EULER CHARACTERISTICS OF GENERAL LINEAR SECTIONS AND POLYNOMIAL CHERN CLASSES

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Abstract. We obtain a precise relation between the Chern-Schwartz-MacPherson class of a subvariety of projective space and the Euler characteristics of its general linear sections. In the case of a hypersurface, this leads to simple proofs of formulas of Dimca-Papadima and Huh for the degrees of the polar map of a homogeneous polynomial, extending these formula to any algebraically closed field of characteristic 0, and proving a conjecture of Dolgachev on 'homaloidal' polynomials in the same context. We generalize these formulas to subschemes of higher codimension in projective space.

We also describe a simple approach to a theory of 'polynomial Chern classes' for varieties endowed with a morphism to projective space, recovering properties analogous to the Deligne-Grothendieck axioms from basic properties of the Euler characteristic. We prove that the polynomial Chern class defines homomorphisms from suitable relative Grothendieck rings of varieties to $\mathbb{Z}[t]$.

1. Introduction

1.1. Let X be a projective variety over an algebraically closed field k of characteristic 0, endowed with a specific embedding in projective space. If $k = \mathbb{C}$, one of the most important invariants of X is its topological Euler characteristic, $\chi(X)$. There is a natural generalization of this invariant to arbitrary algebraically closed fields of characteristic zero: if X is nonsingular, we may take $\chi(X)$ to equal the degree of $c(TX) \cap [X]$; and the singular case may be dealt with by using resolution of singularities (see §2 for details). The resulting invariant has the expected properties of the topological Euler characteristic: it is multiplicative on products, it satisfies inclusion-exclusion, and in particular it can be consistently defined for any locally closed subset of $\mathbb{P}^n$.

While $\chi(X)$ does not depend on the embedding of X into $\mathbb{P}^n$, we can access more refined invariants of the embedding by considering general linear sections. We let $X_r = X \cap H_1 \cap \cdots \cap H_r$ be the intersection of X with r general hyperplanes, and we assemble the Euler characteristics of these loci into a generating polynomial of degree $\leq n$:

$$\chi_X(t) := \sum_{r\geq 0} \chi(X_r) \cdot (-t)^r \quad .$$

This polynomial may be defined for any locally closed subset of $\mathbb{P}^n$. Our main theme in this note is an interpretation of $\chi_X(t)$ in terms of the *Chern-Schwartz-MacPherson* class $c_{\mathrm{SM}}(X)$. This is a class in the Chow group of X, generalizing to (possibly) singular varieties the total Chern class of the tangent bundle of X in the nonsingular case, and satisfying a compelling functoriality property, which will be recalled in §2.

Chern-Schwartz-MacPherson classes can also be defined over any algebraically closed field of characteristic 0, and the functoriality property mentioned above implies that if X is complete, then the degree $\int c_{\mathrm{SM}}(X)$ equals $\chi(X)$. In fact, a c_{SM} class may be defined for any constructible function on a variety, and here we will associate with $X \subseteq \mathbb{P}^n$ the class $c_{\mathrm{SM}}(1\!\!1_X) \in A_*\mathbb{P}^n$. (As a template to keep in mind, the information carried by this class for a nonsingular projective $X \subseteq \mathbb{P}^n$ amounts to the degrees of the components of different dimension in $c(TX) \cap [X]$.) We will also write this class as a polynomial of degree $\le n$: we let

$$\gamma_X(t) = \sum_{r \ge 0} \gamma_r \, t^r$$

be the polynomial obtained from $c_{\mathrm{SM}}(1\!\!1_X)$ by replacing $[\mathbb{P}^r]$ with t^r. This is another polynomial defined for any locally closed $X \subseteq \mathbb{P}^n$. (In our template, γ_r is the degree of $c_{\dim X - r}(TX) \cap [X]$ as a class in $\mathbb{P}^n$.)

We will prove that for all locally closed subsets of $\mathbb{P}^n$ the polynomials $\gamma_X(t)$, $\chi_X(t)$ carry precisely the same information. We consider the following linear transformation:

$$p(t) \mapsto \mathscr{I}(p) := \frac{t \cdot p(-t-1) + p(0)}{t+1} \quad .$$

It is clear that if p is a polynomial, then $\mathscr{I}(p)$ is a polynomial of the same degree. Also, it is immediately checked that $\mathscr{I}(p)(0) = p(0)$, and $\mathscr{I}(p)(-1) = p(0) + p'(0)$. Further, $\mathscr{I}$ is an *involution;* in fact, if $p(t) = p(0) + tp_+(t)$, then $\mathscr{I}(p) = p(0) - tp_+(-t-1)$, so that the effect of $\mathscr{I}$ is to perform a sign-reversing symmetry about $t = -1/2$ of the non-constant part of p.

Theorem 1.1. *For every locally closed set $X \subseteq \mathbb{P}^n$, the involution $\mathscr{I}$ interchanges $\chi_X(t)$ and $\gamma_X(t)$:*

$$\gamma_X = \mathscr{I}(\chi_X) \quad , \quad \chi_X = \mathscr{I}(\gamma_X) \quad .$$

Theorem 1.1 is a straightforward exercise for X nonsingular and projective. Its extension to arbitrarily singular quasi-projective varieties is not technically demanding, but appears to carry significant information.

1.2. To see why Theorem 1.1 may be interesting, consider the (very) special case in which X is the complement $D(F)$ of a hypersurface in $\mathbb{P}^n$, given by the vanishing of a homogeneous polynomial $F(x_0, \dots, x_n)$. Using an expression for the c_{SM} class from [Alu03a], it is easy to show that the degree of the 'polar' (or 'gradient') map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by $(\partial F/\partial x_0, \dots, \partial F/\partial x_n)$ equals $(-1)^n \gamma_{D(F)}(-1)$ (see §3.1). By Theorem 1.1,

$$\gamma_{D(F)}(-1) = \chi_{D(F)}(0) + \chi'_{D(F)}(0) = \chi(D(F)) - \chi(D(F) \cap H) = \chi(D(F) \smallsetminus H)$$

for a general hyperplane H. Over $\mathbb{C}$, this formula for the polar degree was obtained by Dimca and Papadima as a consequence of their description of the homotopy type of the complement $D(F)$ ([DP03], Theorem 1). The argument deriving this formula from Theorem 1.1 works over any algebraically closed field of characteristic zero, and hence it also extends to this context the consequence that the degree of the polar map only depends on the *reduced* polynomial F_{red}. In particular, F is 'homaloidal' if and only if F_{red} is; this fact was conjectured by Dolgachev ([Dol00], §3). More generally, the argument extends easily to yield a formula for the *Huh-Teissier-Milnor* numbers $\mu^{(i)}$ defined in [Huh12] in terms of mixed multiplicities. Theorem 1.1 implies

$$\mu^{(i)} = (-1)^i \chi(D(F) \cap (\mathbb{P}^i \smallsetminus \mathbb{P}^{i-1})) \quad ,$$

where $\mathbb{P}^i$ denotes a general linear subspace of dimension i (Corollary 3.7). This formula is given in Theorem 9 (1) of [Huh12], as a consequence of an explicit topological description of the intersection $D(f) \cap \mathbb{P}^i$. The argument sketched above does less, since it only yields the numerical consequence of this more refined topological information; but it proves the validity of this formula over any algebraically closed field of characteristic zero, and is in a sense more straightforward.

This approach also allows us to generalize some of these considerations to arbitrary codimension. We propose a definition of 'polar degrees' for arbitrary subschemes $X \subseteq \mathbb{P}^n$, giving a relation between these degrees and the Euler characteristics of sections of $\mathbb{P}^n \smallsetminus X$ generalizing the Dimca-Papadima-Huh formulas for hypersurfaces recalled above. We prove that these polar degrees only depend on the support $X_{\rm red}$, thus generalizing Dolgachev's conjecture to arbitrary subschemes of $\mathbb{P}^n$. The polar degrees of X are given in terms of generators for an ideal defining X; the fact that they are independent of the choice of the generators would deserve to be understood from a more explicitly algebraic viewpoint. In our approach, the independence follows from the relation with Chern-Schwartz-MacPherson classes.

Providing algebro-geometric proofs of the formula for the polar degree and Dolgachev's conjecture was a natural problem. Fassarella, Pereira, and Medeiros have developed an algebro-geometric approach through foliations ([FP07], [FM12]), and also obtain the Dimca-Papadima formula and Huh's generalization to higher order polar degrees; their results are stated for complex hypersurfaces. We note that some of the beautiful formulas for the polar degrees obtained in [FM12] have a natural explanation when the degrees are viewed in terms of $c_{\rm SM}$ classes. (For example, formula (3) in [FM12] is an expression of the inclusion-exclusion property satisfied by $c_{\rm SM}$ classes.)

Previous work on Dolgachev's conjecture and homaloidal polynomials also includes [KS01]; [Bru07] (for product of linear forms, over fields of arbitrary characteristic); [CRS08]; and [Ahm10]. We are not aware of work on polar degrees in higher codimension. Over $\mathbb{C}$, the relation between $c_{\rm SM}$ classes and the Huh-Teissier-Milnor numbers is also observed in [Huh12] and further employed very effectively in the recent preprint [Huh], which also includes applications to the problem of studying homaloidal polynomials.

1.3. Details for the application to Dimca-Papadina/Huh formulae, and the generalization to higher codimension, are given in §3. In §4 we sketch a general framework suggested by Theorem 1.1. The fact that the information carried by the naive Euler polynomial $\chi_X(t)$ and the more sophisticated Chern class polynomial $\gamma_X(t)$ is precisely the same indicates that it should be possible to give a simple treatment of 'polynomial' $c_{\rm SM}$ classes, based solely on the Euler characteristic. The target of the $c_{\rm SM}$ natural transformation is the Chow functor; while this is a virtue of the full theory, it is an obstacle if one is interested in e.g., studying the motivic nature of these classes. For example, while $c_{\rm SM}$ classes satisfy a scissor relation, they do not factor through the Grothendieck group of varieties, simply because their target depends on the variety. We propose a theory of Chern classes with constant *polynomial* target for varieties endowed with a map to projective space. This theory can be defined solely in terms of Euler characteristics and the involution appearing in Theorem 1.1, and its main covariance property, which parallels closely the functoriality of $c_{\rm SM}$ classes, is a simple cut-and-paste exercise (cf. Lemmata 2.3 and 4.4). It is immediate that the resulting 'polynomial Chern classes' c_* with values in $\mathbb{Z}[t]$ factor through the Grothendieck group of varieties endowed with morphisms to projective space. We prove that the classes c_* also define *ring* homomorphisms, preserving different products one can define on this Grothendieck group (Propositions 4.6 and 4.11). This is useful in concrete computations.

Relative Grothendieck groups of varieties have been applied to a very general theory of characteristic classes for singular varieties in [BSY10].

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2. Euler and Chern

2.1. **The Euler characteristic.** Throughout the paper, we work over a fixed algebraically closed field k of characteristic 0. Part of our goal is to emphasize that familiar notions from complex geometry generalize to this context, so for once we do not encourage the reader to assume that $k = \mathbb{C}$. Morphisms are implicitly assumed to be separable, of finite type; 'point' will mean 'closed point'.

The *Grothendieck group* of k-varieties, $K(\mathrm{Var}_k)$ is the abelian group generated by isomorphism classes of k-varieties modulo the scissor relation

$$[X] = [U] + [Z]$$

for every closed $Z \subseteq X$, $U = X \smallsetminus Z$. The operation defined on generators by $[X] \cdot [Y] := [X \times Y]$ endows $K(\mathrm{Var}_k)$ with a structure of ring.

Lemma 2.1. *There is a unique ring homomorphism $\chi : K(\mathrm{Var}_k) \to \mathbb{Z}$ such that if X is nonsingular and projective, then $\chi([X]) = \int c(TX) \cap [X]$.*

Proof. By Theorem 3.1 in [Bit04], $K(\mathrm{Var}_k)$ admits an alternative presentation as the abelian group generated by isomorphism classes of smooth projective k-varieties modulo the relation $[B\ell_Z X] - [E] = [X] - [Z]$ for X smooth and projective and $Z \subseteq X$ a closed smooth subvariety; here $B\ell_Z X$ is the blow-up of X along Z and E is the exceptional divisor. It suffices therefore to verify that the degree of the top Chern class of the tangent bundle satisfies these relations, and the relation defining the ring structure. This latter check is immediate. As for the blow-up relations, since $p : E \to Z$ may be identified with the projective normal bundle of Z in X, $c(TE)$ is determined by the Euler sequence:

$$0 \longrightarrow \mathscr{O} \longrightarrow p^* N_Z X \otimes \mathscr{O}(1) \longrightarrow TE \longrightarrow p^* TZ \longrightarrow 0 \quad .$$

From this it is straightforward to verify that

$$\int c(TE) \cap [E] = (d+1) \int c(TZ) \cap [Z] \quad ,$$

where $d = \operatorname{rk} N_Z X$ is the codimension of Z in X. Thus, what needs to be checked is that

$$\int c(TB\ell_Z X) \cap [B\ell_Z X] - \int c(TX) \cap [X] = d \int c(TZ) \cap [Z] \quad ,$$

for Z a closed smooth subvariety of a smooth projective variety X. This can be done by using the explicit formula for blowing up Chern classes given in [Ful84], Theorem 15.4. □

We will write $\chi(U)$ for the value taken by the homomorphism χ on $[U]$, and call this number the *Euler characteristic* of U. Every locally closed subset of a complete variety has a class in $K(\mathrm{Var}_k)$, so every such set has a well-defined Euler characteristic. Of course if $k = \mathbb{C}$, then χ agrees with the ordinary topological Euler characteristic (with compact support).

Remark 2.2. The fact that χ defines a homomorphism $K(\mathrm{Var}_k) \to \mathbb{Z}$ captures the usual properties of the ordinary Euler characteristic: inclusion-exclusion ($\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$) and multiplicativity on locally trivial fibrations. Also, if $X \to Y$ is smooth and proper, then $\chi(X) = \chi(Y) \cdot \chi_f$, where χ_f is the Euler characteristic of any fiber.

This seems the appropriate place to point out the following observation, that will be used in §4:

Lemma 2.3. *Let $f : X \to Y$ be a morphism of k-varieties. Then there exist subvarieties $V_1, \dots, V_r$ of Y and integers $m_1, \dots, m_r$ such that $\forall p \in Y$,*

$$\chi(f^{-1}(p)) = \sum_{V_j \ni p} m_j \quad .$$

Further, for every subvariety $W \subseteq Y$,

$$\chi(f^{-1}(W)) = \sum_{i=1}^{r} m_j \, \chi(W \cap V_j) \quad .$$

Indeed, the existence of $V_1, \dots, V_r$ follows from the fact that the Euler characteristic of fibers is constant on a nonempty open set. Both this fact and the second assertion follow from standard techniques: Nagata's embedding theorem, resolution of singularities, generic smoothness, and inclusion-exclusion for χ may be used to reduce to the case of f smooth and proper, for which the assertions are trivial. Details are left to the reader, and may be distilled from [Alu06], §5.4-6. ⌟

2.2. **Chern-Schwartz-MacPherson classes.** For a variety X, we denote by $\mathscr{C}(X)$ the abelian group of $\mathbb{Z}$-valued *constructible functions* on X; thus, every $\varphi \in \mathscr{C}(X)$ may be written as a finite sum $\sum_i n_i 1\!\!1_{Z_i}$ where $n_i \in \mathbb{Z}$, Z_i are subvarieties of X, and $1\!\!1_{Z_i}$ is the function giving 1 for $p \in Z_i$ and 0 for $p \notin Z_i$. The assignment $X \mapsto \mathscr{C}(X)$

defines a covariant functor to abelian groups: if $f : X \to Y$ is a morphism, we may define a push-forward

$$f_* : \mathscr{C}(X) \to \mathscr{C}(Y)$$

by letting $f_*(1\!\!1_Z)(p) = \chi(Z \cap f^{-1}(p))$ for any subvariety $Z \subseteq X$ and $p \in Y$, and extending by linearity. Note that according to this definition

$$f_*(1\!\!1_X) = \sum_i m_i 1\!\!1_{V_i} \quad ,$$

where the varieties V_i are those appearing in Lemma 2.3.

On the category of complete k-varieties and proper morphisms there exists a unique natural transformation $\mathscr{C} \rightsquigarrow A_*$, normalized by the condition that if X is nonsingular and complete, then $1\!\!1_X \mapsto c(TX) \cap [X]$. Over $\mathbb{C}$ and in homology, this fact is due to R. MacPherson ([Mac74]). With suitable positions, the class associated with $1\!\!1_X$ for a (possibly) singular X agrees with the class previously defined by M.-H. Schwartz ([Sch65a], [Sch65b]). The theory was extended to arbitrary algebraically closed fields of characteristic 0 in [Ken90]; the treatment in [Alu06] includes an extension to non-complete varieties and not necessarily proper morphisms. We call the class associated with a constructible function φ on a variety X the 'Chern-Schwartz-MacPherson class' of φ, denoted $c_{\rm SM}(\varphi)$. If $Z \subseteq X$ and the context is clear, we denote by $c_{\rm SM}(Z)$ the class $c_{\rm SM}(1\!\!1_Z) \in A_*X$.

The normalization and functoriality properties of $c_{\rm SM}$:

- $c_{\rm SM}(1\!\!1_X) = c(TX) \cap [X]$ for X nonsingular and complete, and
- $f_* c_{\rm SM}(\varphi) = c_{\rm SM}(f_*\varphi)$ for $f : X \to Y$ a proper morphism

(and linearity) determine $c_{\rm SM}$ uniquely, by resolution of singularities. We note that if $X_1, X_2 \subseteq X$, then $1\!\!1_{X_1 \cup X_2} = 1\!\!1_{X_1} + 1\!\!1_{X_2} - 1\!\!1_{X_1 \cap X_2}$, and hence

$$c_{\rm SM}(X_1 \cup X_2) = c_{\rm SM}(X_1) + c_{\rm SM}(X_2) - c_{\rm SM}(X_1 \cap X_2) \quad \text{in } A_*X:$$

thus, $c_{\rm SM}$ classes (like χ) satisfy inclusion-exclusion. In fact, the degree of $c_{\rm SM}(X)$ agrees with $\chi(X)$: if X is complete, then

$$\int c_{\rm SM}(X) = \chi(X) \quad . \tag{1}$$

To see this, apply functoriality to the constant map from X to a point. This simple observation will be crucial in what follows.

2.3. **The $c_{\rm SM}$ class of a hypersurface.** As this will be needed in a proof given below, we recall an expression for $c_{\rm SM}(X)$ in the case that X is a hypersurface in a nonsingular variety V. We will use the following notation: if $a \in A_{\dim V - i}V$ is a class in codimension i, and $\mathscr{L}$ is a line bundle on V, we let

$$a^\vee := (-1)^i a \quad , \quad a \otimes \mathscr{L} = \frac{a}{c(\mathscr{L})^i} \quad ,$$

and extend these operations to A_*V by linearity. (As $c(\mathscr{L}) = 1 + c_1(\mathscr{L})$, and $c_1(\mathscr{L})$ is nilpotent, $c(\mathscr{L})$ has an inverse as an operator over A_*V; this is what the notation $a/c(\mathscr{L})$ means.) For properties satisfied by these operations, we address the reader to [Alu94], §2. In particular, $(A \otimes \mathscr{L}_1) \otimes \mathscr{L}_2 = A \otimes (\mathscr{L}_1 \otimes \mathscr{L}_2)$ for all line bundles $\mathscr{L}_1$, $\mathscr{L}_2$, and

$$(c(\mathscr{E}) \cap A) \otimes \mathscr{L} = c(\mathscr{E} \otimes \mathscr{L}) \cdot c(\mathscr{L})^{-\operatorname{rk}\mathscr{E}} \cap (A \otimes \mathscr{L})$$

for all $A \in A_*V$ and all vector bundles $\mathscr{E}$ on V. We routinely abuse language and write $A \otimes \mathscr{L}$ for $A \in A_*Y$ if $Y \subseteq V$; if we need to emphasize that the codimension is taken in V, we write $A \otimes_V \mathscr{L}$.

Theorem 2.4 ([Alu99], Theorem I.4)**.** *Let X be a hypersurface in a nonsingular complete variety V. Then*

$$c_{SM}(1\!\!1_X) = c(TV) \cap \left(\frac{[X]}{1+X} + \frac{1}{1+X} \cap (s(JX, V)^\vee \otimes_V \mathscr{O}(X)) \right)$$

in A_*V.

Here JX denotes the *singularity subscheme* of X, locally defined in V by a local generator F for the ideal of X and by ∂F as ∂ ranges over local sections of Der_V. (Informally, JX is defined by F and its partial derivatives.) We use the Segre class $s(-,-)$ in the sense of [Ful84], and omit evident pull-backs and push-forwards.

Remark 2.5. The right-hand side of the formula in Theorem 2.4 makes sense for hypersurfaces with (possibly) multiple components: multiple components of X appear as components of JX. It is a remarkable feature of this expression that it does *not* depend on the multiplicities of the components: the change in $s(JX, V)$ due to the presence of multiplicities is precisely compensated by the other ingredients in the expression. This is observed in [Alu99], §2.1; briefly, the blow-up formula proved in §3 of [Alu99] reduces this fact to the simple normal crossing case, where it can be worked out explicitly.

This is compatible with the fact that the left-hand side, $c_{\mathrm{SM}}(1\!\!1_X)$, should ignore the multiplicities of the components of X because $1\!\!1_X$ is determined by set-theoretic information: if U is the complement of X in V, $1\!\!1_X = 1\!\!1_V - 1\!\!1_U = 1\!\!1_{X_{\mathrm{red}}}$. ⌟

2.4. **$c_{\mathbf{SM}}$ classes and general hyperplane sections.** We now assume that $V = \mathbb{P}^n$, and consider general hyperplane sections of c_{SM} classes of locally closed subsets.

Proposition 2.6. *Let $U \subseteq \mathbb{P}^n$ be any locally closed set (so that $1\!\!1_U$ is a constructible function). Then for a general hyperplane $H \subseteq \mathbb{P}^n$,*

$$c_{SM}(1\!\!1_{U \cap H}) = \frac{H}{1+H} \cdot c_{SM}(1\!\!1_U)$$

in A_*V.

Proof. Since U may be written as a set-difference of two closed sets, we may assume that U is itself closed, by additivity of c_{SM} classes. Since every closed subset may be written as an intersection of hypersurfaces, by inclusion-exclusion we may assume that $U = X$ is a hypersurface of $\mathbb{P}^n$. Let $H \cong \mathbb{P}^{n-1}$ be a general hyperplane and $X' = X \cap H$. By Theorem 2.4,

$$c_{\rm SM}(1\!\!1_X) = c(T\mathbb{P}^n) \cap \left(\frac{[X]}{1+X} + \frac{1}{1+X} \cap \left(s(JX, \mathbb{P}^n)^\vee \otimes_{\mathbb{P}^n} \mathscr{O}(X) \right) \right) \quad \text{in } A_*\mathbb{P}^n, \text{ and}$$

$$c_{\rm SM}(1\!\!1_{X'}) = c(T\mathbb{P}^{n-1}) \cap \left(\frac{[X']}{1+X'} + \frac{1}{1+X'} \cap \left(s(JX', H)^\vee \otimes_H \mathscr{O}(X') \right) \right) \quad \text{in } A_*\mathbb{P}^{n-1}.$$

It is a good exercise in the notation introduced in §2.3 to verify that the equality of these two classes is equivalent to

$$H \cdot s(JX, \mathbb{P}^n) = s(JX', H) \tag{2}$$

(cf. [Alu12a], §3.2). Thus, we are reduced to proving (2). This follows from two observations:

- $H \cdot s(JX, \mathbb{P}^n) = s(H \cap JX, H)$ if H intersects properly the supports of the cone of JX in $\mathbb{P}^n$; and
- For a general hyperplane H, $s(JX', H) = s(H \cap JX, H)$.

The first assertion may be verified by comparing the blow-up of $\mathbb{P}^n$ along JX and the blow-up of H along $H \cap JX$; details may be found in e.g., the proof of Claim 3.2 in [Alu12a]. For the second assertion, the ideals of JX' and $H \cap JX$ have the same integral closure by Teissier's idealistic Bertini, [Tei77], §2.8 (see [Huh12], Lemma 31 for a transparent proof in the homogeneous case that does not depend on complex geometry). Subschemes with the same integral closure have the same Segre class since they have the same normalized blow-up, and Segre classes are birational invariants ([Ful84], Proposition 4.2).

This verifies (2), concluding the proof of the proposition. □

2.5. **Proof of Theorem 1.1.** Let X be a locally closed subset of $\mathbb{P}^n$. As in §1, we let

$$\chi_X(t) := \sum_{r\ge 0} (-1)^r \chi(H_1 \cap \cdots \cap H_r \cap X)\, t^i \quad ,$$

where $H_1, H_2, \dots$ are general hyperplanes. Also, we let

$$\gamma_X(t) := \sum_{r\ge 0} \left(\int H^r \cdot c_{\rm SM}(1\!\!1_X) \right) t^r$$

where H is the hyperplane class. That is, $\gamma_X(t)$ is obtained from $c_{\rm SM}(1\!\!1_X) = \sum_{r\ge 0} c_r[\mathbb{P}^r]$ by replacing $[\mathbb{P}^r]$ by t^r. Iterating Proposition 2.6, we obtain that with notation as above,

$$c_{\rm SM}(1\!\!1_{H_1\cap\cdots\cap H_r \cap X}) = \frac{H^r}{(1+H)^r} \cap c_{\rm SM}(1\!\!1_X)$$

in $A_*\mathbb{P}^n$; by (1),

$$\chi(H_1 \cap \cdots \cap H_r \cap X) = \int \frac{H^r}{(1+H)^r} \cap \left(\sum_{\ell \ge 0} c_\ell [\mathbb{P}^\ell] \right)$$

and hence

$$\begin{aligned}
\chi(t) &= \sum_{r\ge 0}\left(\int \frac{(-H)^r}{(1+H)^r}\cap c_{\mathrm{SM}}(1\!\!1_X)\right) t^r = \sum_{r\ge 0}\left(\int \frac{(-H)^r}{(1+H)^r}\cap \sum_{\ell\ge 0} c_\ell[\mathbb{P}^\ell]\right) t^r \\
&= \sum_{\ell\ge 0} c_\ell \sum_{r\ge 0}\sum_{k\ge 0}\binom{r+k-1}{k}(-H)^{r+k}\cdot[\mathbb{P}^\ell] t^r \\
&= c_0 + t\sum_{\ell\ge 1} c_\ell \sum_{k\ge 0}\binom{\ell-1}{k}(-1)^\ell t^{\ell-1-k} \\
&= c_0 - t\sum_{\ell\ge 1} c_\ell\,(-1)^{\ell-1}\sum_{k\ge 0}\binom{\ell-1}{k} t^{\ell-1-k} \\
&= c_0 - t\sum_{\ell\ge 1} c_\ell\,(-t-1)^{\ell-1} \\
&= \frac{\gamma_X(0)+t\cdot\gamma_X(-t-1)}{t+1} \quad .
\end{aligned}$$

With notation as in §1, this verifies that $\chi_X = \mathscr{I}(\gamma_X)$ and concludes the proof of Theorem 1.1.

2.6. The fact that the polynomial $\gamma_X(t)$ may be recovered from the collection of Euler characteristics of general hyperplane sections follows directly from the good behavior of c_{SM} under general hyperplane sections (verified in Proposition 2.6). If a class has the same behavior, and its degree equals the Euler characteristic, then the class must agree with the c_{SM} class. This strategy was used in [Alu94] to prove a numerical version of Theorem 2.4; it has also been used recently by Liao in studying the relation between the Chern class of the bundle of logarithmic derivations of a free divisors and the c_{SM} class of the complement of the divisor ([Lia]).

Other classes have the same behavior under general hyperplane sections. For example, for a hypersurface X of a nonsingular variety V, let $\pi(X)$ denote the *Parusiński-Milnor number* of X: this is an integer defined for arbitrary hypersurfaces, and agreeing with the sum of the Milnor numbers at the singularities if these are all isolated. This invariant is defined and studied in [Par88] (over $\mathbb{C}$). For a hypersurface X of $\mathbb{P}^n$, we can define the polynomial

$$\pi_X(t) := \sum_{r\ge 0}(-1)^r\pi(H_1\cap\cdots\cap H_r\cap X)\,t^i \quad ,$$

where $H_1, H_2, \dots$ are general hyperplanes. On the other hand, we can consider the *Milnor class* of X, defined in [PP01]; its push-forward to $\mathbb{P}^n$ is a class $\nu_0[\mathbb{P}^0]+\nu_1[\mathbb{P}^1]+\cdots$, with which we associate the polynomial

$$\nu_X(t) := \sum_{r\ge 0}\nu_r\,t^r \quad .$$

Claim 2.7. *With notation as above, ν_X and π_X are mapped to each other by the involution $\mathscr{I}$.*

Indeed, the degree of the Milnor class equals the Parusiński-Milnor number ([Alu99], §4.1); so this follows (as in the proof of Theorem 1.1) from the fact that the Milnor class satisfies the same formula as c_{SM} does with respect to general hyperplanes sections. This in turn follows from the fact that, up to sign, the Milnor class equals the difference between the Chern-Schwartz-MacPherson class and the Chern class of the virtual tangent bundle, which also behaves as prescribed by Proposition 2.6 with respect to hyperplane sections. Details are left to the reader.

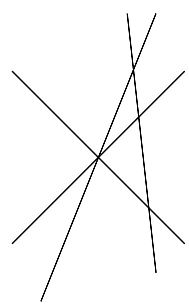
Example 2.8. As an example illustrating Theorem 1.1, we consider a hyperplane arrangement $\mathscr{A}$ in $\mathbb{P}^n$. With $\mathscr{A}$, or more precisely with the corresponding central arrangement $\hat{\mathscr{A}}$ in k^{n+1}, we can associate the *characteristic polynomial* $P_{\hat{\mathscr{A}}}(t)$ (Definition 2.5.2 in [OT92]). This is one of the most important combinatorial invariants of the arrangement; for example, in the case of graphical arrangements it recovers the chromatic polynomial of the corresponding graph (Theorem 2.88 in [OT92]). It immediately follows from the definition that $P_{\hat{\mathscr{A}}}(1) = 0$, and we let $\underline{P}_{\hat{\mathscr{A}}}(t)$ denote the quotient $P_{\hat{\mathscr{A}}}(t)/(t-1)$. In general, $\underline{P}_{\mathscr{A}}(t)$ agrees with the Poincaré polynomial of the complement $M(\mathscr{A})$ of $\mathscr{A}$ in projective space up to a simple coordinate change (Theorem 5.93 in [OT92]).

Corollary 2.9. *With notation as above,*

$$\underline{P}_{\hat{\mathscr{A}}}(t) = \frac{(t-1)\chi_{M(\mathscr{A})}(-t) + \chi_{M(\mathscr{A})}(0)}{t} \quad \textit{and} \quad \chi_{M(\mathscr{A})}(t) = \frac{t\,\underline{P}_{\hat{\mathscr{A}}}(-t) + \underline{P}_{\hat{\mathscr{A}}}(1)}{t+1} \quad .$$

Proof. By Theorem 3.1 in [Alu12b], $\gamma_{M(\mathscr{A})}(t) = \underline{P}_{\hat{\mathscr{A}}}(t+1)$. Applying the involution $\mathscr{I}$ gives the first formula. The second formula is equivalent to the first. □

For example, consider the arrangement $\mathscr{A}$ in $\mathbb{P}^2$ consisting of three incident lines and of a line not containing the point of intersection:



It is immediately verified that $\chi_0(M(\mathscr{A})) = 0$, $\chi_1(M(\mathscr{A})) = -2$, $\chi_2(M(\mathscr{A})) = 1$. (For instance, the intersection of $M(\mathscr{A})$ with a general line consists of the complement of 4 points in $\mathbb{P}^1$, with Euler characteristic -2.) Therefore $\chi_{M(\mathscr{A})}(t) = 2t + t^2$, and hence

$$\underline{P}_{\hat{\mathscr{A}}}(t) = \frac{(t-1)(-2t+t^2)+0}{t} = t^2 - 3t + 2 \quad .$$

Note that c_{SM} classes do not appear directly in the statement of Corollary 2.9, but they streamline the proof considerably, via Theorem 1.1.

J. Huh has recently proved that the coefficients of $P_{\hat{\mathscr{A}}}(t)$ form a *log-concave* sequence, settling long-standing conjectures of Read, Rota, Heron, and Welsh ([Huh12]). The relation of $P_{\hat{\mathscr{A}}}(t)$ with information equivalent to the c_{SM} class of the complement (the *polar degrees,* cf. §3, particularly Remark 3.5) plays an important rôle in Huh's work. ⌟

3. Dimca-Papadima/Huh formulae

3.1. Polar degrees. The application sketched in §1.2 depends on the following result from [Alu03a].

Let X be a hypersurface of $\mathbb{P}^n$, defined by a homogeneous polynomial $F(x_0,\dots,x_n)$. Consider the rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by the partial derivatives of F:

$$p \mapsto \left(\frac{\partial F}{\partial x_0}(p) : \cdots : \frac{\partial F}{\partial x_n}(p)\right) \quad .$$

This is the *polar* map of [Dol00], called the *gradient* map in [DP03]. We denote by Γ the graph of this map. The class of Γ in $\mathbb{P}^n\times\mathbb{P}^n$ determines integers g_i such that

$$[\Gamma] = g_0k^n + g_1hk^{n-1} + \cdots + g_nh^n$$

where h, resp., k is the pull-back of the hyperplane class from the first, resp., second factor. The number g_i are the *projective degrees* of the polar map ([Har92], Example 19.4): g_i is the degree of the restriction of the polar map to a general $\mathbb{P}^i$ in $\mathbb{P}^n$. We will call g_i the *i-th polar degree* of F. In particular, the n-th degree g_n equals the degree of the polar map itself. By definition, the polynomial F is *homaloidal* if $g_n = 1$, i.e., if the polar map is birational ([Dol00]).

All the considerations in this section will be consequences of Theorem 1.1 and the following result.

Theorem 3.1 ([Alu03a], Theorem 2.1)**.** *With notation as above,*

$$c_{SM}(1\!\!1_X) = (1+h)^{n+1} - \sum_{j=0}^{n} g_j\cdot(-h)^j(1+h)^{n-j} \quad . \tag{3}$$

The point is that Γ may be identified with the blow-up of $\mathbb{P}^n$ along the scheme defined by the ideal generated by the partial derivatives of F, that is, the singularity subscheme JX of X. It is then clear that the class of Γ carries essentially the same information as the (push-forward of the) Segre class of JX in $\mathbb{P}^n$. Performing this notational translation in the formula given in Theorem 2.4 yields Theorem 3.1. Further details may be found in [Alu03a].

The following is an immediate consequence of (3):

Corollary 3.2. *The polar degrees depend only on the reduced polynomial F_{red} associated with F. In particular, F is homaloidal if and only if F_{red} is homaloidal.*

Indeed, by means of Theorem 3.1 all the g_j's are determined by $c_{\mathrm{SM}}(1\!\!1_X) = c_{\mathrm{SM}}(1\!\!1_{X_{\mathrm{red}}})$ (cf. Remark 2.5).

As mentioned in the introduction, the last part of this statement had been conjectured by Dolgachev ([Dol00], end of §3). A topological proof (over $\mathbb{C}$) was first given in [Dim01]; several other proofs have appeared in the meanwhile. The argument given above shows that the fact holds over any algebraically closed field of characteristic zero, and is straightforward modulo the result from [Alu99] recalled in §2.3.

3.2. **Formularium.** As mentioned above, the g_i are the projective degrees of the polar map. The term $(1+h)^{n+1}$ in (3) is the Chern class of $\mathbb{P}^n$, so Theorem 3.1 may be reformulated as

$$c_{\mathrm{SM}}(1\!\!1_{\mathbb{P}^n\smallsetminus X}) = \sum_{j=0}^{n} g_j \cdot (-h)^j (1+h)^{n-j} \quad . \tag{4}$$

Using the notation introduced in §2.5 and applying Theorem 1.1 we get:

Corollary 3.3. *Let $D(F)$ be the complement $\mathbb{P}^n \smallsetminus X$. Then*

- $\gamma_{D(F)}(t) = \sum_{j=0}^{n} g_j \cdot (-1)^j (t+1)^{n-j}$.
- $\chi_{D(F)}(t) = (-1)^n \sum_{j=0}^{n} g_j \cdot \dfrac{t^{n-j+1} - (-1)^{n-j+1}}{t+1}$.

Proof. In (4), h^i stands for the class $h^i \cdot [\mathbb{P}^n] = [\mathbb{P}^{n-i}]$, which is replaced by t^{n-i} in the polynomial $\gamma_{D(F)}(t)$. This change may be performed by replacing h by $1/t$ and multiplying through by t^n, yielding the first formula. The second follows immediately by applying the involution $\mathscr{I}$, as prescribed by Theorem 1.1. □

We can assemble the g_j's into yet another polynomial determined by X:

$$g_X(t) := \sum_{j=0}^{n} g_j\, t^{n-j} \quad ,$$

and Corollary 3.3 may then be reformulated as

- $\gamma_{D(F)}(t) = (-1)^n\, g_X(-t-1)$.
- $\chi_{D(F)}(t) = (-1)^n \dfrac{t \cdot g_X(t) + g_X(-1)}{t+1}$.

'Solving for g' in these two formulas gives

Corollary 3.4. *With notation as above,*

$$g_X(t) = (-1)^n\, \gamma_{D(F)}(-t-1) = (-1)^n \frac{(t+1)\cdot \chi_{D(f)}(t) - \chi_{D(f)}(0)}{t} \quad . \tag{5}$$

Remark 3.5. If X is a hyperplane arrangement, Corollary 3.4 shows that $g_X(t)$ agrees with $\underline{P}_{\hat{\mathscr{A}}}(-t)$ up to a sign (cf. Example 2.8). This is observed (at least over $\mathbb{C}$) in Corollary 25 of [Huh12].

In general, the first formula shows that $\gamma_{D(f)}(t)$ and $g_X(t)$ are also related by an involution. ⌟

In particular, (5) computes the Euler characteristic of the complement $D(F)$ as

$$\chi(D(F)) = \gamma_{D(F)}(0) = (-1)^n g_X(-1) = g_0 - g_1 + g_2 - \cdots \pm g_n \quad .$$

Further, the polar degree g_n equals

$$g_X(0) = (-1)^n \gamma_{D(f)}(-1) = (-1)^n \left(\chi_{D(f)}(0) + \chi'_{D(f)}(0)\right) \quad ,$$

as was mentioned in §1.2. That is,

Corollary 3.6 (Dimca-Papadima). *The degree of the polar map determined by the homogeneous polynomial F equals*

$$(-1)^n \left(\chi(D(F)) - \chi(H \cap D(F))\right)$$

where H is a general hyperplane.

Over $\mathbb{C}$, this formula is given in [DP03], Theorem 1. The argument presented above proves it over any algebraically closed field of characteristic 0, where we adopt the definition of Euler characteristic χ recalled in §2.1. Tracing the argument shows that the precise requirement on H is that it should intersect properly the supports of the normal cone of JX (cf. the proof of Proposition 2.6). This should be viewed as an algebro-geometric analog of the condition specified in the paragraph following the statement of Theorem 1 in [DP03].

More generally:

Corollary 3.7 (Huh). *For $j = 0, \dots, n$:*

$$g_j = (-1)^j \, \chi(D(F) \cap (L_j \smallsetminus L_{j-1}))$$

where L_j is a general linear subspace of dimension j in $\mathbb{P}^n$.

(Just read off the coefficient of t^{n-j} in (5).)

Over $\mathbb{C}$, this formula is given in Theorem 9 in [Huh12], where it is obtained from a description of the homotopy type of the general linear sections of $D(F)$.

Remark 3.8. The connection between c_{SM} classes and the polar degrees g_j is mentioned explicitly in [Huh12]. Our only contribution here amounts to the observation that this connection alone suffices for the formula in Corollary 3.7, modulo the rather simple-minded Theorem 1.1. This has the very minor advantage of providing a 'non-topological' interpretation of the formula, which is then shown to hold over any algebraically closed field of characteristic 0. ⌟

3.3. **Higher codimension.** It is natural to ask whether formulas analogous to those reviewed in the previous section may be given for higher codimensional subschemes in $\mathbb{P}^n$. It is not clear *a priori* what should play the role of the 'polar degrees' g_j defined in §3.1. Maybe the most surprising aspect here is that one *can* in fact define these degrees in complete generality.

Let $S \subseteq \mathbb{P}^n$ be any subscheme, and let $F_1, \dots, F_r$ be nonzero homogeneous generators for any ideal I defining X. Denote by $g_i^{(F)}$ the i-th polar degree of the homogeneous polynomial F, defined as in §3.1.

Definition 3.9. We define the *i-th polar degree of S* to be

$$g_i^S := \sum_{\emptyset \neq J \subseteq \{1,\dots,r\}} (-1)^{|J|+1} g_i^{(\prod_{j\in J} F_j)}$$

where $F_1, \dots, F_r$ are any collection of homogeneous polynomials generating an ideal I defining S. ⌟

It is not obvious (to us) that these degrees are well-defined, that is, that they do not depend on the choice of the generators of the ideal I. We will see that they are, and that in fact they only depend on the support S_{red} of the scheme defined by I. Thus, the numbers g_i^S do not change if I is replaced by $\sqrt{I}$ or by the saturation of I. This fact generalizes Dolgachev's conjecture to arbitrary subschemes of $\mathbb{P}^n$, over any algebraically closed field of characteristic 0.

Example 3.10. The ideal of a twisted cubic $C \subseteq \mathbb{P}^3$ is generated by the quadratic polynomials $F_1 = x_0x_3 - x_1x_2$, $F_2 = x_0x_2 - x_1^2$, $F_3 = x_1x_3 - x_2^2$. The polar map of F_1 is given in homogeneous coordinates by $(x_3 : -x_2 : -x_1 : x_0)$, giving $g_0^{(F_1)} = \cdots = g_3^{(F_1)} = 1$. Both F_2 and F_3 are cones over smooth conics, and this gives easily $g_0^{(F_i)} = \cdots = g_2^{(F_i)} = 1$, $g_3^{(F_i)} = 0$ for $i = 1, 2$. An explicit computation (which may be performed with e.g., Macaulay2 [GS]) shows that

$$g_0^{(F_iF_j)} = 1, \quad g_1^{(F_iF_j)} = 3, \quad g_2^{(F_iF_j)} = 5, \quad g_3^{(F_iF_j)} = 3$$

for $i \neq j$, and

$$g_0^{(F_1F_2F_3)} = 1, \quad g_1^{(F_1F_2F_3)} = 5, \quad g_2^{(F_1F_2F_3)} = 10, \quad g_3^{(F_1F_2F_3)} = 6 \quad .$$

It follows that

$$g_0^C = 1, \quad g_1^C = -1, \quad g_2^C = -2, \quad g_3^C = -2 \quad .$$

(For example, $g_3^C = 1 + 0 + 0 - 3 - 3 - 3 + 6 = -2$.)

On the other hand, the twisted cubic C is also the set-theoretic intersection of the quadric $F = x_0x_2 - x_1^2 = 0$ and the cubic $G = x_2(x_1x_3 - x_2^2) - x_3(x_0x_3 - x_1x_2) = 0$. An explicit computation (again performed with Macaulay2) gives

$$g_0^{(F)} = 1, \quad g_1^{(F)} = 1, \quad g_2^{(F)} = 1, \quad g_3^{(F)} = 0$$
$$g_0^{(G)} = 1, \quad g_1^{(G)} = 2, \quad g_2^{(G)} = 3, \quad g_3^{(G)} = 1$$

(so that G is homaloidal; this plays no rôle here) and

$$g_0^{(FG)} = 1, \quad g_1^{(FG)} = 4, \quad g_2^{(FG)} = 6, \quad g_3^{(FG)} = 3 \quad .$$

This gives

$$g_0^C = 1+1-1 = 1, \quad g_1^C = 1+2-4 = -1, \quad g_2^C = 1+3-6 = -2, \quad g_3^C = 0+1-3 = -2\,,$$

with the same result from the ideal for a different scheme structure, as promised. ⌟

To prove that the polar degrees of a subscheme S are well-defined, and in fact only depend on S_{red}, it suffices to observe that they are related with the polynomials $\gamma_{\mathbb{P}^n \smallsetminus S}$, $\chi_{\mathbb{P}^n \smallsetminus S}$ by *the same formulas as in the hypersurface case.* As in the hypersurface case (but now for arbitrary subschemes $S \subseteq \mathbb{P}^n$) we define

$$g_S(t) := \sum_{i=0}^{n} g_i^S\, t^{n-i} \quad .$$

Theorem 3.11. *With notation as above,*

$$g_S(t) = (-1)^n\, \gamma_{\mathbb{P}^n \smallsetminus S}(-t-1) = (-1)^n\, \frac{(t+1)\cdot \chi_{\mathbb{P}^n \smallsetminus S}(t) - \chi_{\mathbb{P}^n \smallsetminus S}(0)}{t} \quad .$$

This proves that Definition 3.9 is indeed independent of the ideal chosen to define S, or of the generators of this ideal (since the other expressions are independent of these choices), and that $g_i^S = g_i^{S_{\mathrm{red}}}$, since S and S_{red} have the same complement in $\mathbb{P}^n$.

Of course the other formulas encountered in §3.2 also hold for arbitrary S, since they may be derived from the equalities given in Theorem 3.11. Thus,

$$\chi(\mathbb{P}^n \smallsetminus S) = (-1)^n g_S(-1) \quad ,$$

and

$$g_j^S = (-1)^j \chi((\mathbb{P}^n \smallsetminus S) \cap (L_j \smallsetminus L_{j-1}))$$

as in Huh's formulas for hypersurfaces (Corollary 3.7). For instance, with C the twisted cubic as in Example 3.10, the intersection of $\mathbb{P}^3 \smallsetminus C$ with a general $\mathbb{P}^2 \smallsetminus \mathbb{P}^1$ consists of the complement in $\mathbb{P}^2$ of a general line and 3 distinct points, hence

$$g_2^C = (-1)^2(\chi(\mathbb{P}^2) - \chi(\mathbb{P}^1) - 3\,\chi(\mathbb{P}^0)) = -2$$

in agreement with the algebraic computation(s) given in Example 3.10.

Proof of Theorem 3.11. Choose any collection of generators $F_1, \dots, F_r$ for any ideal I defining S, as in Definition 3.9, and define $g_S(t)$ as specified above. Also, let X_i be the hypersurface of $\mathbb{P}^n$ defined by F_i. Then

$$\begin{aligned}(-1)^n\, g_S(-t-1) &= (-1)^n \sum_{i=0}^{n} \sum_{\emptyset \neq J \subseteq \{1,\dots,r\}} (-1)^{|J|+1} g_i^{(\prod_{j\in J} F_j)} (-t-1)^{n-i} \\ &= \sum_{\emptyset \neq J \subseteq \{1,\dots,r\}} (-1)^{|J|+1} (-1)^n\, g_{\cup_{j\in J} X_j}(-t-1) \\ &= \sum_{\emptyset \neq J \subseteq \{1,\dots,r\}} (-1)^{|J|+1} \gamma_{\mathbb{P}^n \smallsetminus (\cup_{j\in J} X_j)}(t)\end{aligned}$$

by Corollary 3.4. Now, $\gamma_{\mathbb{P}^n \smallsetminus (\cup_{j\in J} X_j)}(t)$ is the polynomial corresponding to the c_{SM} class of $1\!\!1_{\mathbb{P}^n \smallsetminus (\cup_{j\in J} X_j)}$. Therefore, the end result is the polynomial corresponding to the c_{SM} class of the constructible function

$$\sum_{\emptyset \neq J \subseteq \{1,\dots,r\}} (-1)^{|J|+1} 1\!\!1_{\mathbb{P}^n \smallsetminus (\cup_{j\in J} X_j)} = \sum_{\emptyset \neq J \subseteq \{1,\dots,r\}} (-1)^{|J|+1} 1\!\!1_{\cap_{j\in J}(\mathbb{P}^n \smallsetminus X_j)} \quad .$$

A simple inclusion-exclusion argument shows that this equals

$$1\!\!1_{\cup_{j=1,\dots,r}(\mathbb{P} \smallsetminus X_j)} = 1\!\!1_{\mathbb{P}^n \smallsetminus S} \quad .$$

Therefore,

$$(-1)^n g_S(-t-1) = \gamma_{\mathbb{P}^n \smallsetminus S}(t) \quad ,$$

or equivalently

$$g_S(t) = (-1)^n\, \gamma_{\mathbb{P}^n \smallsetminus S}(-t-1) \quad .$$

The other equality in Theorem 3.11 follows from this, by applying Theorem 1.1. □

It would be desirable to have a more direct argument showing the independence of the degrees g_i^S on the choices used in Definition 3.9 to define them. (This is a reformulation of a problem posed in [Alu03b].) Example 3.10 shows that the polar degrees of a higher codimension subscheme may be negative; in particular, they cannot be directly interpreted as degrees of rational maps as in the hypersurface case. It seems conceivable that they can be expressed as Euler characteristics of complexes determined by the ideal sheaf of S.

4. Polynomial Chern-Schwartz-MacPherson classes

4.1. We now switch our focus to a different question. Our goal is to provide the gist of a theory of characteristic classes for projective varieties (over our fixed algebraically closed field of characteristic 0), with values in $\mathbb{Z}[t]$. Such a theory can be constructed using the theory of Chern-Schwartz-MacPherson classes (Proposition 4.8), but we are going to take a naive approach and not assume the existence of c_{SM} classes. We find it remarkable that the 'polynomial' version of this theory, including an analogue of the key covariance property of c_{SM} classes, can be established using no tools other than the naive considerations on Euler characteristics recalled in §2.1.

4.2. $\mathbb{P}^\infty$**-varieties.** Our objects will be varieties endowed with base-point-free linear systems. We consider a fixed infinite chain of embeddings

$$\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq \cdots \quad ,$$

as a direct system; ι_{nm} is the chosen embedding $\mathbb{P}^m \to \mathbb{P}^n$ for $n > m$. We denote by $\mathbb{P}^\infty$ the limit of this system. A '$\mathbb{P}^\infty$-variety' is represented by a regular morphism $\varphi : X \to \mathbb{P}^m$, where we identify $\varphi : X \to \mathbb{P}^m$ with $\iota_{nm} \circ \varphi : X \to \mathbb{P}^n$ for $m < n$. In particular, we may always assume that any two $\mathbb{P}^\infty$-varieties are represented by morphisms with common target: $\varphi : X \to \mathbb{P}^n$, $\psi : Y \to \mathbb{P}^n$; a morphism between the corresponding $\mathbb{P}^\infty$-varieties is then represented by a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \mathbb{P}^n \arrow[r, "\sim", "\xi"'] & \mathbb{P}^n \end{tikzcd}$$

such that f is a regular morphism and ξ is an isomorphism. Thus, a $\mathbb{P}^\infty$-variety is a variety endowed with a morphism to a projective space, and isomorphisms of $\mathbb{P}^\infty$-varieties are induced by 'automorphisms of $\mathbb{P}^\infty$' (by which we mean automorphisms induced by automorphisms at some finite level). For example, a line $L \hookrightarrow \mathbb{P}^2$ and a conic $C \hookrightarrow \mathbb{P}^2$ are *not* isomorphic as $\mathbb{P}^\infty$-varieties, since the abstract isomorphism $L \cong \mathbb{P}^1 \cong C$ is not induced by an automorphism of $\mathbb{P}^2$; in other words, the corresponding linear systems do not match.

We also consider a 'Chow group' $A_*\mathbb{P}^\infty$, as the direct limit $\varinjlim A_*\mathbb{P}^n$; this is the free abelian group on classes $[\mathbb{P}^i]$ for $i \geq 0$. We identify $A_*\mathbb{P}^\infty$ with $\mathbb{Z}[t]$, where t^i is associated with $[\mathbb{P}^i]$. Of course the *ring* structure of $\mathbb{Z}[t] = A_*\mathbb{P}^\infty$ does not define an intersection product, but it serves as a convenient shorthand for manipulations of classes in $A_*\mathbb{P}^\infty$. We will also relate it with products in a Grothendieck ring of $\mathbb{P}^\infty$-varieties in §4.4.

If φ is proper, then push-forward to $\mathbb{P}^n$ followed by the inclusion in the direct limit defines a push-forward $\varphi_* : A_*X \to \mathbb{Z}[t]$ for every object $\varphi : X \to \mathbb{P}^n$. Concretely, for $a \in A_*X$, the coefficient of t^k in $\varphi_*(a) \in \mathbb{Z}[t]$ equals $\int (\varphi^*(H))^k \cdot a$, where H is the hyperplane class. This push-forward satisfies the evident compatibility property with morphisms: if (f,ξ) defines a morphism $\varphi\to\psi$ as above and all maps are proper, then $\varphi_*=\psi_*\circ f_*$. (Indeed, automorphisms of $\mathbb{P}^\infty$ induce the identity on $A_*\mathbb{P}^\infty$.)

4.3. **Chern classes in $A_*\mathbb{P}^\infty$.** We associate with $\varphi: X\to\mathbb{P}^n$ the group of constructible functions $\mathscr{C}(X)$, and recall that $\mathscr{C}$ is a covariant functor, see §2.2. For $\alpha\in\mathscr{C}(X)$, we seek a 'Chern class' $c_*^\varphi(\alpha)\in A_*\mathbb{P}^\infty=\mathbb{Z}[t]$ with the following properties:

(i) c_*^φ is a group homomorphism $\mathscr{C}(X)\to\mathbb{Z}[t]$;
(ii) If φ is proper and X is nonsingular, then $c_*^\varphi(1\!\!1_X)=\varphi_*(c(TX)\cap[X])$;
(iii) If (f,ξ) defines a morphism $\varphi\to\psi$, and $\alpha\in\mathscr{C}(X)$, then $c_*^\varphi(\alpha)=c_*^\psi(f_*(\alpha))$.

By resolution of singularities, a theory satisfying requirements (i)–(iii) is necessarily unique. We could use c_{SM} classes to provide such a notion (cf. Proposition 4.8 below); but the work involved in proving the existence of c_{SM} classes is itself nontrivial. We want to advertise an alternative, simpler construction, suggested by Theorem 1.1.

By linearity, it suffices to define $c_*^\varphi(1\!\!1_Z)$, for an object $\varphi: X\to\mathbb{P}^n$ and a closed subvariety Z. Given such data, we let $\chi_i^\varphi(Z)$ denote the Euler characteristic (in the sense of §2.1) of $\varphi^{-1}(L)\cap Z$ for a general linear subspace $L\subseteq\mathbb{P}^n$ of codimension i. We let

$$\chi_Z^\varphi(t):=\sum_{r\ge 0}(-1)^r\chi_i^\varphi(Z)\,t^i\quad,$$

and note that this is compatible with the definition given in §1.1, to which it reduces if φ is an embedding.

Definition 4.1. We define $c_*^\varphi(1\!\!1_Z)$ to be $\mathscr{I}(\chi_Z^\varphi)\in\mathbb{Z}[t]=A_*\mathbb{P}^\infty$, where $\mathscr{I}$ is the involution defined in §1.1. Explicitly,

$$c_*^\varphi(1\!\!1_Z)=\frac{t\cdot\chi_Z^\varphi(-t-1)+\chi_Z^\varphi(0)}{t+1}\quad.$$

Example 4.2. The constant term of $c_*^\varphi(1\!\!1_Z)$ equals $\chi(Z)$. Indeed, $c_*^\varphi(1\!\!1_Z)|_{t=0}=\chi_Z^\varphi(0)=\chi_0^\varphi(Z)=\chi(Z)$. This is as it should be expected, given the parallel between the characterizing properties (i)–(iii) for c_*^φ and the Deligne-Grothendieck axioms for c_{SM} classes (cf. (1)). ⌟

We now proceed to verifying properties (i)–(iii). Property (i) is implicit in the construction. Property (ii):

Lemma 4.3 (Normalization)**.** *If φ is proper and X is nonsingular, then*

$$c_*^\varphi(1\!\!1_X)=\varphi_*(c(TX)\cap[X])\quad.$$

Proof. By Bertini's theorem (Corollary 10.9 in [Har77]), if L is a general codimension i subspace of $\mathbb{P}^n$, then $\varphi^{-1}(L)$ is a codimension i nonsingular subvariety of X. The class $[\varphi^{-1}(L)]$ equals $H^i\cdot[X]$, where H is the pull-back of the class of a hyperplane, and the Chern class of the normal bundle $N_{\varphi^{-1}(L)}X$ is (the restriction of) $(1+H)^i$. Thus

$$\chi_i^\varphi(Z)=\int c(T(\varphi^{-1}(L)))\cap[\varphi^{-1}(L)]=\int\frac{H^i}{(1+H)^i}\,c(TX)\cap[X]\quad.$$

Now we argue exactly as in the proof of Theorem 1.1, and obtain

$$\chi_X^{\varphi}(t) = \sum_{r\ge 0}\left(\int \frac{(-H)^r}{(1+H)^r}\, c(TX)\cap [X]\right) t^r = c_0 - t\sum_{\ell\ge 1} c_\ell\,(-t-1)^{\ell-1}$$

where $c_i = \int H^i \cdot c(TX)\cap[X]$. As $c_0 + c_1 t + \cdots + c_n t^n = \varphi_*(c(TX)\cap[X])$, this says

$$\chi_X^{\varphi} = \mathscr{I}(\varphi_* c(TX)\cap[X]) \quad ,$$

and it follows that $\mathscr{I}(\chi_X^{\varphi}) = \varphi_* c(TX)\cap[X]$ as $\mathscr{I}$ is an involution. This is precisely the statement. $\square$

Property (iii):

Lemma 4.4 (Covariance)**.** *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \mathbb{P}^n \arrow[r, "\sim", "\xi"'] & \mathbb{P}^n \end{tikzcd}$$

be a commutative diagram, and let $\alpha \in \mathscr{C}(X)$. Then $c_*^{\varphi}(\alpha) = c_*^{\psi}(f_*(\alpha))$.

Proof. Since ξ is an isomorphism, it is clear that $c_*^{\xi\circ\varphi}(\alpha) = c_*^{\varphi}(\alpha)$; thus we may assume that ξ is the identity, and $\varphi = \psi\circ f$.

By linearity we may assume $\alpha = 1\!\!1_Z$, with $Z \subseteq X$ a closed subvariety. Apply Lemma 2.3 to $f|_Z$ to deduce the existence of subvarieties $V_1,\dots,V_r$ of Y and integers $m_1,\dots,m_r$ such that $f_*(1\!\!1_Z) = \sum_j m_j 1\!\!1_{V_j}$. Then

$$c_*^{\psi}(f_*(1\!\!1_Z)) = c_*^{\psi}(\sum_j m_j 1\!\!1_{V_j}) = \sum_j m_j c_*^{\psi}(1\!\!1_{V_j}) = \sum_j m_j \mathscr{I}(\chi_{V_j}^{\psi}) = \mathscr{I}(\sum_j m_j \chi_{V_j}^{\psi}) \quad .$$

On the other hand, $c_*^{\varphi}(1\!\!1_Z) = \mathscr{I}(\chi_Z^{\varphi})$. This shows that the equality $c_*^{\varphi}(\alpha) = c_*^{\psi}(f_*(\alpha))$ is equivalent to the statement

$$\chi_Z^{\varphi}(t) = \sum_j m_j \chi_{V_j}^{\psi}(t)$$

and hence to

$$\forall i \quad , \quad \chi_i^{\varphi}(Z) = \sum_j m_j \chi_i^{\psi}(V_j) \quad .$$

For each i, the left-hand side is

$$\chi(\varphi^{-1}(L)\cap Z) = \chi(f|_Z^{-1}(\psi^{-1}(L)))$$

where L is a general subspace of $\mathbb{P}^n$ of codimension i. By the second part of Lemma 2.3, this equals

$$\sum_{j=1}^r m_j \chi(\psi^{-1}(L)\cap V_j) = \sum_{j=1}^r m_j \chi_i^{\psi}(V_j) \quad ,$$

concluding the proof. $\square$

4.4. **Polynomial Chern classes and Grothendieck ring(s).** The free abelian group of isomorphism classes of $\mathbb{P}^\infty$-varieties modulo the usual scissor relations defines a 'relative Grothendieck group of varieties over $\mathbb{P}^\infty$', which we will denote $K(\mathrm{Var}_{\mathbb{P}^\infty})$. If $\varphi : X \to \mathbb{P}^m$ is an (understood) embedding, we write $[X]$ for the corresponding element $[\varphi] \in K(\mathrm{Var}_{\mathbb{P}^\infty})$. The part of the Grothendieck group determined by embeddings is essentially the same as the Grothendieck group of 'immersed conical varieties' studied in [AM11]. Also note that $K(\mathrm{Var}_{\mathbb{P}^\infty})$ admits generators factoring through affine space: $\varphi : X \to \mathbb{P}^m$ obtained by composing a morphism $\varphi^\circ : X \to \mathbb{A}^m$ with a standard embedding into $\mathbb{P}^m$. It also admits a description in terms of Bittner's relations.

Isomorphisms of $\mathbb{P}^\infty$ varieties allow for automorphisms of the base $\mathbb{P}^\infty$; the usual context of relative Grothendieck groups (as in e.g., [Bit04], §5) does not. This appears to be advantageous here since then this Grothendieck group carries more interesting products. For example, we can define a product by specifying the operation on generators, as follows: if $[\varphi], [\psi] \in K(\mathrm{Var}_{\mathbb{P}^\infty})$ are represented by $\varphi : X \to \mathbb{P}^{m-1}$ and $\psi : Y \to \mathbb{P}^{n-1}$, define $[\varphi] \star [\psi]$ to be the class represented by the morphism

$$X \times Y \xrightarrow{\varphi\times\psi} \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \xhookrightarrow{s} \mathbb{P}^{mn-1}$$

where s is the Segre embedding. This product is distributive and associative up to automorphisms of $\mathbb{P}^\infty$, so it defines a ring structure on $K(\mathrm{Var}_{\mathbb{P}^\infty})$. A different ring structure will be defined in §4.6. Determining the precise behavior of c_* with respect to these (and possibly other) products is an interesting problem, as knowledge of this behavior is helpful in concrete computations.

The Chern classes defined in §4.3 factor through $K(\mathrm{Var}_{\mathbb{P}^\infty})$, in the sense that $c_*^\varphi(\alpha)$ only depends on the isomorphism class of φ and this assignment satisfies the relations defining $K(\mathrm{Var}_{\mathbb{P}^\infty})$. That is, if we have an object $\varphi : X \to \mathbb{P}^n$, a closed subvariety $i : Z \hookrightarrow X$, and let $j : U = X \smallsetminus Z \hookrightarrow X$ be the complement, then

$$c_*^\varphi(1\!\!1_X) = c_*^{\varphi\circ i}(1\!\!1_Z) + c_*^{\varphi\circ j}(1\!\!1_U) \quad .$$

Indeed, this identity may be verified after applying the involution $\mathscr{I}$, which gives

$$\chi_X^\varphi(t) = \chi_Z^{\varphi\circ i}(t) + \chi_U^{\varphi\circ j}(t) \quad ;$$

and this latter identity is immediate from the additivity of Euler characteristic on disjoint unions.

Proposition 4.5. *The assignment* $(\varphi : X \to \mathbb{P}^n) \mapsto c_*^\varphi(1\!\!1_X)$ *defines a group homomorphism* $\gamma : K(\mathrm{Var}_{\mathbb{P}^\infty}) \to \mathbb{Z}[t]$.

The same assignment has an interesting behavior with respect to products. Consider the $\mathbb{Z}$-module automorphism $\sigma : \mathbb{Q}[t] \to \mathbb{Q}[t]$ defined on generators by $t^i \mapsto \frac{t^i}{i!}$. (This homomorphism could be defined for power series; in the terminology of combinatorics, it turns 'ordinary' generating functions into 'exponential' ones.)

Proposition 4.6. $\sigma \circ \gamma : (K(\mathrm{Var}_{\mathbb{P}^\infty}), \star) \to \mathbb{Q}[t]$ *is a* ring *homomorphism.*

Proof. To verify that $\sigma \circ \gamma$ preserves products, we use Bittner's relations ([Bit04], Remark 3.2): it suffices to verify that if $\varphi : X \to \mathbb{P}^m$ and $\psi : Y \to \mathbb{P}^n$ are morphisms with

X and Y projective and nonsingular, then $\sigma(c_*^{s\circ(\varphi\times\psi)}(1\!\!1_{X\times Y})) = \sigma(c_*^\varphi(1\!\!1_X))\,\sigma(c_*^\psi(1\!\!1_Y))$. This will follow from the normalization property of c_*. Denote by h_1, resp., h_2, H the hyperplane class in $\mathbb{P}^{m-1}$, resp., $\mathbb{P}^{n-1}$, $\mathbb{P}^{mn-1}$. Since X and Y are nonsingular,

$$c_*^\varphi(1\!\!1_X) = \sum_i c_i' t^i, \quad c_*^\psi(1\!\!1_Y) = \sum_j c_j'' t^j$$

where by Lemma 4.3 $c_i' = \int(\varphi^* h_1)^i \cdot c(TX)\cap[X]$ and $c_j'' = \int(\psi^* h_2)^j \cdot c(TY)\cap[Y]$. Likewise,

$$c_*^{s\circ(\varphi\times\psi)}(1\!\!1_{X\times Y}) = \sum_k c_k t^k$$

where $c_k = \int((s\circ(\varphi\times\psi)^*H)^k \cdot c(T(X\times Y))\cap[X\times Y]$. We have $(s\circ(\varphi\times\psi))^*H = (\varphi\circ p_1)^*h_1 + (\psi\circ p_2)^*h_2$, where p_1, resp., p_2 is the first, resp., second projection from $X\times Y$. Also, $T(X\times Y)\cong p_1^*TX\oplus p_2^*TY$. It follows that

$$\begin{aligned}
c_k &= \int((s\circ(\varphi\times\psi))^*H^k \cdot c(T(X\times Y))\cap[X\times Y] \\
&= \int((\varphi\circ p_1)^*h_1 + (\psi\circ p_2)^*h_2)^k \cdot p_1^*c(TX)\cap p_2^*c(TY)\cap[X\times Y] \\
&= \sum_{i+j=k}\binom{k}{i}\int p_1^*(\varphi^*h_1^i\cdot c(TX))\cap p_2^*(\psi^*h_2^j\cdot c(TY))\cap[X\times Y] \\
&= \sum_{i+j=k}\binom{k}{i}\left(\int \varphi^*h_1^i\cdot c(TX)\cap[X]\right)\left(\int \psi^*h_2^j\cdot c(TY)\cap[Y]\right) \\
&= \sum_{i+j=k}\binom{k}{i} c_i' c_j'' \quad ,
\end{aligned}$$

and hence

$$\sigma(c_*^{s\circ(\varphi\times\psi)}(1\!\!1_{X\times Y})) = \sum_k\sum_{i+j=k}\binom{k}{i}c_i'c_j''\frac{t^k}{k!} = \sum_k\sum_{i+j=k}c_i'c_j''\frac{t^i}{i!}\frac{t^j}{j!} = \sigma(c_*^\varphi(1\!\!1_X))\,\sigma(c_*^\psi(1\!\!1_Y))$$

as needed. □

Example 4.7. Consider the class $[\mathbb{P}^1]\in K(\mathrm{Var}_{\mathbb{P}^\infty})$ determined by the identity: $\mathbb{P}^1\to\mathbb{P}^1$. Then $[\mathbb{P}^1]\star[\mathbb{P}^1] = [\mathbb{P}^1\times\mathbb{P}^1]$, where $\iota:\mathbb{P}^1\times\mathbb{P}^1\hookrightarrow\mathbb{P}^3$ embeds $\mathbb{P}^1\times\mathbb{P}^1$ as a nonsingular quadric Q in $\mathbb{P}^3$. The polynomial Chern class of $\mathbb{P}^1$ is $2+t$. According to Proposition 4.6,

$$\sigma(c_*^\iota(1\!\!1_{\mathbb{P}^1\times\mathbb{P}^1})) = (\sigma(2+t))^2 = (2+t)^2 = 4+4t+t^2 \quad .$$

Therefore,

$$c_*^\iota(1\!\!1_{\mathbb{P}^1\times\mathbb{P}^1}) = \sigma^{-1}(4+4t+t^2) = 4+4t+2t^2 \quad .$$

This is as it should: for a nonsingular quadric in $\mathbb{P}^3$ the Euler characteristics of general linear sections are $\chi_0=4, \chi_1=2, \chi_2=2$, and therefore

$$c_*^\iota(1\!\!1_{\mathbb{P}^1\times\mathbb{P}^1}) = \mathscr{I}(4-2t+2t^2) = 4+4t+2t^2$$

according to Definition 4.1. ⌟

There is at least one alternative product that may be considered on $K(\mathrm{Var}_{\mathbb{P}^\infty})$, see §4.6.

4.5. **Polynomial Chern classes and $c_{\mathbf{SM}}$ classes.** The characterizing properties listed in §4.3 imply that the classes are a numerical aspect of the Chern-Schwartz-MacPherson classes:

Proposition 4.8. *If φ is proper, then $c_*^\varphi(\alpha) = \varphi_*(c_{SM}(\alpha))$.*

Indeed, the normalizations of c_* and c_{SM} are compatible, so it suffices to verify that the covariance property of c_{SM} classes implies the third property of the classes defined in §4.3. If $f : X \to Y$ is proper, then $\forall \alpha \in \mathscr{C}(X)$

$$\psi_*(c_{\mathrm{SM}}(f_*\alpha)) = \psi_*(f_*(c_{\mathrm{SM}}(\alpha))) = \varphi_* c_{\mathrm{SM}}(\alpha)$$

as needed. Note that with notation as in §2.5 and Proposition 4.5, we have $\gamma([X]) = \gamma_X(t)$.

Of course we could use the formula in Proposition 4.8 to provide a construction of the classes c_*^φ alternative to the one given in Definition 4.1. Similarly, we could use the Grothendieck group to define c_*^φ: if X is nonsingular and $\varphi : X \to \mathbb{P}^n$ is proper, we could *define* $c_*^\varphi(1\!\!1_X)$ to be $\varphi_*(c(TX) \cap [X])$; and then use Bittner's relations to show this prescription descends to the Grothendieck group (cf. Lemma 2.1), giving a notion for possibly singular or noncomplete sources. The normalization and covariance properties would be immediate; in particular, the resulting class must agree with the one given in Definition 4.1.

Remark 4.9. By Proposition 4.8, the algorithm presented in [Alu03a] computes the polynomial Chern class of a subscheme of projective space, given generators for an ideal defining it set-theoretically. By means of the definition given here (Definition 4.1), any algorithm computing Euler characteristics may be adapted to compute the polynomial Chern class. One such algorithm is presented by Marco-Buzunáriz in [MB12], together with a polynomial generalization of the Euler characteristic. This generalization differs from the polynomial Chern class introduced above by a simple change of coordinates, as follows from Proposition 4.8 and the result proved by Rennemo in the appendix to [MB12]. ⌟

The elementary approach described in §4.3 streamlines the proof of some properties of these polynomial c_{SM} classes. For example, let $\iota : X \hookrightarrow \mathbb{P}^n$ be a projective variety, and consider the cone $\iota' : X' \hookrightarrow \mathbb{P}^{n+1}$ with vertex a point. It is clear that $\chi(X') = 1 + \chi(X)$ and (with notation as in §4.3) $\chi_j^{\iota'}(X') = \chi_{j-1}^{\iota}(X)$ for $j > 0$. That is,

$$\chi_{X'}^{\iota'}(t) = 1 + \chi(X) - t\chi_X^{\iota}(t) \quad ,$$

and hence

$$c_*^{\iota'}(1\!\!1_{X'}) = \mathscr{I}(\chi(X) + 1 - t\chi_X^{\iota}(t)) = \chi(X) + 1 + t\chi_X^{\iota}(-t-1) = (t+1)c_*^{\iota}(t) + 1 \quad .$$

This formula matches the one obtained in Proposition 5.2 in [AM09] by a somewhat more involved argument.

On the other hand, the relation obtained in Proposition 4.8 allows us to interpret results for c_{SM} classes in terms of their polynomial aspect. We illustrate one application in the following, final section.

4.6. **Another ring homomorphism.** We can give a different, and in a sense more natural, multiplication operation on $K(\mathrm{Var}_{\mathbb{P}^\infty})$. We define it on *affine* generators, i.e., morphisms $X \to \mathbb{P}^m$, $Y \to \mathbb{P}^n$ which factor through affine space:

$$\begin{tikzcd} X \arrow[r, "\varphi^0"] \arrow[rr, bend right, "\varphi"'] & \mathbb{A}^m \arrow[r, hook] & \mathbb{P}^m \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\psi^0"] \arrow[rr, bend right, "\psi"'] & \mathbb{A}^n \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

We set $[\varphi]\cdot[\psi]$ to be the class represented by the morphism

$$X \times Y \to \mathbb{A}^{m+n} \hookrightarrow \mathbb{P}^{m+n}$$

defined by

$$\begin{aligned}(x,y) &\mapsto (\varphi_1^\circ(x),\dots,\varphi_m^\circ(x),\psi_1^\circ(x),\dots,\psi_n^\circ(x))\\ &\qquad\qquad \mapsto (1\colon \varphi_1^\circ(x)\colon \ \dots\colon \varphi_m^\circ(x)\colon \psi_1^\circ(x)\colon \ \dots\colon \psi_n^\circ(x)) \quad .\end{aligned}$$

This definition has a counterintuitive aspect to it: Although it does determine (by distributivity) a class $[\varphi]\cdot[\psi]$ for every $[\varphi],[\psi]\in K(\mathrm{Var}_{\mathbb{P}^\infty})$, this class may not have a compelling geometric realization. For example, we have

$$\begin{aligned}[\mathbb{P}^1]\cdot[\mathbb{P}^1] &= ([\mathbb{A}^1]+[\mathbb{A}^0])\cdot([\mathbb{A}^1]+[\mathbb{A}^0]) = [\mathbb{A}^1]\cdot[\mathbb{A}^1]+2[\mathbb{A}^1]\cdot[\mathbb{A}^0]+[\mathbb{A}^0]\cdot[\mathbb{A}^0]\\ &= [\mathbb{A}^2]+2[\mathbb{A}^1]+[\mathbb{A}^0] \quad ,\end{aligned}$$

but this does *not* equal $[\mathbb{P}^1\times\mathbb{P}^1]=[\mathbb{P}^1]\star[\mathbb{P}^1]$ in $K(\mathrm{Var}_{\mathbb{P}^\infty})$: although $\mathbb{P}^1\times\mathbb{P}^1$ admits the same affine decomposition, the morphisms induced on the affine pieces by the Segre map are not the inclusions as dense open sets of the corresponding projective spaces.

Remark 4.10. A partial antidote to this unpleasant feature is through the *join* construction. If $X\subseteq\mathbb{P}^m$ and $Y\subseteq\mathbb{P}^n$, then we can place $\mathbb{P}^m$ and $\mathbb{P}^n$ as disjoint subspaces of $\mathbb{P}^{m+n+1}$ (by acting with an 'automorphism of $\mathbb{P}^\infty$' on $Y\subseteq\mathbb{P}^n\subseteq\mathbb{P}^{m+n+1}$), and let $J(X,Y)$ be the union of the lines joining points of X to points of Y. The complement $J(X,Y)^\circ$ of X and Y in $J(X,Y)$ maps surjectively to $X\times Y$, with k^* fibers. The reader can verify that

$$[J(X,Y)^\circ]=\mathbb{T}\cdot[X]\cdot[Y]$$

in $K(\mathrm{Var}_{\mathbb{P}^\infty})$, where $\mathbb{T}$ denotes the class of the natural embedding $k^*\subseteq\mathbb{P}^1$. Thus, while $[X]\cdot[Y]$ may not have a direct 'geometric' realization, $[X]\cdot[Y]\cdot\mathbb{T}$ does. ⌟

The new product $\cdot$ clearly defines an alternative structure of ring on the group $K(\mathrm{Var}_{\mathbb{P}^\infty})$. Now recall that we have defined a group homomorphism $\gamma: K(\mathrm{Var}_{\mathbb{P}^\infty})\to\mathbb{Z}[t]$, cf. Proposition 4.5.

Proposition 4.11. *γ is a* ring *homomorphism* $(K(\mathrm{Var}_{\mathbb{P}^\infty}),\cdot)\to\mathbb{Z}[t]$.

Proof. We have to verify that if $\varphi: X\to\mathbb{P}^{m-1}$, $\psi: Y\to\mathbb{P}^{n-1}$ are affine generators, then $c_*^{\varphi\times\psi}(1\!\!1_{X\times Y})=c_*^\varphi(1\!\!1_X)c_*^\psi(1\!\!1_Y)$. By Lemma 4.4, this is easily reduced to the case in which φ, ψ are embeddings. Using Remark 4.10, we see that it suffices to verify that

$$\gamma([J(X,Y)^\circ])=\gamma(\mathbb{T})\,\gamma([X])\,\gamma([Y]) \quad . \tag{6}$$

Now, $\gamma(\mathbb{T}) = t$: indeed $[\mathbb{P}^1] = 2 + t$, and k^* is the complement of two distinct points in $\mathbb{P}^1$. Next, the c_{SM} class of a join was computed in Theorem 3.13 in [AM11]:

$$c_{\mathrm{SM}}(1\!\!1_{J(X,Y)}) = ((f(H) + H^m)(g(H) + H^n) - H^{m+n}) \cap [\mathbb{P}^{m+n-1}] \quad , \tag{7}$$

where $c_{\mathrm{SM}}(1\!\!1_X) = f(H) \cap [\mathbb{P}^{m-1}]$, $c_{\mathrm{SM}}(1\!\!1_Y) = g(H) \cap [\mathbb{P}^{n-1}]$, and H denotes the hyperplane class throughout. Using Proposition 4.8, (7) implies a statement on polynomial Chern classes, which translates into

$$t\,\gamma([J(X,Y)]) = (t\,\gamma([X]) + 1)(t\,\gamma([X]) + 1) - 1 \quad ,$$

as the reader may verify. As $[J(X,Y)^\circ] = [J(X,Y)] - [X] - [Y]$, this is immediately seen to imply (6), concluding the proof. □

Example 4.12. By Proposition 4.11,

$$\gamma([\mathbb{P}^1] \cdot [\mathbb{P}^1]) = (2 + t)^2 = 4 + 4t + t^2 \quad . \tag{8}$$

This equals the expression obtained for $\sigma(\gamma([\mathbb{P}^1] \star [\mathbb{P}^1]))$ in Example 4.7, but reflects a very different geometric situation: $[\mathbb{P}^1] \star [\mathbb{P}^1]$ is the class of a nonsingular quadric in $\mathbb{P}^3$; the class $[\mathbb{P}^1] \cdot [\mathbb{P}^1]$ does not appear to be the class of an irreducible variety, but we can realize $\mathbb{T} \cdot [\mathbb{P}^1] \cdot [\mathbb{P}^1]$ as the class of the 'open join' $J(\mathbb{P}^1, \mathbb{P}^1)^\circ$ in $\mathbb{P}^3$. Since $J(\mathbb{P}^1, \mathbb{P}^1) = \mathbb{P}^3$, this gives

$$t\,\gamma([\mathbb{P}^1] \cdot [\mathbb{P}^1]) = \gamma([\mathbb{P}^3] - 2[\mathbb{P}^1]) = (4 + 6t + 4t^2 + t^3) - 2(2 + t) = t(4 + 4t + t^2) \quad ,$$

confirming (8). ⌟

Remark 4.13. If $\iota_X : X \to \mathbb{P}^{m-1}$ is a closed embedding, let $\hat{X} \subseteq \mathbb{A}^m$ be the corresponding affine cone, and denote by $\iota_{\hat{X}} : \hat{X} \to \mathbb{P}^m$ the embedding of this cone in the projectivization of $\mathbb{A}^m$. The polynomial $\gamma([\hat{X}]) = c_*^{\iota_{\hat{X}}}(1\!\!1_{\hat{X}})$ agrees with the polynomial denoted $G_{\hat{X}}$ in [AM11]. Proposition 4.11 is then a mild generalization of Theorem 3.6 in [AM11], which is the key step in the definition of 'polynomial Feynman rules'. ⌟

Remark 4.14. Using the involution $\mathscr{I}$, Propositions 4.6 and 4.11 yield expressions for the Euler characteristics of general linear sections of products and joins of embedded varieties X, Y in terms of the same information for X and Y. For example, as we have seen above we have

$$\gamma_{J(X,Y)}(t) = t\,\gamma_X(t)\,\gamma_Y(t) + \gamma_X(t) + \gamma_Y(t) \quad ;$$

applying $\mathscr{I}$ gives

$$\begin{aligned}\chi_{J(X,Y)}(t) &= -t\,\gamma_X(-t-1)\,\gamma_Y(-t-1) + \chi_X(t) + \chi_Y(t)\\ &= \frac{1}{t}\left((t+1)\chi_X(t) - \chi(X)\right)\left((t+1)\chi_Y(t) - \chi(Y)\right) + \chi_X(t) + \chi_Y(t) \quad ,\end{aligned}$$

and reading off the coefficient of t^ℓ we get

$$\chi_\ell^{J(X,Y)} = \sum_{j+k=\ell-1} (\chi_j^X - \chi_{j+1}^X)(\chi_k^Y - \chi_{k+1}^Y) + \chi_\ell^X + \chi_\ell^Y$$

for all $\ell > 0$. These expressions interpolate between the $\ell = 1$ case, stating that the Euler characteristic of a general hyperplane section of $J(X,Y)$ equals

$$(\chi_0^X - \chi_1^X)(\chi_0^Y - \chi_1^Y) + \chi_1^X + \chi_1^Y$$

(which is straightforward) and the $\ell = \dim X + \dim Y + 1$ case, stating that the degree of $J(X,Y)$ is the product of the degrees of X and Y ([Har92], Example 18.7).

Similarly, for the product $X \times Y$ embedded via the Segre embedding, the corresponding expressions interpolate between the well-known formulas $\chi(X \times Y) = \chi(X)\chi(Y)$ and $\deg(X \times Y) = \binom{\dim X + \dim Y}{\dim X}(\deg X)(\deg Y)$. The Euler characteristic of a general hyperplane section is

$$\sum_{j \geq 1} \left(\chi_j^X(\chi_{j-1}^Y - \chi_j^Y) + \chi_j^Y(\chi_{j-1}^X - \chi_j^X)\right)$$

and formulas for general linear sections of higher codimension are progressively more complicated. The relative complexity of these formulas hides the simplicity of their source, that is the homomorphism statement in Theorem 4.6. ⌟

References

[Ahm10] Imran Ahmed. Polar Cremona transformations and monodromy of polynomials. *Studia Sci. Math. Hungar.*, 47(1):81–89, 2010.

[Alu94] Paolo Aluffi. MacPherson's and Fulton's Chern classes of hypersurfaces. *Internat. Math. Res. Notices*, (11):455–465, 1994.

[Alu99] Paolo Aluffi. Chern classes for singular hypersurfaces. *Trans. Amer. Math. Soc.*, 351(10):3989–4026, 1999.

[Alu03a] Paolo Aluffi. Computing characteristic classes of projective schemes. *J. Symbolic Comput.*, 35(1):3–19, 2003.

[Alu03b] Paolo Aluffi. Inclusion-exclusion and Segre classes. *Comm. Algebra*, 31(8):3619–3630, 2003. Special issue in honor of Steven L. Kleiman.

[Alu06] Paolo Aluffi. Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. *Pure Appl. Math. Q.*, 2(4):915–941, 2006.

[Alu12a] Paolo Aluffi. Chern classes of graph hypersurfaces and deletion-contraction. *Moscow Math. J.*, 12:671–700, 2012.

[Alu12b] Paolo Aluffi. Grothendieck classes and Chern classes of hyperplane arrangements. *Int. Math. Res. Not.*, 2012.

[AM09] Paolo Aluffi and Matilde Marcolli. Feynman motives of banana graphs. *Commun. Number Theory Phys.*, 3(1):1–57, 2009.

[AM11] Paolo Aluffi and Matilde Marcolli. Algebro-geometric Feynman rules. *Int. J. Geom. Methods Mod. Phys.*, 8(1):203–237, 2011.

[Bit04] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.

[Bru07] Andrea Bruno. On homaloidal polynomials. *Michigan Math. J.*, 55(2):347–354, 2007.

[BSY10] Jean-Paul Brasselet, Jörg Schürmann, and Shoji Yokura. Hirzebruch classes and motivic Chern classes for singular spaces. *J. Topol. Anal.*, 2(1):1–55, 2010.

[CRS08] Ciro Ciliberto, Francesco Russo, and Aron Simis. Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian. *Adv. Math.*, 218(6):1759–1805, 2008.

[Dim01] Alexandru Dimca. On polar Cremona transformations. *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.*, 9(1):47–53, 2001.

[Dol00] Igor V. Dolgachev. Polar Cremona transformations. *Michigan Math. J.*, 48:191–202, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[DP03] Alexandru Dimca and Stefan Papadima. Hypersurface complements, Milnor fibers and higher homotopy groups of arrangments. *Ann. of Math. (2)*, 158(2):473–507, 2003.

[FM12] Thiago Fassarella and Nivaldo Medeiros. On the polar degree of projective hypersurfaces. *J. Lond. Math. Soc. (2)*, 86(1):259–271, 2012.

[FP07] Thiago Fassarella and Jorge Vitório Pereira. On the degrees of polar transformations. An approach through logarithmic foliations. *Selecta Math. (N.S.)*, 13(2):239–252, 2007.

[Ful84] William Fulton. *Intersection theory*. Springer-Verlag, Berlin, 1984.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har92] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. A first course.

[Huh] June Huh. The maximum likelihood degree of a very affine variety. To appear in Compositio Mathematica.

[Huh12] June Huh. Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. *J. Amer. Math. Soc.*, 25:907–927, 2012.

[Ken90] Gary Kennedy. MacPherson's Chern classes of singular algebraic varieties. *Comm. Algebra*, 18(9):2821–2839, 1990.

[KS01] Hanspeter Kraft and Gerald W. Schwarz. Rational covariants or reductive groups and homaloidal polynomials. *Math. Res. Lett.*, 8(5-6):641–649, 2001.

[Lia] Xia Liao. Chern classes of logarithmic vector fields for locally quasi-homogeneous free divisors. arXiv:1205.3843.

[Mac74] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[MB12] Miguel Ángel Marco Buzunáriz. A polynomial generalization of the Euler characteristic for algebraic sets. *J. Singul.*, 4:114–130, 2012. With an appendix by Jørgen Vold Rennemo.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[Par88] Adam Parusiński. A generalization of the Milnor number. *Math. Ann.*, 281(2):247–254, 1988.

[PP01] Adam Parusiński and Piotr Pragacz. Characteristic classes of hypersurfaces and characteristic cycles. *J. Algebraic Geom.*, 10(1):63–79, 2001.

[Sch65a] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. I. *C. R. Acad. Sci. Paris*, 260:3262–3264, 1965.

[Sch65b] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. II. *C. R. Acad. Sci. Paris*, 260:3535–3537, 1965.

[Tei77] Bernard Teissier. The hunting of invariants in the geometry of discriminants. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 565–678. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

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