solution of the Navier-Stokes system with viscosity coefficient $\nu$ by $\mathbf{u}^\nu$, while the solution of the Euler system will be denoted by $\mathbf{u}^0$. For simplicity, we will take the same initial condition for both $\mathbf{u}^\nu$ and $\mathbf{u}^0$, which we will denote by $\mathbf{u}_0$. This choice can be relaxed.

The special type of parallel pipe flow that we investigate in this manuscript satisfies the following **ansatz** for the Navier-Stokes solution:

$$\mathbf{u}^\nu = u^\nu_\phi(t,r)\mathbf{e}_\phi + u^\nu_x(t,r,\phi)\mathbf{e}_x, \quad p^\nu = p^\nu(t,r), \tag{1.1}$$

where $\mathbf{u}^\nu, p^\nu$ are the velocity and pressure field respectively, and $\mathbf{e}_\phi, \mathbf{e}_x, \mathbf{e}_r$ are the unit vector in the azimuthal direction, $x$ direction, and radial direction respectively.

Observe that such flow satisfying the incompressibility condition automatically, and the Navier-Stokes system with viscosity $\nu$, external body force $\mathbf{f}$ and the boundary shear velocity $\boldsymbol{\beta}$ reduces to the following weakly nonlinear system under the ansatz (1.1)

$$-(u^\nu_\phi)^2 + r\partial_r p^\nu = 0,$$

$$
\begin{aligned}
&\partial_t u_\phi^\nu = \frac{\nu}{r}\partial_r(r\partial_r u_\phi^\nu) - \frac{\nu}{r^2}u_\phi^\nu + f_1(t,r),\\
&\partial_t u_x^\nu + \frac{u_\phi^\nu}{r}\partial_\phi u_x^\nu = \frac{\nu}{r}\partial_r(r\partial_r u_x^\nu) + \frac{\nu}{r^2}\partial_{\phi\phi}u_x^\nu + f_2(t,r,\phi),
\end{aligned} \tag{1.2}
$$

with the following boundary and initial data

$$
\begin{aligned}
&\mathbf{u}^\nu|_{r=1} = \boldsymbol{\beta} := \beta_\phi(t)\mathbf{e}_\phi + \beta_x(t,\phi)\mathbf{e}_x,\\
&\mathbf{u}^\nu \text{ is periodic in } x \text{ direction},\\
&\mathbf{u}^\nu|_{t=0} = \mathbf{u}_0 := a(r)\mathbf{e}_\phi + b(r,\phi)\mathbf{e}_x.
\end{aligned} \tag{1.3}
$$

It is remarkable that the pressure term $p^\nu$ can be uniquely (up to a constant) recovered from the first equation in system (1.2). Therefore the second equation and third equation of (1.2) form a closed weakly coupled parabolic system, written in Cartesian coordinates as the following:

$$
\begin{aligned}
&\partial_t \mathbf{u}_v^\nu - \nu\Delta_v \mathbf{u}_v^\nu = \mathbf{F}_1,\\
&\partial_t u_x^\nu + (\mathbf{u}_v^\nu \cdot \nabla_v)u_x^\nu - \nu\Delta_v u_x^\nu = F_2,
\end{aligned} \tag{1.4}
$$

with the same boundary and initial conditions as (1.3). It follows in particular that the ansatz (1.1) is preserved by the evolution of the flow.
Here $\mathbf{u}_v^\nu = (-u_\phi \sin\phi, u_\phi\cos\phi)$, $\mathbf{F}_1 = \big(-f_1(t,r)\sin\phi, f_1(t,r)\cos\phi\big)$, $\mathbf{u}^\nu = (\mathbf{u}_v^\nu, u_x^\nu)$, $F_2 = f_2(t,r,\phi)$, $\Delta_v = \partial_{x_1x_1} + \partial_{x_2x_2}$, $\nabla_v = (\partial_{x_1}, \partial_{x_2})$.

Similar to the ansatz (1.1), we also assume

$$
\mathbf{u}^0 = u_\phi^0(t,r)\mathbf{e}_\phi + u_x^0(t,r,\phi)\mathbf{e}_x, \quad p^0 = p^0(t,r). \tag{1.5}
$$

Then the Euler system reduces to the following system:

$$
\begin{aligned}
&-(u_\phi^0)^2 + r\partial_r p^0 = 0,\\
&\partial_t u_\phi^0 = f_1,\\
&\partial_t u_x^0 + \frac{u_\phi^0}{r}\partial_\phi u_x^0 = f_2,
\end{aligned} \tag{1.6}
$$

with initial condition

$$
\mathbf{u}^0|_{t=0} = a(r)\mathbf{e}_\phi + b(r,\phi)\mathbf{e}_x, \tag{1.7}
$$

We observe that the no-penetration condition at the walls for the Euler solution is automatically satisfied in this case.

Due to the disparity of boundary conditions between the reduced Navier-Stokes system (1.2) and the reduced Euler system (1.6), a boundary layer must exist outside of which the flow is expected to be well approximated by the Euler solution $\mathbf{u}^0$. Inside the layer, a flow *corrector* is needed, which approximates $\mathbf{u}^\nu - \mathbf{u}^0$. At leading order, the corrector $\boldsymbol{\theta}^{\mathbf{0}}$ is formally governed by the Prandtl-type equation (2.4) (see the next section for a formal derivation). The goal of this manuscript is to investigate the mathematical validity of the Prandtl-type approximation for this special type of flow in a pipe. More precisely, we investigate whether $\mathbf{u}^\nu - \mathbf{u}^{\mathbf{0}} - \boldsymbol{\theta}^{\mathbf{0}}$ converges to zero in various norms. Our main result is the rigorous verification of the Prandtl theory in the sense of the following theorem.

**Theorem 1.1.** *Under appropriate smoothness and compatibility assumptions on the initial and boundary data, we have, for some constant $c$ independent of the viscosity $\nu$,*

$$
\begin{aligned}
\|\mathbf{u}^\nu - \mathbf{u}^{\mathbf{0}} - \boldsymbol{\theta}^{\mathbf{0}}\|_{L^\infty(0,T;L^2(\Omega))} &\leq c\nu^{\frac{3}{4}}, && (1.8)\\
\|\mathbf{u}^\nu - \mathbf{u}^{\mathbf{0}} - \boldsymbol{\theta}^{\mathbf{0}}\|_{L^\infty(0,T;H^1(\Omega))} &\leq c\nu^{\frac{1}{4}}, && (1.9)\\
\|\mathbf{u}^\nu - \mathbf{u}^{\mathbf{0}} - \boldsymbol{\theta}^{\mathbf{0}}\|_{L^\infty(\Omega\times[0,T])} &\leq c\nu^{\frac{1}{2}}, && (1.10)\\
\|p^\nu - p^0\|_{L^\infty(\Omega\times[0,T])} &\leq c\nu^{\frac{1}{2}}, && (1.11)\\
\|p^\nu - p^0\|_{L^\infty(0,T;H^1(\Omega))} &\leq c\nu^{\frac{1}{4}}. && (1.12)
\end{aligned}
$$

Flows with the special symmetry (1.1) were first investigated in [30], where the convergence in the $L^\infty(L^2)$-norm of the viscous solution $\mathbf{u}^\nu$ to the inviscid solution $\mathbf{u}^0$ as $\nu \to 0$ was established via a Kato-Hopf type approach without referring to the Prandtl theory. Mazzucato and Taylor [18] have recently carried out an analysis of the boundary layer using semiclassical teachniques and layer potentials. This approach does not rely as well on the Prandtl theory and does not require any type of compatibility conditions between the intial and boundary data. However, it yields only convergence in $L^\infty(L^p)$ with $p \in [1, +\infty]$ and does not provide any estimate on normal gradients at the boundary. Convergence in $L^\infty(L^2)$ and $L^2(H^1)$ norm was formally derived and announced in [31]. We believe that the result presented here is the first rigorous result on the validity of the Prandtl boundary-layer theory for the Navier-Stokes system in a nonlinear setting in a domain with curved boundaries. The curvature effect can be discerned from the pressure estimates which is different from the flat boundary case (see for instance [19]). The curved boundary also motivated us to further develop certain classical anisotropic estimates and embeddings. (See Temam and Wang [26, 28] for this idea applied to boundary layer associated with the linear and nonlinear Navier-Stokes equations with Dirichlet boundary conditions with flat boundary.) In particular, a novel coupled boundary layer and interior domain approach is developed in order to derive the $L^\infty(H^1)$ estimate in our curved geometry.

We also remark that there exist abundant literature on boundary layer analysis as well as the related vanishing viscosity limit problem associated with the Navier-Stokes system equipped with *different (non-Dirichlet)* boundary conditions. For instance, for the case of Navier-slip (and the simpler free-slip) boundary condition, there are many interesting works on the related vanishing viscosity limit as well as the analysis of the (secondly) boundary layer. (See for example [1, 2, 6, 9, 10, 12, 15, 16, 17, 29, 32, 33, 34, 36] among many others). However it is beyond the scope of this paper to survey results associated with various kinds of boundary conditions (non no-slip no-penetration).

The rest of paper is organized as follows. We provide a formal derivation of the Prandtl-type equation for the leading order corrector $\boldsymbol{\theta}^0$ utilizing the Prandtl-type ansatz in Section 2. The well-posedness of the Prandtl-type boundary-layer system as well as appropriate decay properties is briefly discussed in Appendix Appendix A. An approximate solution to the reduced Navier-Stokes system (1.2) is constructed in the second part of Section 2 utilizing the inviscid solution $\mathbf{u}^0$ and the leading order boundary-layer type corrector $\boldsymbol{\theta}^0$. The validity of the approximation proposed in Section 2 is rigorously established in Section 3 under various norms. Higher-order asymptotic expansions are considered in Section 4. The regularity of solutions to Euler equations as well as the compatibility conditions needed to ensure the smoothness of the Navier-Stokes system are mentioned in Appendix Appendix B.

## 2. Prandtl type equation and approximate solution

### 2.1. Prandtl-type equation for the corrector

According to the Prandtl boundary layer theory as proposed in [21], the viscous solution and the inviscid solution are close to each other outside a boundary layer of thickness proportional to $\sqrt{\nu}$. Moreover, the viscous solution must make a sharp transition to the inviscid main flow at the boundary within the boundary layer because of the no-slip boundary no-penetration condition of the viscous flow. Therefore, we postulate that the solution to the Navier-Stokes system can be approximated by

$$\mathbf{u}^\nu(t,r,\phi) \approx \mathbf{u}^{\mathbf{0}}(t,r,\phi) + \boldsymbol{\theta}^{\mathbf{0}}(t, \frac{1-r}{\sqrt{\nu}}, \phi), \tag{2.1}$$

$$p^\nu(t,r,\phi) \approx p^0(t,r) + q^0(t, \frac{1-r}{\sqrt{\nu}}), \tag{2.2}$$

where $\mathbf{u^0}(t,r,\phi) = u^0_\phi(t,r)\mathbf{e}_\phi + u^0_x(t,r,\phi)\mathbf{e}_x$ is the inviscid solution to the Euler system, and the (boundary-layer-type) **corrector** $\boldsymbol{\theta^0}(t, \frac{1-r}{\sqrt{\nu}}, \phi) = \theta^0_\phi(t, \frac{1-r}{\sqrt{\nu}})\mathbf{e}_\phi + \theta^0_x(t, \frac{1-r}{\sqrt{\nu}}, \phi)\mathbf{e}_x$, thanks to our flow ansatz (1.1).

Introducing the stretched variable $Z = \frac{1-r}{\sqrt{\nu}}$, we notice that the corrector must satisfy the following matching conditions

$$\boldsymbol{\theta^0} \to 0 \text{ as } Z \to \infty, \quad \boldsymbol{\theta}^0(t,\cdot,\phi)|_{Z=0} = \boldsymbol{\beta}(t,\phi) - \mathbf{u}^0(t,1,\phi). \tag{2.3}$$

It is then convenient to work with the following domain for the corrector $\boldsymbol{\theta}^0$:

$$\Omega_\infty := [0,2\pi] \times [0,\infty).$$

Introducing (2.1) and (2.2) into (1.2) and (1.3), utilizing the Euler equation (1.6) and keeping the leading order terms in $\nu$, we deduce the following **Prandtl-type equation** for the leading-order of the boundary-layer profile (corrector) $\boldsymbol{\theta^0}$:

$$\begin{aligned}
&\partial_t\theta^0_\phi - \partial_{ZZ}\theta^0_\phi = 0,\\
&\partial_t\theta^0_x + \theta^0_\phi\partial_\phi u^0_x(t,1,\phi) + \theta^0_\phi\partial_\phi\theta^0_x + u^0_\phi(t,1)\partial_\phi\theta^0_x = \partial_{ZZ}\theta^0_x,\\
&(\theta^0_\phi,\theta^0_x)|_{Z=0} = \big(\beta_\phi(t) - u^0_\phi(t,1), \beta_x(t,\phi) - u^0_x(t,1,\phi)\big),\\
&(\theta^0_\phi,\theta^0_x)|_{Z=\infty} = 0,\ (\theta^0_\phi,\theta^0_x)|_{t=0} = (0,0).
\end{aligned} \tag{2.4}$$

The well-posedness of the system is trivial. The decay, as $Z \to \infty$, of the solution can be derived in a straightforward manner just as in the case of the linearized compressible Navier-Stokes system studied by Xin and Yanagisawa [34], assuming appropriate compatibility conditions between the initial and boundary data. These are discussed in Appendix Appendix B. Decay estimates as well as the main idea of the proof are presented in Appendix A.

It is also easy to realize that the leading-order correction $q^0$ to the pressure term satisfies

$$\partial_Z q^0 \equiv 0, \tag{2.5}$$

and hence we can conveniently set

$$q^0 \equiv 0. \tag{2.6}$$

*2.2. Approximate Solution*

With the leading-order corrector $\boldsymbol{\theta}^0$ and the inviscid solution $\mathbf{u}^0$ in hand, we are now in a position to construct an approximate solution to the Navier-Stokes system (1.2) with the given ansatz (1.1).

As in Temam and Wang [26, 28] and Mazzucato, Niu and Wang [19], we introduce a cut-off function to ensure that that the approximate Navier-Stokes solution $\tilde{\mathbf{u}}^{app}$, given below, satisifes the same boundary conditions as the true Navier-Stokes solution $\mathbf{u}^\nu$. Let $\rho(r)$ be a smooth function defined on $[0, 1]$ such that

$$\rho(r) = \begin{cases} 1 & r \in [\frac{1}{2}, 1], \\ 0 & r \in [0, \frac{1}{4}]. \end{cases} \tag{2.7}$$

Because of (1.1), the **approximate solution to the Navier-Stokes equation** must have the form:

$$\tilde{\mathbf{u}}^{app} = \tilde{u}_\phi^{app}(t, r)\mathbf{e}_\phi + \tilde{u}_x^{app}(t, r, \phi)\mathbf{e}_x, \tag{2.8}$$

$$\tilde{u}_\phi^{app}(t, r) = u_\phi^0(t, r) + \rho(r)\theta_\phi^0(t, \frac{1-r}{\sqrt{\nu}}), \tag{2.8a}$$

$$\tilde{u}_x^{app}(t, r, \phi) = u_x^0(t, r, \phi) + \rho(r)\theta_x^0(t, \frac{1-r}{\sqrt{\nu}}, \phi). \tag{2.8b}$$

In view of (2.6), we take the pressure associated with the approximate velocity to be:

$$p^{app} = p^0. \tag{2.9}$$

It is straightforward to verify that the approximate solution $\tilde{\mathbf{u}}^{app}$ constructed above satisfies the Navier-Stokes system with (small) extra body force:

$$\begin{aligned} &- (\tilde{u}_\phi^{app})^2 + r\partial_r p^{app} = A, \\ &\partial_t \tilde{u}_\phi^{app} - \frac{\nu}{r}\partial_r(r\partial_r \tilde{u}_\phi^{app}) + \frac{\nu}{r^2}\tilde{u}_\phi^{app} = B + C + f_1, \\ &\partial_t \tilde{u}_x^{app} + \frac{\tilde{u}_\phi^{app}}{r}\partial_\phi \tilde{u}_x^{app} - \frac{\nu}{r}\partial_r(r\partial_r \tilde{u}_x^{app}) - \frac{\nu}{r^2}\partial_{\phi\phi}\tilde{u}_x^{app} = D + E + F + f_2, \end{aligned} \tag{2.10}$$

where the (small) extra body forces are given by

$$\begin{aligned} A &= -(\rho\theta_\phi^0)^2 - 2\rho u_\phi^0 \theta_\phi^0, \\ B &= \nu \left[ -\frac{1}{r}\partial_r(r\partial_r u_\phi^0) + \frac{1}{r^2}u_\phi^0 - \frac{1}{r}\rho'(r)\theta_\phi^0 + \frac{1}{r^2}\rho\theta_\phi^0 - \rho''(r)\theta_\phi^0 \right], \end{aligned}$$

$$
\begin{aligned}
C &= \sqrt{\nu}\left[\frac{1}{r}\rho\partial_Z\theta_\phi^0 + 2\rho'(r)\partial_Z\theta_\phi^0\right],\\
D &= \rho\left(\frac{\rho}{r}-1\right)\theta_\phi^0\partial_\phi\theta_x^0 + \left(\frac{u_\phi^0(t,r)}{r} - u_\phi^0(t,1)\right)\rho\partial_\phi\theta_x^0 \\
&\quad + \left(\frac{1}{r}\partial_\phi u_x^0(t,r,\phi) - \partial_\phi u_x^0(t,1,\phi)\right)\rho\theta_\phi^0,\\
E &= -\nu\left[\frac{1}{r}\partial_r(r\partial_r u_x^0) + \frac{1}{r}\rho'(r)\theta_x^0 + \rho''(r)\theta_x^0\right.\\
&\quad \left. + \frac{1}{r^2}\partial_{\phi\phi}u_x^0 + \frac{1}{r^2}\rho\partial_{\phi\phi}\theta_x^0\right],\\
F &= \sqrt{\nu}\left[2\rho'(r)\partial_Z\theta_x^0 + \frac{1}{r}\rho\partial_Z\theta_x^0\right].
\end{aligned}
\tag{2.11}
$$

This approximate solution satisfies the desired boundary and initial conditions in the sense that

$$
\begin{aligned}
\tilde{\mathbf{u}}^{app}|_{r=1} &= \beta_\phi(t)\mathbf{e}_\phi + \beta_x(t,\phi)\mathbf{e}_x,\\
\tilde{\mathbf{u}}^{app}|_{t=0} &= a(r)\mathbf{e}_\phi + b(r,\phi)\mathbf{e}_x.
\end{aligned}
\tag{2.12}
$$

## 3. Error Estimates and Convergence Rates

We are now ready to prove our main result, that is, estimates on the error $\mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}$. We observe that the convergence of $\tilde{\mathbf{u}}^{app}$ to $\mathbf{u}^\nu$ also implies the convergence of $\mathbf{u}^\nu - \mathbf{u}^0 - \boldsymbol{\theta}^0$ to zero due to the choice of the cut-off function $\rho$ in (2.7) and the decay property of the boundary layer function $\boldsymbol{\theta}^0$.

For the purpose of convergence analysis, we introduce the **error solution** $\mathbf{u}^{err} = \mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}$, with associated pressure $p^{err} = p^\nu - p^{app}$. (We recall that, due to the symmetry of the flow, the pressure appears only in the equations for the cross-sectional components of the velocity, which are linear.) The error solution satisfies the following system of equations:

$$
(u_\phi^{err})^2 + 2u_\phi^{err}\tilde{u}_\phi^{app} - r\partial_r p^{err} = A, \tag{3.1a}
$$

$$
\partial_t u_\phi^{err} - \frac{\nu}{r}\partial_r(r\partial_r u_\phi^{err}) + \frac{\nu}{r^2}u_\phi^{err} = -B - C, \tag{3.1b}
$$

$$
\begin{aligned}
&\partial_t u_x^{err} + \frac{u_\phi^\nu}{r}\partial_\phi u_x^{err} + \frac{u_\phi^{err}}{r}\partial_\phi\tilde{u}_x^{app} - \frac{\nu}{r}\partial_r(r\partial_r u_x^{err})\\
&\qquad - \frac{\nu}{r^2}\partial_{\phi\phi}u_x^{err} = -D - E - F,
\end{aligned}
\tag{3.1c}
$$

where the body forcing terms $A$ through $F$ are given in (2.11), and the boundary conditions and initial data are specified as:

$$\mathbf{u}^{err}|_{r=1} = 0,$$
$$\mathbf{u}^{err}|_{t=0} = 0. \tag{3.2}$$

Our goal in this section is to show that $\mathbf{u}^{err}, p^{err}$ converge to zero in different s norms as $\nu$ tends to zero. More precisely, we aim at proving the following result.

**Theorem 3.1.** *Suppose the initial data $\mathbf{u}_0$, the boundary data $\boldsymbol{\beta}$, and the external forces $\mathbf{F}$ are given as in Proposition Appendix B.1. Then there exist positive constants $c$s independent of $\nu$, such that for any solution $\mathbf{u}^\nu$ of the system* (1.2)-(1.3),

$$||\mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}||_{L^\infty(0,T;L^2(\Omega))} \leq c\nu^{\frac{3}{4}}, \tag{3.3}$$
$$||\mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}||_{L^2(0,T;H^1(\Omega))} \leq c\nu^{\frac{1}{4}}, \tag{3.4}$$
$$||\mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}||_{L^\infty(0,T;H^1(\Omega))} \leq c\nu^{\frac{1}{4}}, \tag{3.5}$$
$$||\mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}||_{L^\infty(\Omega\times[0,T])} \leq c\nu^{\frac{1}{2}}, \tag{3.6}$$
$$||p^\nu - p^0||_{L^\infty(\Omega\times[0,T])} \leq c\nu^{\frac{1}{2}}, \tag{3.7}$$
$$||p^\nu - p^0||_{L^\infty(0,T;H^1(\Omega))} \leq c\nu^{\frac{1}{4}}. \tag{3.8}$$

Our main result, Theorem 1.1, follows from the theorem above and the decay property of the boundary layer corrector $\boldsymbol{\theta}^0$, once a choice of cut-off function $\rho$ has been made.

In view of the estimate

$$||\boldsymbol{\theta}^0||_{L^\infty(0,T;L^2(\Omega))} \approx c\nu^{\frac{1}{4}},$$

and (3.3), by the triangle inequality we can derive sharp convengence rates in viscosity as an immediate corollary of Theorem 3.1.

**Corollary 3.2.** *Under the hypotheses of Theorem 3.1, the following optimal convergence rate holds:*

$$c_1\nu^{\frac{1}{4}} \leq ||\mathbf{u}^\nu - \mathbf{u}^0||_{L^\infty(0,T;L^2(\Omega))} \leq c_2\nu^{\frac{1}{4}}, \tag{3.9}$$

*where $c_1$ and $c_2$ are positive constants, independent of $\nu$.*

The proof of Theorem 3.1 consists of several parts. We first show that the extra body force terms are small. The $L^\infty(L^2)$ and $L^2(H^1)$ estimates then follows directly. Estimates in $L^\infty(\Omega \times [0,T])$ are derived, instead, via the maximum principle and the anisotropic embedding theorem. The $L^\infty(H^1)$ estimate requires a different approach, which entails two distinct bounds, one near boundary, the other in the interior, obtained by introducing a further cut-off function. The convergence of the pressure follows from the convergence of the velocity field.

*3.1. Smallness of the extra body forcing terms*

We first verify that the extra body forcing terms $A$-$F$ in the right-hand-side of the equations in (3.1) are all small in some appropriate sense. Here and below, with a slight abuse of notation, $c$ denotes a generic constant, independent of the viscosity $\nu$, which may change from line to line. Also, we set $\langle Z \rangle = \sqrt{1+Z^2}$.

**Lemma 3.3.** *Suppose the initial data $\mathbf{u}_0$, the boundary data $\boldsymbol{\beta}$, and the forces $\mathbf{F}$ are given as in Proposition Appendix B.1 in Appendix B. Then the following estimates for $A$-$F$ given in* (2.11) *hold:*

$$||\frac{A}{r^2}||_{L^\infty(0,T;L^1(\Omega))} \leq c\nu^{\frac{1}{2}}, \tag{3.10a}$$
$$||\frac{A}{r}||_{L^\infty(0,T;L^2(\Omega))} \leq c\nu^{\frac{1}{4}}, \tag{3.10b}$$
$$||B+C||_{L^\infty(0,T;L^2(\Omega))} \leq c\nu^{\frac{3}{4}}, \tag{3.10c}$$
$$||D+E+F||_{L^\infty(0,T;L^2(\Omega))} \leq c\nu^{\frac{3}{4}}, \tag{3.10d}$$
$$||\partial_\phi(D+E+F)||_{L^\infty(0,T;L^2(\Omega))} \leq c\nu^{\frac{3}{4}}, \tag{3.10e}$$
$$||B+C||_{L^\infty(\Omega\times[0,T])} \leq c\nu^{\frac{1}{2}}, \tag{3.10f}$$
$$||D+E+F||_{L^\infty(\Omega\times[0,T])} \leq c\nu^{\frac{1}{2}}. \tag{3.10g}$$
$$||B+C||_{L^\infty(0,T;L^2(\Omega'))} \leq c\nu, \tag{3.10h}$$
$$||D+E+F||_{L^\infty(0,T;L^2(\Omega'))} \leq c\nu, \tag{3.10i}$$

*for any subset $\Omega'$ of $\Omega$ such that the closure $\overline{\Omega'} \subset \Omega$.*

PROOF. We first observe that inequality (3.10a) follows from the estimate:

$$||\frac{A}{r^2}||_{L^1(\Omega)} = \int_{\frac{1}{4}}^{1} \frac{(\rho\theta_\phi^0)^2 + 2|\rho u_\phi^0 \theta_\phi^0|}{r^2}\, r dr$$

$$
\begin{aligned}
&\leq \quad c\big(1+||u_\phi^0||_{L^\infty(\Omega)}\big)\int_{\frac14}^1 \big(\theta_\phi^0(t,\frac{1-r}{\sqrt{\nu}})\big)^2+|\theta_\phi^0(t,\frac{1-r}{\sqrt{\nu}})|\,dr\\
&\leq \quad c\nu^{\frac12}\big(1+||u_\phi^0||_{L^\infty(\Omega)}\big)\int_0^\infty \big(\theta_\phi^0(t,Z)\big)^2+|\theta_\phi^0(t,Z)|\,dZ\\
&\leq \quad c\nu^{\frac12},
\end{aligned}
\tag{3.11}
$$

where we have utilized the regularity and decay properties of the corrector $\theta_\phi^0$ and the fact that the term $A$ contains the cut-off function $\rho$. (See Lemma Appendix A.2 and Remark Appendix A.1 in Appendix Appendix A for further details.) Estimate (3.10b) is deduced in the similar fashion. The constants $c$ in (3.10a) and (3.10b) depend on the norms of $||\mathbf{u}_0||_{H^2(\Omega)}$, $||\mathbf{F}||_{L^\infty(0,T;H^2(\Omega))}$ and $||\boldsymbol{\beta}||_{L^\infty(0,T;H^2(\Omega))}$.

We now turn to estimates $||B+C||_{L^\infty(0,T;L^2(\Omega))}$ and $||E+F||_{L^\infty(0,T;L^2(\Omega))}$. Making the change of variable $Z=\frac{1-r}{\sqrt{\nu}}$ in computing $||\theta_\phi^0||_{L^2(\Omega_\infty)}$ yields a factor of $\nu^{\frac14}$ in the bounds below, which follow from similar arguments as before:

$$
\begin{aligned}
&||B||_{L^\infty(0,T;L^2(\Omega))}\leq c\nu||\Delta(u_\phi^0\mathbf{e}_\phi)||_{L^\infty(0,T;L^2(\Omega))}+c\nu^{\frac54}||\theta_\phi^0||_{L^\infty(0,T;L^2(0,\infty))},\\
&||C||_{L^\infty(0,T;L^2(\Omega))}\leq c\nu^{\frac34}||\partial_Z\theta_\phi^0||_{L^\infty(0,T;L^2(0,\infty))},\\
&||E||_{L^\infty(0,T;L^2(\Omega))}\leq c\nu||\Delta u_x^0||_{L^\infty(0,T;L^2(\Omega))}\\
&\qquad\qquad\qquad\qquad +c\nu^{\frac54}\big(||\theta_x^0||_{L^\infty(0,T;L^2(0,\infty))}+||\partial_{\phi\phi}\theta_x^0||_{L^\infty(0,T;L^2(0,\infty))}\big),\\
&||F||_{L^\infty(0,T;L^2(\Omega))}\leq c\nu^{\frac34}||\partial_Z\theta_\phi^0||_{L^\infty(0,T;L^2(0,\infty))}.
\end{aligned}
\tag{3.12}
$$

These in turn give immediately (3.10c) and (3.10d).

To estimate the norm of $D$, we decompose $D$ into three parts $D=I_1+I_2+I_3$, with

$$
\begin{aligned}
\|I_1\|^2_{L^2(\Omega)}&=||\rho(\frac{\rho}{r}-1)(\theta_\phi^0)\partial_\phi\theta_x^0||^2_{L^2(\Omega)}=\int_0^{2\pi}\int_0^1\rho^2(\frac{\rho}{r}-1)^2(\theta_\phi^0)^2(\partial_\phi\theta_x^0)^2\,rdrd\phi\\
&\leq c\int_0^{2\pi}\left(\int_{\frac14}^{\frac12}(\theta_\phi^0)^2(\partial_\phi\theta_x^0)^2\,rdr+\int_{\frac12}^1(r-1)^2(\theta_\phi^0)^2(\partial_\phi\theta_x^0)^2\,rdr\right)d\phi\\
&\leq c\int_0^{2\pi}\int_{\frac{1}{2\sqrt{\nu}}}^{\frac{3}{4\sqrt{\nu}}}\nu^{\frac32}(\theta_\phi^0)^2(\partial_\phi\theta_x^0)^2Z^2\,dZd\phi+\\
&\qquad\qquad +c\int_0^{2\pi}\int_0^{\frac{1}{2\sqrt{\nu}}}\nu^{\frac32}(\theta_\phi^0)^2(\partial_\phi\theta_x^0)^2Z^2\,dZd\phi
\end{aligned}
$$

$$\leq c\nu^{\frac{3}{2}}||\theta_\phi^0||^2_{L^\infty(0,+\infty)}||\langle Z\rangle\,\partial_\phi\theta_x^0||^2_{L^2(\Omega_\infty)}, \tag{3.13}$$

and

$$\begin{aligned}
\|I_2\|^2_{L^2(\Omega)} &= ||\rho\big(\frac{u_\phi^0(t,r)}{r} - u_\phi^0(t,1)\big)\partial_\phi\theta_x^0||^2_{L^2(\Omega)} \\
&= \int_0^{2\pi}\int_0^1 \left[\rho\big(\frac{u_\phi^0(t,r)}{r} - u_\phi^0(t,1)\big)\partial_\phi\theta_x^0\right]^2 rdrd\phi \\
&= \int_0^{2\pi}\int_0^1 \left[\rho\frac{r-1}{r}\big(\partial_r u_\phi^0(t,\xi) - u_\phi^0(t,1)\big)\partial_\phi\theta_x^0\right]^2 rdrd\phi, \\
&\leq c\nu^{\frac{3}{2}}\big(||u_\phi^0||_{L^\infty(\Omega)} + ||\partial_r u_\phi^0||_{L^\infty(\Omega)}\big)^2||\langle Z\rangle\,\partial_\phi\theta_x^0||^2_{L^2(\Omega_\infty)}, \qquad (3.14)
\end{aligned}$$

and finally

$$\begin{aligned}
\|I_3\|^2_{L^2(\Omega)} &= ||\rho\big(\frac{\partial_\phi u_x^0(t,r,\phi)}{r} - \partial_\phi u_x^0(t,1,\phi)\big)\theta_\phi^0||^2_{L^2(\Omega)} \\
&= \int_0^{2\pi}\int_0^1 \left[\rho\big(\frac{\partial_\phi u_x^0(t,r,\phi)}{r} - \partial_\phi u_x^0(t,1,\phi)\big)\theta_\phi^0\right]^2 rdrd\phi \\
&= \int_0^{2\pi}\int_0^1 \left[\rho\frac{r-1}{r}\big(\partial_{r\phi} u_x^0(t,\xi,\phi) - \partial_\phi u_x^0(t,1,\phi)\big)\theta_\phi^0\right]^2 rdrd\phi, \\
&\leq c\nu^{\frac{3}{2}}\big(||\partial_\phi u_x^0||_{L^\infty(\Omega)} + ||\partial_{r\phi} u_x^0||_{L^\infty(\Omega)}\big)^2||\langle Z\rangle\,\theta_\phi^0||^2_{L^2(0,+\infty)}. \qquad (3.15)
\end{aligned}$$

We remark that we have imposed enough regularity to ensure the validity of the computations above (see Lemma Appendix A.2, Appendix A.3 and Appendix B.2 in Appendices Appendix A and Appendix B.) Inequalities (3.10c) and (3.10d) then follow from (3.12)-(3.15) with constants $c$ depending on $||\mathbf{u}_0||_{H^4(\Omega)}$, $||\mathbf{F}||_{L^\infty(0,T;H^4(\Omega))}$, $||\boldsymbol{\beta}||_{L^\infty(0,T;H^4(\Omega))}$.

Estimates (3.10h) and (3.10i) contain only the forcing terms $C$, $D$ and $F$. We suppose that $\bar{\Omega}' \subset B(0,\sigma)$ with $B(0,\sigma)$ being a ball of radius $\sigma < 1$. We discuss in detail how to bound the first term in $C$, all other terms can be bounded in a similar fashion:

$$\begin{aligned}
||\frac{1}{r}\rho\partial_Z\theta_\phi^0||^2_{L^2(\bar{\Omega}')} &\leq c\int_0^\sigma \big(\partial_Z\theta_\phi^0(t,\frac{1-r}{\sqrt{\nu}})\big)^2\,dr \qquad (3.16) \\
&\leq \frac{c\nu^{\frac{1}{2}}}{1-\sigma}\int_0^\sigma \frac{1-r}{\sqrt{\nu}}\big(\partial_Z\theta_\phi^0(t,\frac{1-r}{\sqrt{\nu}})\big)^2\,dr \\
&= c\nu\int_{\frac{1-\sigma}{\sqrt{\nu}}}^{\frac{1}{\sqrt{\nu}}} Z(\partial_Z\theta_\phi^0)^2\,dZ \leq c\nu||\langle Z\rangle\,\partial_Z\theta_\phi^0||^2_{L^2(0,\infty)}
\end{aligned}$$

Finally, (3.10f) is a direct consequence of the estimates for the corrector $\boldsymbol{\theta}^0$ contained in Lemma Appendix A.2 and Lemma Appendix A.3, as well as the regularity of solutions to the Euler equations stated in Lemma Appendix B.2. The constant $c$ here depends on $||\mathbf{u}_0||_{H^4(\Omega)}$, $||\mathbf{F}||_{L^\infty(0,T;H^4(\Omega))}$, $||\boldsymbol{\beta}||_{L^\infty(0,T;H^4(\Omega))}$. One can derive (3.10g) similarly to (3.10d) employing the $L^\infty$ norm instead. The constant $c$ in (3.10g) depends , however, on more regular data in $H^7(\Omega)$, see Lemma Appendix A.3.

**Remark 3.1.** *It is mentioned that the interior estimates can be improved up to any order of $\nu$ for terms $C$, $D$, $F$. However, the interior estimates* (3.10h) *and* (3.10i) *are optimal because of the appearance of $\Delta u_\phi^0$ and $\Delta u_x^0$ in terms $B$ and $E$.*

**Remark 3.2.** *We did not try to optimize the regularity condition we imposed on the data $\mathbf{u}_0$, $\mathbf{F}$ and $\boldsymbol{\beta}$, because the boundary layer exists even if the data is assumed smooth.*

*3.2. The $L^\infty(L^2)$ and $L^2(H^1)$ convergence*

We recall that the error solution $\mathbf{u}^{err} = \mathbf{u}^\nu - \tilde{\mathbf{u}}^{app}$, $p^{err} = p^\nu - p^0$ satisfies the system (3.1)-(3.2).

It will be convenient here to work in Cartesian rather than cylindrcal coordinated. We observe that equations (3.1b), (3.1c) together with the initial boundary conditions (3.2) form a closed weakly coupled parabolic system which can be rewritten in Cartesian coordinates as

$$\partial_t \mathbf{v}_v^{err} - \nu\Delta_v \mathbf{v}_v^{err} = \mathbf{g_2}, \tag{3.17a}$$

$$\partial_t v_3^{err} + (\mathbf{u}_v^\nu \cdot \nabla_v) v_3^{err} - \nu\Delta_v v_3^{err} = g_3, \tag{3.17b}$$

where $\mathbf{v}^{err} \equiv \mathbf{u}^{err}$ in Cartesian coordinates, that is,

$$\mathbf{v}^{err}(t, x_1, x_2) = (v_1^{err}, v_2^{err}, v_3^{err}) := u_r^{err}\mathbf{e}_r + u_\phi^{err}\mathbf{e}_\phi + u_x^{err}\mathbf{e}_x$$

with

$$\begin{aligned}
&v_1^{err}(t, x_1, x_2) = -u_\phi^{err}\sin\phi,\\
&v_2^{err}(t, x_1, x_2) = u_\phi^{err}\cos\phi,\\
&v_3^{err}(t, x_1, x_2) = u_x^{err},\\
&\mathbf{v}_v^{err} = v_1^{err}\mathbf{e}_{x_1} + v_2^{err}\mathbf{e}_{x_2} = u_r^{err}\mathbf{e}_r + u_\phi^{err}\mathbf{e}_\phi,
\end{aligned}$$

together with homogeneous initial and boundary conditions

$$\begin{aligned} \mathbf{v}^{err}|_{r=1} &= 0, \\ \mathbf{v}^{err}|_{t=0} &= 0. \end{aligned} \tag{3.18}$$

The forcing terms $\mathbf{g_2}, g_3$ are given by

$$\begin{aligned} \mathbf{g_2} &= -(B+C)\big(-\tfrac{x_2}{\sqrt{x_1^2+x_2^2}}, \tfrac{x_1}{\sqrt{x_1^2+x_2^2}}\big), \\ g_3 &= -(D+E+F) - (\mathbf{v}_v^{err}\cdot\nabla_v)\tilde{u}_x^{app}. \end{aligned} \tag{3.19}$$

We notice that the cross-sectional component $\mathbf{v}_v^{err}$ satisfies a two-component (scalar) heat equation (3.17a). Therefore standard energy estimates and the maximum principle together with the estimates (3.10c) and (3.10f) in Lemma 3.3 yields

$$\begin{aligned} ||\mathbf{v}_v^{err}||_{L^\infty(0,T;L^2(\Omega))} &\leq c\nu^{\frac{3}{4}}, \\ ||\mathbf{v}_v^{err}||_{L^2(0,T;H^1(\Omega))} &\leq c\nu^{\frac{1}{4}}, \\ ||\mathbf{v}_v^{err}||_{L^\infty(0,T;H^1(\Omega))} &\leq c\nu^{\frac{1}{4}}, \\ ||\mathbf{v}_v^{err}||_{L^\infty(\Omega\times[0,T])} &\leq c\nu^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

For later use, we also derive an interior estimate on $||\mathbf{v}_v^{err}||_{L^\infty(0,T;L^2(\Omega'))}$ for $\Omega' \subset\subset \Omega$. Let $\eta(r)$ be a smooth function with compact support in $\Omega$. Multiplying equation (3.17a) by $\eta^2\mathbf{v}_v^{err}$, and integrating the resulting equations by parts leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\eta\mathbf{v}_v^{err}||^2_{L^2(\Omega)} + \nu||\eta\nabla_v\mathbf{v}_v^{err}||^2_{L^2(\Omega)} \\ &\leq c||B+C||_{L^2(\Omega')}||\eta\mathbf{v}_v^{err}||_{L^2(\Omega)} - \nu\int_\Omega \nabla_v\mathbf{v}_v^{err}\cdot(2\eta\nabla_v\eta)\cdot\mathbf{v}_v^{err}\,dx \\ &\leq c\nu||\eta\mathbf{v}_v^{err}||_{L^2(\Omega)} + c\nu^{\frac{7}{4}}||\eta\nabla_v\mathbf{v}_v^{err}||_{L^2(\Omega)}, \end{aligned} \tag{3.21}$$

where we have employed (3.10h) in Lemma 3.3 and the $L^\infty(L^2)$ estimate in (3.20). Applying first Cauchy's inequality, and then Grönwall's inequality, we then obtain

$$||\eta\mathbf{v}_v^{err}||_{L^\infty(0,T;L^2(\Omega))} + \nu^{\frac{1}{2}}||\eta\nabla_v\mathbf{v}_v^{err}||_{L^2(0,T;L^2(\Omega))} \leq c\nu. \tag{3.22}$$

We now notice that the last term in $g_3$ can be rewritten as:

$$(\mathbf{v}_v^{err} \cdot \nabla_v)\tilde{u}_x^{app} = \frac{u_\phi^{err}}{r}\partial_\phi \tilde{u}_x^{app}.$$

We then conclude again from the definition of $\mathbf{u}^{app}$ given in (2.8), the decay properties of the corrector $\boldsymbol{\theta}^0$ found in Appendix Appendix A, and the regularity of solutions to the Euler system in Lemma Appendix B.2, that

$$||\frac{1}{r}\partial_\phi \tilde{u}_x^{app}||_{L^\infty(\Omega\times[0,T])} \leq c, \tag{3.23}$$

with a constant $c$ depending on $||\mathbf{u}_0||_{H^3(\Omega)}$, $||\mathbf{F}||_{L^\infty(0,T;H^3(\Omega))}$, and $||\boldsymbol{\beta}||_{L^\infty(0,T;H^3(\Omega))}$, but independent of $\nu$. Therefore one has the following uniform estimates by (3.20) and (3.23):

$$||(\mathbf{v}_v^{err} \cdot \nabla_v)\tilde{u}_x^{app}||_{L^\infty(\Omega\times[0,T])} \leq c\nu^{\frac{1}{2}}. \tag{3.24}$$

Applying the same energy argument to equation (3.17b) gives

$$\begin{aligned} ||v_3^{err}||_{L^\infty(0,T;L^2(\Omega))} &\leq c\nu^{\frac{3}{4}}, \\ ||v_3^{err}||_{L^2(0,T;H^1(\Omega))} &\leq c\nu^{\frac{1}{4}}, \\ ||v_3^{err}||_{L^\infty(0,T;L^2(\Omega'))} &\leq c\nu. \end{aligned} \tag{3.25}$$

by inequalities (3.10d), (3.10i) in Lemma 3.3 and estimates (3.20), (3.22), (3.24).

### *3.3. Uniform in space and time convergence*

We begin by observing that the uniform convergence of the tangential component $\mathbf{v}_v^{err}$ has been already derived in the previous subsection via the maximum principle. Similar uniform estimates on $v_3^{err}$ can be derived via maximum principle as well since $v_3^{err}$ satisfies a (scalar) advection-diffusion equation with source term. For this purpose, we define the differential operator $L$ by

$$L = \partial_t + \mathbf{u}_v^\nu \cdot \nabla_v - \nu\Delta.$$

A simple calculation shows that

$$L(v_3^{err}) \leq L(\textstyle\int_0^t ||g_3(s)||_{L^\infty(\Omega)}\, ds), \text{ and}$$

$$v_3^{err} \leq \int_0^t ||g_3(s)||_{L^\infty(\Omega)}\, ds, \quad \text{on } \mathscr{P}\Omega$$

where $\mathscr{P}\Omega$ is the parabolic boundary of the domain $\Omega \times [0,T]$. Then the comparison principle for linear parabolic equations (see e.g. [14]) implies that $v_3^{err} \leq \int_0^t ||g_3||_{L^\infty(\Omega)}\, ds$ in $\Omega \times [0,T]$. Similarly, we have $v_3^{err} \geq -\int_0^t ||g_3||_{L^\infty(\Omega)}\, ds$. One then concludes from estimates (3.10g), (3.20) and (3.23) that

$$||v_3^{err}||_{L^\infty(\Omega\times[0,T])} \leq \Big| \int_0^t ||g_3||_{L^\infty(\Omega)}\, ds \Big| \leq T||g_3||_{L^\infty(\Omega\times[0,T])} \leq c\nu^{\frac{1}{2}}, \tag{3.26}$$

with a constant $c$ depending on $T$, $||\mathbf{u}_0||_{H^7(\Omega)}$, $||\mathbf{F}||_{L^\infty(0,T;H^7(\Omega))}$, and $||\boldsymbol{\beta}||_{L^\infty(0,T;H^7(\Omega))}$.

**Remark 3.3.** *An alternative proof of uniform bounds in $L^\infty(\Omega \times [0,T])$ is based on the use of anisotropic Sobolev-type embedding (see for instance [26, 28] for the case of flat boundary.) In the case, as our setting, of curved boundaries, the main idea is to perform separate estimates, one valid next the boundarty, the other in the interior. Near the boundary, curvilinear coordinates allow to generalize the flat case result (see Lemma 3.4 below, which is a counterpart of Remark 4.2 in [26]). Away from the boundary, on the other hand, we expect to employ a direct energy estimate due to the absence of the boundary layer. This alternative approach has the advantage that it can handle systems where the maximum principle may be invalid. This dual approach will be utilized to derive $L^\infty(H^1)$ estimates.*

**Lemma 3.4.** *Suppose the domain $D$ is an annulus $D = \{(r,\theta)|0 < R_1 < r < R_2, \theta \in (0, 2\pi)\}$. Then for any function $u \in H^1(\Omega)$ satisfying either $u|_{r=R_1} = 0$ or $u|_{r=R_2} = 0$, there exists a constant $C$ depending only on $R_1$ such that*

$$\begin{aligned} ||u||_{L^\infty(D)} \leq\ & c\Big( ||u||^{\frac{1}{2}}_{L^2(D)} ||\frac{\partial u}{\partial r}||^{\frac{1}{2}}_{L^2(D)} + ||\frac{\partial u}{\partial r}||^{\frac{1}{2}}_{L^2(D)} ||\frac{\partial u}{\partial \theta}||^{\frac{1}{2}}_{L^2(D)} \\ & + ||u||^{\frac{1}{2}}_{L^2(D)} ||\frac{\partial^2 u}{\partial r \partial \theta}||^{\frac{1}{2}}_{L^2(D)} \Big). \end{aligned} \tag{3.27}$$

The proof is straightforward via Agmon type embedding in the azimuthal direction together with embedding (interpolation) in the radial direction. Generalization to general curvilinear coordinates as well as high dimension can be considered as well.

### 3.4. Converegence in $L^\infty(H^1)$

The goal of this section is to derive $L^\infty(H^1)$ estimate for $v_3^{err}$, given that an $L^\infty(H^1)$ estimate of $\mathbf{v}_v^{err}$ was already obtained in (3.20). This estimate is the most interesting given that it involves normal gradients of the error solution.

We employ again the the two-step approach described above: first, we derive an estimate near the boundary based on the better control we have on tangential derivatives even in the presence of a boundary layer; second, we derive a standard interior energy estimate away from the boundary layer. In order to separate the boundary layer from the interior, we introduce a further cut-off function $\psi(r)$ with an appropriately chosen support in $\Omega$ (to be specified below).

Let us denote $w = \psi u_x^{err} = \psi v_3^{err}$. Then $w$ satisfies the following equation written in polar coordinates:

$$\partial_t w + \frac{u_\phi^\nu}{r}\partial_\phi w - \frac{\nu}{r}\partial_r(r\partial_r w) - \frac{\nu}{r^2}\partial_{\phi\phi} w = -\frac{\psi u_\phi^{err}}{r}\partial_\phi \tilde{u}_x^{app} - \\ \psi(D+E+F) - \nu u_x^{err}\Delta_v\psi - 2\nu\psi'(r)\partial_r u_x^{err}, \tag{3.28}$$

with homogeneous initial and boundary conditions

$$w|_{r=1} = 0,$$
$$w|_{t=0} = 0.$$

#### 3.4.1. Estimate near the boundary

To emphasize that this is a construction near the boundary, we will $\psi_b(r)$ for $\psi(r)$ and $w_b$ for $w$ in (3.28). We take $\psi_b(r)$ to be a smooth function defined on $[0,1]$ such that

$$\psi_b(r) = \begin{cases} 1 & r \in [\frac{1}{2}, 1], \\ 0 & r \in [0, \frac{1}{3}]. \end{cases} \tag{3.29}$$

First, we multiply equation (3.28) by $-\partial_{\phi\phi} w_b \cdot r$ and then integrate in $r$ and $\phi$, in light of estimate (3.10e) in Lemma 3.3,

$$\frac{1}{2}\frac{d}{dt}||\partial_\phi w_b||^2_{L^2(\Omega)} + \nu||\partial_{\phi r} w_b||^2_{L^2(\Omega)} + \nu||\frac{\partial_{\phi\phi} w_b}{r}||^2_{L^2(\Omega)}$$
$$\leq c\Big(||\psi_b\partial_\phi D||_{L^2(\Omega)} + ||\psi_b\partial_\phi E||_{L^2(\Omega)} + ||\psi_b\partial_\phi F||_{L^2(\Omega)} +$$
$$(||\Delta u_x^0||_{L^\infty(\Omega)} + ||\partial_{\phi\phi}\theta_x^0||_{L^\infty(\Omega_\infty)})||\psi u_\phi^{err}||_{L^2(\Omega)}\Big)||\partial_\phi w_b||_{L^2(\Omega)} +$$
$$\big(\nu||u_x^{err}\Delta_v\psi_b||_{L^2(\Omega)} + 2\nu||\psi_b'(r)\partial_r u_x^{err}||_{L^2(\Omega)}\big)||\frac{\partial_{\phi\phi} w_b}{r}||_{L^2(\Omega)}$$

$$\leq c\nu^{\frac{3}{4}}||\partial_\phi w_b||_{L^2(\Omega)} + c\nu||\frac{\partial_{\phi\phi}w_b}{r}||_{L^2(\Omega)}. \tag{3.30}$$

Then it follows from Grönwall's inequality and estimate (3.25) that

$$\begin{aligned}
&||\partial_\phi w_b||_{L^\infty(0,T;L^2(\Omega))}+ \\
&\qquad \sqrt{\nu}\big(||\partial_{\phi r} w_b||_{L^2(0,T;L^2(\Omega))} + ||\frac{\partial_{\phi\phi}w_b}{r}||_{L^2(0,T;L^2(\Omega))}\big) \leq c\nu^{\frac{3}{4}}.
\end{aligned} \tag{3.31}$$

In order to obtain an estimate for $\partial_r w_b$, we multiply by $-\frac{1}{r}\partial_r(r\partial_r w_b)\cdot r$ on both sides of equation (3.28) and integrate it by parts

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\partial_r w_b||^2_{L^2(\Omega)} + \nu||\frac{1}{r}\partial_r(r\partial_r w_b)||^2_{L^2(\Omega)} + \nu||\frac{\partial_{r\phi}w_b}{r}||^2_{L^2(\Omega)} \\
&\leq c\Big((||u_\phi^{err}||_{L^\infty(\Omega)} + ||\tilde{u}_\phi^{app}||_{L^\infty(\Omega)})||\partial_\phi w_b||_{L^2(\Omega)} + \nu||u_x^{err}||_{L^2(\Omega)}+ \\
&\quad \nu||\psi_b'(r)\partial_r u_x^{err}||_{L^2(\Omega)} + ||\psi_b(D+E+F)||_{L^2(\Omega)}+ \\
&\qquad ||\partial_\phi \tilde{u}_x^{app}||_{L^\infty(\Omega)}||\psi_b u_\phi^{err}||_{L^2(\Omega)}\Big)||\frac{1}{r}\partial_r(r\partial_r w_b)||_{L^2(\Omega)}+ \\
&\qquad\quad c\nu||\partial_\phi w_b||_{L^2(\Omega)}||\frac{\partial_{r\phi}w_b}{r}||_{L^2(\Omega)}.
\end{aligned} \tag{3.32}$$

Young's inequality and Grönwall's inequality then yield

$$\begin{aligned}
&||\partial_r w_b||_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\nu}||\frac{1}{r}\partial_r(r\partial_r w_b)||_{L^2(0,T;L^2(\Omega))} \\
&\qquad\qquad + \sqrt{\nu}||\frac{\partial_{r\phi}w_b}{r}||_{L^2(0,T;L^2(\Omega))} \leq c\nu^{\frac{1}{4}}.
\end{aligned} \tag{3.33}$$

where we have used estimates (3.23) and (3.31).

*3.4.2. Interior estimate*

We now turn to the estimates in the interior of $\Omega$. To this end, we let $\psi_i(r) = 1 - \psi_b(r)$ so that

$$\psi_i(r) = \begin{cases} 1 & r \in [0, \frac{1}{2}], \\ 0 & r \in [\frac{2}{3}, 1]. \end{cases} \tag{3.34}$$

We rewrite equation (3.28) in Cartesian coordinates as

$$\begin{aligned}
\partial_t w_i + (\mathbf{u}_h^\nu \cdot \nabla_v)w_i - \nu\Delta_v w_i = -\psi_i(D+E+F) - \nu v_3^{err}\Delta_v\psi_i \\
- 2\nu\psi_i'(r)\partial_r v_3^{err} - \big((\psi_i \mathbf{v}_v^{err})\cdot\nabla_v\big)\tilde{u}_x^{app},
\end{aligned} \tag{3.35}$$

with homogeneous initial and boundary conditions.

Multiplying (3.35) by $w_i$ and integrating the resulting equation over $\Omega$ gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||w_i||^2_{L^2(\Omega)} + \nu||\nabla_v w_i||^2_{L^2(\Omega)} \\
&\leq ||\psi_i(D+E+F)||_{L^2(\Omega)}||w_i||_{L^2(\Omega)} + c\nu||v_3^{err}||_{H^1(\Omega)}||w_i||_{L^2(\Omega)} \\
&\qquad + c||\frac{\partial_\phi \tilde{u}_x^{app}}{r}||_{L^\infty(\Omega)}||\psi_i u_\phi^{err}||_{L^2(\Omega)}||w_i||_{L^2(\Omega)}. \qquad (3.36)
\end{aligned}$$

By utilizing the estimates (3.10i), (3.22), (3.25), and the tangential estimate on the approximate solution (3.23), we deduce that

$$||w_i||_{L^\infty(0,T;L^2(\Omega))} + \nu^{\frac{1}{2}}||\nabla_v w_i||_{L^2(0,T;L^2(\Omega))} \leq c\nu. \qquad (3.37)$$

In particular,

$$||\nabla_v w_i||_{L^2(0,T;L^2(\Omega))} \leq c\nu^{\frac{1}{2}}. \qquad (3.38)$$

Furthermore, by multiplying equation (3.35) by $-\Delta_v w_i$ and integrating over the domain $\Omega$, one has that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla_v w_i||^2_{L^2(\Omega)} + \nu||\Delta_v w_i||^2_{L^2(\Omega)} \\
&\leq ||\psi_i(D+E+F)||_{L^2(\Omega)}||\Delta_v w_i||_{L^2(\Omega)} + \nu||v_3^{err}||_{H^1(\Omega)}||\Delta_v w_i||_{L^2(\Omega)} \\
&\qquad + c||\frac{\partial_\phi \tilde{u}_x^{app}}{r}||_{L^\infty(\Omega)}||\psi_i u_\phi^{err}||_{L^2(\Omega)}||\Delta_v w_i||_{L^2(\Omega)} \\
&\qquad\qquad + \int_\Omega (\mathbf{u}_v^\nu \cdot \nabla_v) w_i \Delta_v w_i \, dx. \qquad (3.39)
\end{aligned}$$

We now recall that $\tilde{\mathbf{u}}_v^{app} = \tilde{u}_\phi^{app}\mathbf{e}_\phi$, that $\nabla_v \cdot \mathbf{u}_v^{app} = 0$, and that $\mathbf{u}^\nu = \mathbf{v}^{err} + \tilde{\mathbf{u}}^{app}$. Consequently, all terms in the right hand side of equation (3.39) except the last one can be estimated in the same way as in (3.36)-(3.37). We deal with the last term as follows:

$$\begin{aligned}
&\int_\Omega (\mathbf{u}_v^\nu \cdot \nabla_v) w_i \Delta_v w_i \, dx \\
&= \int_\Omega (\mathbf{v}_v^{err} \cdot \nabla_v) w_i \Delta_v w_i \, dx + \int_\Omega (\tilde{\mathbf{u}}_v^{app} \cdot \nabla_v) w_i \Delta_v w_i \, dx \\
&\leq c\nu^{\frac{1}{2}}||\nabla_v w_i||_{L^2(\Omega)}||\Delta_v w_i||_{L^2(\Omega)} - \int_\Omega (\nabla_v \tilde{\mathbf{u}}_v^{app} \cdot \nabla_v w_i) \cdot \nabla_v w_i \, dx
\end{aligned}$$

$$
\begin{aligned}
&\le c||\nabla_v w_i||^2_{L^2(\Omega)} + \frac{\nu}{4}||\Delta_v w_i||^2_{L^2(\Omega)} + ||\nabla_v \tilde{\mathbf{u}}_v^{app}||_{L^\infty(\Omega')}||\nabla_v w_i||^2_{L^2(\Omega)} \\
&\le c\big(||u_\phi^0 \mathbf{e}_\phi||_{H^{2+s}(\Omega)} + ||Z\partial_Z \theta_\phi^0||_{L^\infty(0,\infty)}\big)||\nabla_v w_i||^2_{L^2(\Omega)} \\
&\qquad + c||\nabla_v w_i||^2_{L^2(\Omega)} + \frac{\nu}{4}||\Delta_v w_i||^2_{L^2(\Omega)},
\end{aligned}
\tag{3.40}
$$

where $\Omega' = \{r \le \frac{2}{3}\}$ by the definition of the the cut-off function (3.34). By introducing (3.40) back into (3.39), applying Young's inequality, integrating in time $t$, we finally obtain, utilizing (3.38),

$$
||\nabla_v w||_{L^\infty(0,T;L^2(\Omega))} \le c\nu^{\frac{1}{2}}. \tag{3.41}
$$

Combining estimates (3.25), (3.31), (3.33) and (3.41) gives the desired estimate

$$
||v_3^{err}||_{L^\infty(0,T;H^1(\Omega))} \le c\nu^{\frac{1}{4}}. \tag{3.42}
$$

**Remark 3.4.** *An alternative way of deriving the $L^\infty(H^1)$ estimate is to include higher-order terms in the asymptotic expansion* (2.1)-(2.2)*. We address this point in Section 4.*

*3.5. Convergence of the pressure*

We first recall the following calculus formula for a vector function $\mathbf{u} = v(r)\mathbf{e}_\phi$

$$
\nabla \mathbf{u}_v = \begin{pmatrix} -\partial_r v \sin\phi\, \mathbf{e}_r - \frac{v}{r}\cos\phi\, \mathbf{e}_\phi \\ \partial_r v \cos\phi\, \mathbf{e}_r - \frac{v}{r}\sin\phi\, \mathbf{e}_\phi \end{pmatrix}. \tag{3.43}
$$

Then it follows directly from equation (3.1a) that

$$
\begin{aligned}
||\partial_r p^{err}||_{L^2(\Omega)} &\le ||\frac{(u_\phi^{err})^2}{r}||_{L^2(\Omega)} + ||\frac{2u_\phi^{err}\tilde{u}_\phi^{app}}{r}||_{L^2(\Omega)} + ||\frac{A}{r}||_{L^2(\Omega)} \\
&\le (||u_\phi^{err}||_{L^\infty(\Omega)} + 2||\tilde{u}_\phi^{app}||_{L^\infty(\Omega)})||\frac{u_\phi^{err}}{r}||_{L^2(\Omega)} + c\nu^{\frac{1}{4}} \\
&\le c||\nabla_v v_h^{err}||_{L^2(\Omega)} + c\nu^{\frac{1}{4}} \le c\nu^{\frac{1}{4}}
\end{aligned}
\tag{3.44}
$$

where we used the estimates (3.10b) and (3.20) as well as the calculus identity above.

Next, we integrate equation (3.1a) to find that, assuming $p^{err}(1) = 0$

$$
-p^{err} = \int_r^1 \frac{(u_\phi^{err})^2}{s} + \frac{2u_\phi^{err}\tilde{u}_\phi^{app}}{s} - \frac{A}{s}\, ds. \tag{3.45}
$$

Therefore estimates (3.10a) and (3.20) yield

$$
\begin{aligned}
&||p^{err}||_{L^\infty(\Omega)} \\
&\leq ||\frac{u_\phi^{err}}{s}||^2_{L^2(\Omega)} + c(||\frac{u_\phi^0}{r}||_{L^\infty(\Omega)} + ||\theta_\phi^0||_{L^\infty(0,+\infty)})||u_\phi^{err}|| + c||\frac{A}{s^2}||_{L^1(\Omega)} \\
&\leq ||\nabla_v v_h^{err}||^2_{L^2(\Omega)} + c_1\nu^{\frac{1}{2}} + c_2\nu^{\frac{1}{2}} \leq c\nu^{\frac{1}{2}}. \qquad (3.46)
\end{aligned}
$$

## 4. Improved convergence rate

We ask whether the rates of converegence in viscosity presented in our main theorem, Theorem 1.1, are optimal. A heuristic argument using the order of the expansion in $\nu$ indicates that some of the rates are suboptimal. Optimal rate of convergence can be deduced by formally expanding the Nevier-Stokes solution to higher orders as it is classically done (see for instance [27, 19, 34] among others). However, expanding to higher order requires correspondingly more stringent compatibility conditions between the initial and boundary data, as discussed in Appendix Appendix B. Below, we present an asymptotic expansion up the first order (which is the next order) to illustrate the point and for the sake of simplicity.

### *4.1. Formal asymptotics*

Similarly to (2.1) and (2.2), we now assume that the approximate Navier-Stokes solution has the form:

$$\mathbf{u}^{\mathbf{app},\mathbf{1}}(t,r,\phi) := \mathbf{u}^{\mathbf{ou}}(t,r) + \mathbf{u}^{\mathbf{c}}(t, \frac{1-r}{\sqrt{\nu}}, \phi), \qquad (4.1)$$

$$p^{app,1}(t,r) := p^0(t,r) + \sqrt{\nu}p^1(t,r) + \sqrt{\nu}q^1(t, \frac{1-r}{\sqrt{\nu}}). \qquad (4.2)$$

where

- $\mathbf{u}^{\mathbf{ou}}(t,r,\phi) = \mathbf{u}^{\mathbf{0}}(t,r) + \sqrt{\nu}\mathbf{u}^{\mathbf{1}}(t,r)$ is the outer solution, valid in $\Omega$;
- $\mathbf{u}^{\mathbf{c}}(t, \frac{1-r}{\sqrt{\nu}}, \phi) = \boldsymbol{\theta}^{\mathbf{0}}(t, \frac{1-r}{\sqrt{\nu}}, \phi) + \sqrt{\nu}\boldsymbol{\theta}^{\mathbf{1}}(t, \frac{1-r}{\sqrt{\nu}}, \phi)$ is the corresponding boundary layer solution, which is valid in $\Omega_\infty$.

In terms of the stretched coordinate $Z = \frac{1-r}{\sqrt{\nu}}$ the corrector satisfies the following matching conditions

$$\boldsymbol{\theta}^{\boldsymbol{i}} \to 0 \text{ as } Z \to \infty, \qquad (4.3)$$

where $i = 0, 1$.

The equations satisfied by the outer solutions and correctors can be easily derived by keeping only terms with the same order in $\nu$:

1. The leading order $\mathbf{u^0}(t,r,\phi) = (0, u_\phi^0(t,r), u_x^0(t,r,\phi))$ satisfies reduced Euler equation (1.6) with initial data (1.7).
2. The first order of outer solution $\mathbf{u}^1(t,r,\phi) = (u_\phi^1(t,r), u_x^1(t,r,\phi))$ satisfies the following equations

$$
\begin{aligned}
&-2u_\phi^0 u_\phi^1 + r\partial_r p^1 = 0,\\
&\partial_t u_\phi^1 = 0,\\
&\partial_t u_x^1 + \frac{u_\phi^0}{r}\partial_\phi u_x^1 + \frac{u_\phi^1}{r}\partial_\phi u_x^0 = 0,\\
&(u_\phi^1, u_x^1)|_{t=0} = (0,0).
\end{aligned} \tag{4.4}
$$

   Since $(u_\phi^1, u_x^1)$ satisfies transport equations with homogeneous initial data, it follows that $(u_\phi^1, u_x^1) \equiv 0$, and consequently, we can take $p^1 = 0$ for convenience.
3. The leading order of the boundary layer profile $\boldsymbol{\theta^0}(t,Z,\phi)$ satisfies system (2.4) in Section 2.
4. The first order of the boundary layer profile $\boldsymbol{\theta^1}(t,Z,\phi) = (0, \theta_\phi^1, \theta_x^1)$ satisfies the following system:

$$
\begin{aligned}
&(\theta_\phi^0)^2 + 2u_\phi^0(t,1)\theta_\phi^0 = -\partial_Z q^1,\\
&\partial_t\theta_\phi^1 = \partial_{ZZ}\theta_\phi^1 - \partial_Z\theta_\phi^0,\\
&\partial_t\theta_x^1 + u_\phi^0(t,1)\partial_\phi\theta_x^1 + \theta_\phi^0\partial_\phi\theta_x^1 - \partial_{ZZ}\theta_x^1 = -\theta_\phi^1\partial_\phi\theta_x^0\\
&\quad - \theta_\phi^1\partial_\phi u_x^0(t,1,\phi) + Z\theta_\phi^0(\partial_\phi\partial_r u_x^0(t,1,\phi) - \partial_\phi u_x^0(t,1,\phi))\\
&\quad + Z(\partial_r u_\phi^0(t,1) - u_\phi^0(t,1))\partial_\phi\theta_x^0 - Z\theta_\phi^0\partial_\phi\theta_x^0 - \partial_Z\theta_x^0,\\
&(\theta_\phi^1, \theta_x^1)|_{Z=0} = (0,0),\\
&(\theta_\phi^1, \theta_x^1)|_{Z=\infty} = 0,\ (\theta_\phi^1, \theta_x^1)|_{t=0} = 0.
\end{aligned} \tag{4.5}
$$

The existence, regularity, and decay properties of solutions to system (4.5) can be derived in a manner similar to that for the system satisfied by the zeroth-order expansion under higher regularity assumptions and higher compatibility conditions between the initial data and boundary data , as illustrated in Appendix Appendix A.

*4.2. Approximate solution*

The formal expansion $\mathbf{u^{app,1}}$ presented in the previous subsection cannot be directly used to accommodate for the fact that the decay properties of the corrector arise in an infinite domain. As in Section 2, we remedy this point by introducing a truncation factor in the radial direction. We then define a truncated approximation $\tilde{\mathbf{u}}^{\mathbf{app},\mathbf{1}}(t,r,\phi) = (\tilde{u}_\phi^{app,1}(t,r), \tilde{u}_x^{app,1}(t,r,\phi))$ with

$$
\begin{aligned}
&\tilde{u}_\phi^{app,1}(t,r) := u_\phi^0(t,r) + \rho(r)(\theta_\phi^0 + \sqrt{\nu}\theta_\phi^1)(t, \frac{1-r}{\sqrt{\nu}}), \\
&\tilde{u}_x^{app,1}(t,r,\phi) := u_x^0(t,r,\phi) + \rho(r)(\theta_x^0 + \sqrt{\nu}\theta_x^1)(t, \frac{1-r}{\sqrt{\nu}}, \phi), \\
&\tilde{p}^{app}(t,r) = p^0(t,r) + \sqrt{\nu} q^1(t, \frac{1-r}{\sqrt{\nu}}),
\end{aligned}
\tag{4.6}
$$

where $\rho$ is defined in Section 2.

Then $\tilde{\mathbf{u}}^{app,1}$ satisfies the following system

$$
\begin{aligned}
&-(\tilde{u}_\phi^{app,1})^2 + r\partial_r \tilde{p}^{app,1} = \hat{A}, \\
&\partial_t \tilde{u}_\phi^{app,1} - \frac{\nu}{r}\partial_r(r\partial_r \tilde{u}_\phi^{app,1}) + \frac{\nu}{r^2}\tilde{u}_\phi^{app,1} = f_1 + \hat{B} + \hat{C} + G, \\
&\partial_t \tilde{u}_x^{app,1} + \frac{\tilde{u}_\phi^{app,1}}{r}\partial_\phi \tilde{u}_x^{app,1} - \frac{\nu}{r}\partial_r(r\partial_r \tilde{u}_x^{app,1}) - \frac{\nu}{r^2}\partial_{\phi\phi}\tilde{u}_x^{app,1} \\
&= f_2 + \hat{D} + \hat{E} + \hat{F} + H,
\end{aligned}
\tag{4.7}
$$

where $\hat{A}, \hat{B}, \hat{C}, G, \hat{D}$, $\hat{E}$, $\hat{F}$ and $H$are given by

$$
\begin{aligned}
\hat{A} &= (r-\rho^2)(\theta_\phi^0)^2 + 2\theta_\phi^0(ru_\phi^0(t,1) - u_\phi^0) - 2\sqrt{\nu}(u_\phi^0 + \rho\theta_\phi^0)\theta_\phi^1 - \nu\rho^2(\theta_\phi^1)^2, \\
\hat{B} &= \nu\left[-\frac{1}{r}\partial_r(r\partial_r u_\phi^0) + \frac{1}{r^2}u_\phi^0 - \frac{1}{r}\rho'(r)\theta_\phi^0 + \frac{1}{r^2}\rho\theta_\phi^0 - \rho''(r)\theta_\phi^0 \right. \\
&\qquad \left. + \frac{\rho}{r}\partial_Z\theta_\phi^1 + 2\rho'(r)\partial_Z\theta_\phi^1\right], \\
\hat{C} &= \sqrt{\nu}\left[\rho(\frac{1}{r} - 1)\partial_Z\theta_\phi^0 + 2\rho'(r)\partial_Z\theta_\phi^0\right], \\
G &= \nu^{\frac{3}{2}}\left[\frac{\rho}{r^2}\theta_\phi^1 - \frac{\rho'(r)}{r}\theta_\phi^1 - \rho''(r)\theta_\phi^1\right], \\
\hat{D} &= \rho\left[\frac{\partial_\phi u_x^0}{r} - \partial_\phi u_x^0(t,1,\phi) + \sqrt{\nu}Z\left(\partial_{r\phi}u_x^0(t,1,\phi) - \partial_\phi u_x^0(t,1,\phi)\right)\right]\theta_\phi^0 \\
&\qquad + \rho(r)\left[\frac{u_\phi^0}{r} - u_\phi^0(t,1) + \sqrt{\nu}Z\left(\partial_r u_\phi^0(t,1) - u_\phi^0(t,1)\right)\right]\partial_\phi\theta_x^0
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \rho(r)\left(\frac{\rho}{r} - 1 - \sqrt{\nu} Z\right)\theta_\phi^0 \partial_\phi \theta_x^0,\\
\hat{E} &= \sqrt{\nu}\left[\rho\left(\frac{\rho}{r}-1\right)\left(\theta_\phi^0\partial_\phi\theta_x^1 + \theta_\phi^1\partial_\phi\theta_x^0\right) + \rho\left(\frac{1}{r}-1\right)\partial_Z\theta_x^0 + 2\rho'\partial_Z\theta_x^0\right. \qquad (4.8)\\
&\quad \left. +\rho\left(\frac{\partial_\phi u_x^0}{r} - \partial_\phi u_x^0(t,1,\phi)\right)\theta_\phi^1 + \rho\left(\frac{u_\phi^0}{r} - u_\phi^0(t,1)\right)\partial_\phi\theta_x^1\right],\\
H &= \nu\left[-\frac{1}{r}\partial_r(r\partial_r u_x^0) - \left(\frac{\rho'}{r} + \rho''\right)\theta_x^0 + \left(\frac{\rho}{r} + 2\rho'\right)\partial_Z\theta_x^1\right.\\
&\quad \left. +\frac{\rho^2}{r}\theta_\phi^1\partial_\phi\theta_x^1 - \frac{1}{r^2}\partial_{\phi\phi}u_x^0 - \frac{\rho}{r^2}\partial_{\phi\phi}\theta_x^0\right],\\
\hat{F} &= \nu^{\frac{3}{2}}\left[-\frac{\rho'}{r}\theta_x^1 - \rho''\theta_x^1 - \frac{\rho}{r^2}\partial_{\phi\phi}\theta_x^1\right].
\end{aligned}
$$

The corresponding boundary conditions and initial data are imposed as

$$
\begin{aligned}
\tilde{\mathbf{u}}^{app,1}|_{t=0} &= (0, a(r), b(r,\phi)),\\
\tilde{\mathbf{u}}^{app,1}|_{r=1} &= (0, \beta_\phi(t), \beta_x(t,\phi)).
\end{aligned}
$$

*4.3. Convergence*

We define again an error solution $\hat{\mathbf{u}}^{err}(t,r,\phi) :=$ $(\hat{u}_\phi^{err}(t,r), \hat{u}_x^{err}(t,r,\phi))$ and $\hat{p}^{err}$, where

$$
\begin{aligned}
\hat{u}_\phi^{err}(t,r) &= u_\phi^\nu(t,r) - \tilde{u}_\phi^{app,1}(t,r),\\
\hat{u}_x^{err}(t,r,\phi) &= u_x^\nu(t,r,\phi) - \tilde{u}_x^{app,1}(t,r,\phi),\\
\hat{p}^{err} &= p^\nu(t,r) - \tilde{p}^{app,1}(t,r).
\end{aligned} \tag{4.9}
$$

Then the error solution satisfies the following system

$$
\begin{aligned}
&(\hat{u}_\phi^{err})^2 + 2\hat{u}_\phi^{err}\tilde{u}_\phi^{app,1} - r\partial_r\hat{p}^{err} = \hat{A},\\
&\partial_t\hat{u}_\phi^{err} - \frac{\nu}{r}\partial_r(r\partial_r\hat{u}_\phi^{err}) + \frac{\nu}{r^2}\hat{u}_\phi^{err} = -\hat{B} - \hat{C} - G,\\
&\partial_t\hat{u}_x^{err} + \frac{u_\phi^\nu}{r}\partial_\phi\hat{u}_x^{err} + \frac{\hat{u}_\phi^{err}}{r}\partial_\phi\tilde{u}_x^{app,1} - \frac{\nu}{r}\partial_r(r\partial_r\hat{u}_x^{err}) - \frac{\nu}{r^2}\partial_{\phi\phi}\hat{u}_x^{err}\\
&\quad = -(\hat{D} + \hat{E} + \hat{F} + H),
\end{aligned} \tag{4.10}
$$

with corresponding boundary and initial conditions

$$
\hat{\mathbf{u}}^{err}|_{r=1} = 0,
$$

$$\hat{\mathbf{u}}^{err}|_{t=0} = 0. \tag{4.11}$$

One can verify that the extra body force terms $\hat{B}, \cdots, \hat{F}$ are small in the following sense:

$$\begin{aligned} &||\hat{B} + \hat{C} + G||_{L^\infty(0,T;L^2(\Omega))} \le c\nu, \\ &||\hat{D} + \hat{E} + \hat{F} + H||_{L^\infty(0,T;L^2(\Omega))} \le c\nu, \\ &||\hat{B} + \hat{C} + G||_{L^\infty(0,T;L^\infty(\Omega))} \le c\nu, \\ &||\hat{D} + \hat{E} + \hat{F} + H||_{L^\infty(0,T;L^\infty(\Omega))} \le c\nu. \end{aligned} \tag{4.12}$$

Utilizing the new expansion (4.1) and applying exactly the same technique as in the proof of Theorem 3.1, we are able to improve the convergence rate of Theorem 3.1 as follows:

**Theorem 4.1.** *Assume that the initial data $a(r), b(r,\phi)$ and the boundary data $(\beta_\phi, \beta_x)$ satisfy appropriate high order compatibility conditions as described in Appendix Appendix B. In addition, we assume that $\mathbf{u}^0 \in H^m(\Omega), \boldsymbol{\beta} \in H^2(0,T;H^m(\Omega)), m \ge 9$. Then we have that*

$$\|\mathbf{u}^\nu - \mathbf{u^0} - \rho(r)(\boldsymbol{\theta}^\mathbf{0} + \sqrt{\nu}\boldsymbol{\theta}^\mathbf{1})\|_{L^\infty(0,T;H^1(\Omega))} \le \mathcal{O}(\nu^{\frac{1}{2}}), \tag{4.13}$$

$$\|\mathbf{u}^\nu - \mathbf{u^0} - \rho(r)(\boldsymbol{\theta}^\mathbf{0} + \sqrt{\nu}\boldsymbol{\theta}^\mathbf{1})\|_{L^\infty((0,T)\times\Omega)} \le \mathcal{O}(\nu), \tag{4.14}$$

*where the cut-off function $\rho(r)$ is defined in Section 2.*

**Remark 4.1.** *Estimate* (4.14) *is sharper than the corresponding result for plane-parallel flows (inequality (6.13) in theorem 6.1 of [19]), since we employ here the maximum principle instead of the anisotropic Sobolev embedding, and we impose more compatibility and regularity conditions on the data. Therefore we can reach optimal convergence rates in viscosity.*

As a corollary, we deduce the following optimal convergence rates for the zeroth order approximation.

**Corollary 4.2.** *Under the same assumption as Theorem 4.1, we have*

$$\begin{aligned} c_3\nu^{\frac{1}{4}} \le \|\mathbf{u}^\nu - \mathbf{u^0} - \rho(r)\boldsymbol{\theta}^\mathbf{0}\|_{L^\infty(0,T;H^1)} \le c_4\nu^{\frac{1}{4}}, \\ c_5\nu^{\frac{1}{2}} \le \|\mathbf{u}^\nu - \mathbf{u^0} - \rho(r)\boldsymbol{\theta}^\mathbf{0}\|_{L^\infty(\Omega\times[0,T])} \le c_6\nu^{\frac{1}{2}}, \end{aligned}$$

*where $c_3$, $c_4$, $c_5$ and $c_6$ are generic constants depending on $\mathbf{u}_0$ and $\boldsymbol{\beta}$ but independent of viscosity $\nu$.*

## Appendix A. Corrector estimates

In this appendix, we discuss the decay properties and the regularity of the correctors $\boldsymbol{\theta}^0$ and $\boldsymbol{\theta}^1$ governed by the Prandtl-type equations (2.4) and (4.5) respectively. For this purpose, we introduce the following general Prandtl-type boundary-layer problem for a corrector-like function $\theta$ over the domain $\Omega_\infty$:

$$\begin{aligned} &\partial_t\theta + a(t,Z)\partial_\phi\theta - \partial_{ZZ}\theta = h(t,\phi,Z) \quad \text{in } \Omega_\infty \times [0,T], \\ &\theta\big|_{Z=0} = 0, \quad \theta\big|_{Z=\infty} = 0, \\ &\theta\big|_{t=0} = 0, \end{aligned} \tag{A.1}$$

where $a(t,Z)$ and $h(t,\phi,Z)$ are given functions with the following regularity:

$$\begin{aligned} \partial_t^k a \in L^\infty(\Omega_\infty \times [0,T]), \quad &\langle Z\rangle^l\, \partial_t^k\partial_\phi^p h \in C(0,T;L^2(\Omega_\infty)) \\ &\text{for } k+p \le n, \quad k = 0,1, \end{aligned} \tag{A.2}$$

where $\langle Z\rangle = \sqrt{1+Z^2}$, $l$ $n$, and $m$ are given positive integers.

Moreover, we impose the following compatibility conditions on the data in problem (A.1):

$$\partial_t^k h(0,\phi,Z) = 0, \quad k = 0,1. \tag{A.3}$$

Then one can modify the approach in Xin and Yanagisawa [34] (Theorem 4.1) to obtain the following result.

**Proposition Appendix A.1.** *Assume conditions* (A.2)-(A.3) *hold. Then the Prandtl type boundary layer equation* (A.1) *has a unique solution such that*

$$\begin{aligned} &\langle Z\rangle^l\, \partial_\phi^{\alpha_1}\partial_Z^{\alpha_2}\theta \in C(0,T;L^2(\Omega_\infty)), \\ &\textit{for } \alpha_1 + [\frac{\alpha_2+1}{2}] \le n-1, \quad \alpha_2 \le 2, \end{aligned} \tag{A.4}$$

*and*

$$\begin{aligned} &\langle Z\rangle^l\, \partial_t^k\partial_\phi^p\theta \in C(0,T;L^2(\Omega_\infty)), \\ &\textit{for } k+p \le n-1, \quad k = 0,1. \end{aligned} \tag{A.5}$$

*In addition,*

$$\partial_t^k\theta(0,\phi,Z) = 0, \quad k = 0,1. \tag{A.6}$$

We now apply Proposition Appendix A.1 to equations (2.4). First, we notice that the $m$-th order compatibility conditions (B.2) on the data in equations (1.3)-(1.4) imply the following compatibility conditions on the data in equation (2.4):

$$\partial_t^p \big(\boldsymbol{\beta} - \mathbf{u}^0|_{r=1}\big)\Big|_{t=0} = 0, \quad p = 0, 1, \dots, [m/2], \tag{A.7}$$

where $[z]$ denoted the greatest integer in $z$. Now, define $j_\phi = [\beta_0(t) - u_\phi^0(t,1)]e^{-Z^2}$. Then one finds that $\theta_1 = \theta_\phi^0 - j_\phi$, where $\theta_\phi^0$ is defined below equation (2.2), satisfies equation (A.1) with $a(t,Z) = 0$ and

$$h(t,Z) = -[\beta_0'(t) - \partial_t u_\phi^0(t,1)]e^{-Z^2} + [\beta_0(t) - u_\phi^0(t,1)](4Z^2 - 2)e^{-Z^2}.$$

It is easy to verify that conditions (A.2)-(A.3) are satisfied with $p = 0$ if we assume $m \geq 4$. Therefore the conclusion of Proposition Appendix A.1 holds for $\theta_1$. Then it follows that

$$\langle Z \rangle^l \, \partial_Z^\alpha \theta_1 \in L^\infty([0,T] \times [0,+\infty)), \quad \alpha = 0, 1,$$

from the interpolation inequality

$$||\theta(t,Z)||_{L_t^\infty(L^\infty(0,+\infty))} \leq K ||\theta||^{\frac{1}{2}}_{L_t^\infty(L^2(0,+\infty))} ||\theta||^{\frac{1}{2}}_{L_t^\infty(H^1(0,+\infty))}.$$

The lemma below then follows from the definition of $\theta_1$ given above.

**Lemma Appendix A.2.** *Under the same conditions as in Proposition Appendix B.1 with $m \geq 4$, $\theta_\phi^0 \in \cap_{j=0}^{[m/2]} C^j([0,T]; H^{m-2j}(0,+\infty))$ and*

$$\langle Z \rangle^l \, \partial_Z^\alpha \theta_\phi^0 \in C(0,T; L^2(0,+\infty)), \quad \alpha \leq 2, \tag{A.8}$$

$$\langle Z \rangle^l \, \partial_Z^\alpha \theta_\phi^0 \in L^\infty([0,T] \times [0,+\infty)), \quad \alpha = 0, 1, \tag{A.9}$$

*and*

$$\partial_t^k \theta_\phi^0 = 0, \quad k = 0, 1. \tag{A.10}$$

**Remark Appendix A.1.** *An alternative way of deriving $L^\infty$ estimate in time and space for $\theta_\phi^0$ is to use a comparison principle of parabolic equation (see e.g. [19].) In fact, one can show that $\partial_t \theta_\phi^0 \in L^\infty([0,T] \times [0,+\infty))$ by the same method. Moreover, by integrating equation* (A.1) *one finds that $\theta_\phi^0 \in L^\infty(0,T; L^1(\Omega_\infty))$.*

We now similarly define $j_x = [\beta_x(t,\phi) - u_x^0(t,1,\phi)]e^{-Z^2}$ and $\theta_2 = \theta_x^0 - j_x$, where $\theta_x^0$ is also defined below equation (2.2). Then one easily verifies that $\theta_2$ satisfies equation (A.1) with

$$a(t,Z) = \theta_\phi^0 + u_\phi^0(t,1), \tag{A.11}$$
$$h(t,\phi,Z) = -\theta_\phi^0 \partial_\phi u_x^0(t,1,\phi) + [\beta_x(t,\phi) - u_x^0(t,1,\phi)](4Z^2-2)e^{-Z^2}$$
$$- a[\partial_\phi \beta_x(t,\phi) - \partial_\phi u_x^0(t,1,\phi)]e^{-Z^2} - [\partial_t \beta_x(t,\phi) - \partial_t u_x^0(t,1,\phi)]e^{-Z^2}. \tag{A.12}$$

It follows that $h \in C^1(0,T;H^{m-2})$ given the regularity of the data, that of the Euler solution $\mathbf{u}^0$, and the regularity of $\theta_\phi^0$, we established above. Therefore conditions (A.2)-(A.3) are satisfied with $n = m-2$ and $p \le m-2$ by Lemma Appendix A.2, Remark Appendix A.1 and compatibility condition (A.7), if we assume $m \ge 7$. We thus have the following lemma.

**Lemma Appendix A.3.** *Under the same condition as Proposition Appendix B.1 with $m \ge 7$, one has*

$$\begin{aligned} &\langle Z\rangle^l \, \partial_\phi^{\alpha_1}\partial_Z^{\alpha_2}\theta \in C(0,T;L^2(\Omega_\infty)), \\ &\textit{for } \alpha_1 + [\frac{\alpha_2+1}{2}] \le 4, \quad 0 \le \alpha_2 \le 2, \end{aligned} \tag{A.13}$$

$$\langle Z\rangle^l \, \partial_\phi^{\alpha_1}\partial_Z^{\alpha_2}\theta_x^0 \in L^\infty(\Omega_\infty \times [0,T]), \quad 0 \le \alpha_1 \le 2, 0 \le \alpha_2 \le 1, \tag{A.14}$$

$$\begin{aligned} &\langle Z\rangle^l \, \partial_t^k \partial_\phi^p \theta_x^0 \in C(0,T;L^2(\Omega_\infty)), \\ &\textit{for } p \le m-3, \quad k = 0,1, \end{aligned} \tag{A.15}$$

*and*

$$\partial_t^k \theta_x^0 = 0, \quad k = 0,1. \tag{A.16}$$

We note that estimate (A.14) follows from (A.13) and the following anisotropic Sobolev embedding result for the domain $\Omega_\infty$, which can be derived in the same way as Lemma 3.4

$$\begin{aligned} ||\theta||_{L^\infty(\Omega_\infty)} \le C\big(&||\theta||_{L^2}^{\frac{1}{2}}||\partial_\phi\theta||_{L^2}^{\frac{1}{2}} + ||\partial_Z\theta||_{L^2}^{\frac{1}{2}}||\partial_\phi\theta||_{L^2}^{\frac{1}{2}} \\ &+ ||\theta||_{L^2}^{\frac{1}{2}}||\partial_{\phi Z}\theta||_{L^2}^{\frac{1}{2}}\big). \end{aligned} \tag{A.17}$$

Finally, we notice that conclusions of Lemma Appendix A.2 and Lemma Appendix A.3 apply to $\theta_\phi^1$ and $\theta_x^1$, as long as we impose $m \ge 9$.

## Appendix B. Compatibility condition and regularity

To analyze the boundary layer for the pipe flows under consideration, we need to assume that the initial data, boundary conditions and body forcing term in (1.3)-(1.4) satisfy appropriate compatibility conditions so that the viscous solution is sufficiently regular.

Following Xin and Yanagisawa [34] (see also Temam [24]), we define the $p-$Cauchy data of (1.3) and (1.4) inductively by

$$\begin{aligned}
&\partial_t^0 \mathbf{u}^\nu\big|_{t=0} = \mathbf{u}_0,\\
&\partial_t^p \mathbf{u}_v^\nu\big|_{t=0} = \big(\nu\Delta_v \partial_t^{p-1}\mathbf{u}_v^\nu + \partial_t^{p-1}\mathbf{F}_1\big)\big|_{t=0},\\
&\partial_t^p u_x^\nu\big|_{t=0} = \Big(-\sum_{s=0}^{p-1}\binom{p-1}{s}\partial_t^s \mathbf{u}_v^\nu\cdot\nabla_v\partial_t^{p-1-s}u_x^\nu +\\
&\qquad\qquad \nu\Delta_v\partial_t^{p-1}u_x^\nu + \partial_t^{p-1}F_2\Big)\Big|_{t=0}.
\end{aligned} \tag{B.1}$$

Then $\boldsymbol{\beta}, \mathbf{u}_0, \mathbf{F} = (\mathbf{F}_1, F_2)$ are said to satisfy the compatibility condition of order $m$ for the initial boundary value problem (1.3)-(1.4) for any $\nu > 0$ if

$$\partial_t^p \mathbf{u}^\nu\big|_{t=0,r=1} = \partial_t^p \boldsymbol{\beta}\big|_{t=0}, \quad p = 0, 1, \ldots, m, \text{ for any } \nu > 0. \tag{B.2}$$

These compatibility conditions prevent the formation of an initial layer in the Navier-Stokes equation (1.2)-(1.3) due to the possible mismatch of the boundary and initial data. The zeroth order compatibility condition simply takes the form:

$$a(1) = \beta_\phi(0), \qquad b(1,\phi) = \beta_x(0,\phi), \tag{B.3}$$

and the first order compatibility condition are given by:

$$\begin{aligned}
&\partial_t\beta_\phi(0) = \nu\big(a'(1) + a''(1) - a(1)\big) + f_1(0,1),\\
&\partial_t\beta_x(0,\phi) = \nu\big(\partial_r b(1,\phi) + \partial_{rr} b(1,\phi) + \partial_{\phi\phi} b(1,\phi)\big) - a(1)\partial_\phi b(1,\phi)\\
&\qquad\qquad + f_2(0,1,\phi).
\end{aligned} \tag{B.4}$$

We notice that the first order compatibility condition involves the viscosity $\nu$. This undesirable dependence, however, can be eliminated if we impose that $\partial_r b(1,\phi) + \partial_{rr} b(1,\phi) + \partial_{\phi\phi} b(1,\phi) = 0$ and $\partial_t\beta_x(0,\phi) = -a(1)\partial_\phi b(1,\phi) + f_2(0,1,\phi)$.

Since we are working in a domain that is periodic in the $x$ direction, we employ the following Sobolev spaces for $m \in \mathbb{Z}_+$,:

$$H^m(Q) = \{f \in L^2(Q), D^\alpha f \in L^2(Q), |\alpha| \le m, \\ f \text{ periodic in the } x \text{ direction and the azimuthal direction } \phi\}.$$

We denote the subspace of functions in $H^m(Q)$ that are constant in $x$ by $H^m(\Omega)$. We also use $H^m(Q)$ to denote $(H^m(Q))^3$ for vector functions. Concerning the existence and regularity of the solution $\mathbf{u}^\nu$ to the initial boundary value problem (1.3)-(1.4) for fixed $\nu$, the following result is classical (see page 219 of [13] and [24], for instance.)

**Proposition Appendix B.1.** *Let $\nu > 0$ be a constant. Let $m$ be an integer. Suppose the forces and boundary data are smooth, $\mathbf{F}, \beta \in C^\infty(\bar{\Omega} \times [0,T])$, and the initial data $\mathbf{u}_0 \in H^m(\Omega)$ satisfies the compatibility condition of order $[m/2]$ for the initial boundary value problem* (1.3)-(1.4)*. Then there exists a unique solution $\mathbf{u}^\nu$ in the space $\bigcap_{j=0}^{[m/2]} C^j([0,T]; H^{m-2j}(\Omega))$.*

**Remark Appendix B.1.** *The requirement $\mathbf{F}, \boldsymbol{\beta} \in C^\infty(\bar{\Omega} \times [0,T])$ is purely for the ease of simplifying notation. In fact, the same conclusion holds under much less regularity on $\mathbf{F}$ and $\boldsymbol{\beta}$. We refer to [24] for details.*

The solution of (1.6)-(1.7) can be obtained by solving an ordinary differential equation and a transport equation. Therefore the well-posedness of $\mathbf{u}^0$ is readily established. For example, if $\mathbf{u}_0 \in H^m(Q)$, and $\mathbf{F} \in C(0,T; H^m(Q))$, then $\mathbf{u}^0 \in C(0,T; H^m(Q))$. (See Temam [25] for results concerning the existence of smooth solution to the full Euler equations. )Since we are also working in cylindrical coordiantes, we gather the regularity of solutions to the Euler equation in cylindrical coordinates in the following lemma.

**Lemma Appendix B.2.** *Suppose $\mathbf{u}_0 \in H^m(Q)$, $\mathbf{F} \in C(0,T; H^m(Q))$ with $m \ge 4$. Then one has, in polar coordinates':*

$$u_\phi^0, \partial_r u_\phi^0, \frac{u_\phi^0}{r}, \left(-\frac{1}{r}\partial_r(r\partial_r u_\phi^0) + \frac{u_\phi^0}{r^2}\right) \in L^\infty([0,T] \times \Omega), \tag{B.5}$$

$$u_x^0, \partial_r u_x^0, \partial_\phi u_x^0, \partial_{r\phi} u_x^0, \left(\frac{1}{r}\partial_r(r\partial_r u_x^0) + \frac{\partial_{\phi\phi} u_x^0}{r^2}\right) \in L^\infty([0,T] \times \Omega). \tag{B.6}$$

PROOF. Recall the **ansatz** (1.5):

$$\mathbf{u}^0 = u_\phi^0(t,r)\mathbf{e}_\phi + u_x^0(t,r,\phi)\mathbf{e}_x = \left(-u_\phi^0 \sin\phi, u_\phi^0 \cos\phi, u_x^0\right). \tag{B.7}$$

Noticing that $\mathbf{u}^0$ is independent of variable $x$, one concludes by Sobolev imbedding that

$$\mathbf{u}^0, \nabla\mathbf{u}^0, \Delta\mathbf{u}^0 \in L^\infty([0,T]\times\Omega) \tag{B.8}$$

Then (B.6) follows directly, given that

$$\begin{aligned}\frac{\partial}{\partial r} &= \frac{x_1}{r}\frac{\partial}{\partial x_1} + \frac{x_2}{r}\frac{\partial}{\partial x_2},\\ \frac{\partial}{\partial \phi} &= -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2},\end{aligned}$$

and using the fact $\Omega$ is bounded.

Next, the bound $\left(-\frac{1}{r}\partial_r(r\partial_r u_\phi^0) + \frac{u_\phi^0}{r^2}\right) \in L^\infty([0,T]\times\Omega)$ follows, since

$$\Delta(u_\phi^0 \sin\phi) = \left(\frac{1}{r}\partial_r(r\partial_r u_\phi^0) - \frac{u_\phi^0}{r^2}\right)\sin\phi \in L^\infty([0,T]\times\Omega)$$

In order to deduce $\partial_r u_\phi^0, \frac{u_\phi^0}{r} \in L^\infty([0,T]\times\Omega)$, we differentiate $u_\phi^0 \sin\phi$ and $u_\phi^0\cos\phi$ with respect to $x_1$ and $x_2$ respectively. One has

$$\partial_{x_1}\left(u_\phi^0\sin\phi\right) = \left(\partial_r u_\phi^0 - \frac{u_\phi^0}{r}\right)\cos\phi\sin\phi \in L^\infty([0,T]\times\Omega), \tag{B.9}$$

$$\partial_{x_2}\left(u_\phi^0\sin\phi\right) = \partial_r u_\phi^0(\sin\phi)^2 + \frac{u_\phi^0}{r}(\cos\phi)^2 \in L^\infty([0,T]\times\Omega), \tag{B.10}$$

$$\partial_{x_1}\left(u_\phi^0\cos\phi\right) = \partial_r u_\phi^0(\cos\phi)^2 + \frac{u_\phi^0}{r}(\sin\phi)^2 \in L^\infty([0,T]\times\Omega), \tag{B.11}$$

It follows from (B.9) that $\left(\partial_r u_\phi^0 - \frac{u_\phi^0}{r}\right) \in L^\infty([0,T]\times\Omega)$. On the other hand, the addition of (B.10) and (B.11) implies $\partial_r u_\phi^0 + \frac{u_\phi^0}{r} \in L^\infty([0,T]\times\Omega)$. Therefore, one obtains $\partial_r u_\phi^0, \frac{u_\phi^0}{r} \in L^\infty([0,T]\times\Omega)$.

**Acknowledgments**

This work was initiated while the last three authors were visiting the Institute for Mathematics and its Applications (IMA) at the University of Minnesota during the spring of 2010. The hospitality and support from IMA is greatly appreciated. The IMA receives major funding from the National Science Foundation and the University of Minnesota.

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