Categorical group invariants of 3-manifolds

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Abstract

For a given class $\mathcal{G}$ of groups, a closed topological n-manifold M^n is of $\mathcal{G}$-category $\leq k$ if it can be covered by k open subsets such that for each path-component W of the subsets the image of its fundamental group $\pi_1(W) \to \pi(M^n)$ belongs to $\mathcal{G}$. The smallest number k such that M^n admits such a covering is the $\mathcal{G}$-category, $cat_{\mathcal{G}}(M^n)$. For $n = 3$, M^3 has $\mathcal{G}$-category ≤ 4. We characterize all closed 3-manifolds of $\mathcal{G}$-category 1, 2, and 3 for various classes $\mathcal{G}$.[1] [2]

1 Introduction

In [3], M. Clapp and D. Puppe proposed the following generalization of the Lusternik-Schnirelman category $cat(M)$ for a manifold M: Let $\mathcal{A}$ be a non-empty class of spaces. A subset W of M is $\mathcal{A}$-*contractible* (in M) if the inclusion $\iota : W \to M$ factors homotopically through some $X \in \mathcal{A}$, i.e. there exist maps $f : W \to X$, $\alpha : X \to M$, such that ι is homotopic to $\alpha \cdot f$. (W and X need not be connected). The $\mathcal{A}$-category $cat_{\mathcal{A}}(M)$ of M is the smallest number of open $\mathcal{A}$-contractible subsets of M that cover M. If no such finite cover exists, $cat_{\mathcal{A}}(M)$ is infinite. When the family $\mathcal{A}$ contains just one space X, we write X-category instead of $\mathcal{A}$-category. In particular, if X is a single point, then the X-category of M is its classical Lusternik-Schnirelman category $cat(M)$. An extensive survey for this category can be found in [4]. For closed n-manifolds, $1 \leq cat_{\mathcal{A}}(M) \leq cat(M) \leq n + 1$.

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For each path-component W' and each basepoint $* \in W'$ of an $\mathcal{A}$-contractible set W, the image $\iota_*(\pi_1(W',*)) \subset \pi_1(M,*)$ is a quotient of a subgroup of $\pi_1(X', f(*))$ (for some basepoint), where X' is a path-component of some $X \in \mathcal{A}$. For example, if W is contractible in M, then all these images are trivial and one is led to consider open overs of M by sets V such that for each component V' of V and basepoint $* \in V$, $\iota_*(\pi_1(V',*)) = 1$. The smallest number of these sets that cover M is the π_1-category of M, studied in [7]. More generally we say that for any given class of groups $\mathcal{G}$, a subset W of M is *$\mathcal{G}$-contractible* (in M) if, for every basepoint $* \in W$, the image $\iota_*(\pi(W,*)) \subset \pi(M,*)$ belongs to $\mathcal{G}$, and define $cat_{\mathcal{G}}(M)$ to be the smallest number of open $\mathcal{G}$-contractible subsets of M that cover M.

If $\mathcal{G} = ame$, the class of amenable groups, a $\mathcal{G}$-contractible set is called an *amenable* set (in M). Gromov [12] showed that the simplicial volume $|M^n|$ of a closed n-manifold vanishes if M^n is covered by n amenable open sets. By Perelman (see e.g. [1]), $|M^3| = 0$ if and only if M^3 is a connected sum of graph manifolds. In [10] we classified all closed 3-manifolds of *ame*-category ≤ 3.

Little is known for other classes $\mathcal{G}$ of groups. In this paper we study $cat_{\mathcal{G}}(M^3)$ for (among other classes) the classes $hunf$ (groups that do not contain a free group of rank 2 as a subgroup), *vsolv* (virtually solvable), *solv* (solvable), *abel* (abelian).

First we obtain in Proposition 1 a relation between the geometric $\mathcal{A}$-*cat* and the algebraic $\mathcal{G}$-*cat*. It turns out (Proposition 2) that the $\mathcal{G}$-categories of a manifold M agree for the classes $hunf$, *ame*, and *vsolv*. Observing that for $\mathcal{G}_1 \supset \mathcal{G}_2$ we have $cat_{\mathcal{G}_1}(M) \leq cat_{\mathcal{G}_2}(M)$, we consider (Theorem 1 in section 4) the smallest class *spec* for which $cat_{spec}(M^3) = cat_{vsolv}(M^3) = 2$. This class is studied in detail in section 3; building blocks are the fundamental groups of compact 3-manifolds with solvable fundamental group and with boundary containing projective planes. Still smaller classes are $\mathcal{G}_T$, the set consisting of the cyclic groups and the torus group $\mathbb{Z} \times \mathbb{Z}$, and $\mathcal{G}_K$, the set consisting of cyclic groups, the torus and Kleinbottle groups, and $\mathbb{Z}_2 * \mathbb{Z}_2$. As an application we obtain in Corollary 4 another characterization of graph-manifolds in terms of categories: A closed orientable 3-manifold M is a connected sum of graph manifolds if and only if $cat_{abel}(M) \leq 3$ if and only if $cat_{solv}(M) \leq 3$. Finally, in Theorem 4 we obtain a complete classification of closed 3-manifolds for which $cat_{\mathcal{G}}(M) \leq 3$, where $hunf \supset \mathcal{G} \supset \mathcal{G}_K$. In particular we list in Corollary 5 all closed prime 3-manifolds M for which $cat_{solv}(M) = 1$, 2, or 3. The only non-prime closed 3-manifold with $cat_{solv}(M) = 1$, is $M = P^3 \# P^3$; the non-prime closed 3-manifolds with $cat_{solv}(M) = 2$ or 3 are described by Lemma 3.

2 $cat_{\mathcal{A}}(M)$ and $cat_{\mathcal{G}}(M)$

By a *closed* manifold we mean a compact manifold without boundary. We also assume that a closed manifold is connected unless stated otherwise.

Definition 1. (a) Let $\mathcal{A}$ be a non-empty class of spaces and let Y be topological space. A subset W of Y is $\mathcal{A}$-*contractible* (in Y) if for some space $X \in \mathcal{A}$ there exist maps $f : W \to X$, $\alpha : X \to Y$, such that the inclusion $\iota : W \to Y$ is homotopic to $\alpha \cdot f$.
(b) $cat_{\mathcal{A}}(Y)$ is the smallest number of open $\mathcal{A}$-contractible subsets of Y that cover Y. If no such finite cover exists, we say that $cat_{\mathcal{A}}(Y)$ is infinite.

The most interesting case is when $Y = M$, a manifold. For a $\mathcal{A}$-contractible subset W (in M) the diagram below commutes up to homotopy and it follows that the image $\iota_*(\pi(W,*) \subset \pi(M,*))$ is isomorphic to a quotient of a subgroup of $\pi(X, f(*))$, for every basepoint $* \in W$.

$$\begin{tikzcd} W \arrow[rr, "\iota"] \arrow[dr, "f"'] & & M \\ & X \arrow[ur, "\alpha"'] & \end{tikzcd}$$

If $\mathcal{A} = \{X\}$, we write $cat_X(M)$ instead of $cat_{\mathcal{A}}(M)$. The Lusternik-Schnirelman category $cat(M)$ is the same as $cat_{point}(M)$. We have that for any $\mathcal{A}$ and any connected n-manifold M,

$$\frac{cat(M)}{sup_{X \in \mathcal{A}}\{cat(X)\}} \leq cat_{\mathcal{A}}(M) \leq cat(M) \leq n+1.$$

The algebraic version is as follows:

Definition 2. Let $\mathcal{G}$ be a nonempty class of groups and let M be a manifold. A subset W of M is $\mathcal{G}$-*contractible* (in M) if, for every basepoint $* \in W$, the image $\iota_*(\pi(W,*) \subset \pi(M,*)$ belongs to $\mathcal{G}$.
(b) $cat_{\mathcal{G}}(M)$ is the smallest number of open $\mathcal{G}$-contractible subsets of M that cover M. If no such finite cover exists, $cat_{\mathcal{G}}(M) = \infty$.

Note that, if $\mathcal{G}$ is closed under subgroups, a subset of a $\mathcal{G}$-contractible set is $\mathcal{G}$-contractible.

There is a correspondence between the geometric $\mathcal{A} - cat$ and the algebraic $\mathcal{G} - cat$ as follows. For a given $\mathcal{G}$ let $\mathcal{A}_{\mathcal{G}} = \{X : \pi(X,*) \in \mathcal{G}$, for all $* \in X\}$.

Proposition 1. *If $\mathcal{G}$ is closed under subgroups and quotients, then $cat_{\mathcal{G}}(M) = cat_{\mathcal{A}_{\mathcal{G}}}(M)$.*

Proof. Suppose W is $\mathcal{G}$-contractible in M. Attach sufficiently many 2-cells to W along loops that are null-homotopic in M to obtain a space W_X such that

$\pi(W_X, *)$ is isomorphic to the image of $\iota_* : \pi(W, *) \to \pi(M, *)$ for any base point $*$ in W (for details see the proof of Theorem 2 in [8]). ι can be extended to a map $\alpha : W_X \to M$ because the loops along which the 2-cells were attached are null-homotopic in M. Then α induces an isomorphism from $\pi(W_X, *)$ onto $im(\iota_*)$ for any base point $*$ in W, hence W_X belongs to $\mathcal{A}_\mathcal{G}$. For the the natural inclusion $f : W \to W_X$ we have $\alpha f = \iota$ and therefore W is $\mathcal{A}_\mathcal{G}$-contractible. It follows that $cat_{\mathcal{A}_\mathcal{G}}(M) \leq cat_\mathcal{G}(M)$. (Note that no closure properties of $\mathcal{G}$ were used here.)

Now suppose that W is $\mathcal{A}_\mathcal{G}$-contractible so we have a homotopy commutative diagram as in Definition 1, with $\pi(X, f(*)) \in \mathcal{G}$. The image $\iota_*(\pi(W, *))$ is a quotient of a subgroup of $\pi(X, f(*))$ for any $*$ in W, and therefore belongs to $\mathcal{G}$, since $\mathcal{G}$ is closed under subgroups and quotients. Hence $cat_\mathcal{G}(M) \leq cat_{\mathcal{A}_\mathcal{G}}(M)$. $\square$

When $\mathcal{G}$ consists of the trivial group only, then $cat_\mathcal{G}(M) = cat_{\pi_1}(M)$, the π_1-category of M. This has been calculated in [7] (Corollary 4.2) for closed 3-manifolds M^3. The result is that $cat_{\pi_1}(M^3) = 1$ (resp. 2, resp. 4), if $\pi_1(M^3) = 1$ (resp. free non-trivial, resp. non-free). Thus by Perelman,

$$cat_{\pi_1}(M^3) = \begin{cases} 1 & \text{if and only if } M \text{ is the 3-sphere} \\ 2 & \text{if and only if } M \text{ is a connected sum of } S^2\text{-bundles over } S^1 \\ 4 & \text{otherwise.} \end{cases}$$

Some well-known classes of groups are the following classes $\mathcal{G}$:

$hunf$, the class of groups that do not contain the free group F_2 of rank 2 as a subgroup ($hunf$ stands for "hereditarily unfree")
The classes of of amenable, virtually solvable, virtually solvable 3-manifold groups, solvable, abelian groups, are denoted respectively by *ame*, *vsolv*, $vsolv_3$, *solv*, *abel*.

Here G is *amenable* if it a has finitely additive, left-invariant probability measure μ, i.e. $\mu(gS) = \mu(S)$ for all subsets $S \subset G$, $g \in G$; $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint subsets $A, B \subset G$; $\mu(G) = 1$.
A 3-*manifold group* is a group that is isomorphic to the fundamental group of a 3-manifold.

We have the following inclusions

$$\begin{array}{ccccccccc} hunf & \supset & ame & \supset & vsolv & \supset & solv & \supset & abel \\ & & & & \cup & & & & \\ & & & & vsolv_3 & & & & \end{array}$$

Note that if $\mathcal{G} \supset \mathcal{B}$ then $cat_\mathcal{G}(M) \leq cat_\mathcal{B}(M)$.

If $\mathcal{G} = ame$, (resp. $solv, \dots$), we say that W is *amenable*, (resp. *solvable*, ...), (in M) instead of W is *ame-contractible*, (resp. *solv-contractible*, ...).

Lemma 1. *Let $\mathcal{G}$ be a nonempty class of groups closed under subgroups and let $\mathcal{G}^f$ be the class consisting of the finitely generated members of $\mathcal{G}$. Then, for any compact manifold M, $cat_{\mathcal{G}}(M) = cat_{\mathcal{G}^f}(M)$.*

Proof. Clearly $cat_{\mathcal{G}}(M) \leq cat_{\mathcal{G}^f}(M)$.
Write $k = cat_{\mathcal{G}}(M)$ and let $\{U_1, \ldots, U_k\}$ be an open cover of M by $\mathcal{G}$ - contractible subspaces. Since M is normal there is an open cover $\{V_1, ..., V_k\}$ of M such that $\overline{V_i} \subset U_i$ $(i = 1, \ldots, k)$ (shrinking Lemma) and, using topological transversality (see [19] and [24]), it follows that there are compact codimension 0 submanifolds W_i such that $\overline{V_i} \subset int(W_i) \subset W_i \subset U_i$ (here $int(W_i)$ means the maximal open set contained in W_i). Since $\mathcal{G}$ is closed under subgroups $int(W_i)$ is $\mathcal{G}$ - contractible. Compact topological manifolds are ANR's and are dominated by finite simplicial complexes (Borsuk [2]), hence they have finitely generated fundamental groups. It follows that $\pi(int(W_i))$ is finitely generated and therefore $int(W_i)$ is $\mathcal{G}^f$ - contractible. Therefore $cat_{\mathcal{G}^f}(M) \leq cat_{\mathcal{G}}(M)$. $\square$

Proposition 3 in section 3 below, proved in [10] (Proposition 3), lists all compact 3-manifolds whose fundamental group does not contain F_2. It turns out that all these groups are virtually solvable. From this we obtain the following relation between the various $\mathcal{G}$-categories for compact 3-manifolds:

Proposition 2. *If M^3 is a compact 3-manifold, then $cat_{hunf}(M^3) = cat_{ame}(M^3) = cat_{vsolv}(M^3) = cat_{vsolv_3}(M^3)$.*

Proof. Clearly $cat_{hunf}(M^3) \leq cat_{ame}(M^3) \leq cat_{vsolv}(M^3) \leq cat_{vsolv_3}(M^3)$.
We show $cat_{solv_3}(M^3) \leq cat_{hunf}(M^3)$. Note that by lemma 1, $cat_{hunf}(M^3) = cat_{hunf^f}(M^3) := k$. Let $\{W_1, \ldots, W_k\}$ be an open cover of M^3 by $hunf^f$ - contractible subsets . It suffices to show that W_i is $vsolv_3$ - contractible.
The image of $\pi(W_i, *)$ in $\pi(M^3, *)$, for any base point $*$ in W_i, is finitely generated and therefore, by Scott's theorem [25] it is a compact-three-manifold group. Since it does not contain F_2, it belongs to $vsolv_3$ (Proposition 3 and [10] Proposition 2). Hence W_i is $vsolv_3$ - contractible. $\square$

Corollary 1. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ contain $vsolv_3$ and are contained in $hunf$, then $cat_{\mathcal{G}_1}(M) = cat_{\mathcal{G}_2}(M)$ for any compact 3-manifold M.*

This follows from Proposition 2, since $cat_{hunf}(M) \leq cat_{\mathcal{G}_i}(M) \leq cat_{solv_3}(M) = cat_{hunf}(M)$, $(i = 1, 2)$.

3 The dipus, quadripus, bipod, tetrapod, octopod and their groups.

We use the following notations:

The manifold that is obtained from a 3-manifold W by filling in all boundary spheres with 3-balls is denoted by $\hat{W}$.

$T\tilde{\times}I$, $K\tilde{\times}I$, $S^1\tilde{\times}D^2$, $S^1\tilde{\times}S^2$ denote, resp., an I-bundle over the torus, an I-bundle over the Klein bottle K, a D^2-bundle over S^1, an S^2-bundle over S^1. The bundles may be trivial (i.e. product bundles) or non-trivial. In particular, by $(K\tilde{\times}I)_0$ we denote the unique orientable I-bundle over K.

A *twisted double* of $(K\tilde{\times}I)_0$ is a closed 3-manifold obtained by gluing two copies of $(K\tilde{\times}I)_0$ along their boundary components.

The *geminus* M_G is the disk sum of two copies of $(P^2\times I)$:
$M_G = (P^2\times I)\#_b(P^2\times I)$. The boundary ∂M_G consists of 2 projective planes and a Klein bottle.

The *dipus* M_D is obtained from the geminus $M_G = (P^2\times I)\#_b(P^2\times I)$ and the solid Klein bottle $m\times I$ (where m is the Moebius band) by gluing a non-sparating annulus A_1 in the Klein bottle boundary of M_G to the incompressible annulus $A_2 = \partial m \times I$. The boundary ∂M_D consists of 2 projective planes and an incompressible Klein bottle $\partial_K M_D$.

The *quadripus* M_Q is the orbit manifold of $M = S^1\times S^1\times I$ under the orientation-reversing involution $\tau(z_1, z_2, t) = (\bar{z_1}, \bar{z_2}, 1-t)$ with the interiors of invariant 3-ball neighborhoods of the four fixed points removed (see [17], [16]). Its boundary consists of 4 projective planes and one incompressible torus. Any self-homeomorphism of the torus boundary T_0 of the quadripus M_Q extends to a homeomorphism of M_Q.

The *octopod* $M_O = M_Q \cup_{T_0} M_Q$ is the union of two copies of M_Q along the torus boundary. Its boundary consists of 8 projective planes.

The *tetrapod* M_T is the union of two copies of the dipus along the Kleinbottle boundary. Its boundary consists of 4 projective planes. The tetrapod may also be viewed as $M_Q \cup_{T_0} T\tilde{\times}I$ (where $T\tilde{\times}I$ is the non-orientable twisted I-bundle) and as $M_Q \cup_{T_0} (K\tilde{\times}I)_0$.

The *bipod* $M_B = M_D \cup (K\tilde{\times}I)$ is the union of the dipus M_D and the non-orientable I-bundle over the Kleinbottle $K\tilde{\times}I$, along the Kleinbottle boundaries: $M_D \cap (K\tilde{\times}I) = \partial_K M_D = \partial(K\tilde{\times}I)$. (This construction of M_B is independent of the gluing homeomorphism $\partial_K M_D \to \partial(K\tilde{\times}I)$).

We now consider the collection $\mathcal{M}_{\mathcal{S}}$ of these manifolds, together with torus and Kleinbottle bundles over S^1:

Definition 3. $\mathcal{M}_{\mathcal{S}}$ denotes the set of 3-manifolds listed in (1)-(6) below:
(1) A torus bundle or Kleinbottle bundle over S^1.

(2) A twisted double of $(K\tilde{\times}I)_0$.
(3) $T\tilde{\times}I$, $K\tilde{\times}I$, $S^1\tilde{\times}D^2$.
(4) $P^2\times I$, $(P^2\times I)\#P^3$, $(P^2\times I)\#(P^2\times I)$.
(5) The quadripus M_Q, dipus M_D, geminus M_G.
(6) The octopod M_O, tetrapod M_T, bipod M_B.

$\mathcal{M}_0$ denotes the set of closed Seifert manifolds with non-negative orbifold Euler characteristic.

Here the orbifold Euler characteristic of a Seifert manifold is $\chi(S)-\sum_{i=1}^{k}(1-1/\alpha_i)$, where $\chi(S)$ is the usual Euler characteristic of the orbit surface S and the α_i are the multiplicities of the exceptional fibers.

By the Proposition below, proved in [10] (Proposition 3), $\mathcal{M}_{\mathcal{S}}\cup\mathcal{M}_0$ consists precisely of all the compact (connected) 3-manifolds whose fundamental groups belong to $hunf$:

Proposition 3. *Let W be a compact connected 3-manifold. Then $\pi_1(W)$ does not contain F_2 if and only if $\hat{W}$ belongs to $\mathcal{M}_0$ or $\mathcal{M}_{\mathcal{S}}$.*

The Kleinbottle bundles in case (1) of $\mathcal{M}_S$ and the non-orientable twisted doubles of $K\tilde{\times}I$ appear also in $\mathcal{M}_0$. As pointed out in [10], all fundamental groups of the manifolds in $\mathcal{M}_{\mathcal{S}}\cup\mathcal{M}_0$ are solvable with the exception of those Seifert manifolds in $\mathcal{M}_0$ that are covered by the dodecahedral manifold, which are the finite groups $SL(2,5)\times Z_m$, with $gcd(m,30)=1$.

We now consider the structure of the fundamental groups of the manifolds in $\mathcal{M}_{\mathcal{S}}$.

Starting with the natural presentation $\langle a_1,a_2\,:\,a_1^2=a_2^2=1\rangle$ of $\pi(M_G)$, we see that $\pi(M_G)$ has a unique infinite cyclic subgroup of index 2, namely $\langle a_1a_2\rangle$.

For the fundamental group of the dipus $M_D=M_G\cup(m\times I)$ we obtain the presentation presentation $\pi(M_D)=\langle a_1,a_2,b\,:\,a_1^2=a_2^2=1,\,a_1a_2=b^2\rangle$. With $a=a_1$ the subgroup corresponding to the Klein bottle boundary is generated by $(ba)^2$ and b and

$$\pi(M_D)=\langle a,b\,:\,a^2=1,a^{-1}b^2a=b^{-2}\rangle,\text{ with }\pi(\partial_K(M_D))=\langle(ba)^2,b\rangle.$$

The fundamental groups of the dipus and the bipod are characterized as follows:

Proposition 4. *(1) The fundamental group of M_D is the unique free product with amalgamation $\mathbb{Z}*_{\mathbb{Z}}(\mathbb{Z}_2*\mathbb{Z}_2)$, where the index of the amalgamating group $\mathbb{Z}$ in both factors is 2.*
(2) The fundamental group of M_B is the unique free product with amalgamation $\pi(M_D)*_{\mathrm{K}_1}\mathrm{K}$, where K, K_1 are Kleinbottle groups and the index of the amalgamating group K_1 in both factors is 2.

Proof. (1) If H is a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a_1, a_2 \,:\, a_1^2 = a_2^2 = 1\rangle$ not containing elements of finite order, then a_1 and a_2 do not belong to H, but if H has index 2, then $a_1a_2 \in H$, hence $(\mathbb{Z}_2 * \mathbb{Z}_2)/H \subset (\mathbb{Z}_2 * \mathbb{Z}_2)/\langle a_1a_2\rangle = \mathbb{Z}_2$. So $H = \langle a_1a_2\rangle$ is the unique infinite cyclic subgroup of index 2.

(2) Recalling that $M_B = M_D \cup (K\tilde{\times}I)$, it suffices to show that $G := \pi(M_D)$ (resp. K) contains a unique (up to equivalence) Kleinbottle group of index 2, i.e. if K_1, K_2 are Kleinbottle subgroups of G (resp. K), then there is an automorphism $\varphi : G \to G$ with $\varphi(K_1) = K_2$.

(i) First note that $G = \langle a, b \,:\, a^2 = 1, a^{-1}b^2a = b^{-2}\rangle$ has exactly three subgroups of index 2: If H is such a subgroup and $a \in H$, then $b \notin H$, but $b^2 \in H$. So $\mathbb{Z}_2 = G/H \subset G/\langle a, b^2\rangle = \mathbb{Z}_2$, and it follows that $H = \langle a, b^2\rangle$. Similarly, if $b \in H$ then $a \notin H$ and it follows that $H =< b >$ (the subgroup normally generated by b). Finally, if $a \notin H$, $b \notin H$, then $ba \in H$ and $H =< ba >$.

Note also that the first H is not a Kleinbottle group, (since a has order 2), but $K_1 :=< ba >= \langle b^2, ba\rangle$ is a normal subgroup of index 2 in G that is isomorphic to the Klein bottle group, where $(ba)^{-1}b^2(ba) = b^{-2}$. We show that $K_2 :=< b >$ is a Kleinbottle group equivalent to K_1.

From $a^{-1}b^2a = b^{-2}$ it follows that $b^2ab^2a = 1$, hence $bab = b^{-1}ab^{-1}$, and $abab = ab^{-1}ab^{-1}$. Then $a^{-1}(ba)^2a = ababaa = abab = ab^{-1}ab^{-1} = (ba)^{-2}$. Hence $\varphi : a \to a, b \to ba$ defines a homomorphism from G to G and, since φ^2 =identity, φ is an automorphism.

So $\phi(\langle b^2, ba\rangle) = \langle \phi(b^2), \phi(ba)\rangle = \langle (ba)^2, b\rangle = K_2$ is also a normal subgroup of index 2 isomorphic to the Kleinbottle group, where $b^{-1}(ba)^2b = (ba)^{-2}$.

(ii) As above we see that the Kleinbottle group $\mathrm{K} = \langle x, y : y^{-1}xy = x^{-1}\rangle$ has exactly three subgroups of index 2. Now, $\mathrm{K}_1 := \langle y, x^2\rangle$ is a normal subgroup of index 2 isomorphic to the Kleinbottle group, where $y^{-1}x^2y = x^{-2}$. Let $\varphi : x \to x^{-1}, y \to xy$. Since $(xy)^{-1}x^{-1}(xy) = x = (x^{-1})^{-1}$, φ defines a homomorphism from K to K and, since φ^2 =identity, φ is an automorphism.

So $K_2 := \varphi(\langle y, x^2\rangle) = \langle \phi(y), \phi(x^2)\rangle = \langle xy, x^{-2}\rangle$ is also a normal subgroup of index 2 isomorphic to the Kleinbottle group, where $(xy)^{-1}x^{-2} = x^2$.

The third subgroup is $< x, y^2 >$ which is abelian and therefore not a Kleinbottle group. □

We now characterize the fundamental groups of the manifolds in $\mathcal{M}_{\mathcal{S}}$.

Definition 4. $\mathcal{G}_S$ denotes the set of groups consisting of the trivial group and the fundamental groups of the manifolds in $\mathcal{M}_{\mathcal{S}}$.

Example 1. *hunf* $\supset$ *ame* $\supset$ *solv* $\supset \mathcal{G}_S$.

Example 2. If C_1, C_2 are as in cases (3) or (4) of $\mathcal{M}_S$ and $C = C_1 \cup_F C_2$ is a union along an incompressible torus or Klein bottle boundary component F, then $C \in \mathcal{M}_S$, hence $\pi_1(C) \in \mathcal{G}_S$.

For if $C_i = T\tilde{\times}I$ or $K\tilde{\times}I$ we may assume that these are not product bundles. Since C_i is irreducible and F is incompressible and $\pi(C)$ is solvable, Theorem 4.5 of [6] applies to show that C is a torus or Klein bottle bundle over S^1 or a

twisted double of $(K\tilde{\times}I)_o$. Hence $C \in \mathcal{M}_S$.
The remaining cases are (remembering that ∂C_i is incompressible):
$T\tilde{\times}I \cup_F M_Q = K\tilde{\times}I \cup_F M_Q = M_T$, $M_Q \cup_F M_Q = M_O$, $K\tilde{\times}I \cup M_D = M_B$, $M_D \cup_F M_D = M_T$. All these manifolds belong to $\mathcal{M}_S$.

Definition 5. (a) *polyZ*(n) consists of all groups G that have a normal series with infinite cyclic quotients and of length at most n, i.e. $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$ where each $G_{i+1}/G_i \cong \mathbb{Z}$ and $m \leq n$.
(b) Denote by $N \rtimes H$ a semi-direct product of H acting on N, that is an extension of the group N with quotient H and let
spolyc $= \{N \rtimes H \mid N \in polyZ(3),\ H$ a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2\} - \{\mathbb{Z} \times \mathbb{Z}_2\}$.
(c) $spolyc_3 = spolyc \cap \{3\text{−manifold groups}\}$.

Note that every group in *spolyc* is finitely generated and so the groups of $spolyc_3$ are fundamental groups of *compact* 3-manifolds.

Lemma 2. *$spolyc_3$ is closed under subgroups.*

Proof. It is easy to see that extensions of *polyZ*(3)-groups by a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are closed under subgroups. Since the set of 3-manifold groups is closed under subgroups, it follows that the intersection $\mathcal{I}$ of these two classes is closed under subgroups. By Epstein ([5], Theorem 9.5), any 3-manifold group containing properly $\mathbb{Z}\times\mathbb{Z}_2$ is a free product $(Z\times Z_2)*H$ (for some group $H \neq 1$). This group is not solvable and therefore does not belong to $\mathcal{I}$. It follows that $spolyc_3 = \mathcal{I} - \{\mathbb{Z} \times \mathbb{Z}_2\}$ is still subgroup closed. □

Proposition 5. $\mathcal{G}_S = spolyc_3$.

Proof. We first show that the groups (1)-(6) of $\mathcal{G}_S$ belong to *spolyc*.

(1) Clearly these groups are in *polyZ*(3).
(2) There is a 2-sheeted covering of a torus bundle over S^1 over M=twisted double of $(K\tilde{\times}I)_0$. Hence $\pi(M) = N \rtimes \mathbb{Z}_2$, for $N \in polyZ(3)$.
(3) Clearly these groups are in *polyZ*(3).
(4) $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1\rangle = \mathbb{Z} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}$ is the (normal) subgroup generated by ab.
(5) The 2-sheeted orientable cover of M_Q (resp. M_D) is a 4-times (resp. 2-times) punctured $S^1\times S^1\times I$ (resp. $(K\tilde{\times}I)_0$).
Hence $\pi(M_Q) = (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}_2$ and $\pi(M_D) = \mathrm{K} \rtimes \mathbb{Z}_2$ (K=Kleinbottle group).
(6) The 2-sheeted orientable cover of M_O and M_T is a punctured torus-bundle over S^1, hence $\pi(M_O)$ and $\pi(M_T) \cong N \rtimes \mathbb{Z}_2$, for $N \in polyZ(3)$.
The 2-sheeted orientable cover of M_B is a punctured twisted double of $(K\tilde{\times}I)_0$, which is covered by a punctured torus bundle over S^1 as in (2). So M_B is covered by a punctured torus bundle over S^1 with group of covering transformation $\mathbb{Z}_2 \times \mathbb{Z}_2$ and it follows that $\pi(M_B) = N \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, with $N \in polyZ(3)$.

We now show that $spolyc_3 \subset \mathcal{G}_S$.

Since $spolyc_3 \subset hunf$, Proposition 3 implies that a group in $spolyc_3$ is the fundamental group of a manifold in $\mathcal{M}_0$ or $\mathcal{M}_{\mathcal{S}}$, and it suffices to show that any member of $spolyc_3$ with fundamental group of a manifold in $\mathcal{M}_0$ belongs to $\mathcal{G}_{\mathcal{S}}$. So suppose that $G = \pi_1(M) \in spolyc_3$ where $M \in \mathcal{M}_0$.

If $M \in \mathcal{M}_0$ is a fiber bundle over S^1 then the fiber is a torus, a Kleinbottle, P^2, or S^2. Since $\mathbb{Z} \times \mathbb{Z}_2 \notin spolyc_3$, $G = \pi_1(M^3)$ for M as in (1) or (3) of $\mathcal{M}_{\mathcal{S}}$.

By [23] (7.2) or [27], every $M \in \mathcal{M}_0$ fibers over S^1 with fiber a torus, a Kleinbottle, P^2, or S^2, except in the following cases:

(a) $\pi_1(M)$ is finite.

(b) M has orbit surface S^2 and non-zero Euler number $e = -(b+\sum_{i=1}^{k} \beta_i/\alpha_i)$.

(c) $M = \{b; (n_2, 1); (2, 1), (2, 1)\}$.

We first note (*): Except for the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$, every member of *spolyc* has a subgroup of index ≤ 4 which is $polyZ(3)$ of positive length and therefore has infinite abelianization.

Case (a): It follows from (*) that $\pi_1(M)$ is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a 3-manifold group, $G = 1$ or $G = \mathbb{Z}_2 \in \mathcal{G}_{\mathcal{S}}$.

Case (b): Seifert's argument in his proof of Theorem 12 in [26] shows that if the orbit surface of M is S^2, then $H_1(M)$ is finite if and only if the Euler number $e(M) \neq 0$. If $e(M) \neq 0$, then by Scott ([25], Theorem 3.6), every finite cover $\tilde{M}$ of M has $e(\tilde{M}) \neq 0$ and hence (since $\tilde{M}$ also has orbit surface S^2) finite homology group. As in cases (a) it follows from (*) that $G = 1$ or $G = \mathbb{Z}_2$.

Case (c) Let $\tilde{M} = \{2b; (o_1, 0); (2, 1), (2, 1), (2, 1), (2, 1)\}$ be the 2-sheeted orientable cover of M. If $b \neq -1$ then $\tilde{M}$ is as in case (b), so if $\pi_1(M) \in spolyc$ then $\pi_1(\tilde{M}) = 1$ or $\mathbb{Z}_2$, which is not true. If $b = -1$ then M contains a horizontal Klein bottle that splits M into two $(K\tilde{\times}I)_0$'s (M is the manifold M_6 in [13] p. 30), and so M is as in (2) of $\mathcal{M}_{\mathcal{S}}$. $\square$

Another description of $\mathcal{G}_{\mathcal{S}}$ is as the following class *spec*. (Here K or K_i denotes the Kleinbottle group).

Definition 6. The special class *spec* is the class of subgroups of the following:
(a) $\pi_1(M_B)$, the fundamental group of the bipod
(b) all semidirect products $\mathrm{K} \rtimes \mathbb{Z}$ and $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$
(c) all free products with amalgamation $\mathrm{K}_1 *_{\mathbb{Z}\times\mathbb{Z}} \mathrm{K}_2$ where $[\mathrm{K}_i : \mathbb{Z} \times \mathbb{Z}] = 2$, $(i = 1, 2)$.

Proposition 6. $\mathcal{G}_S = spec$.

Proof. The first part of the proof of Proposition 5 shows that $\mathcal{G}_S \subset spec$.

To show that $spec \subset \mathcal{G}_S$, note that $\pi_1(M_B) \in \mathcal{G}_S$. Every semidirect product $\mathrm{K} \rtimes \mathbb{Z}$ and $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$ can be realized as the fundamental group of a Kleinbottle bundle, resp. torus bundle, over S^1 and so is as in case (1) of $\mathcal{G}_S$. Every free product with amalgamation $\mathrm{K}_1 *_{\mathbb{Z}\times\mathbb{Z}} \mathrm{K}_2$ with $[\mathrm{K}_i : \mathbb{Z} \times \mathbb{Z}] = 2$ can be realized as the fundamental group of a union of two twisted I-bundles over the Kleinbottle and is as in case (2) of $\mathcal{G}_S$. $\square$

4 $Cat_{\mathcal{G}}(M^3) \leq 2$

The next two lemmas were proved in [10] for the class of amenable groups. A similar proof applies to more general classes $\mathcal{G}$ of groups.

Lemma 3. *Let $M = M_1 \# M_2$ be a connected sum of 3-manifolds and $k \geq 2$.*

(a) If $cat_{\mathcal{G}}(M_i) \leq k$ for $i = 1, 2$ then $cat_{\mathcal{G}}(M) \leq k$.

(b) If $\mathcal{G}$ is closed under subgroups and $cat_{\mathcal{G}}(M) \leq k$, then $cat_{\mathcal{G}}(M_i) \leq k$ for $i = 1, 2$.

Proof. There are 3-balls $B_i \subset M_i$ so that $M = M_1' \cup M_2'$, where $M_i' = M_i - intB_i$ and $M_1' \cap M_2' = \partial B_1 = \partial B_2$.

Suppose $cat_{\mathcal{G}}(M_i) \leq k$ for $i = 1, 2$. Deleting a ball from an open $\mathcal{G}$-contractible subset does not change $\mathcal{G}$-contractibility, so we may assume $M_i = W_{i1} \cup \cdots \cup W_{ik}$ is a $\mathcal{G}$-contractible cover such that $B_i \subset W_{ii}$ and $B_i \cap W_{ij} = \emptyset$ for $j \neq i$ and $B_1 \cap \overline{W}_{12} = \emptyset$. Note that $W_{11} - intB_1$ is $\mathcal{G}$-contractible in $M_1 - intB_1$ and therefore in M. Let N be an open product neighborhood of ∂B_1 in $W_{11} - intB_1$ with $N \cap W_{12} = \emptyset$. Then $M = W_1 \cup \cdots \cup W_k$, where $W_1 = (W_{11} - B_1) \cup W_{21}$, $W_2 = W_{12} \cup (W_{22} - B_2) \cup N$, $W_j = W_{1j} \cup W_{2j}$ for $3 \leq j \leq k$ are $\mathcal{G}$-contractible in M. (Note that the assumption that $\mathcal{G}$ is closed under subgroups is not used here).

Now suppose that $cat_{\mathcal{G}}(M) \leq k$ and let $M = W_1 \cup \cdots \cup W_k$ be a cover by $\mathcal{G}$-contractible sets. Consider M_1' and let $V_i = W_i \cap M_1'$. For a fixed index i let $V_i \xrightarrow{\iota_1} M_1' \xrightarrow{j} M$ and $V_i \xrightarrow{\kappa} W_1 \xrightarrow{\iota} M$ be the inclusions. Then $ji_1 = \iota\kappa$ and $(ji_1)_*\pi(V_i, *) \subset \iota_*\pi(W_i, \kappa(*)) \in \mathcal{G}$, for any basepoint $* \in V_i$. Since j_* is injective and $\mathcal{G}$ is closed under subgroups, $\iota_{1*}\pi(V_i) \cong (ji_1)_*\pi(V_i, *) \in \mathcal{G}$. Hence $M_1' = V_1 \cup \cdots \cup V_k$ is a cover by $\mathcal{G}$-contractible subsets. $\square$

For the case that $\mathrm{cat}_{\mathcal{G}} M^n = 2$ we may choose two compact $\mathcal{G}$-contractible submanifolds that intersect along their boundaries:

Lemma 4. *Let $\mathcal{G}$ be closed under subgroups and let M be a closed n-manifold. Then $cat_{\mathcal{G}}(M) \leq 2$ if and only if there are compact $\mathcal{G}$-contractible n-submanifolds W_i of M so that $M = W_1 \cup W_2$ and $W_1 \cap W_2 = \partial W_1 = \partial W_2$.*

Proof. If $cat_{\mathcal{G}}(M) \leq 2$ there are open $\mathcal{G}$-contractible subsets U_0 and U_1 of M whose union is M. By Lemma 1 of [8], there exist compact n-submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^n$, $W_0 \cap W_1 = \partial W_0 = \partial W_1$ and $W_i \subset U_i$ $(i = 0, 1)$. Since $\mathcal{G}$ is closed under subgroups, W_i is $\mathcal{G}$-contractible. $\square$

For a *surface* F in a 3-manifold M, we assume that F is properly embedded, but not necessarily connected, with a regular neighborhood $N(F) \approx F \times I$.

For a closed connected surface F define the complexity $c(F) = 2g(F) - 1$, where $g(F)$ is its (orientable or non-orientable) genus. If F is not connected, we define $c(F)$ to be the sum of the complexities of the components of F.

Lemma 5. *Let $\mathcal{G}$ be closed under subgroups and let M be a closed 3-manifold with $\mathrm{cat}_{\mathcal{G}} M \leq 2$. Then there is a closed surface F in M such that F and*

$\overline{M - N(F)}$ *are* $\mathcal{G}$*-contractible and every component of* F *is a* 2*-sphere or incompressible.*

Proof. We write $M = W_1 \cup W_2$ as in Lemma 4, with $F = W_1 \cap W_2$. For each component F' of F, $im(\pi(F') \to \pi(M))$ is contained in $im(\pi(W_i') \to \pi(M))$, where W_i' is a component of W_i, and it follows that F and $\overline{M - N(F)}$ are $\mathcal{G}$-contractible. Now assume that F is a closed surface in M of minimal complexity such that F and $\overline{M - N(F)}$ are $\mathcal{G}$-contractible.

If a non-sphere component F' of F is not incompressible, let D be a compressing disk for F'. Let $D \times I$ be a regular neighborhood such that $(D \times I) \cap F = \partial D \times I$ and $\partial D \times 0$ is an essential curve in F'. For the component F_1' of $F_1 = (F - \partial D \times I) \cup (D \times \partial I)$ that contains $D \times \{0\}$ or $D \times \{1\}$, $\mathrm{im}(\pi(F_1') \to \pi(M))$ is a subgroup of $\mathrm{im}(\pi(F') \to \pi(M))$. Since F' is $\mathcal{G}$-contractible, so is F_1.

Furthermore, if M' is the component of $\overline{M - N(F)}$ that contains F' but not D and if M_1' is the component $M' \cup D \times I$ of $\overline{M - N(F_1)}$, then $\pi(M')$ and $\pi(M_1')$ have the same image in $\pi(M)$. Since M' is $\mathcal{G}$-contractible, so is M_1'.

Hence F_1 and $\overline{M - N(F_1)}$ are $\mathcal{G}$-contractible and $c(F_1) < c(F)$, a contradiction. $\square$

Theorem 1. *Suppose that* $\mathcal{G}$ *is closed under subgroups and* $hunf \supset \mathcal{G} \supset spec$*. If* M *is a closed* 3*-manifold with* $cat_{\mathcal{G}}(M) \leq 2$*, then there is a disjoint collection* F *of* 2*-spheres and projective planes in* M *such that every component* C *of* $\overline{M - N(F)}$ *has fundamental group in* $\mathcal{G}$ *and* $\hat{C}$ *is either a closed manifold or* $\hat{C}$ *is a bipod, tetrapod, or octopod.*

Proof. By Lemma 5, there is a closed surface F of minimal complexity in M such that F and $\overline{M - N(F)}$ are $\mathcal{G}$-contractible and every component of F is a 2-sphere or incompressible. It follows that for every component E' of $E = \overline{M - F \times [0,1]}$ the inclusion $E' \to M$ is π_1- injective (e.g. [11] Lemma 2.2). Since E is $\mathcal{G}$-contractible and $\mathcal{G} \subset hunf$, it follows that the components of E have fundamental groups belonging to $\mathcal{G}$ and so are as in Proposition 3. Furthermore, since for each component F' of F, the inclusion $F' \to E$ is π_1-injective, all non-sphere and non-projective plane components of F have non-negative Euler characteristic.

Now suppose a component F_0 of F is a torus or Klein bottle. Let $F_0 \times [0,1]$ be the component of $F \times [0,1]$ containing F_0. If a component C of E contains $\partial(F_0 \times [0,1])$ then $\hat{C}$ is homeomorphic to $F_0 \times I$ (only case (3) of $\mathcal{M}_{\mathcal{S}}$ applies) so $C \cup (F_0 \times [0,1])$ is a punctured F_0-bundle over S^1 and its fundamental group belongs to $spec \subset \mathcal{G}$; hence $F - F_0$ and its complement in M are $\mathcal{G}$-contractible and $c(F - F_0) < c(F)$, contradicting the minimality of $c(F)$.

If no component of E contains $\partial(F_0 \times [0,1])$, let C_1, C_2 be the two components of E intersecting $F_0 \times [0,1]$. Note that C_i is not a trivial I-bundle because of the minimal complexity condition. Then $C_1 \cup F_0 \times [0,1] \cup C_2$ is as in Example 1, i.e. $\mathcal{G}$-contractible. So again $F - F_0$ and its complement are $\mathcal{G}$-contractible with $c(F - F_0) < c(F)$, a contradiction.

Thus the boundary of every component C of $\overline{M - N(F)}$ consists of 2-spheres and projective planes and C has fundamental group belonging to $\mathcal{G}$.

If $\partial(\hat{C}) \neq \emptyset$ then $\hat{C}$ is as in cases (5) and (6) of $\mathcal{M}_{\mathcal{S}}$. If $\hat{C} = P^2 \times I$, $(P^2 \times I)\#(P^2 \times I)$ or $(P^2 \times I)\#P^3$, let P be a projective plane component of F parallel to a boundary component of C, and (if $C \neq P^2 \times I$) let S be a 2-sphere in C splitting it into two punctured copies of $P^2 \times I$ (resp. into a punctured $P^2 \times I$ and a punctured P^3) (that is, the 2-sphere used for the connected sum #). Then $F_1 = (F - P) \cup S$ and $M - F_1$ are $\mathcal{G}$-contractible and $c(F_1) < c(F)$, a contradiction. Hence $\hat{C}$ is a bipod, tetrapod, or octopod. □

A compact prime 3-manifold contains a *complete* system (possibly empty) $\mathcal{P}$ of projective planes of M, unique up to isotopy, such that no two projective planes of $\mathcal{P}$ are parallel in M and every projective plane in $\overline{M - N(\mathcal{P})}$ is parallel (in M) to a component of $\mathcal{P}$ (see [22]).

Corollary 2. *Suppose that $\mathcal{G}$ is closed under subgroups and $hunf \supset \mathcal{G} \supset spec$. Let M be a closed 3-manifold. Then $cat_{\mathcal{G}}(M) \leq 2$ if and only if $M = M_1 \# M_2 \# \ldots \# M_m$, where for each $1 \leq i \leq m$, either*

(a) $\pi_1(M_i) \in \mathcal{G}$, or

(b) The complete system $\mathcal{P}_i$ of projective planes of M_i is non-empty and every component of $\overline{M_i - N(\mathcal{P}_i)}$ is a bipod, tetrapod, or octopod.

Proof. If $cat_{\mathcal{G}}(M) \leq 2$ then this follows from Theorem 1. Conversely, if M is as in the Corollary, let W_1 be a regular neighborhood of the union of the connected sum 2-spheres and the projective planes, and let $W_2 = \overline{M - W_1}$. Then W_1 and W_2 are $\mathcal{G}$-contractible. □

We note in particular the case when M is prime:

Corollary 3. *Suppose that $\mathcal{G}$ is closed under subgroups and $hunf \supset \mathcal{G} \supset spec$. Let M be a closed prime 3-manifold. Then $cat_{\mathcal{G}}(M) = 2$ if and only if M is non-orientable with non-empty complete system $\mathcal{P}$ of projective planes and every component of $\overline{M - N(\mathcal{P})}$ is a bipod, tetrapod, or octopod.*

Proof. Since $cat_{\mathcal{G}}(M) \neq 1$, i.e. $\pi(M) \notin \mathcal{G}$, the result follows from Corollary 2. □

As an application we obtain the result that *there are no closed, orientable, prime 3-manifolds with $cat_{solv}(M) = 2$.*

5 $Cat_{\mathcal{G}}(M^3) \leq 3$

In this section we classify the closed 3-manifolds M with $cat_{\mathcal{G}}(M) \leq 3$, for some classes of groups $\mathcal{G} \subset hunf$. The classes that we conisider are, besides $\mathcal{G}_S$, $solv$, $abel$, the following:

$\mathcal{G}_T = \{cyclic\, groups,\ \mathbb{Z} \times \mathbb{Z}\}$
$\mathcal{G}_K = \{cyclic\, groups,\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}_2 * \mathbb{Z}_2,\ \mathrm{K}\}$

A *graph manifold* is a closed 3-manifold that contains a disjoint collection (possibly empty) of 2-sided incompressible tori and Klein bottles which splits M into Seifert fiber spaces.

Example 3. Let M be a Seifert fiber space with non-empty boundary. Then $M = W_1 \cup W_2$, where W_1 is a fibered solid torus, each component of W_2 is a fibered solid torus and each component of $W_1 \cap W_2$ is a fibered annulus (see e.g. [10] proof of Lemma 8). If $\mathcal{G}$ is a class of groups that contains $\mathbb{Z}$, then W_i is $\mathcal{G}$-contractible and hence $cat_{\mathcal{G}}(M) \leq 2$; in particular $cat_{\mathcal{G}_T}(M) \leq 2$ and $cat_{\mathcal{G}_K}(M) \leq 2$.

Example 4. Let M be a closed Seifert fiber space. Let W_3 be a trivially fibered solid torus in M. Then $\overline{M - intW_3} = W_1 \cup W_2$, as in Example 3. If $\mathcal{G}$ is a class of groups that contains the cyclic groups, then W_i is $\mathcal{G}$-contractible and hence $cat_{\mathcal{G}}(M) \leq 3$; in particular $cat_{\mathcal{G}_T}(M) \leq 3$ and $cat_{\mathcal{G}_K}(M) \leq 3$.

Example 5. Let M be a graph manifold that contains a non-empty collection of 2-sided incompressible tori or Kleinbottles that splits M into Seifert fiber spaces, each with non-empty boundary. Let W_3 be a regular neighborhood (in M) of this collection of tori and Kleinbottles, let $\overline{M - W_3} = M_1 \cup M_2 \cup \cdots \cup M_n$, and let $M_i = W_{i1} \cup W_{i2}$, as in Example 3, so that for $W_1 = \bigcup_i W_{i1}$, $W_2 = \bigcup_i W_{i2}$ we have $M = W_1 \cup W_2 \cup W_3$. If $\mathcal{G}$ is a class of groups that contains $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, and K, then each W_i is $\mathcal{G}$-contractible, hence $cat_{\mathcal{G}}(M) \leq 3$; in particular $cat_{\mathcal{G}_K}(M) \leq 3$. If M is orientable, then there are no incompressible Kleinbottles and $cat_{\mathcal{G}_T}(M) \leq 3$.

For easy reference we state the following theorem, which is a combination of theorems 4 and 5 of [10].

Theorem 2. *Let M be a closed 3-manifold M.*
(a) If M is orientable, then $cat_{ame}M \leq 3$ if and only if M is a connected sum of graph manifolds.
(b) If M is nonorientable and $\tilde{M}$ is its orientable 2-fold cover, then the following are equivalent :
i) $cat_{ame}(M) \leq 3$
ii) $cat_{ame}(\tilde{M}) \leq 3$
iii) $\tilde{M}$ is a connected sum of graph manifolds
iv) M contains a disjoint collection F of 2-spheres, projective planes, tori, and Kleinbottles (all 2-sided), such that every component of $\overline{M - N(F)}$ is a punctured S^1-bundle or geminus.

Theorem 3. *Assume that $hunf \supset \mathcal{G} \supset \mathcal{G}_T$ and let M be a closed orientable 3-manifold. Then $cat_{\mathcal{G}}(M) \leq 3$ if and only if M is a connected sum of graph manifolds.*

Proof. By Poposition 2 we have $cat_{ame}(M) = cat_{hunf}(M) \leq cat_{\mathcal{G}}(M) \leq 3$ and it follows from Theorem 2(a) that M is a connected sum of graph manifolds.

Conversely, if $M = M_1 \# \ldots \# M_n$, where each M_i is a graph manifold, then from Example 5, $cat_{\mathcal{G}}(M_i) \leq cat_{\mathcal{G}_T}(M_i) \leq 3$ and by Lemma 3(a), $cat_{\mathcal{G}}(M) \leq 3$. $\square$

Corollary 4. *let M be a closed orientable 3-manifold. Then $cat_{abel}(M) \leq 3$ if and only if $cat_{solv}(M) \leq 3$ if and only if M is a connected sum of graph manifolds.*

Proof. This follows since $hunf \supset solv \supset abel \supset \mathcal{G}_T$. $\square$

We now consider the case that M is non-orientable.

Lemma 6. *Assume $\mathcal{G}$ is closed under subgroups. Let $p : \tilde{M} \to M$ be a covering map. Then $cat_{\mathcal{G}}(\tilde{M}) \leq cat_{\mathcal{G}}(M)$.*

Proof. We show that if W is $\mathcal{G}$-contractible in M then $\tilde{W} = p^{-1}(W)$ is $\mathcal{G}$-contractible in $\tilde{M}$. Assume W, $\tilde{W}$ are connected (otherwise we look at components). Let $\iota : W \to M$ and $\tilde{\iota} : \tilde{W} \to \tilde{M}$ be the inclusions and let $p' : \tilde{W} \to W$ be the restriction of p to $\tilde{W}$. Then $p_* \tilde{\iota}_*(\pi(\tilde{W})) = \iota_* p'_*(\pi(\tilde{W}))$ is a subgroup of $\iota_*(\pi(W))$, which is $\mathcal{G}$-contractible. Now $p_* : \tilde{\iota}_*(\pi(\tilde{W})) \to p_* \tilde{\iota}_*(\pi(\tilde{W}))$ is an isomorphism, hence $\tilde{\iota}_*(\pi(\tilde{W}))$ is $\mathcal{G}$-contractible. $\square$

Theorem 4. *Assume that $hunf \supset \mathcal{G} \supset \mathcal{G}_K$, let M be a closed non-orientable 3-manifold and let $\tilde{M}$ be its orientable 2-fold cover. The following are equivalent:*
(i) $cat_{\mathcal{G}}(M) \leq 3$
(ii) $cat_{\mathcal{G}_K}(M) \leq 3$
(iii) M contains a disjoint collection F of 2-spheres, projective planes, tori, and Kleinbottles (all 2-sided), such that every component of $\overline{M - N(F)}$ is a punctured S^1-bundle or geminus.
(iv) $cat_{\mathcal{G}}(\tilde{M}) \leq 3$
(v) $cat_{\mathcal{G}_T}(\tilde{M}) \leq 3$
(vi) $\tilde{M}$ is a connected sum of graph manifolds.

Proof. $(i) \Rightarrow (vi)$: $cat_{hunf}(M) \leq cat_{\mathcal{G}}(M) \leq 3$, since $hunf \supset \mathcal{G}$. Since $hunf$ is closed under subgroups, $cat_{hunf}(\tilde{M}) \leq cat_{hunf}(M) \leq 3$, by Lemma 6. Now (vi) follows from Theorem 3, since $\mathcal{G} \supset \mathcal{G}_K \supset \mathcal{G}_T$.

$(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$, by Theorem 3, since $hunf \supset \mathcal{G} \supset \mathcal{G}_K \supset \mathcal{G}_T$.

$(iii) \Leftrightarrow (vi)$, by Theorem 2(b).

$(iii) \Rightarrow (ii)$: Let V_T (respectively V_S) be the union of the components of $\overline{M - N(F)}$ that are (respectively are not) gemini. There is an S^1-fibration $p : \hat{V}_S \to B$ where B is a compact 2-manifold.

For every component of B with empty boundary take an annulus embedded in it and let A be the union of these annuli. We may assume that $p^{-1}(A) \subset int\, V$. Let $W_1 = N(F) \cup p^{-1}(A)$.

Now, since every component of $\overline{B - A}$ has nonempty boundary we obtain a decomposition $\overline{B - A} = D \cup D'$ where D and D' are disjoint unions of disks and $D \cap D' = \partial D \cap \partial D'$. We may assume that $p(\overline{\hat{V} - V}) \subset int\, D'$. Let $W_2 = p^{-1}(D) \cup V_T$ and $W_3 = p^{-1}(D') \cup V_S$.

Then $M = W_1 \cup W_2 \cup W_3$. The components of W_1 are regular neighborhoods of 2-spheres, projective planes, tori or Klein bottles. The components of W_2 are solid tori, solid Klein bottles or gemini and the components of W_3 are punctured solid tori or solid Kleinbottles.

Let W be a component of W_i, $1 \leq i \leq 3$. Denote by $image(W)$ the $image(\pi_1(W,*) \to \pi_1(M,*))$, for some basepoint $* \in W$. If W is a solid torus, solid Kleinbottle, $S^2 \times I$, or $P^2 \times I$, then $image(W)$ is cyclic. If W is a neighborhood of an incompressible torus or Kleinbottle then the image is $\mathbb{Z} \times \mathbb{Z}$ or K. If W is a neighborhood of a compressible torus or Kleinbottle F, then for a compressing disk D of F the inclusion $F \to M$ factors through $F \to F \cup D \to M$. If F is a torus then $\pi_1(F \cup D) \approx \mathbb{Z}$, so $image(W) = image(\pi_1(F \cup D) \to \pi_1(M))$ is cyclic. All of these belong to $\mathcal{G}_K$.

If F is a Kleinbottle, then either $\pi_1(F \cup D) \approx \mathbb{Z}$ (if ∂D does not separate F), in which case $image(W)$ is cyclic, or $\pi_1(F \cup D) \approx \mathbb{Z}_2 * \mathbb{Z}_2$ (if ∂D separates F). In the latter case, as well as in the remaining case when W is a geminus, $image(W)$ is a 3-manifold group that is a quotient of $\mathbb{Z}_2 * \mathbb{Z}_2$ and we claim that it is either $\mathbb{Z}_2 * \mathbb{Z}_2$ or cyclic, hence belongs to $\mathcal{G}_K$.

First we show that a proper quotient of $\mathbb{Z}_2 * \mathbb{Z}_2$ is a finite dihedral group D_{2m}.

Present $\mathbb{Z}_2 * \mathbb{Z}_2$ as the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2 = \langle r, s : s^2 = 1, srs^{-1} = r^{-1} \rangle$ and let N be a nontrivial normal subgroup of $\mathbb{Z} \rtimes \mathbb{Z}_2$. If N is not contained in the infinite cyclic group $\langle r \rangle$ generated by r, then there is some integer m such that $\langle r^m s \rangle \subset N$, and $G/N \subseteq G/\langle r^m s \rangle = \mathbb{Z}_2$. If $N = \langle r^m \rangle$ for some integer $m \neq 0$ then G/N is the finite dihedral group of order $2m$.

Now, if N^3 is a 3-manifold with finite fundamental group D_{2m} then N^3 can be taken to be compact ([14], Thm 8.1) and ∂N^3 is a union of 2-spheres, so N^3 can be assumed closed. By Milnor [21], every element of order 2 of N^3 is central. However this is not true for D_{2m}, for $m > 2$. If $m = 2$, then it is well known that D_{2m}, the four group, is not a 3-manifold group (see, for example, [14], Thm 9.13).

$(ii) \Rightarrow (i)$, since $\mathcal{G} \supset \mathcal{G}_K$. $\square$

The manifolds in (iii) may described in more detail as follows. Suppose that M is irreducible.

Let $p : \tilde{M} \to M$ be the 2-fold covering, where $\tilde{M}$ is a connected sum of graph manifolds, and let $h : \tilde{M} \to \tilde{M}$ be the covering involution. By [18] there is a collection $\tilde{S}$ of disjoint h-invariant 2-spheres in $\tilde{M}$ with an h-invariant neighborhood $N(\tilde{S})$. Since M is irreducible, $h(S') = S'$, for every component S' of $\tilde{S}$ and $p(S')$ is a projective plane P in M. If P is one-sided, then $M = \mathbb{P}^3$ and $\pi_1(M) \in \mathcal{G}$. So assume that each projective plane in M is 2-sided.

Every component M' of $\overline{\tilde{M} - N(\tilde{S})}$ is an h-invariant punctured graph manifold and since M is irreducible, $h(M') = M'$. Extend h to an involution $\hat{h} : \hat{M}' \to \hat{M}'$, with fixed points of $\hat{h}$ the centers of ball components of $\hat{M}' - M'$.

(a) If $\hat{M}'$ is a torus bundle, then by [17] $p(M')$ is the tetrapod or octopod.

(b) If $\hat{M}'$ is not a torus bundle, then by [20] there is an $\hat{h}$-invariant disjoint collection T' of tori in $\hat{M}'$, such that the components of $\overline{\hat{M}' - N(T')}$ are Seifert fibered. If a component of T' intersects $Fix(\hat{h})$ replace it by the two boundary components of an $\hat{h}$-invariant product neighborhood of this component to get a collection T that is an h-invariant union of tori in $intM'$. Let C_i be a component of $\overline{\hat{M}' - N(T)}$. If $h(C_i) \cap C_i = \emptyset$, then $p(C_i)$ is a Seifert fiber space. If $h(C_i) = C_i$, then $p(C_i)$ is non-orientable and $\pi_1(p(C_i))$ contains a non-trivial cyclic normal subgroup. By Theorem 1 of [15] there is a disjoint collection K of 2-sided Klein bottles in $p(C_i)$ such that every component of $\overline{p(C_i) - N(\mathrm{K})}$ is a geminus or a Seifert bundle. (Here a Seifert bundle is a compact manifold that admits a decomposition into disjoint circle-fibers each having a regular neighborhood that is either a fibered solid torus or a fibered solid Klein bottle).

Thus if M is irreducible, the statement in case (iii) may be replaced by:

(iii_*) There is a disjoint collection of 2-sided projective planes, Kleinbottles and tori that splits M into tetrapods, octopods, gemini, and Seifert bundles.

In particular, for $\mathcal{G} = solv$, we note

Corollary 5. *Let M be a closed prime 3-manifold. Then*

(a) $cat_{solv}(M) = 1$ if and only if M is a torus or Kleinbottle bundle over S^1, or a union of two orientable I-bundles over the Kleinbottle, or a closed Seifert fiber space with non-negative orbifold Euler characteristic and not covered by the dodecahedral manifold.

(b) $cat_{solv}(M) = 2$ if and only if M is a union of bipods, tetrapods, and octopods along their projective plane boundaries.

(c) $cat_{solv}(M) = 3$ if and only if M is not as in case (b) and M is a union of Seifert bundles, gemini, bipods, tetrapods, and octopods along their boundaries.

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