UNIT DISTANCE PROBLEMS

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Abstract. We study some discrete and continuous variants of the following problem of Erdős: given a finite subset P of $\mathbb{R}^2$ or $\mathbb{R}^3$, what is the maximum number of pairs (p_1, p_2) with $p_1, p_2 \in P$ and $|p_1 - p_2| = 1$?

1. Introduction and Statement of Results

In 1946 Paul Erdős [3] posed the following question: given a finite subset P of $\mathbb{R}^2$ or $\mathbb{R}^3$, what is the maximum number of pairs (p_1, p_2) with $p_1, p_2 \in P$ and $|p_1 - p_2| = 1$? The Erdős unit distance conjecture in $\mathbb{R}^2$ is the estimate

$$\left|\{(p_1, p_2) \in P^2 : |p_2 - p_1| = 1\}\right| \le C\,|P|\sqrt{\log(|P|)}. \tag{1.1}$$

(We will use $|\cdot|$ for the cardinality of a finite set as well as Lebesgue measure on $\mathbb{R}^d$.) In two dimensions the best currently-known partial result, due to Spencer, Szemerédi, and Trotter [10], is

$$\left|\{(p_1, p_2) \in P^2 : |p_2 - p_1| = 1\}\right| \le C\,|P|^{4/3},$$

while the current best estimate for the analogous problem in $\mathbb{R}^3$ has the exponent $3/2 + \epsilon$ (for any $\epsilon > 0$ and C depending on ϵ) in place of $4/3$ - see Clarkson *et al.* [1]. In four or more dimensions it follows from an example we learned in [6] that one cannot significantly improve the trivial $|P|^2$ bound: let $\tilde{P}$ be any set of N points $\tilde{x}_n$ in $\mathbb{R}^2$ satisfying $|\tilde{x}_n| = 2^{-1/2}$. Let P be the subset of $\mathbb{R}^4$ given by

$$P \doteq \{(\tilde{x}_n; 0, 0), (0, 0; \tilde{x}_m) : \tilde{x}_n, \tilde{x}_m \in \tilde{P}\}.$$

Then the left hand side of (1.1) is at least N^2 while $|P|^2 = 4N^2$. Our first result shows that if we ban a salient feature of this example - many points in low-dimensional subspaces - then a nontrivial estimate is still possible:

Theorem 1.1. *Fix $d \ge 2$. There is a positive constant C_d such that if $P \subset \mathbb{R}^d$ and if every d-element subset of P is affinely independent, then*

$$\left|\{(p_1, p_2) \in P^2 : |p_2 - p_1| = 1\}\right| \le C_d\,|P|^{(2d-1)/d}. \tag{1.2}$$

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(The proofs of the results described in this section can be found in §2.)

Another famous problem of Erdős is his distinct distance conjecture, the estimate

$$\big|\{|p_1-p_2| : (p_1,p_2)\in P^2\}\big| \ge c\,\frac{|P|}{\sqrt{\log(|P|)}}. \tag{1.3}$$

An easy pigeon-hole argument shows that (1.1) implies (1.3). But while the conjecture (1.1) is still far from resolved, Guth and Katz [5] have recently come very close to (1.3) by showing that

$$\big|\{|p_1-p_2| : (p_1,p_2)\in P^2\}\big| \ge c\,\frac{|P|}{\log(|P|)}.$$

This distinct distance problem has a continuous analog known as the Falconer distance set problem ([4]): if K is a compact subset of $\mathbb{R}^d$ and if we define the distance set $\Delta(K)$ by

$$\Delta(K)=\{|k_1-k_2| : (k_1,k_2)\in K^2\},$$

then what can we say about lower bounds for $\dim\big(\Delta(K)\big)$ in terms of $\dim(K)$? For example, Wolff proves in [11] that if $K\subset\mathbb{R}^2$ and $\dim(K)>4/3$ then $\Delta(K)$ has positive Lebesgue measure and so dimension one, while Erdoğan [2] contains analogous results in $\mathbb{R}^d$.

The primary purpose of this paper is to study the following continuous analog of the **unit** distance problem: if

$$D=D(K)=\{(k_1,k_2)\in K^2 : |k_2-k_1|=1\},\ K\subset\mathbb{R}^d,$$

find

$$g_d(\alpha)\doteq\sup\{\dim(D) : K \text{ is a compact subset of } \mathbb{R}^d \text{ with } \dim(K)=\alpha\}. \tag{1.4}$$

When $d=1$ this is trivial: the projection $(k_1,k_2)\mapsto k_1$ is at most two-to-one on D and so it follows that $\dim(D)\le\alpha$. If $\tilde{K}\subset\mathbb{R}$, $\dim(\tilde{K})=\alpha$, and if $K=\tilde{K}\cup(\tilde{K}+1)$, then $\dim(D)=\alpha=\dim(K)$. Thus $g_1(\alpha)=\alpha$.

Here is a trivial bound in higher dimensions: the map

$$(k_1,k_2)\mapsto(k_1,k_2-k_1)$$

shows that D and

$$G\doteq\{(k,y) : k\in K,\ y\in S^{d-1},\ k+y\in K\} \tag{1.5}$$

have the same dimension. This gives the bound

$$\dim(D)\le\alpha+d-1. \tag{1.6}$$

More interestingly, D is the intersection of $K\times K$ with the variety

$$\{(x_1,x_2)\in\mathbb{R}^d\times\mathbb{R}^d : |x_2-x_1|=1\}.$$

Thus one might conjecture that

$$\dim(D)\le 2\alpha-1$$

and so

$$g_d(\alpha) \leq 2\alpha - 1. \tag{1.7}$$

Of course this cannot always be correct since $g_d(\alpha) \geq \alpha$ if $0 \leq \alpha \leq 1$ (because $g_d(\alpha) \geq g_1(\alpha)$ since $\mathbb{R}^d$ contains a copy of $\mathbb{R}$). But here is an example related to (1.7): suppose $C \subset B(0, 1/2) \subset \mathbb{R}^{d-1}$ has $\dim(C) = \gamma$ and put $K = C \times [0, 2] \subset \mathbb{R}^d$. Then $\alpha = \dim(K) = 1 + \gamma$. Also

$$D = \{(c_1, t_1; c_2, t_2) : c_1, c_2 \in C,\ t_1, t_2 \in [0, 2],\ |t_1 - t_2| = \sqrt{1 - |c_1 - c_2|^2}\}.$$

Since for each fixed $(c_1, t_1; c_2)$ with $c_1, c_2 \in C,\ 0 \leq t_1 \leq 1$ there is a $t_2 \in [0, 2]$ which works in $|t_1 - t_2| = \sqrt{1 - |c_1 - c_2|^2}$, it follows that

$$\dim(D) = \dim(C \times C) + 1 \geq 2\gamma + 1 = 2\alpha - 1.$$

Thus when $\alpha \geq 1$ it is at least not possible to do better than (1.7). This example has another implication too: there are sets $C \subset \mathbb{R}$ with $\dim(C) = 0$ and $\dim(C \times C) = 1$. (That is a manifestation of the fact that Hausdorff dimension does not always behave well when forming Cartesian products.) It follows that there are sets $K \subset \mathbb{R}^2$ with $\dim(K) = 1$ and $\dim(D) = 2$, discouraging news when looking for something better than the trivial estimate (1.6). To rule out this sort of degeneracy we will assume that our α-dimensional sets K have a certain regularity - defining $K_\delta = K + B(0, \delta)$, we will assume for the remainder of the paper that K_δ is a δ-discrete α-set in the sense of Katz and Tao [8]. This means that

$$|K_\delta \cap B(x, r)| \leq C(K)\,(r/\delta)^\alpha \delta^d \tag{1.8}$$

for any $x \in \mathbb{R}^d$ and $r \geq \delta$. In particular, we will now assume that the α-dimensional sets figuring in (1.4) all satisfy (1.8). With the assumption (1.8) in place we will obtain some nontrivial estimates on the upper Minkowski dimension $\dim_M(D)$ of D. But first we record another trivial estimate. Since $|K_\delta| \lesssim \delta^{d-\alpha}$ by (1.8), it follows from $D \subset K \times K$ that $|D_\delta| \lesssim \delta^{2d-2\alpha}$. Thus $\dim_M(D) \leq 2\alpha$ and so

$$g_d(\alpha) \leq 2\alpha. \tag{1.9}$$

Our first nontrivial bound for g_d concerns large values of α:

Theorem 1.2. *If $(d+1)/2 \leq \alpha \leq d$, then $g_d(\alpha) = 2\alpha - 1$; if $\alpha \leq (d+1)/2$, then $g_d(\alpha) \leq \alpha + (d-1)/2$.*

The proof uses the Fourier transform. The second statement of Theorem 1.2 is only interesting when $\alpha + (d-1)/2$ is less than the 2α in (1.9) and so only when $\alpha > (d-1)/2$. On the other hand, the first statement of Theorem 1.2 shows that the conjecture (1.7) is correct for $\alpha \geq (d+1)/2$. In particular, and in contrast to the discrete unit distance problem, when α is sufficiently large there are positive results available in $\mathbb{R}^d$ even when $d \geq 4$. But the same example which rules out positive results on the discrete unit distance problem for $d \geq 4$ can be easily modified to show that there are no nontrivial results on the continuous problem when $d \geq 4$ and α is small. In particular we have the following statement.

$$\text{If } d \geq 4 \text{ and } \alpha \leq \lfloor d/2 \rfloor - 1, \text{ then } g_d(\alpha) = 2\alpha. \tag{1.10}$$

(To see why (1.10) is true, first note that the inequality $g_{d+1}(\alpha) \geq g_d(\alpha)$ shows that it is enough to consider only the case when d is even. In this case let $\tilde{K}$ be an appropriate α-dimensional subset of $S^{d/2-1} \subset \mathbb{R}^{d/2}$ and define K by

$$K = 2^{-1/2}\{(\tilde{k}_1, 0),\, (0, \tilde{k}_2) \in \mathbb{R}^{d/2} \times \mathbb{R}^{d/2} : \tilde{k}_1, \tilde{k}_2 \in \tilde{K}\}.)$$

If $d \geq 4$ and $\alpha \in (\lfloor \frac{d}{2} \rfloor - 1, \frac{d-1}{2})$ we do not know if the trivial estimate (1.9) can be improved.

For $d = 2$ or $d = 3$ we have the following theorems, which contain nontrivial results for small α.

Theorem 1.3. *For $0 < \alpha \leq 1$ we have*

$$\frac{3\alpha}{2} \leq g_2(\alpha) \leq \min\left\{\frac{5\alpha}{3}, \frac{\alpha(2+\alpha)}{1+\alpha}\right\}.$$

Additionally, for $1 \leq \alpha \leq 3/2$ we have $g_2(\alpha) = \alpha + 1/2$ and for $3/2 \leq \alpha \leq 2$ we have $g_2(\alpha) = 2\alpha - 1$.

Except for the fact that $g_2(\alpha) \geq \alpha + 1/2$ when $1 \leq \alpha \leq 3/2$, the second statement here is a consequence of Theorem 1.2. Parts of the proofs of Theorem 1.3 and of Theorem 1.4 below employ incidence geometry in the continuous setting - see [9] for other examples.

Theorem 1.4. *We have $g_3(\alpha) \leq \frac{15\alpha}{8}$.*

We note that, in addition to improving (1.9) and improving (1.6) for $\alpha < 16/7$, the estimate in Theorem 1.4 improves the second bound in Theorem 1.2 when $\alpha \leq 8/7$.

2. Proofs

Proof of Theorem 1.1: Modifying (1.5) to fit the context of Theorem 1.1 gives

$$G = \{(p, b) : p \in P,\, b \in S^{d-1},\, p + b \in P\}.$$

The correspondence $(p_1, p_2) \longleftrightarrow (p_1, b) \doteq (p_1, p_2 - p_1)$ shows that (1.2) is equivalent to

$$\big|G\big| \leq C_d\, |P|^{(2d-1)/d}. \tag{2.1}$$

Define

$$V \doteq \{(p, b_1, \ldots, b_d) : (p, b_j) \in G,\, j = 1, \ldots, d\}.$$

Then (2.1) is a consequence of the two inequalities

$$\frac{|G|^d}{|P|^{d-1}} \leq |V| \tag{2.2}$$

and

$$|V| \leq C_d\,|P|^d. \tag{2.3}$$

Inequality (2.2) follows from a Hölder's inequality argument in the spirit of [7]:

$$|G| = \sum_{p\in P, |b|=1} \chi_G(p,b) \leq \Big(\sum_{p\in P}\big(\sum_{|b|=1}\chi_G(p,b)\big)^d\Big)^{1/d}\,|P|^{(d-1)/d}.$$

To see (2.3), write V as the disjoint union $V' \cup V''$ where V' is the subset of V consisting of all $(p, b_1, \dots, b_d)$ for which $b_i = b_j$ for some $i \neq j$. Since $(p,b) \in G$ implies $b \in P - p$, it is clear that

$$|V'| \leq C_d\,|P|^d.$$

To obtain a similar estimate for V'', consider the mapping

$$\Phi : (p, b_1, \dots, b_d) \mapsto (p + b_1, \dots, p + b_d)$$

of V'' into P^d. It will be enough to show that Φ is at most two-to-one. Since $(p,b) \in G$ implies $b \in P - p$, it follows from $(p, b_1, \dots, b_d) \in V''$ that there are distinct $p_1, \dots, p_d \in P$ such that

$$(b_2 - b_1, \dots, b_d - b_1) = (p_2 - p_1, \dots, p_d - p_1) \doteq (a_2, \dots, a_d).$$

Our hypothesis concerning affine independence implies that the vectors $a_2, \dots, a_d$ are linearly independent. Next, suppose that

$$\Phi(p', b_1', \dots, b_d') = \Phi(p, b_1, \dots, b_d).$$

Then

$$b_j' - b_1' = (p' + b_j') - (p' + b_1') = (p + b_j) - (p + b_1) = a_j$$

for $j = 2, \dots, d$. The desired multiplicity estimate for Φ now follows from Lemma 2.1 below (an analog of the fact that there are at most two chords of a circle which are congruent under translation).

Lemma 2.1. *Suppose that $a_2, \dots, a_d \in \mathbb{R}^d$ are linearly independent. Then there are at most two d-tuples $(b_1, \dots, b_d)$ with $b_j \in \mathbb{R}^d$ such that*

$$|b_1| = \cdots = |b_d| = 1, \text{ and } b_j - b_1 = a_j,\ j = 2, \dots, d. \tag{2.4}$$

Proof. Let H be the hyperplane in $\mathbb{R}^d$ spanned by $a_2, \dots, a_d$ and fix a nonzero vector v with $v \perp H$. Our first goal is to prove the following statement:

(2.5) there is $\{t_1, t_2\} \subset \mathbb{R}$ depending only on $\{a_2, \dots, a_d\}$ and v such that if (2.4) holds, then $\{b_1, \dots, b_d\} \subset (tv + H) \cap S^{d-1}$ for some $t \in \{t_1, t_2\}$.

To see (2.5) we begin by noting that if $w \in \mathbb{R}^d$ then the intersection

$$(tw + H) \cap S^{d-1}$$

is either a $(d-2)$-sphere or empty. In particular, if r_0 is the radius of the $(d-2)$-sphere determined by $\{0, a_2, \ldots, a_d\}$ (the linear independence of $a_2, \ldots, a_d$ guarantees that there is at most one such sphere) then there is $\{t_1, t_2\} \subset \mathbb{R}$, depending only on $a_2, \ldots, a_d$ and v, such that if $(tv+H) \cap S^{d-1}$ is a $(d-2)$-sphere of radius r_0 exactly when $t \in \{t_1, t_2\}$. Suppose that (2.4) holds. Since $(b_1 + H) \cap S^{d-1}$ contains b_1, $(b_1 + H) \cap S^{d-1}$ is a $(d-2)$-sphere. Since

$$b_1 + \{0, a_2, \ldots, a_d\} \subset (b_1 + H) \cap S^{d-1},$$

it follows that $(b_1 + H) \cap S^{d-1}$ is a $(d-2)$-sphere of radius r_0. Then (by (2.4))

$$\{b_1, \ldots, b_d\} = b_1 + \{0, a_2, \ldots, a_d\} \subset (b_1 + H) \cap S^{d-1} = (tv + H) \cap S^{d-1}$$

for some $t \in \{t_1, t_2\}$. This establishes (2.5).

Given (2.5), the proof of the lemma will be complete if we show that for fixed $t \in \mathbb{R}$ there is at most one d-tuple $(b_1, \ldots, b_d)$ such that both (2.4) and

$$\{b_1, \ldots, b_d\} \subset (tv + H) \cap S^{d-1} \tag{2.6}$$

hold. So suppose that (2.4) and (2.6) hold for $(b_1, \ldots, b_d)$ and also for $(b_1', \ldots, b_d')$. Let r_0 and c be the radius and center of the $(d-2)$-sphere $(tv+H) \cap S^{d-1}$. Then the points $\{0, a_2, \ldots, a_d\}$ are contained in the $(d-2)$-spheres of radius r_0 centered at $c-b_1$ and $c-b_1'$. Again appealing to the fact that $\{0, a_2, \ldots, a_d\}$ determine a unique $(d-2)$-sphere we see that $b_1 = b_1'$ and hence $(b_1, \ldots, b_d) = (b_1', \ldots, b_d')$. □

Proof of Theorem 1.2: Recalling the definition (1.4) of g_d, we will bound g_d by estimating $\dim_M(D)$. Since $\dim_M(D) \le \gamma$ will follow from $|D_\delta| = |D + B(0,\delta)| \lesssim \delta^{2d-\gamma-\epsilon}$ for all $\epsilon > 0$ and since $D_\delta \subset D^\delta$ where D^δ is defined by

$$D^\delta \doteq \{(k_1, k_2) \in K_\delta \times K_\delta : 1 - 2\delta \le |k_2 - k_1| \le 1 + 2\delta\},$$

we will be interested in estimating $|D^\delta|$. Without loss of generality assume that $K = -K$, and write

$$|D^\delta| = \int_{K_\delta}\int_{K_\delta} 1_{A(0,\delta)}(x_2 - x_1)\, dx_1\, dx_2 = \langle 1_{K_\delta} * 1_{K_\delta}, 1_{A(0,\delta)}\rangle \tag{2.7}$$

where, for $c \in \mathbb{R}^d$, $A(c,\delta) = \{x \in \mathbb{R}^d : 1 - 2\delta \le |x - c| \le 1 + 2\delta\}$. Let ρ be a symmetric Schwartz function with

$$1_{B(0,C)} \lesssim |\hat\rho| \lesssim 1_{B(0,2C)},\ 1_{B(0,C')}(x) \le \rho(x) \lesssim \sum_{j=1}^{\infty} 2^{-jd}\, 1_{B(0,2^j)}(x). \tag{2.8}$$

Write σ for Lebesgue measure on S^{d-1}. If $\rho_r(x) = r^{-d}\rho(x/r)$, and if C' is chosen appropriately, then

$$|D^\delta| \lesssim \delta\, \langle 1_{K_\delta} * 1_{K_\delta}, \rho_\delta * \rho_\delta * \sigma\rangle = \delta\, \langle (1_{K_\delta} * \rho_\delta) * (1_{K_\delta} * \rho_\delta), \sigma\rangle \lesssim \delta \int_{B(0,2C/\delta)} \left|\widehat{1_{K_\delta} * \rho_\delta}(\xi)\right|^2 \frac{d\xi}{1 + |\xi|^{(d-1)/2}}. \tag{2.9}$$

We will control the last integral by estimating $\|1_{K_\delta} * \rho_\delta\|_2$ and we begin by estimating $\|1_{K_\delta} * \chi_{B(0,r)}\|_2$ for $r \geq \delta$. Using (1.8) we have

$$\|1_{K_\delta} * 1_{B(0,r)}\|_1 \lesssim r^d\, \delta^{d-\alpha},\ \ \|1_{K_\delta} * 1_{B(0,r)}\|_\infty \lesssim r^\alpha \delta^{d-\alpha},$$

and so

$$\|1_{K_\delta} * 1_{B(0,r)}\|_2 \lesssim r^{(d+\alpha)/2}\, \delta^{d-\alpha},\ \ r \geq \delta. \tag{2.10}$$

Then (2.8) and (2.10) show that for $r \geq \delta$ we have

$$\|1_{K_\delta} * \rho_r\|_2 \lesssim r^{(\alpha-d)/2} \delta^{d-\alpha}. \tag{2.11}$$

Since $\hat{\rho}_r$ is supported on $B(0, C/r)$, (2.11) implies

$$\int\limits_{\frac{C}{2r} \leq |\xi| \leq \frac{C}{r}} \left|\widehat{1_{K_\delta} * \rho_\delta}(\xi)\right|^2 d\xi \lesssim \int\limits_{\frac{C}{2r} \leq |\xi| \leq \frac{C}{r}} \left|\widehat{1_{K_\delta} * \rho_r}(\xi)\right|^2 d\xi \lesssim r^{\alpha-d}\, \delta^{2(d-\alpha)},\ \ r \geq \delta.$$

Thus

$$\int \left|\widehat{1_{K_\delta} * \rho_\delta}(\xi)\right|^2 \frac{d\xi}{|\xi|^{d-\alpha}} = \int\limits_{|\xi| \leq \frac{2C}{\delta}} \left|\widehat{1_{K_\delta} * \rho_\delta}(\xi)\right|^2 \frac{d\xi}{|\xi|^{d-\alpha}} = \tag{2.12}$$

$$\int\limits_{\{|\xi| \leq C\}} \left|\widehat{1_{K_\delta} * \rho_\delta}(\xi)\right|^2 \frac{d\xi}{|\xi|^{d-\alpha}} + \sum_{\frac{1}{2} \leq 2^j \leq \frac{1}{\delta}} \int\limits_{\frac{C}{2^{j+1}\delta} \leq |\xi| \leq \frac{C}{2^j \delta}} \left|\widehat{1_{K_\delta} * \rho_\delta}(\xi)\right|^2 \frac{d\xi}{|\xi|^{d-\alpha}} \lesssim$$

$$\delta^{2(d-\alpha)} + \sum_{\frac{1}{2} \leq 2^j \leq \frac{1}{\delta}} \left(\frac{C}{2^j \delta}\right)^{\alpha-d} \left(2^j \delta\right)^{\alpha-d} \delta^{2(d-\alpha)} \lesssim \log\left(\frac{1}{\delta}\right) \delta^{2(d-\alpha)}.$$

(So normalized Lebesgue measure on K_δ behaves like an α-dimensional measure from the Fourier transform point of view.) We will use (2.12) to estimate $|D^\delta|$ via (2.9) and thus to obtain the upper bounds on g_d in Theorem 1.2. If $\alpha \geq (d+1)/2$, so that $(d-1)/2 \geq d - \alpha$, then the integral in (2.9) is dominated by the integral estimated in (2.12). That leads to $|D^\delta| \lesssim \log(\frac{1}{\delta})\, \delta^{2(d-\alpha)+1}$ and so, by the remarks at the beginning of this proof, to $g_d(\alpha) \leq 2\alpha - 1$. With the example described after (1.7), this gives $g_d(\alpha) = 2\alpha - 1$. If $\alpha \leq (d+1)/2$, then when $|\xi| \leq C/\delta$ we have

$$\frac{1}{|\xi|^{(d-1)/2}} \lesssim \frac{\delta^{\alpha-(d+1)/2}}{|\xi|^{d-\alpha}}$$

which leads as above to $g_d(\alpha) \leq \alpha + (d-1)/2$.

Proof of Theorem 1.3: We begin by claiming that it is enough to prove the upper bounds for $g_2(\alpha)$ under the additional assumption that

$$\mathrm{diam}(K) \leq 2 - \eta$$

for some fixed $\eta > 0$. (The purpose of this restriction is to avoid the possibility of external tangencies of certain annuli and thus to allow the use of estimates like (2.23) below.) To see that this reduction is legitimate, let

$\{C_1, \dots, C_7\}$ be a partition of the unit circle into arcs each having length less than .9 and let

$$G_i = B(0, 1/100) \cup \big(B(0, 1/100) + C_i\big).$$

Then

$$D \subset \cup_{k \in K} \cup_{1 \le i \le 7} \{(k_1, k_2) \in (k + G_i)^2\}.$$

As D is compact, it is contained in some finite union of sets

$$\{(k_1, k_2) \in (k + G_i)^2\}.$$

Since $\operatorname{diam}\big(k + G_i\big) \le 2 - \eta$ for some fixed $\eta > 0$, our claim is established.

By renaming η and assuming that $\delta > 0$ is small enough, we can (and do) assume for the remainder of this proof that

$$\operatorname{diam}(K_\delta) \le 2 - \eta. \tag{2.13}$$

We now turn to the proof of the upper bound

$$g_2(\alpha) \le \frac{\alpha(2+\alpha)}{1+\alpha} \tag{2.14}$$

Under the assumption that K satisfies (1.8) for $d = 2$, it is enough to establish the estimate

$$|D^\delta| \lesssim \log(1/\delta)\, \delta^{4 - \alpha(2+\alpha)/(1+\alpha)}.$$

(Throughout this argument the constants implied by the symbol $\lesssim$ depend only on K). With

$$G^\delta = \{(k, y) : k \in K_\delta,\ 1 - 2\delta \le |y| \le 1 + 2\delta,\ k + y \in K_\delta\},$$

the correspondence $(k_1, k_2) \longleftrightarrow (k_1, y) \doteq (k_1, k_2 - k_1)$ shows that $|D^\delta| = |G^\delta|$. Thus it suffices to show that

$$|G^\delta| \lesssim \log(1/\delta)\, \delta^{4 - \alpha(2+\alpha)/(1+\alpha)}. \tag{2.15}$$

Recall that $A(0, \delta) = \{x \in \mathbb{R}^d : 1 - 2\delta \le |x| \le 1 + 2\delta\}$. For $k \in K_\delta$ we will write $(G^\delta)_k$ for the k-section of G^δ given by $\{y \in A(0, \delta) : k + y \in K_\delta\}$. Since K_δ is a δ-discrete α-set we can assume that the two-dimensional Lebesgue measure of $(G^\delta)_k$ satisfies $\delta^2 \lesssim |(G^\delta)_k| \lesssim \delta^{2-\alpha}$. Find $M \lesssim \log(1/\delta)$ positive numbers $\{\lambda_m\}_{m=1}^M$ such that $\lambda_{m+1} = 2\,\lambda_m$ and such that for each $k \in K_\delta$ we have $\lambda_m \le |(G^\delta)_k| \le \lambda_{m+1}$ for some m. Then define

$$K^m = \{k \in K_\delta : \lambda_m \le |(G^\delta)_k| \le \lambda_{m+1}\}.$$

The estimate (2.15) (of the four-dimensional Lebesgue measure of G^δ) will follow from the following estimate of the two-dimensional Lebesgue measure of K^m:

$$\lambda_m\, |K^m| \lesssim \delta^{4 - \alpha(2+\alpha)/(1+\alpha)}. \tag{2.16}$$

So fix m. Choose $N = N(m)$ disjoint balls $B(c_n, \delta)$ with $c_n \in K_m$ for which

$$\lambda \doteq \lambda_m \le |A(c_n, \delta) \cap K_\delta| \tag{2.17}$$

and such that

$$|K^m| \lesssim N\,\delta^2. \tag{2.18}$$

Our goal is the estimate

$$\lambda\, N\, \delta^2 \lesssim \frac{\delta^4}{\lambda^\alpha} \tag{2.19}$$

which when interpolated with the trivial estimate

$$\lambda\, N\, \delta^2 \lesssim \lambda\, \delta^{2-\alpha} \tag{2.20}$$

gives (2.16) via (2.18).

To prove (2.19) we begin by fixing $r = C\delta^2/\lambda$. Since $|K_\delta \cap B(x,r)| \lesssim (r/\delta)^\alpha \delta^2$, any $B(x,r)$ contains $\lesssim (r/\delta)^\alpha$ of the $B(c_n,\delta)$'s. Thus there is an r-separated subcollection $\{\tilde{c}_n\}$ containing $\tilde{N}$ of the of the c_n's, where

$$N\delta^\alpha/r^\alpha \lesssim \tilde{N}. \tag{2.21}$$

The bound (2.19) will follow from a certain estimate from below of the two-dimensional Lebesgue measure

$$|\cup_n \big(A(\tilde{c}_n,\delta) \cap K_\delta\big)|.$$

Part of the strategy here is the general estimate

$$|\cup_n E_n| \geq \sum_n |E_n| - \sum_{n_1<n_2} |E_{n_1} \cap E_{n_2}|. \tag{2.22}$$

We will take $E_n = A(\tilde{c}_n,\delta) \cap K_\delta$ and use the estimate

$$|A(\tilde{c}_{n_1},\delta)\ \cap\ A(\tilde{c}_{n_2},\delta)| \lesssim \frac{\delta^2}{\delta + |\tilde{c}_{n_1} - \tilde{c}_{n_2}|} \tag{2.23}$$

(in which the implied constant depends on η in (2.13)) to bound $|E_{n_1} \cap E_{n_2}|$. For this reason we are interested in controlling the quantity

$$\sum_{n \neq n_0} \frac{\delta^2}{|\tilde{c}_n - \tilde{c}_{n_0}|}.$$

We are assuming that the sets $K_{\delta'}$ are unifomly δ'-discrete - that they satisfy (1.8) uniformly in δ' - and so, in particular, K_r is r-discrete. Thus for each $\tilde{c}_{n_0}$ there are at most $C_2\, 2^{k\alpha}$ of the r-separated $\tilde{c}_n$' s within distance $2^k r$ of $\tilde{c}_{n_0}$. Therefore, since $\alpha < 1$,

$$\sum_{n \neq n_0} \frac{\delta^2}{|\tilde{c}_n - \tilde{c}_{n_0}|} \lesssim \delta^2 \sum_{k=1}^{\infty} \frac{2^{k\alpha}}{2^k r} \lesssim \frac{\delta^2}{r}$$

and so

$$\sum_{n \neq n_0} \frac{\delta^2}{|\tilde{c}_n - \tilde{c}_{n_0}|} \leq c\, \lambda \tag{2.24}$$

by our choice of r. Thus (2.23) and (2.24) imply

$$\sum_{n_1<n_2} |A(\tilde{c}_{n_1},\delta) \cap \ A(\tilde{c}_{n_2},\delta)| \le C'\tilde{N}c\,\lambda = \tilde{N}c'\lambda.$$

On the other hand, because of (2.17) and (2.21) we have

$$\sum_n |A(\tilde{c}_n,\delta)\cap K_\delta| \ge \tilde{N}\lambda$$

and so, by (2.22),

$$|\cup_n \big(A(\tilde{c}_n,\delta)\cap K_\delta\big)| \ge (1-c')\tilde{N}\lambda \gtrsim (1-c')\Big(\frac{N\delta^\alpha}{r^\alpha}\Big)\lambda.$$

If C (figuring in the choice of r) is large enough, then $1-c'>0$ and so this last estimate and the fact that $|K_\delta| \lesssim \delta^{2-\alpha}$, together with our choice of r, yield (2.19). This completes the proof of (2.14).

Next we give the proof of the upper bound

$$g_2(\alpha) \le \frac{5\alpha}{3}. \tag{2.25}$$

Part of the argument is analogous to the proof of Theorem 1.1. Let K_m, $\lambda=\lambda_m$, and the $B(c_n,\delta)$, $1\le n\le N$, be as in the proof of (2.14). Instead of (2.19) we will now prove

$$\lambda\, N\,\delta^2 \lesssim \frac{\delta^{2+3(2-\alpha)}}{\lambda^2}. \tag{2.26}$$

As above, interpolation with (2.20) will then lead to

$$|D^\delta| \lesssim \log(1/\delta)\,\delta^{4-5\alpha/3}$$

and so to (2.25).

Choose a maximal δ-separated subset J of K_δ. For each c_n let

$$S_{c_n} = \{a\in J : 1-3\delta \le |a-c_n| \le 1+3\delta\},$$

so that S_{c_n} is like a discretized c_n-section of D^δ. Define

$$V = \Big\{(c_n,a_1,a_2) : 1\le n\le N,\ a_1,a_2\in S_{c_n},\ |a_1-a_2| \ge c\,\Big(\frac{\lambda}{\delta^{2-\alpha}}\Big)^{1/\alpha}\Big\},$$

where c is a small positive constant. We will prove (2.26) by comparing upper and lower estimates for $|V|$.

Since

$$|\{k\in K_\delta : 1-2\delta \le |k-c_n| \le 1+2\delta\}| \ge \lambda$$

by the choice of c_n it follows that $|S_{c_n}| \gtrsim \lambda/\delta^2$. Since (1.8) implies that

$$|K_\delta \cap B\big(a, c(\lambda/\delta^{2-\alpha})^{1/\alpha}\big)| \lesssim c^\alpha\,\lambda$$

for any a, it follows that

$$|V| \gtrsim N\,\Big(\frac{\lambda}{\delta^2}\Big)^2 \tag{2.27}$$

if c is small enough.

To obtain an upper bound for $|V|$ we begin by noting that if

$$(c_{n_0}, a_1, a_2) \in V$$

then c_{n_0} is in

$$A(a_1, 3\delta) \cap A(a_2, 3\delta). \tag{2.28}$$

Because $|a_1 - a_2| \leq 2 - \eta < 2$ it follows that if $|a_1 - a_2| \gtrsim \delta$ then (2.28) is a union of two connected components, one on either side of the line through a_1 and a_2 and each having diameter bounded above by

$$C \frac{\delta}{|a_1 - a_2|} \lesssim \Big(\frac{\delta^2}{\lambda}\Big)^{1/\alpha}, \tag{2.29}$$

where the inequality comes from the definition of V. The hypothesis (1.8) then implies that each connected component of (2.28) contains $\lesssim \delta^{2-\alpha}/\lambda$ points from $\{c_n\}$. Thus the projection

$$(c_n, a_1, a_2) \mapsto (a_1, a_2)$$

of V into $J \times J$ has multiplicity at most $C\,\delta^{2-\alpha}/\lambda$. Therefore

$$|V| \lesssim |J|^2 \frac{\delta^{2-\alpha}}{\lambda} \lesssim \delta^{-2\alpha} \frac{\delta^{2-\alpha}}{\lambda}. \tag{2.30}$$

Comparison of (2.27) and (2.30) yields (2.26). This completes the proof of (2.25).

To complete the proof of Theorem 1.3 we need to establish the two lower bounds on $g_2(\alpha)$

$$g_2(\alpha) \geq 3\alpha/2 \text{ if } 0 < \alpha \leq 1 \tag{2.31}$$

and

$$g_2(\alpha) \geq \alpha + 1/2 \text{ if } 1 < \alpha \leq 3/2. \tag{2.32}$$

These will be consequences of the following lemma.

Lemma 2.2. *Suppose* $0 < \beta, \gamma < 1$ *are rational and let* $\alpha' = \beta + \gamma$. *There is a compact set* $K \subset \mathbb{R}^2$ *which satisfies* (1.8) *with* α' *instead of* α *and for which we have* $|D^\delta| \gtrsim \delta^{4-(\beta+3\gamma/2)}$ *for some sequence of* δ*'s tending to* 0.

To deduce (2.31), approximate α by α' with β very close to 0; to deduce (2.32), approximate α by α' with γ very close to 1.

Proof. We will require compact subsets $A, B \subset [0,1]$ which satisfy (1.8) with α replaced by β in the case of A and by γ in the case of B. We will also need A and B to satisfy the two lower bounds

$$\int_{A_{\delta_n}} \int_{A_{\delta_n}} 1_{\{2\delta_n \leq |x_1 - x_2| \leq 5\delta_n/2\}}\, dx_1\, dx_2 \gtrsim \delta_n^{2-\beta} \tag{2.33}$$

and

$$\int_{B_{\delta_n}} \int_{B_{\delta_n}} 1_{\{\sqrt{7\delta_n/2} \leq |t_1 - t_2| \leq 2\sqrt{\delta_n}\}}\, dt_1\, dt_2 \gtrsim \delta_n^{2-2\gamma} \delta_n^{\gamma/2}. \tag{2.34}$$

for a sequence δ_n's tending to 0. (At the end of this proof we will say a few words about how to obtain A and B.) Put $F = A \cup (A+1)$. Then

$$\int_{F_{\delta_n}} \int_{F_{\delta_n}} 1_{\{2\delta_n \le 1 - |x_1 - x_2| \le 5\delta_n/2\}}\, dx_1\, dx_2 \gtrsim \delta_n^{2-\beta}. \tag{2.35}$$

Let $K = F \times B$. Then (1.8) holds with $\alpha' = \beta + \gamma$ in place of α by our choices of F and B.

Now

$$1 - \delta \le \sqrt{(x_1 - x_2)^2 + (t_1 - t_2)^2} \le 1 + \delta$$

is equivalent to

$$\sqrt{(1-\delta)^2 - |x_1 - x_2|^2} \le |t_1 - t_2| \le \sqrt{(1+\delta)^2 - |x_1 - x_2|^2}.$$

If

$$2\delta \le 1 - |x_1 - x_2| \le 5\delta/2$$

then

$$2\delta \le 1 - |x_1 - x_2|^2 \le 5\delta$$

and so if $\delta < 1/2$ some algebra shows that

$$\sqrt{(1-\delta)^2 - |x_1 - x_2|^2} \le \sqrt{7\delta/2} < 2\sqrt{\delta} \le \sqrt{(1+\delta)^2 - |x_1 - x_2|^2}.$$

Thus if

$$2\delta \le 1 - |x_1 - x_2| \le 5\delta/2 \text{ and } \sqrt{7\delta/2} \le |t_1 - t_2| \le 2\sqrt{\delta}$$

it follows that

$$\sqrt{(1-\delta)^2 - |x_1 - x_2|^2} \le |t_1 - t_2| \le \sqrt{(1+\delta)^2 - |x_1 - x_2|^2}.$$

With (2.35) and (2.34) this gives

$$\int_{F_{\delta_n}} \int_{F_{\delta_n}} \int_{B_{\delta_n}} \int_{B_{\delta_n}} 1_{\{1-\delta_n \le \sqrt{(x_1-x_2)^2 + (t_1-t_2)^2} \le 1+\delta_n\}}\, dt_1\, dt_2\, dx_1\, dx_2 \gtrsim \delta_n^{4-(\beta+3\gamma/2)}$$

and so $|D^{\delta_n}| \gtrsim \delta_n^{4-(\beta+3\gamma/2)}$.

We conclude the proof of this lemma by describing a construction (which, though tedious, we include for the sake of completeness) of the required sets F and B. For positive integers $p < q$ consider the Cantor set $C = C(p,q)$ constructed by removing $(2^p - 1)$ equally spaced intervals open intervals from $C_0 = [0,1]$ to obtain $C_1 = [0, 2^{-q}] \cup \cdots \cup [1 - 2^{-q}, 1]$ and then continuing in the usual way, so that at the jth stage of the construction we have a set C_j which is the union of 2^{jp} closed intervals of length 2^{-jq}. Then (1.8) holds with $C = \cap C_j$ instead of K and with $\alpha = p/q$. Also, since $C_j \subset C + B(0, 2^{-qj}) = C_{2^{-qj}}$, for any $0 < \kappa_1 < \kappa_2 < 1$ we have

$$\int_{C_{2^{-qj}}} \int_{C_{2^{-qj}}} 1_{\{\kappa_1 2^{-qj} \le |x_1 - x_2| \le \kappa_2 2^{-qj}\}}\, dx_1\, dx_2 \gtrsim (2^{-qj})^{(2-p/q)}$$

and then also

$$\int_{C_{2^{-qj-2}}} \int_{C_{2^{-qj-2}}} 1_{\{\kappa_1 2^{-qj} \le |x_1 - x_2| \le \kappa_2 2^{-qj}\}}\, dx_1\, dx_2 \gtrsim (2^{-qj})^{(2-p/q)}, \tag{2.36}$$

where the implied constant depends on κ_1 and κ_2. One then sees that

$$\int_{C_{2^{-2qj-2}}}\int_{C_{2^{-2qj-2}}} 1_{\{\kappa 2^{-qj}\le|x_1-x_2|\le 2^{-qj}\}}\,dx_1\,dx_2 \gtrsim \tag{2.37}$$
$$2^{2(p-q)j}(2^{-qj})^{(2-p/q)} = (2^{-2qj})^{(2-\frac{3}{2}\frac{p}{q})}.$$

If p_2 and q_2 are chosen so that $\gamma = p_2/q_2$, if $B = C(p_2,q_2)$, and if

$$\delta_n = 2^{-2q_1q_2n-2}$$

then

$$\sqrt{\frac{7\delta_n}{2}} = \sqrt{\frac{7}{8}}\,2^{-q_1q_2n},\ 2\sqrt{\delta_n} = 2^{-q_1q_2n}$$

and so (2.37) with $q = q_2$, $j = nq_1$, and $\kappa = \sqrt{7/8}$ shows that (2.34) holds.
If p_1 and q_1 are chosen so that $\beta = p_1/q_1$ and if $A = C(p_1,q_1)$, then

$$2\delta_n = \frac{1}{2}2^{-2q_1q_2n},\ \frac{5}{2}\delta_n = \frac{5}{8}2^{-2q_1q_2n}$$

and so (2.36) with $q = q_1$, $j = nq_2$, $\kappa_1 = 1/2$, and $\kappa_2 = 5/8$ shows that (2.33) holds. This completes the proof of the lemma.

□

Proof of Theorem 1.4: The proof is similar to the proof of the bound $g_2(\alpha) \le 5\alpha/3$ of Theorem 1.3. We begin by letting K_m, $\lambda = \lambda_n$, and the balls $B(c_n,\delta)$, $1 \le n \le N$ be the three-dimensional analogs of the quantities defined in the proof of Theorem 1.3. Instead of (2.26) we will now establish

$$\lambda\, N\, \delta^3 \lesssim \frac{\delta^{5(3-\alpha)}\delta^{\alpha/2}}{\lambda^3}. \tag{2.38}$$

Interpolation with the trivial bound

$$\lambda\, N\, \delta^3 \lesssim \lambda\, \delta^{3-\alpha} \tag{2.39}$$

gives

$$\lambda\, N\, \delta^3 \lesssim \lambda\, \delta^{6-15\alpha/8}. \tag{2.40}$$

Then an argument completely analogous to the one in the proof of Theorem 1.3 leads to $g_3(\alpha) \le 15\alpha/8$.

Again choose a maximal δ-separated subset J of K_δ. For each c_n let

$$S_{c_n} = \{a \in J : 1-3\delta \le |a-c_n| \le 1+3\delta\} = J \cap A(c_n, 3\delta)$$

Define

$$V =$$
$$\Big\{(c_n,a_1,a_2,a_3) : 1 \le n \le N,\ a_1,a_2,a_3 \in S_{c_n},\ |a_i - a_j| \ge c\left(\frac{\lambda}{\delta^{3-\alpha}}\right)^{1/\alpha} \text{ if } 1 \le i < j \le 3\Big\},$$

where c is a small positive constant. We will prove (2.38) by again comparing upper and lower estimates for $|V|$. Before continuing we note that it suffices to prove (2.38) under the assumption that

$$\left(\frac{\lambda}{\delta^{3-\alpha}}\right)^{1/\alpha} \gtrsim \delta^{1/2-\epsilon} \tag{2.41}$$

for some small $\epsilon > 0$ - otherwise (2.40) follows from (2.39). By using (1.8) just as in the proof of (2.27) we get the lower bound

$$|V| \gtrsim N\left(\frac{\lambda}{\delta^3}\right)^3. \tag{2.42}$$

As before we will obtain an upper bound for $|V|$ by controlling the multiplicity of the projection

$$(c_n, a_1, a_2, a_3) \mapsto (a_1, a_2, a_3)$$

of V into J^3. In fact we will show that
(2.43)

$$(c_n, a_1, a_2, a_3) \mapsto (a_1, a_2, a_3) \text{ has multiplicity bounded by } C\,\delta^{-\alpha/2}\frac{\delta^{3-\alpha}}{\lambda}.$$

Since $|J| \lesssim \delta^{-\alpha}$ it will then follow that

$$|V| \lesssim \delta^{-3\alpha}\delta^{-\alpha/2}\frac{\delta^{3-\alpha}}{\lambda}.$$

Comparing this with (2.42) then gives (2.38). Thus the proof of Theorem 1.4 will be complete when (2.43) is established.

We will establish (2.43) by estimating the diameter of an intersection

$$I \doteq A(a_1, 3\delta) \cap A(a_2, 3\delta) \cap A(a_3, 3\delta). \tag{2.44}$$

To begin, the intersection $A(a_1, 0) \cap A(a_2, 0)$ of the unit spheres centered at a_1 and a_2 is a circle contained in the hyperplane

$$P_{1,2} = \big\{x \in \mathbb{R}^3 : (x - a_1)\cdot(a_2 - a_1) = \frac{1}{2}|a_2 - a_1|^2\big\}. \tag{2.45}$$

If $x \in A(a_i, 3\delta)$ then $x \in A(a_i + e_i, 0)$ with $|e_i| \leq 3\delta$. It follows that if $x \in A(a_1, 3\delta) \cap A(a_2, 3\delta)$, then

$$\big|(x - a_1)\cdot(a_2 - a_1) - \frac{1}{2}|a_2 - a_1|^2\big| \lesssim \delta,$$

and so

$$A(a_1, 3\delta) \cap A(a_2, 3\delta) \subset P_{1,2} + B\big(0, C\frac{\delta}{|a_2 - a_1|}\big).$$

Similarly,

$$A(a_1, 3\delta) \cap A(a_3, 3\delta) \subset P_{1,3} + B\big(0, C\frac{\delta}{|a_3 - a_1|}\big).$$

If the a_i are affinely independent, it follows that the intersection (2.44) is contained in an extrusion (in the direction perpendicular to $a_2 - a_1$ and $a_3 - a_1$) of a parallelogram P contained in the plane $a_1 + \mathrm{span}(a_2 - a_1, a_3 -$

a_1). This parallelogram has two sides of length $\frac{C\delta}{|a_2-a_1|\sin(\theta)}$ perpendicular to a_3-a_1 and two sides of length $\frac{C\delta}{|a_3-a_1|\sin(\theta)}$ perpendicular to a_2-a_1 where $\theta \in (0,\pi)$ is the angle between a_2-a_1 and a_3-a_1.

We will need the estimate

$$\sin(\theta) \gtrsim |a_3 - a_2|. \tag{2.46}$$

This is the point at which (2.41) will come into play: we will be assuming that $(c_n, a_1, a_2, a_3) \in V$ and so it will follow that

$$|a_2 - a_1|,\ |a_3 - a_1|,\ |a_3 - a_2| \gtrsim \delta^{1/2-\epsilon} \tag{2.47}$$

for some $\epsilon > 0$. With no loss of generality we can write $a_1 = (0,0,0)$, $a_2 = (x_2,0,0)$, $a_3 = (x_3,y_3,0)$ and then assume that these points lie in the first octant, that $y_3 > 0$, and that $|a_2| \geq |a_3|$. We will now observe that if $\sin(\theta)$ and therefore $\tan(\theta) = y_3/x_3$ are small compared to $|a_2 - a_3|$, then the extrusion fails to intersect the shells $A(a_i, 3\delta)$. To show this we begin by observing that the center p of the parallelogram P is the point of intersection of the perpendicular bisectors of the segments $[a_1,a_2]$ and $[a_1,a_3]$ and has y coordinate equal to

$$p_y \doteq \frac{y_3}{2} - \frac{x_3}{2y_3}(x_2 - x_3) = \frac{y_3}{2} - \frac{1}{2\tan(\theta)}(x_2 - x_3).$$

If $\tan(\theta)$ is small compared to $|a_2 - a_3|$, then $x_2 - x_3 \geq |a_2 - a_3|/2$. Thus, it follows that $|p_y|$ is large. Since we have assumed that $|a_2 - a_1| \geq |a_3 - a_1|$, the diameter of P is bounded by

$$\frac{2C\delta}{|a_3 - a_1|\sin(\theta)} \lesssim \frac{\delta^{1/2+\epsilon}}{\sin(\theta)},$$

where we have used (2.47). This will be small compared to $|p_y|$ (since

$$\frac{|x_2 - x_3|}{\tan(\theta)} \gtrsim \frac{\delta^{1/2-\epsilon}}{\sin(\theta)},$$

again by (2.47)). In this case the distance ρ from P to the x-axis will be comparable to $|p_y|$. But if $\rho > 2$, say, the extrusion will miss the shells $A(a_i, 3\delta)$ (whose centers lie in the xy-plane above the x-axis).

With (2.46) it now follows from the definition of V that the diameter of P is bounded by $C\delta\big(\frac{\delta^{3-\alpha}}{\lambda}\big)^{2/\alpha}$. The following estimate is a consequence of the subadditivity of the function $\sqrt{\cdot}$ on $(0,\infty)$:

$$\Big|\sqrt{(1+\epsilon_1)^2 - (x_1^2 + y_1^2)} - \sqrt{(1+\epsilon_2)^2 - (x_2^2 + y_2^2)}\Big| \leq \sqrt{\big|2\epsilon_1 + \epsilon_1^2 - 2\epsilon_2 - \epsilon_2^2\big|} + \sqrt{\big|x_2^2 + y_2^2 - (x_1^2 + y_2^2)\big|}.$$

With $|\epsilon_1|, |\epsilon_2| \leq 2\delta$ this shows that if $(x_i, y_i, z_i) \in A(0,\delta)$ for $i = 1,2$ and $|(x_1,y_1) - (x_2,y_2)| \leq \kappa$ then

$$|(x_1,y_1,z_1) - (x_2,y_2,z_2)| \lesssim \max(\delta^{1/2}, \kappa^{1/2}).$$

Thus it follows from our bound on the diameter of P that

$$\text{diam}(I) \leq C\left(\delta\big(\frac{\delta^{3-\alpha}}{\lambda}\big)^{2/\alpha}\right)^{1/2} = C\,\delta\,\delta^{-1/2}\big(\frac{\delta^{3-\alpha}}{\lambda}\big)^{1/\alpha}. \tag{2.48}$$

Now if $(c_n, a_1, a_2, a_3), (c_{n'}, a_1, a_2, a_3) \in T$, we have $c_n, c_{n'} \in I$. Thus (2.48), the fact that the c_n's are δ-separated, and (1.8) together yield (2.43). This completes the proof of Theorem 1.4.

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