ON ASSOCIATED GRADED RINGS HAVING ALMOST MAXIMAL DEPTH

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Abstract. We generalize a recent result of Rossi and Valla, and independently Wang, about the depth of $G(m)$ where m is the maximal ideal of a d-dimensional Cohen-Macaulay local ring R having embedding dimesion $e_0(m)+d-2$. The generalization removes the restriction on the embedding dimension and replaces it with the condition that $\lambda(m^3/Jm^2)\leq 1$ where J is a d-generated minimal reduction of m. The main theorem also applies to m-primary ideals I satisfying $J\cap I^2=JI$ and $\lambda(I^3/JI^2)\leq 1$, where J is a d-generated reduction of I. An example of such an I in a 5-dimensional regular local ring is included as a nontrivial illustration of the theorem.

1. Introduction

The purpose of this article is to generalize a recent result proved independently by Rossi and Valla in [RV], and Wang in [W]. Their result answered a question posed by J.D. Sally in [S3] about the nature of depth($G(m)$) where (R,m) is a Cohen-Macaulay local ring of embedding dimension $e_0(m)+d-2$, $e_0(m)$ denotes the multiplicity of R, and d is the dimension of R. It was first shown in [A] that if (R,m) is a d-dimensional Cohen-Macaulay local ring then the embedding dimension $\lambda(m/m^2)$ satisfies the inequality $\lambda(m/m^2)\leq e_0(m)+d-1$. In fact, if J is a d-generated *reduction* of m (that is, $Jm^n=m^{n+1}$ for $n>>0$) then by a length computation it holds that

$$\lambda(m/m^2)=e_0(m)+d-1-\lambda(m^2/Jm)$$

(a d-generated reduction usually exists; always if R/m is infinite). Sally's work on arithmetic properties of $G(m)$ covered the case $\lambda(m^2/Jm)=0$ (i.e., "maximal embedding dimension" or "minimal multiplicity") and gave a sequence of partial results involving the next case, $\lambda(m^2/Jm)=1$. Sally proved in [S1] that if $\lambda(m^2/Jm)=0$ then $G(m)$ is Cohen-Macaulay (more generally this holds for an m-primary ideal I and d-generated reduction J of I satisfying $\lambda(I^2/JI)=0$ by [VV, Proposition 3.1]). In [S3] Sally showed that if $\lambda(m^2/Jm)=1$ then depth($G(m)$) $\geq d-1$ often holds and used those results to help describe the Hilbert function of R. Sally conjectured that if $\lambda(m^2/Jm)=1$ (equivalently that $\lambda(m/m^2)=e_0(m)+d-2$) then depth($G(m)$) $\geq d-1$. Some progress was made recently on settling this conjecture (see [G], [P], and [VP] where some new approaches were initiated) and it was finally solved in the affirmative with two very nice proofs in [RV] and [W].

In this paper we remove the restriction that $\lambda(m^2/Jm)=1$ and replace it with the condition that $\lambda(m^3/Jm^2)\leq 1$. In other words we show that depth($G(m)$) $\geq d-1$ if

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$\lambda(m/m^2) = e_0(m) + d - \lambda(m^2/Jm) - \lambda(m^3/Jm^2)$ (see Corollary 2.8). That there exists such Cohen-Macaulay local rings satisfying $\lambda(m^2/Jm) > 1$ is easy to see by taking a 1-dimensional example (e.g., $R = k[[t^6, t^7, t^8]]$ with $J = (t^6)R$) and adding variables. An interesting bonus of our approach (which was inspired by the proof given in [RV]) is that we can prove a more general result, one that applies to certain (non-maximal) m-primary ideals of R. By assuming only that I is m-primary, and that I satisfies the two conditions $\lambda(I^3/JI^2) \leq 1$ and $J \cap I^2 = JI$, for some d-generated reduction J of I, we are able to prove that $\operatorname{depth}(G(I)) \geq d-1$ (see Theorem 2.4). It is easy to see that in the maximal ideal case $J \cap m^2 = Jm$, but more generally it is true by a result of Huneke [Hun] and Itoh [I] that if I is integrally closed then $J \cap I^2 = JI$ for any reduction J of I. Therefore Theorem 2.4 also applies to integrally closed m-primary ideals of a Cohen-Macaulay local ring (see Corollary 2.7). We give an example in (2.11) of an m-primary ideal of a 5-dimensional regular local ring such that $\lambda(I^2/JI) = 2$, $\lambda(I^3/JI^2) = 1$, and $J \cap I^2 = JI$, thus providing a nontrivial illustration of Theorem 2.4.

Before proceeding we will review some definitions and terminology. If (R, m) is any local ring and I is an m-primary ideal of R then a reduction of I is an ideal J contained in I such that $JI^n = I^{n+1}$ for $n >> 0$ (see [NR]). A *minimal reduction* of I is a reduction of I that is minimal with respect to inclusion. Minimal reductions always exist, and if R/m is infinite then any minimal reduction of I is generated by d elements where d is the dimension of R. An element $x \in I \backslash I^2$ is said to be *superficial* (of order one) for I if there is a positive integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. By the Artin-Rees lemma if x is R-regular and superficial for I then $(I^n : x) = I^{n-1}$ for all $n >> 0$. Superficial elements for I exist if R/m is infinite ([N] or [ZS]). If $y_1, \ldots, y_s \in I$ such that the image of y_i in $I/(y_1, \ldots, y_{i-1})$ is superficial for $I/(y_1, \ldots, y_{i-1})$ for each i, then $\{y_1, \ldots, y_s\}$ is called a *superficial sequence* for I. The associated graded ring of R with respect to I will be denoted by $G(I)$. If $M = m/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$ denotes the maximal homogeneous ideal of $G(I)$, then by $\operatorname{depth}(G(I))$ we mean the depth of the local ring $G(I)_M$. The *Hilbert-Samuel function* of I is the function $H_I(n) = \lambda(R/I^n)$. It follows from the theory of Hilbert functions applied to the associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I that there is an integer-valued polynomial $P_I(n)$ of degree d with rational coefficients such that $P_I(n) = H_I(n)$ for all $n >> 0$. $P_I(n)$ is called the *Hilbert-Samuel polynomial* of I and it can be written in the form

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}e_{d-1}(I)n + (-1)^d e_d(I).$$

The coefficients $e_i(I)$ are called the Hilbert-coefficients of I and $e_1(I)$ plays a decisive role in determining bounds on $\operatorname{depth}(G(I))$ (see [Huc, Thoerem 3.1]). An important property of superficial elements is that if $x \in I$ is superficial for I then $e_i(I) = e_i(I/(x))$ for $0 \leq i \leq d-1$.

A useful tool for studying depth properties of $G(I)$ is the filtration $\{\widetilde{I^n}\}_{n \geq 0}$ introduced by Ratliff and Rush in [RR]. If K is an ideal then $\widetilde{K}$ is defined to be the union of the ascending chain of ideals $\{K^{n+1} : K^n\}$ (for a nice study of general properties of $\widetilde{K}$ see the paper [HLS]). It turns out that $\operatorname{depth}(G(K)) \geq 1$ if and only if $\widetilde{K^n} = K^n$ for all $n \geq 0$. There are two formulas involving $e_1(I)$ that will be used in this paper. The first is a special case of [Huc, Corollary 2.10] and the second appears in [HM, Corollary 4.13]. The following proposition contains both formulas.

Proposition 1.1. *Let R be $d-$dimensional Cohen-Macaulay local ring, I an m-primary ideal of R, and J a d-generated reduction of I.*

(1) *If $d = 2$ then $e_1(I) = \sum_{n\geq 0} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n})$.*
(2) *If $d = 1$ then $e_1(I) = \sum_{n\geq 0} \lambda(I^{n+1}/JI^n)$.*

Finally we recall the definition of reduction number. If I is an ideal of R and J is a reduction of I then the *reduction number* of I with respect to J is the non-negative integer $r_J(I)$ =min$\{n \geq 0 \mid JI^n = I^{n+1}\}$. The notation $r_J(I/(x))$ where $x \in I$ means the reduction number of $I/(x)$ with respect to $J/(x)$.

2. Main result

This section is devoted to proving our main result. We start with some preliminary lemmas that are analogous to lemmas appearing in one or both of [RV] and [W]. The first lemma expands on [RV, Lemma 1.1 and Proposition 1.2].

Lemma 2.1. *Let (R, m) be a Cohen-Macaulay local ring and let I be an m-primary ideal of R. Assume that J is a minimal reduction of I such that $\lambda(I^3/JI^2) = 1$. Then either there exists $z \in I$ such that $I^{n+1} = JI^n + (z^{n+1})$ for all $n \geq 2$, or $JI^3 = I^4$. In particular, $\lambda(I^{n+1}/JI^n) \leq 1$ for all $n \geq 2$.*

Proof. Write $I = J + (z_1, ..., z_l)$ and suppose there is no $z \in I$ such that $I^3 = JI^2 + (z^3)$. Then for each i, $z_i^3 \in JI^2$. Suppose that for some $i \neq j$, $z_i^2 z_j \notin JI^2$. Without loss of generality assume $i = 1$ and $j = 2$. Then $I^3 = JI^2 + (z_1^2 z_2)$, thus $I^4 = JI^3 + z_1^2 z_2(z_1, ..., z_l)$. We have for each i, j, k that $z_i z_j z_k \in JI^2 + (z_1^2 z_2)$, hence $z_1^2 z_2(z_1, ..., z_k) \subseteq z_1(JI^2 + (z_1^2 z_2)) \subseteq JI^3 + (z_1^3 z_2) = JI^3$, therefore $I^4 = JI^3$. On the other hand if $z_i^2 z_j \in JI^2$ for all i, j, then we may assume $I^3 = JI^2 + (z_1 z_2 z_3)$, which implies that $I^4 = JI^3 + (z_1 z_2 z_3)(z_1, ..., z_l)$. Because $z_2 z_3 z_j \in JI^2 + (z_1 z_2 z_3)$ for all $j, 1 \leq j \leq l$ we conclude that $I^4 = JI^3 + (z_1^2 z_2 z_3) = JI^3$. To finish the proof we must show that if $I^3 = JI^2 + (z^3)$ for some $z \in I$, then $I^{n+1} = JI^n + (z^{n+1})$ for all $n \geq 2$. Given such a z, it must hold that $z^2 z_i \in JI^2 + (z^3)$ for each $i, 1 \leq i \leq l$. Therefore

$$I^4 = JI^3 + z^3(z_1, ..., z_l) \subseteq JI^3 + z(JI^2 + (z^3)) \subseteq JI^3 + (z^4),$$

and by using induction the statement follows. For the last statement note that $z^3 m \subseteq JI^2$ because $\lambda(I^3/JI^2) \leq 1$, thus $z^{n+1} m \subseteq JI^n$ for every $n \geq 2$.

The next lemma expands on [W, Lemmas 2.6 and 2.10] and [RV, Corollary 2.3].

Lemma 2.2. *Let (R, m) be a 2-dimensional Cohen-Macaulay local ring and let I be an m-primary ideal of R. Let J be a minimal reduction of I, assume that $J = (x, y)$ where both x and y are superficial for I, and set $r = r_J(I)$, $s = r_J(I/(x)$. If $\lambda(I^3/JI^2) \leq 1$ and $J \cap I^2 = JI$, then the following statements are true.*

(1) *$(I^{n+1} : x) = (I^{n+1} : y) = I^n$ for $1 \leq n \leq s - 1$. In particular, $e_1(I) = \sum_{n=0}^{s-1} \lambda(I^{n+1}/JI^n)$.*
(2) *If $s \leq 2$ and $I = \widetilde{I}$ then depth$(G(I)) \geq 1$.*
(3) *depth$(G(I)) \geq 1$ if and only if $s = r$.*

Proof. (1) It suffices to prove this for x, and we do so by using induction on n. If $n=1$ then the statement holds by the assumption $J\cap I^2=JI$. By Lemma 2.1 we know that $\lambda(I^{n+1}/JI^n)\leq 1$ if $n\geq 2$. Additionally, for each n there is an isomorphism $(I/(x))^{n+1}/J(I/(x))^n\cong I^{n+1}/JI^n+x(I^{n+1}:x)$. Therefore

$$0<\lambda((I/(x))^{n+1}/J(I/(x))^n)=\lambda(I^{n+1}/JI^n+x(I^{n+1}:x))\leq\lambda(I^{n+1}/JI^n)\leq 1$$

for $2\leq n\leq s-1$. This implies that $x(I^{n+1}:x)\subseteq JI^n$ if $2\leq n\leq s-1$. If $ax\in I^{n+1}$ we may write $ax=a_1x+a_2y$ for some $a_1,a_2\in I^n$. Then there exists $b\in R$ such that $a-a_1=by$ and $a_2=bx$. By the induction hypothesis $b\in I^{n-1}$, therefore $a\in I^n$. The second statement holds because $e_1(I)=e_1(I/(x))$, using that x is superficial for I, thus $e_1(I)=\sum_{n=0}^{s-1}\lambda(I^{n+1}/JI^n+x(I^{n+1}:x))$ by Proposition 1.1(2).

(2) If $s\leq 2$ then $e_1(I)=\lambda(I/J)+\lambda(I^2/JI)$ from (1). Also, $e_1(I)=\sum_{n\geq 0}\lambda(\widetilde{I^{n+1}}/J\widetilde{I^n})$ by Proposition 1.1(1). We claim that $I^{n+1}=\widetilde{I^{n+1}}$ for all $n\geq 1$ and prove the claim by induction on n. Using that $I=\widetilde{I}$ we obtain

$$0\geq\lambda(I^2/JI)-\lambda(\widetilde{I^2}/JI)=\sum_{n\geq 2}\lambda(\widetilde{I^{n+1}}/J\widetilde{I^n})\geq 0$$

which yields that $I^2=\widetilde{I^2}$ and $J\widetilde{I^n}=\widetilde{I^{n+1}}$ for all $n\geq 2$. If $n\geq 2$ then $\widetilde{I^{n+1}}=J\widetilde{I^n}=JI^n\subseteq I^{n+1}$, therefore $\widetilde{I^{n+1}}=I^{n+1}$. The claim shows that $\operatorname{depth}(G(I))\geq 1$.

(3) By [Huc, Theorem 3.1] $\operatorname{depth}(G(I))\geq 1$ if and only if $e_1(I)=\sum_{n=0}^{r-1}\lambda(I^{n+1}/JI^n)$. Using this and (1) the statement follows.

Theorem 2.4 is the main result. Its proof was inspired by the argument given in [RV, Theorem 2.5], and in particular it makes important use of the following technical result, Proposition 2.3, which is a slightly generalized version of [RV, Proposition 2.4]. The proof of Proposition 2.3 is essentially identical to the proof of [RV, Proposition 2.4] so we omit the details. Notice however that the statement has been somewhat reformulated.

Proposition 2.3. *[RV, Propostion 2.4] Let R be a commutative Noetherian ring, $t\geq 1$ and integer, I an ideal of R, and J an ideal contained in I. Suppose for each n, $1\leq n\leq t$, ideals I_n are given such that $I_n=(a_{1n},\dots,a_{\nu_n n})\subseteq\widetilde{I^n}$. Assume that $\widetilde{I^{t+1}}=J\widetilde{I^t}$, and for $0\leq n\leq t-1$, $\widetilde{I^{n+1}}=J\widetilde{I^n}+I_{n+1}+I^{n+1}$. Let $z\in I$ and set $\nu=\sum_{i=1}^{t}\nu_i$. Then there exists $\sigma\in JI^{\nu-1}$ such that*

$$(z^\nu-\sigma)a_{in}\in I^{\nu+n}$$

for all n, $1\leq n\leq t$, and for all i, $1\leq i\leq\nu_n$.

Theorem 2.4. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and I an m-primary ideal of R. Assume that J is a d-generated minimal reduction of I satisfying the conditions $\lambda(I^3/JI^2) \leq 1$ and $J \cap I^2 = JI$. Then depth$(G(I)) \geq d-1$.*

Proof. By passing to $R(X) = R[X]_{m[X]}$ and $I(X) = IR(X)$ we may assume that R/m is infinite. The first step is to factor out a superficial sequence $x_1, ..., x_{d-2} \in J$ for I, therby reducing to the case $d = 2$. The assumption $\lambda(I^3/JI^2) \leq 1$ is clearly preserved under this operation, and by [HM, Lemma 2.2] the conclusion of the theorem holds in R if and only if it holds in $R/(x_1, ..., x_{d-2})$ (see also the explanation given in [RV], or [S2] where this dimension-reducing technique was introduced). We must also verify that the assumption $J \cap I^2 = JI$ is preserved modulo $(x_1, ..., x_{d-2})$, and for that it suffices to show that $(J/(x)) \cap (I/(x))^2 = JI + (x)/(x)$ where $x \in J$. If $\bar{a} \in (J/(x)) \cap (I/(x))^2$ then there exists $a_1 \in J, a_2 \in I^2$, such that $a_1 - a_2 \in (x)$ and $\bar{a} = a_1 + (x) = a_2 + (x)$ in $R/(x)$. Thus $a_2 \in J \cap I^2 = JI$, hence $\bar{a} \in JI + (x)/(x)$.

Next we choose $x, y \in J$, each superficial for I, such that $J = (x, y)$. Set $s = r_J(I/(x))$ and $r = r_J(I)$. If $r \leq 3$ then either $s \leq 2$ or $s = r$. If $s = r$ then the result follows from Lemma 2.3(3). Assume that $s \leq 2$. If we can verify that $I = \widetilde{I}$, then it will follow from Lemma 2.2(2) that depth$(G(I)) \geq 1$. By Lemma 2.2(1) and Proposition 1.1(1) we have that

$$e_1(I) = \lambda(I/J) + \lambda(I^2/JI) = \sum_{n \geq 0} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}),$$

thus

$$0 \geq \lambda(I/J) - \lambda(\widetilde{I}/J) = \lambda(\widetilde{I^2}/J\widetilde{I}) - \lambda(I^2/JI) + \sum_{n \geq 2} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}). \tag{2.5}$$

We claim that

$$\lambda(\widetilde{I^2}/J\widetilde{I}) \geq \lambda(I^2/JI). \tag{2.6}$$

By rewriting $\lambda(\widetilde{I^2}/J\widetilde{I}) - \lambda(I^2/JI)$ as $\lambda(\widetilde{I^2}/I^2) - \lambda(J\widetilde{I}/JI)$, the claim follows from the injection $\phi : J\widetilde{I}/JI \to \widetilde{I^2}/I^2$ defined by $\phi(a + JI) = a + I^2$, which is one-one because $J \cap I^2 = JI$. Inserting the inequality from (2.6) into (2.5) yields that $I = \widetilde{I}$, therefore depth$(G(I)) \geq 1$.

We have now reduced to the case where $r \geq 4$, and therefore by Lemma 2.1 there exists $z \in I$ such that $I^{n+1} = JI^n + (z^{n+1})$ for all $n \geq 2$. From this point the proof is going to follow that given in [RV, Theorem 2.5], but we will need to make adjustments to handle the situation in which I is not maximal, and in which $\lambda(I^2/JI) \neq 1$. Let t denote the smallest non-negative integer n such that $J\widetilde{I^n} = \widetilde{I^{n+1}}$. We first show that $t \leq s$. By Lemma 2.1(1) and Proposition 1.1(1) we know that

$$e_1(I) = \sum_{n=0}^{s-1} \lambda(I^{n+1}/JI^n) = \sum_{n \geq 0} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}),$$

thus

$$\begin{aligned}\lambda(I/J) - \lambda(\widetilde{I}/J) + \lambda(I^2/JI) - \lambda(\widetilde{I^2}/J\widetilde{I}) &= -\sum_{n=2}^{s-1} \lambda(I^{n+1}/JI^n) + \sum_{n \geq 2} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}) \\ &= 2 - s + \sum_{n \geq 2} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}),\end{aligned}$$

the last equality because $\lambda(I^{n+1}/JI^n) = 1$ for $2 \le n \le s-1$. But $\lambda(I/J) - \lambda(\widetilde{I}/J) \le 0$, and by (2.6), $\lambda(I^2/JI) - \lambda(\widetilde{I^2}/J\widetilde{I}) \le 0$. Therefore

$$0 \ge 2 - s + \sum_{n \ge 2} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}),$$

and it follows that $t \le s$.

Let k be the least non-negative integer n such that $I^{n+1} \subseteq J\widetilde{I^n}$. Notice that $k \le t$ because $I^{n+1} \subseteq \widetilde{I^{n+1}}$. Using that $r \ne 1$ and $J \cap I^2 = JI$ it also holds that $k \ge 2$. Therefore $2 \le k \le t \le s$. For each $n \ge 1$ let ν_n denote the size of a minimal generating set of the R-module $\widetilde{I^n}/(J\widetilde{I^{n-1}} + I^n)$. Then $\nu_n \le \lambda(\widetilde{I^n}/(J\widetilde{I^{n-1}} + I^n)) \le \lambda(\widetilde{I^n}/(J\widetilde{I^{n-1}})$, $\nu_n < \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})$ if $1 \le n \le k$, and $\nu_n < \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})$ if $n \ge k+1$. To set up an application of Proposition 2.3 we let $I_n = (a_{1n}, ..., a_{\nu_n n})$ where $\{a_{1n}, ..., a_{\nu_n n}\}$ is a minimal generating set for $\widetilde{I^n}/(J\widetilde{I^{n-1}} + I^n)$. For $0 \le n \le t-1$ we have $\widetilde{I^{n+1}} = J\widetilde{I^n} + I_{n+1} + I^{n+1}$. Setting $\nu = \sum_{i=1}^{t} \nu_i$, it follows by Proposition 2.3 that there exists an element $\sigma \in JI^{\nu-1}$ such that $(z^\nu - \sigma)a_{in} \in I^{\nu+n}$ for each a_{in}. We show, by reproducing a calculation from [RV], that $I^{\nu+k+1} = JI^{\nu+k}$. From the definition of k, $\widetilde{I^{k+1}} \subseteq J\widetilde{I^k}$, hence

$$z^{k+1} \in J\widetilde{I^k} = JI^k + J^k I_1 + J^{k-1} I_2 + ... + JI_k.$$

Therefore there exists elements $c_{ij} \in J^{k-j+1}$ and $b \in JI^k$ satisfying

$$z^{k+1} = b + \sum_{j=1}^{k} \sum_{i=1}^{\nu_k} c_{ij} a_{ij}.$$

Note that

$$\sum_{j=1}^{k} \sum_{i=1}^{\nu_k} c_{ij} a_{ij} \in I^{k+1}, \quad \text{thus,} \quad \sigma \sum_{j=1}^{k} \sum_{i=1}^{\nu_k} c_{ij} a_{ij} \in JI^{\nu+k}.$$

Considering $z^{\nu+k+1}$, we have

$$z^{\nu+k+1} = bz^\nu + \sum_{j=1}^{k} \sum_{i=1}^{\nu_k} c_{ij} a_{ij} z^\nu.$$

For each a_{ij} write $z^\nu a_{ij} = \sigma a_{ij} + d_{ij}$, where $d_{ij} \in I^{\nu+j}$. Then

$$z^{\nu+k+1} = bz^\nu + \sum_{j=1}^{k} \sum_{i=1}^{\nu_k} (c_{ij} \sigma a_{ij} + c_{ij} d_{ij}).$$

It holds that $c_{ij} d_{ij} \in J^{k-j+1} I^{\nu+j} \subseteq JI^{\nu+k}$ and $bz^\nu \in JI^{\nu+k}$, and from above $\sigma \sum_{j=1}^{k} \sum_{i=1}^{\nu_k} c_{ij} a_{ij} \in JI^{\nu+k}$, therfore $z^{\nu+k} \in JI^{\nu+k}$. Thus it follows from Lemma 2.1 that $JI^{\nu+k} = I^{\nu+k+1}$.

What remains now is to show that $\nu + k \le s$. First we claim that

$$\nu_2 \le \lambda(\widetilde{I^2}/J\widetilde{I}) - \lambda(I^2/JI).$$

By definition $\nu_2 \le \lambda(\widetilde{I^2}/J\widetilde{I} + I^2)$, thus $\nu_2 \le \lambda(\widetilde{I^2}/I^2) - \lambda(J\widetilde{I} + I^2/I^2)$. Using the isomorphism $J\widetilde{I} + I^2/I^2 \cong J\widetilde{I}/J\widetilde{I} \cap I^2$ along with the assumption $J \cap I^2 = JI$ (which forces $J\widetilde{I} \cap I^2 = JI$) it follows that $\nu_2 \le \lambda(\widetilde{I^2}/I^2) - \lambda(J\widetilde{I}/JI)$, and appropriately rewriting these lengths proves the claim. Combining this inequality with the inequality $\nu_1 \le \lambda(\widetilde{I}/I) = \lambda(\widetilde{I}/J) - \lambda(I/J)$ produces

$$\nu_1 + \nu_2 \le \lambda(\widetilde{I}/J) - \lambda(I/J + \lambda(\widetilde{I^2}/J\widetilde{I}) - \lambda(I^2/JI).$$

Next we insert ν_2 into the equalities

$$e_1(I) = \sum_{n=0}^{s-1} \lambda(I^{n+1}/JI^n) = \sum_{n\ge 0} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n})$$

from Proposition 1.1, to obtain the inequality

$$\nu_1 + \nu_2 \le \sum_{n=2}^{s-1} \lambda(I^{n+1}/JI^n) - \sum_{n\ge 2} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}) = s - 2 - \sum_{n\ge 2} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}),$$

the last equality because $\lambda(I^{n+1}/JI^n) = 1$ for $2 \le n \le s-1$. Recall that $\nu_n < \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})$ if $n \ge k+1$. Therefore, using the inequality yields

$$\nu = \sum_{n=1}^{t} \nu_n \le s - 2 + \sum_{n=3}^{k} [\nu_n - \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})] - \sum_{n\ge t} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}) \le$$
$$s - 2 - (k-2) - \sum_{n\ge t} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}) \le s - k.$$

It follows that $\nu + k \le s$. An application of Lemma 2.3(3) completes the proof of Theorem 2.4.

We list several consequences of Theorem 2.4.

Corollary 2.7. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring. If I is an integrally closed m-primary ideal of R having a d-generated minimal reduction J satisfying $\lambda(I^3/JI^2) \le 1$ then depth$(G(I)) \ge d-1$.*

Proof. If I is integrally closed then $J \cap I^2 = JI$ by either [Hun] or [I, Theorem 1] so it follows from Theorem 2.4.

Using the equality $\lambda(m/m^2) = e_0(m) + d - 1 - \lambda(m^2/Jm)$, which always holds, we get the following extension of the main results of [RV] and [W].

Corollary 2.8. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and assume that m has a d-generated minimal reduction J satisfying*

$$\lambda(m/m^2) = e_0(m) + d - \lambda(m^2/Jm) - \lambda(m^3/Jm^2).$$

Then depth$(G(m)) \geq d - 1$.

Proof. The maximal ideal m is integrally closed, and from above we have $\lambda(m^3/Jm^2) = e_0(m) + d - \lambda(m/m^2) - \lambda(m^2/Jm) = 1$, thus it follows from Corollary 2.7.

By adopting the terminology of [RV] we can rephrase Corollary 2.8. Let $h = \lambda(m/m^2) - d$ denote the *embedding codimension* of R. If J is a d-generated minimal reduction of m then $e_0(m) = h + 1 + \lambda(m^2/Jm)$.

Corollary 2.9. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and assume that m has a d-generated minimal reduction J. If*

$$e_0(m) = h + \lambda(m^2/Jm) + \lambda(m^3/Jm^2)$$

then depth$(G(m)) \geq d - 1$.

It was shown in [Huc,Corollary 2.11] that the coefficients of the Hilbert-Samuel polynomial $P_I(n)$ can be described in terms of the lengths of I^{n+1}/JI^n if depth$(G(I)) \geq d - 1$. The next Corollary records that for the ideals of Theorem 2.4.

Corollary 2.10. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and I an m-primary ideal of R. Let J be a d-generated minimal reduction of I satisfying the conditions $\lambda(I^3/JI^2) \leq 1$ and $J \cap I^2 = JI$. If $r = r_J(I)$ then*

(1) $e_1(I) = \lambda(I/J) + \lambda(I^2/JI) + r - 2$

(2) $e_2(I) = \lambda(I^2/JI) + \sum_{n=2}^{r-1} n$

(3) $e_i(I) = \sum_{n=i-1}^{r-1} \binom{n}{i-1}$ *for* $3 \leq i \leq d$.

Proof. The result follows from [Huc, Corollary 2.11] because depth$(G(I)) \geq d - 1$ (by Theorem 2.4) and $\lambda(I^{n+1}/JI^n) = 1$ for $2 \leq n \leq r - 1$ (by Lemma 2.1).

We end by giving an example to illustrate Theorem 2.4. This example was discovered with the help of Macaulay [BS].

Example 2.11. Let $T = k[a, b, c, d, e]$ (where $k = \mathbb{Z}/(31, 991)\mathbb{Z}$) be a polynomial ring in 5 variables. Set $N = (a, b, c, d, e)T, R = T_N, m = NT_N$, and

$$I = (a^2, b^2, c^2, d^2, e^2, ab + ac + bd + cd + de, abc, abd, abe, bcd, bce).$$

Then $J = (a^2, b^2, c^2, d^2, e^2)$ is a minimal reduction of I satisfying the conditions that $J \cap I^2 = JI$ and $\lambda(I^3/JI^2) = 1$. Therefore depth$(G(I)) \geq 4$ by Theorem 2.4. In this example it holds that $\lambda(I^2/JI) = 2$. It is worth pointing out that our version of Macaulay failed (because of inadequate memory) to complete the task of verifying the depth condition via a projective resolution of the Rees algebra $R[It]$, yet using Macaulay it takes only seconds to verify the two assumptions of Theorem 2.4.

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