CHERN CLASSES OF LOGARITHMIC VECTOR FIELDS FOR LOCALLY QUASI-HOMOGENEOUS FREE DIVISORS

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Abstract. Let X be a nonsingular complex projective variety and D a locally quasi-homogeneous free divisor in X. In this paper we study a numerical relation between the Chern class of the sheaf of logarithmic derivations on X with respect to D, and the Chern-Schwartz-MacPherson class of the complement of D in X. Our result confirms a conjectural formula for these classes, at least after push-forward to projective space; it proves the full form of the conjecture for locally quasi-homogeneous free divisors in $\mathbb{P}^n$. The result generalizes several previously known results. For example, it recovers a formula of M. Mustata and H. Schenck for Chern classes for free hyperplane arrangements. Our main tools are Riemann-Roch and the logarithmic comparison theorem of Calderon-Moreno, Castro-Jimenez, Narvaez-Macarro, and David Mond. As a subproduct of the main argument, we also obtain a schematic Bertini statement for locally quasi-homogeneous divisors.

1. INTRODUCTION

Let X be a nonsingular variety defined over $\mathbb{C}$, D a reduced effective divisor and $U = X \smallsetminus D$ is the hypersurface complement. After a conjecture of Aluffi [Alu12], we raised the following question in [Lia]:

Question 1.1. *In the Chow ring $A_*(X)$, under what conditions is the formula*

$$c_{SM}(1_U) = c(\mathrm{Der}_X(-\log D)) \cap [X] \tag{1}$$

true?

The left hand side of the formula is the Chern-Schwartz-MacPherson class of the open subvariety U, and the right hand side is the total Chern class of the sheaf of logarithmic vector fields along D. In [Lia], we have proven that in the case where X is a complex surface (not necessarily a projective surface), the above formula is true if and only if the divisor D has only quasi-homogeneous singularities. This result strongly overlaps a result by Adrian Langer in [Lan03] Proposition 6.1 and Corollary 6.2.

This paper is a continuation of the investigation for conditions which guarantee the above formula. The main result of this paper is:

Theorem 1.2. *Let $X \subseteq \mathbb{P}^N$ be a nonsingular projective variety defined over $\mathbb{C}$, and let D be a locally quasi-homogeneous free divisor in X. Then the two classes appearing in* (1) *agree after push-forward to $\mathbb{P}^N$, that is:*

$$\int_X c_1(\mathscr{O}(1))^i \cap c_{SM}(1_U) = \int_X c_1(\mathscr{O}(1))^i \cap \Big(c(\mathrm{Der}_X(-\log D)) \cap [X]\Big) \tag{2}$$

for all $i \geq 0$.

Corollary 1.3. *If $X = \mathbb{P}^n$, formula* (1) *holds in the Chow ring for all locally quasi-homogeneous free divisors.*

As a special case when $i=0$, both sides of formula (2) give the topological Euler characteristic of U. For a general i, this formula may be seen as an analogue for Chern classes of the computation of the cohomology of U done by Castro et al. in [CJNMM96].

In [Alu], Aluffi proves that formula (1) is true for free arrangements of hypersurfaces that are locally analytically isomorphic to hyperplane arrangements; this includes the case of normal crossing divisors and of free hyperplane arrangements, verified earlier ([Alu99], [Alu12]). Aluffi's results along with the results in [Lia] have given convincing evidences why the freeness condition has to enter the picture when comparing these two classes.

It should be noted that in the context of hyperplane arrangement, the Chern polynomial of the sheaf of logarithmic vector fields has been compared with other naturally defined polynomials by several authors. Mustaţă and Schenck related the Poincaré polynomial of a free arrangement to $c_t(\Omega_X(\log D))$, the Chern polynomial of the sheaf of logarithmic differentials[MS01]. Aluffi related the characteristic polynomial of an arrangement to the Chern-Shwartz-MacPherson class of the arrangement complement, as well as the class in the Grothendieck ring of varieties $K_0(Var)$ of this complement[Alu12]. The Poincaré polynomial and the characteristic polynomial of an arrangement are closely related[OT92]. All these polynomials carry essentially the same information for free hyperplane arrangements. By Corollary 1.3, the result in this paper recovers Aluffi's result for hyperplane arrangements [Alu12], and generalizes it to arbitrary quasi-homogeneous free hypersurfaces arrangements in projective space. Taking into account the relation between Poincaré polynomial and CSM class polynomial stated above, the result from this paper also gives a generalization of Mustaţă and Schenck's result (Theorem 4.1 in [MS01]) independent of the original proof.

While the freeness of the divisor is a nice relevant condition, it alone does not guarantee the truth of formula (1), as pointed out by our previous study[Lia]. The additional quasi-homogeneity condition on the divisor gives further control over the singularities. One famous result is: the Tjurina number being equal to the Milnor number characterizes quasi-homogeneity for isolated hypersurface singularities[Sai71]. More generally, The quasi-homogeneity of divisors can be described by certain properties of the Tjurina Algebras[XY96]. In this paper, the quasi-homogeneity, together with the freeness of the divisor, are utilized to apply the logarithmic comparison theorem (LCT)[CJNMM96], which implies the classes in (1) have the same degree (the case $i=0$ in (2)).

As a result of our local analysis, we obtain a schematic version of Bertini's theorem for locally quasi-homogeneous divisors (see Corollary 3.12), strengthening Teissier's idealistic Bertini theorem ([Tei77] section 2.8) in this context.

The remaining of this paper will prove theorem 1.2 along the following roadmap:

(1) With the help of Riemann-Roch and LCT, we show the classes appearing in (1) have the same degree.
(2) We show a result of Bertini type: the freeness and the quasi-homogeneity of the pair (X,D) are preserved by intersecting a general hyperplane H. So this allows us to use LCT inductively on the new pair $(X\cap H, D\cap H)$.
(3) We study the relation between $\mathrm{Der}_{X\cap H}(-\log D\cap H)$ and $c_1(\mathscr{O}(1))\cdot c(\mathrm{Der}_X(-\log D))$, using an exact sequence which relates the sheaf of the logarithmic vector fields and the normal bundle. We also show $c_{SM}(1_U)$ behaves in a similar fashion.
(4) We conclude our proof by an induction on the dimension of X.

We will stick to the complex analytic category for all our discussions in section 2 and 3, while we will go the category of schemes when we apply some intersection theory in section 4. Throughout this paper, $\mathcal{O}_X$ refers to the sheaf of holomorphic functions on X and $\mathscr{O}_X$ refers to the structure sheaf of the scheme X.

I am very grateful to Jörg Schürmann, who showed me the connection between LCT and question 1.1. I feel greatly indebted to my advisor Paolo Aluffi, whose constant help and warm encouragement finally lead to this work.

2. The first step of the proof

Let D be a quasi-homogeneous free divisor in a complex algebraic projective variety X of dimension n, and U be the hypersurface complement $X \smallsetminus D$. In this section we prove $c_{SM}(1_U)$ and $c(\mathrm{Der}_X(-\log D)) \cap [X]$, as cycle classes in the Chow ring of X, have the same degree. For general information on Chern-Schwartz-MacPherson classes, see MacPherson's original paper [Mac74] and Example 19.1.7 in [Ful84].

We briefly recall the definition of free divisors and some of their basic properties[Sai80]. Over an open subset U (in the complex topology), let f be a defining equation of the divisor D. In this open set, a logarithmic vector field along D is a derivation $\theta \in \mathrm{Der}_X(U)$ satisfying $\theta f \in (f)$, and a logarithmic p-form with poles along D is a meromorphic p-form $\omega \in \Omega^p_X(\star D)(U)$ satisfying $f \cdot \omega \in \Omega^p_X(U)$ and $f \cdot \mathrm{d}\omega \in \Omega^p_X(U)$. This local information globalizes into coherent $\mathcal{O}_X$-modules $\mathrm{Der}_X(-\log D)$ and $\Omega^p_X(\log D)$. There is a natural dual pairing between $\mathrm{Der}_X(-\log D)$ and $\Omega^1_X(\log D)$, making them reflexive with respect to each other. A divisor is free if $\mathrm{Der}_X(-\log D)$ (and so $\Omega^1_X(\log D)$) is locally free. In this case, its rank equals $\dim X = n$.

Notice all objects discussed in the above paragraph can be defined more broadly either in the category of complex analytic manifold, or in the category of algebraic varieties over an algebraically closed field k. Similar properties hold. In this paper we are content with using these notions in the smaller category of projective complex varieties, in which the familiar GAGA principle applies. The GAGA principle allows us to pass from the Zariski topology to the complex topology, without changing any essential information of coherent sheaves. In particular, D is free as an algebraic divisor if and only if it is free as an analytic divisor in our setting.

According to the second of the defining properties of the logarithmic p-forms along D, the exterior differential makes the sheaves of logarithmic forms a complex $\Omega^\bullet_X(\log D)$. It is a subcomplex of $\Omega^\bullet_X(\star D)$, the complex of the sheaves of meromorphic forms with poles along D. When D is free, the sheaf of logarithmic p-forms is isomorphic to the pth exterior product of the sheaf of logarithmic 1-forms: $\Omega^p_X(\log D) \cong \Lambda^p \Omega^1_X(\log D)$.

Next we recall the definition of locally quasi-homogeneous divisors[CMNM09] (See also [CJNMM96]).

Definition 2.1. A germ of divisor $(D,p) \subset (X,p)$ is quasi-homogeneous if there are local coordinates $x_1, \dots, x_n \in \mathcal{O}_{X,p}$ with respect to which (D,p) has a weighted homogeneous defining equation (with strictly positive weights). We also say D is quasi-homogenous at p. A divisor D in an n-dimensional complex manifold X is locally quasi-homogeneous if the germ (D,p) is quasi-homogeneous for each point $p \in D$. A germ of divisor $(D,p) \subset (X,p)$ is locally quasi-homogeneous if the divisor D is locally quasi-homogeneous in a neighborhood of p.

Example 2.2 ([CJNMM96]). Consider the surface $D \subset \mathbb{C}^3$ given by $f(x,y,z) = x^5z + x^3y^3 + y^5z$. The germ of divisor $(D,0)$ is quasi-homogeneous with respect to the weight (1,1,1). However, $(D,0)$ is not locally quasi-homogeneous. In fact, (D,p) is not quasi-homogeneous for any point p on the z-axis other than the origin.

One benefit of considering locally quasi-homogenous free divisors is, that this class of divisors enjoys the following logarithmic comparison theorem (LCT):

Theorem 2.3 ([CJNMM96], see also [CMNM09] for generalizations)**.** *Let D be a locally quasi-homogeneous free divisor in the complex manifold X. Then the inclusion of complexes $\Omega^\bullet_X(\log D) \to \Omega^\bullet_X(\star D)$ is a quasi-isomorphism.*

Remark 2.4. Denote $j : U \to X$ the inclusion. The Grothendieck comparison theorem states the De Rham morphism:

$$\Omega^\bullet_X(\star D) \to \mathbf{R}j_*\mathbf{C}_U$$

is a quasi-isomorphism. Grothendieck's result together with theorem 2.3 imply that, for locally quasi-homogeneous free divisors, the logarithmic De Rham complex computes the cohomology of U.

Remark 2.5. In [CMNM09], it is pointed out that the Jacobian ideal J_D of linear type, with the freeness of the divisor, is enough to imply LCT. This condition, which is purely algebraic, may be used for generalizing theorem 1.2 in the future.

Now, we can prove:

Proposition 2.6. *For locally quasi-homogeneous free divisors in nonsingular projective complex varieties, $c_{SM}(1_U)$ and $c(\mathrm{Der}_X(-\log D)) \cap [X]$ have the same degree.*

Proof. By the functoriality of Chern-Schwartz-MacPherson transformation, we have:

$$\int_X c_{SM}(1_U) = \chi_c(U)$$

where $\int_X$ denotes degree of the dimension 0 component of the given class.

In the context of complex algebraic varieties, or compactifiable complex analytic manifold, It is well known that the Euler characteristic with compact support equals the usual Euler characteristic ([Sch03] Proposition 2.0.2), and the latter is computed by the logarithmic De Rham complex by remark 2.4:

$$\begin{aligned}
\chi_c(U) &= \chi(U) \\
&= \sum_i (-1)^i H^i\left(\mathbf{R}\Gamma(X; \Omega^\bullet_X(\log D)\right) \\
&= \sum_{p,q} (-1)^{p+q} H^p(X; \Omega^q_X(\log D)) \\
&= \sum_q (-1)^q \int_X \mathrm{ch}(\Omega^q_X(\log D)) \cdot \mathrm{Td}(X) \\
&= \int_X \sum_q (-1)^q \mathrm{ch}(\Lambda^q \Omega^1_X(\log D)) \cdot \mathrm{Td}(X) \\
&= \int_X c_n(\mathrm{Der}_X(-\log D)) \cdot \mathrm{Td}(\mathrm{Der}_X(-\log D))^{-1} \cdot \mathrm{Td}(X) \\
&= \int_X c_n(\mathrm{Der}_X(-\log D)) \cap [X] \\
&= \int_X c(\mathrm{Der}_X(-\log D)) \cap [X]
\end{aligned}$$

The fourth of these equalities comes from Riemann-Roch; the fifth one comes from the fact that $\Omega^p_X(\log D) \cong \Lambda^p \Omega^1_X(\log D)$ for free divisors; the sixth uses a formula showed in [Ful84]

Example 3.2.5; for the seventh, just notice that $c_n(\mathrm{Der}_X(-\log D)) \cap [X]$ is a 0-dimensional cycle and the Todd class starts from $1 + \dots$ for any vector bundle.

□

3. The second step of the proof

In this section we prove the freeness and the quasi-homogeneity of the pair (X, D) is preserved by intersecting with a general hyperplane of the ambient projective space $\mathbb{P}^N$. For this purpose, we first recall the "triviality lemma", which renders an idealistic test of local "Cartesian product with trivial factors" structures of divisors[Sai80][dJP00].

Lemma 3.1 (Triviality lemma)**.** *Let (S, p) locally be isomorphic to $(\mathbb{C}^{n+m}, 0)$ and denote $\mathcal{O}_{S,p} = \mathbb{C}\{x_1, \dots, x_n, y_1, \dots, y_m\}$ and let $h(x_1, \dots, x_n, y_1, \dots, y_m)$ be an element of $\mathcal{O}_{S,p}$. Then the following are equivalent:*

1. *The ideal $(\partial_{y_1} h, \dots, \partial_{y_m} h)$ is contained in the ideal $(h, \partial_{x_1} h, \dots, \partial_{x_n} h)$.*
2. *There exists a local biholomorphic map $\phi : (\mathbb{C}^{m+n}, 0) \to (\mathbb{C}^{m+n}, 0)$ and a holomorphic $v(x, y) \in \mathcal{O}^*_{S,p}$ such that*

$$\phi(x, y) = (\phi_1(x, y), \dots, \phi_n(x, y), y_1, \dots, y_m),$$

$$\phi(x, 0) = (x, 0), v(x, 0) \equiv 1 \text{ and } h \circ \phi(x, y) = v(x, y)h(x, 0).$$

This means that $D = \{h(x, y) = 0\}$ is locally at p isomorphic to some $(D' \times \mathbb{C}^m, (0, 0))$ where D' is a divisor in an open subset of $\mathbb{C}^n$.

The equivalent condition (1) implies that at a neighborhood U of p, we can find m vector fields, tangent to the divisor D at all its smooth points, and generate an m-dimensional linear subspace of TX_q at every point q of U. In fact, if $\partial_{y_i} h = a_{i1}(x, y)\partial_{x_1} h + \dots + a_{in}(x, y)\partial_{x_n} h + g_n(x, y)h$, then $\partial_{y_i} - a_{i1}\partial_{x_1} - \dots - a_{in}\partial_{x_n}$ gives such a vector field. Conversely, starting with m such vector fields, by integrating along these vector fields we find m functions (the parameters of the integral flows) $y'_1, \dots, y'_m$ which can be embedded to a complete set of local coordinates, and that $\partial_{y'_i} h \in (h)$.

In view of this comment, we can introduce for each point $p \in D$ a subspace TD_p of TX_p, which is the same as the tangent space of D at its smooth points:

Definition 3.2. The subspace TD_p is generated by all θ_p where θ is a locally defined non vanishing vector field tangent to D at all its smooth point. We call the dimension of TD_p the triviality index of D at p, and denote it by $t(p)$. Locally $(D, p) \cong (D' \times \mathbb{C}^{t(p)}, (0, 0))$, and D' has no further local Cartesian product factorization containing trivial factors.

Example 3.3. Consider the divisor $D : f(x, y, z) = z^2 - xy = 0$ in $\mathbb{C}^3$. A direct computation shows $\partial_x f \notin (\partial_y f, \partial_z f, f)$. Neither are similar relations obtained by permuting x, y, z true. According to condition (1) of theorem 3.1, at the origin there is no way to factor this divisor into a product containing trivial factors. So the triviality index at the origin is 0. The triviality index at other points are 2 because D is smooth except at the origin.

With this way of describing the tangent space of D (at singular points), we can now define transversal intersection of D with nonsingular varieties.

Definition 3.4. Let X and Y be complex analytic submanifolds of some ambient space, and D is a divisor in X. The divisor D intersect Y transversally at $p \in D \cap Y$ if TD_p intersect TY_p transversally. The divisor D intersect Y transversally if at all points belonging to $D \cap Y$ they intersect transversally.

Remark 3.5. E. Faber has recently studied transversality for singular hypersurfaces in [Fab]. It turns out that our definition of transversal intersection overlaps her definition of "splayed divisors" when one "splayed component" is a nonsingular hypersurface.

The following proposition is seen in a slightly different version in [CJNMM96] Lemma 2.3.

Proposition 3.6. *Let $X \subset \mathbb{P}^N$ be an n-dimensional nonsingular complex projective variety and D a locally quasi-homogeneous divisor in X. Let H be a hyperplane of $\mathbb{P}^N$ intersecting both D and X transversally. Then $D \cap H$ is a locally quasi-homogeneous divisor in the nonsingular $X \cap H$.*

Proof. First, that the space $X \cap H$ is nonsingular follows from the classical Bertini theorem. At any point $p \in H \cap D$, there exists a choice of local coordinates on X, such that $(X,p) \cong (\mathbb{C}^n, 0)$ and $(D,p) \cong (D', 0)$, where D' is defined by a quasi-homogeneous germ $f(x_0, \ldots, x_n)$. The nonsingular $X \cap H$ becomes a nonsigular germ $(X', 0)$ under this new coordinate system, with X' transversal to D' at 0. The transversality indicates that we can find a non vanishing vector field θ at a neighborhood of 0, tangent to D' at all its smooth points (that is $\theta f \in (f)$) and its value at the origin $\theta_0 \notin TX'_0$. There is at least one coordinate hyperplane (say $x_0 = 0$) which does not contain the vector θ_0. Therefore, the integral flow along the vector field θ gives a local isomorphism from X' to the coordinate hyperplane $x_0 = 0$. Clearly the integral flow stays inside D' if the initial point of the flow is in D'. Thus the described isomorphism takes $D' \cap X'$ to the set $\{q \in \mathbb{C}^n \mid f(x_0(q), \ldots, x_n(q)) = x_0(q) = 0\}$. In other words, the pair $(X', D' \cap X')$ is isomorphic to the pair $(V(x_0), V(f, x_0))$, where $V(\cdot)$ denotes the zero locus of the functions in the parenthesis. Finally, $f(0, x_2, \ldots, x_n)$ is of course quasi-homogeneous at 0 of the coordinate hyperplane $V(x_0)$. This proves that $D' \cap X'$ is quasi-homogeneous in X' at 0. □

Remark 3.7. The projective space $\mathbb{P}^N$ provides a global environment to apply the classical Bertini theorem, but it and the hyperplane H play no role in the local analysis.

By the spirit of the local analysis, if X_1 and X_2 are two nonsingular hypersurfaces of $\mathbb{C}^n$ containing 0, and θ is a vector field transversal to both at 0, then this vector field induces an isomorphism between X_1 and X_2 in a neighborhood of 0. Moreover, if $(D, 0) \subset (\mathbb{C}^n, 0)$ is a germ of divisor and θ is tangent to all smooth points of D, then we have the isomorphism of the pairs $(X_1, X_1 \cap D) \cong (X_2, X_2 \cap D)$ in a neighborhood of 0.

With a similar strategy, we also prove:

Proposition 3.8. *Let $X \subset \mathbb{P}^N$ be an n-dimensional nonsingular complex projective variety, D a free divisor in X, and H a hyperplane of $\mathbb{P}^N$ intersecting both D and X transversally. Then $D \cap H$ is a free divisor in the nonsingular $X \cap H$.*

Proof. Let $p \in H \cap D$ be any point in the intersection of the hyperplane and the divisor. According to the hypothesis, in a neighborhood U of p there exists a non vanishing vector field θ such that $\theta_p \notin TH_p$ and θ is tangent to D at all its smooth points. Lemma 3.1 implies that there is an isomorphism $(X,p) \cong (\mathbb{C}^n, 0)$ such that $(D,p) \cong (D' \times \mathbb{C}, (0,0))$, and that $(H \cap X, p) \cong (X', 0)$, where D' is a divisor in an open subset U of $\mathbb{C}^{n-1}$ and X' is a complex manifold of dimension $n-1$ transversal to the trivial factor. The flow along the trivial factor gives an isomorphism $(X', 0) \cong (\mathbb{C}^{n-1}, 0)$. It is also easy to see D' gets mapped to $X' \cap (D' \times \mathbb{C})$ under this flow. From this analysis we see that $D \cap H$ is free at p if and only if D' is free at 0. On the other hand we already know $D' \times \mathbb{C}$ is free at $(0,0)$ from the hypothesis. This condition is equivalent to the freeness of D' at 0 as a consequence of the "Saito's Criterion"[Sai80]. In fact, $\mathrm{Der}_{U\times\mathbb{C}}(-\log D'\times\mathbb{C}) \cong \mathrm{Der}_U(-\log D') \oplus \mathcal{O}_U$. The direct summand $\mathcal{O}_U$ corresponds to the trivial factor.(cf. [CJNMM96]) $\square$

We need the next proposition in our dimension counting argument.

Proposition 3.9. *Set $D_i = \{p \in D \mid t(p) = i\}$ for $i = 0, \dots n-1$. Then for a locally quasi-homogeneous divisor D, D_i is an analytic set of dimension at most i.*

Proof. Pick any point $p \in D$, There is no loss in assuming p is the origin $0 \in \mathbb{C}^n$. In a neighborhood U of the origin, D is defined by a quasi-homogeneous polynomial $f(x_0, \dots, x_n)$ with weight $(a_0, \dots, a_n; d)$. The equation $df = a_0x_0\partial_{x_0}f + \dots + a_nx_n\partial_{x_n}f$ implies that $t(q) \geq 1$ for $q \neq 0$. In fact, if $q \neq 0$, one of the coordinate functions (say x_0) must take non zero value at q. So x_0 is a unit in $\mathcal{O}_q$ and we get $\partial_{x_0}f = \frac{d}{x_0}f - \frac{a_1x_1}{x_0}\partial_{x_1}f - \dots - \frac{a_nx_n}{x_0}\partial_{x_n}f$. Applying lemma 3.1 we immediately see there is at least one trivial factor at q, which means only the origin itself may have triviality index 0. As a result, the set D_0 must be isolated and thus has dimension 0. Because the singularity locus of reduced surface divisors has dimension 0, we have proved the theorem for X a complex surface.

Assuming this proposition is proved for all complex varieties of dimension less than k, we now prove the proposition for k dimensional complex varieties. Consider any point $p \in D\backslash D_0$. Because $t(p) \geq 1$, $(D,p) \cong (D'\times\mathbb{C}, 0)$. Notice D' is a transversal section of $D'\times\mathbb{C}$. The proof of proposition 3.6 shows D' is a locally quasi-homogeneous divisor in a open neighborhood of $0 \in \mathbb{C}^{n-1}$. By induction hypothesis $D' = \coprod D'_i$ and $dim(D'_i) \leq i$. This decomposition allows us to write $D'\times\mathbb{C} = \coprod(D'_i\times\mathbb{C})$. Taking into account the coordinate change, the sets $D'_i\times\mathbb{C}$ correspond to D_{i+1} in a neighborhood of p, and clearly $dim(D'_i\times\mathbb{C}) \leq i+1$. $\square$

So far we have shown if the hyperplane $H \subset \mathbb{P}^N$ cuts both X and the locally quasi-homogenous free divisor D transversally, then we can produce a nonsingular $X\cap H$ with locally quasi-homogeneous free diviosr $D\cap H$. To obtain the Bertini type result, we only need to count the dimension of the hyperplanes which fail to intersect D or X transversally. The set of "bad" hyperplanes which fail to cut X transversally has dimension $N-1$: the result of the classical Bertini theorem. We now show:

Proposition 3.10. *For any i, $0 \leq i \leq n-1$, the set of hyperplanes that fail to intersect D transversally at points from D_i has dimension at most $N-1$.*

Proof. Denote by P^N the projective space of hyperplanes in $\mathbb{P}^N$. Consider the subspace W_i of $D_i \times P^N$ consisting of pairs (p, H) such that H does not meet D transversally at p. Also set π_{ij} the projection from W_i to the jth factor, $j = 1, 2$. For any $p \in D_i$, the fiber $\pi^{-1}(p)$ consists of hyperplanes of $\mathbb{P}^N$ containing TD_p. Here we identify a hyperplane with its tangent space at any of its points. Because $dim(TD_p) = i$, we get $dim(\pi^{-1}(p)) = N-i-1$. Combining with proposition 3.9, we get $dim(W_i) \leq (N-1-i)+i = N-1$. So $dim(\pi_{i2}(W)) \leq N-1$. $\square$

Corollary 3.11. *The general hyperplanes in $\mathbb{P}^N$ intersect X and D transversally at the same time. So the local quasi-homogeneity and the freeness of divisors is preserved by intersecting with general hyperplanes.*

Proof. $dim(\cup\pi_{i2}(W_i)) \leq N-1$ $\square$

As an application of the ideas discussed in this section, we prove a stronger version of Teissier's idealistic Bertini's theorem in the context of locally quasi-homogeneous divisors. Recall the following version of idealistic Bertini theorem stated in [Huh12] Lemma 30. Let

$X = \mathbb{P}^n$ and D is a hypersurface in X. For a sufficient general hyperplane H of X, the ideal of $D^s \cap H$ is integral over the ideal of $(D \cap H)^s$. Here $(\cdot)^s$ denotes the singular analytic subspace (or subscheme) of the given analytic space (or scheme). For a hypersurface with a local equation f, the ideal of the singular subspace is locally generated by all partial derivatives of f as well as f itself. In Teissier's original approach, however, the equation of the hypersurface f did not appear in the definition of the singular analytic subspace. The seemingly difference occurs because Teissier treated the singular subspace as a subspace of D whereas we treat the singular subspace as a subspace of X. For a general analytic space (or scheme), the ideal of the singular subspace can be described in terms of the fitting ideals of the sheaf of regular differentials. Note in particular the idealistic Bertini theorem implies that $D^s \cap H$ and $(D \cap H)^s$ have identical underlying topological spaces, but are not identical as ringed spaces.

Corollary 3.12. *Let $X \subset \mathbb{P}^N$ be an n-dimensional nonsingular complex projective variety and D a locally quasi-homogeneous divisor in X. Let H be a sufficient general hyperplane of $\mathbb{P}^N$. Then $D^s \cap H = (D \cap H)^s$ as analytic subspaces (or subschemes) of D.*

Proof. By the previous corollary we know that a sufficient general hyperplane intersect D transversally. Thus locally we reduce to the case $D = D' \times \mathbb{C}$ and H is "perpendicular" to the trivial factor. It is clear then $D^s = (D')^s \times \mathbb{C}$. So $D^s \cap H = (D')^s = (D \cap H)^s$. The statement about subschemes is obtained from GAGA principle. □

4. The third step of the proof

In this section, if not otherwise specified, $X \subset \mathbb{P}^N$ is a complex projective variety, D is a free divisor in X, U is the complement of D in X, and H is a hyperplane in $\mathbb{P}^N$ intersecting X and D transversally. Also denote by X', D' and U' the intersection of H with X, D and U respectively. We derive a formula relating $c(\mathrm{Der}_X(-\log D)) \cap [X]$ and $c(\mathrm{Der}_{X'}(-\log D') \cap [X']$. We prove that by replacing $c(\mathrm{Der}_X(-\log D)) \cap [X]$ by $c_{SM}(1_U)$, $c(\mathrm{Der}_{X'}(-\log D')) \cap [X']$ by $c_{SM}(1_{U'})$, the same formula is true.

For closed embeddings of nonsingular varieties, we know the normal bundle on the subvariety is the quotient of the tangent bundles. It turns out in our current setting, the quotient of the sheaves of logarithmic vector fields also defines the normal bundle.

Proposition 4.1. *Denote N the normal bundle on X' to X, and $i : X' \to X'$ the closed embedding. We have the exact sequence of vector bundles*

$$0 \to \mathrm{Der}_{X'}(-\log D') \to i^*(\mathrm{Der}_X(-\log D)) \to N \to 0$$

Proof. First note all the arrows in the sequence are naturally defined. The map $\mathrm{Der}_{X'}(-\log D') \to i^*(\mathrm{Der}_X(-\log D))$ is induced by $TX' \to i^*TX$, and $i^*(\mathrm{Der}_X(-\log D)) \to N$ is the restriction of $i^*TX \to N$ to $i^*(\mathrm{Der}_X(-\log D))$. This sequence of analytic coherent sheaves is exact because locally analytically the sheaf in the middle is the direct sum of the sheaves on the sides, as was pointed out by the proof of proposition 3.8. The same sequence is also exact in the category of algebraic coherent sheaves as the GAGA principle applies. □

Taking Chern classes of this exact sequence and then pushing forward to X, we get:

$$\begin{aligned} i_*\left(c(\mathrm{Der}_{X'}(-\log D')) \cdot c(N) \cap [X']\right) &= c(\mathrm{Der}_X(-\log D)) \cap i_*[X'] \\ &= c_1(\mathscr{O}_X(1)) \cdot c(\mathrm{Der}_X(-\log D)) \cap [X] \end{aligned}$$

As we have advertised before, there is a similar formula for CSM classes.

Proposition 4.2.

$$i_* \left(c(N) \cap c_{SM}(1_{U'})\right) = c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_U))$$

Remark 4.3. Under the basic setting of this section, X' is a hyperplane section of X, so the normal bundle $N \cong \mathscr{O}_{X'}(1)$.

Proof of proposition 4.2. Aluffi proved a formula for CSM classes of hypersurfaces in [Alu94] Claim 1 (p. 461):

$$i_* \left(c(N) \cap c_{SM}(1_{D'})\right) = c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_D)).$$

Our proposition follows immediately from this. In fact,

$$\begin{aligned} i_* \left(c(N) \cap c_{SM}(1_{U'})\right) &= i_*\Big(c(N) \cap \Big(c_{SM}(1_{X'}) - c_{SM}(1'_D)\Big)\Big) \\ &= i_*\Big(c(N) \cap (c(TX') \cap [X'])\Big) - i_*\Big(c(N) \cap c_{SM}(1'_D)\Big) \\ &= i_*\Big(i^* c(TX) \cap [X']\Big) - c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_D)) \\ &= c_1(\mathscr{O}_X(1)) \cap \Big(c(TX) \cap [X]\Big) - c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_D)) \\ &= c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_X)) - c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_D)) \\ &= c_1(\mathscr{O}_X(1)) \cap (c_{SM}(1_U)) \end{aligned}$$

□

5. Coda: The fourth step of the proof

With all the preparations in previous sections, we can now prove theorem 1.2 by an induction of the dimension of X.

Proof of theorem 1.2. Let X be a nonsingular complex projective curve, D be a reduced effective divisor in X. Since in this case D itself is nonsingular, therefore normal crossing, thus the classes in equation (1) are rationally equivalent[Alu99]. Alternatively, we can get the same conclusion by a direct computation. If $D = \sum P_i$, then $c_{SM}(U) = c(TX) \cap [X] - \sum [P_i]$. To calculate the Chern class of the logarithmic vector field, we use the following exact sequences:

$$0 \to \mathrm{Der}_X(-\log D) \to TX \to \mathscr{O}_D(D) \to 0$$

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(D) \to \mathscr{O}_D(D) \to 0$$

Taking Chern classes of these exact sequences we get $c(\mathrm{Der}_X(-\log D)) \cap [X] = (c(TX) \cdot c(\mathscr{O}_X(D)^{-1}) \cap [X] = c(TX) \cap [X] - \sum [P_i]$.

Assume the theorem is proved for all $k-1$ dimensional nonsingular complex projective varieties, we now finish the inductive step. Let X be a k dimensional nonsingular complex projective variety, D be a locally quasi-homogeneous free divisor in X, U be the complement of D in X, and H is a hyperplane of the ambient projective space intersecting both X and D transversally. Denote by X', D' and U' respectively the intersections of the corresponding spaces with H.

According to proposition 2.6, the degrees of the CSM class and the Chern class of the logarithmic vector field are always the same. For other numerical relations, we observe as following:

$$
\begin{aligned}
&\int c_1(\mathscr{O}_X(1))^i \cap c_{SM}(1_U)\\
=&\int c_1(\mathscr{O}_X(1))^{i-1} \cap \Big(c_1(\mathscr{O}_X(1)) \cap c_{SM}(1_U)\Big) \quad (i \geq 1)\\
=&\int c_1(\mathscr{O}_X(1))^{i-1} \cap i_*\Big(c(\mathscr{O}_{X'}(1)) \cap c_{SM}(1_{U'})\Big) \quad \text{(proposition 4.2)}\\
=&\int c_1(\mathscr{O}_X(1))^{i-1} \cap \Big(c(\mathscr{O}_X(1) \cap i_*c_{SM}(1'_U)\Big) \quad \text{(projection formula)}\\
=&\int c_1(\mathscr{O}_X(1))^{i-1} \cap i_*c_{SM}(1'_U) + \int c_1(\mathscr{O}_X(1))^{i} \cap i_*c_{SM}(1'_U)\\
=&\int c_1(\mathscr{O}_{X'}(1))^{i-1} \cap c_{SM}(1'_U) + \int c_1(\mathscr{O}_{X'}(1))^{i} \cap c_{SM}(1'_U) \quad \text{(projection formula)}
\end{aligned}
$$

Applying the analogous formula for logarithmic vector fields instead of proposition 4.2 we have:

$$
\begin{aligned}
&\int c_1(\mathscr{O}_X(1))^i \cap \Big(c(\mathrm{Der}_X(-\log D) \cap [X]\Big)\\
=&\int c_1(\mathscr{O}_{X'}(1))^{i-1} \cap \Big(c(\mathrm{Der}_{X'}(-\log D') \cap [X']\Big) +\\
&\int c_1(\mathscr{O}_{X'}(1))^{i} \cap \Big(c(\mathrm{Der}_{X'}(-\log D') \cap [X']\Big) \quad (i \geq 1)
\end{aligned}
$$

Now the theorem follows from inductive hypothesis.

□

Proof of corollary 1.3. For $X = \mathbb{P}^n$, the morphism from $A_i(X)$ to $\mathbb{Z}$

$$\alpha \mapsto \int \alpha \cap c_1(\mathscr{O}(1))^i$$

is an isomorphism. □

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