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Quantum cohomology

at the Mittag-Leffler Institute

Edited by P. Aluffi

Preface

These are transcripts of notes taken at (some of) the lectures given at the Mittag-Leffler institute during the first semester of the 1996/97 year on *Enumerative geometry and its interaction with theoretical physics.* The first part of this collection consists of notes from talks on the basics of quantum cohomology, as developed in [F-P]. These talks formed the main body of the Tuesday seminar series at the Institute. The second part treats more advanced topics in quantum cohomology, which were primarily addressed in the Thursday seminar series. The third part consists of background material and related topics and contains material from both of these two series. An appendix, kindly provided by A. Kresch, gives a description of his C-program `farsta` for quantum cohomology computations.

These notes are meant as a series of snapshots of quantum cohomology as seen by the speakers at the time of their lectures. The reader should bear in mind that quantum cohomology is a growing and rapidly changing field; as any snapshot of a moving target, these notes are unavoidably a little blurry. Many of the writeups have been left in the form of the original talks, which were usually more concerned with giving motivations and a point of view, rather than conveying detailed proofs or attempting to survey the considerably extensive literature on the subject. Also, a glance at the references will show that many of the talks were based on preliminary (and hence not yet refereed) versions of papers on the subject. The published versions of these papers should be consulted for the definitive statements of the results, for the details of their proofs, and for more references.

Most of the notes were taken by myself. Missing and complementary notes were contributed by P. Belorousski, C. Faber, B. Fantechi, W. Fulton, and S. di Rocco, all of whom are warmly thanked here. Many thanks are due to the speakers, both for preparing and delivering the talks, and also for glancing through a preliminary version of these notes and suggesting a number of corrections. Some of the speakers took the trouble of reworking the files for their talks themselves, and deserve a particular note of gratitude both for saving me a great deal of work and for infinitely improving the final result.

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Part I
Stable Maps and Quantum Cohomology

1. Introduction—P. Belorousski, 9/10/96

The aim is to go rather carefully through the definition of the quantum cohomology ring $QH^*(X)$ of a variety X (satisfying suitable hypotheses). This is a ring supported on $H^*(X) \otimes \mathbb{Q}[[y]]$, where y stands for a set of variables, and whose product is defined to reflect sophisticated enumerative information about X. The definition of this product relies on Gromov-Witten invariants, obtained by pulling back classes from X to suitable moduli spaces and intersecting them there. One of our main objectives will be to state precise assumptions and results about the existence and properties of these moduli spaces, postponing the proofs of these properties till after we have seen some enumerative applications.

We will be following [F-P] rather closely. The paper combines Fulton's notes on quantum cohomology [FultonSC] and Rahul Pandharipande's notes on the moduli spaces of stable maps; it is organized as follows:

0: Introduction;

1–6: Construction of the moduli spaces of maps; proof of the properties;

7: Gromov-Witten invariants;

8: Definition of quantum cohomology;

9,10: Applications to enumerative geometry and more.

We will go through 0, then jump to 7–10 extracting properties from 1–6 without proof, then go back to 1–6 and the proofs of those properties.

Background on moduli problems. One seeks to represent a contravariant functor $F : (\text{Schemes}) \to (\text{Sets})$ associating to each S the set $F(S)$ of equivalence classes of families of a certain kind of objects over S.

In the strongest possible sense, one would look for a *Fine moduli space*, that is a scheme X equipped with a universal family $U \to X$ and representing the functor. That is, "every" family of the prescribed objects would be obtained by pull-back of U from X.

This requirement is in general too strong. For example, the functor $\mathcal{M}_g$ for smooth genus-g curves does not have a fine moduli space.

The *Coarse moduli space* would be a scheme Y with a natural transformation from the functor F to the functor $h_Y = \mathrm{Hom}(-, Y)$, such that

(1) $F(\mathrm{Spec}\mathbb{C}) \cong h_Y(\mathbb{C})$ as sets; and
(2) Y satisfies the universal property of mapping uniquely to any other candidate Z (so that there exists a factorization $F \to h_Y \to h_Z$).

Coarse moduli spaces for interesting objects do exist: e.g., $\mathcal{M}_g$ has a coarse moduli space M_g (of dim. $3g-3$ for $g>1$).

We will deal with a whole hierarchy of moduli spaces, prescribing points on the curves ($M_{g,n}$); or parametrizing maps of curves to a given X and satisfying properties as detailed in the next lecture. We will construct and study good compactifications of these moduli spaces.

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2. Stable maps—T. Graber, 9/10/96

Defining the product in the quantum cohomology ring will require counting objects of the form $(C, p_1, \dots, p_n, f)$ where C is a smooth curve of a given genus, p_i are prescribed smooth distinct points of C, and $f: C \to X$ is a map such that $f(p_i) \in$ prescribed loci, and $f_*[C] =$ prescribed $\beta \in A_1X$. Quantum cohomology can be used for example to compute the number of degree-d rational curves in $X = \mathbb{P}^2$ which contain $3d-1$ general points.

The general plan for these computations is the following:
(1) Construct a compactification of the appropriate moduli space;
(2) Do intersection theory on that space;
(3) Use this to define $QH^*(X)$ and solve enumerative problems.

DEFINITION. An n-pointed quasi-stable curve of genus g, $(C, p_1, \dots, p_n)$, will be a projective, reduced, connected, at worst nodal curve C, with $h^1(\mathcal{O}_C) = g$, labeled with n marked distinct smooth points p_i.

A family of n-pointed quasi-stable curves over a scheme S will be a flat, projective map $\mathcal{C} \to S$ with n sections $p_1, \dots, p_n$: $S \to \mathcal{C}$ such that each geometric fiber $(\mathcal{C}_s, p_1(s), \dots, p_n(s))$ is quasi-stable. A family of maps to X will be a map $\mathcal{C} \to X$, and isomorphisms of families are defined in the obvious way respecting the data (and inducing the identity on the base space).

DEFINITION. A *stable* map of an n-pointed quasi-stable curve to X consists of the data $(C, p_1, \dots, p_n, f)$ where $(C, p_1, \dots, p_n)$ is as above, and $f: C \to X$ is such that

(1) all smooth irreducible rational components contracted to points in X have at least 3 'special' points; and
(2) all irreducible genus-1 components contracted to points have at least 1 special point.

Here 'special' means either marked, or on the intersection of the component with the closure of its complement. Condition (2) is nearly vacuous, in that it is a restriction only on 0-pointed, genus-1 curves which map to a point.

The definition is devised so that every $f: C \to X$ will have only finite automorphism group (where the notion of isomorphism of stable maps is defined in the obvious way).

We work over $\mathbb{C}$. The following theorems will be proved in these lectures:

THEOREM 1. *For any projective algebraic scheme X and every $\beta \in A_1X$ there exists a coarse (compactified) projective moduli space, denoted $\overline{M}_{g,n}(X,\beta)$, parametrizing stable maps $(C, p_1, \dots, p_n, f)$ (with C of genus g) such that $f_*[C] = \beta$.*

For the moduli space to be reasonably well-behaved, some condition on X is required.

DEFINITION. A smooth projective variety X is called *convex* if for all maps $f: \mathbb{P}^1 \to X$, $H^1(f^*T_X) = 0$.

Important example: all homogeneous spaces are convex; so projective spaces, Grassmannians, flag manifolds, etc. are convex. (To see that homogeneous $\implies$ convex: if X is homogeneous, then TX is generated by global sections, so f^*TX is generated by global sections in $\mathbb{P}^1$; writing $f^*TX = \oplus \mathcal{O}(d_i)$, we see all $d_i \geq 0$, and this implies that there is no H^1.)

THEOREM 2. *If X is a smooth projective convex variety, and $\beta \in A_1X$, then $\overline{M}_{0,n}(X,\beta)$ is a variety of pure dimension*

$$\dim(X) + \int_\beta c_1(TX) + n - 3$$

(if nonempty), with at worst finite quotient singularities.

THEOREM 3. *The general element of $\overline{M}_{0,n}(X,\beta)$ does correspond to a map of an irreducible curve. The locus corresponding to reducible curves is a divisor with normal crossings (up to finite quotient singularities).*

Note: a priori, $\overline{M}_{0,n}(X,\beta)$ might be disconnected. The above statements hold for every irreducible component; these are necessarily disjoint. For X homogeneous, it can be shown that $\overline{M}_{0,n}(X,\beta)$ is in fact irreducible (Kim–Pandharipande, [Thomsen]).

EXAMPLES. —The simplest example in all X: $\beta = 0$. In this case, $\overline{M}_{g,n}(X,0) = X \times \overline{M}_{g,n}$.

—If X is an abelian variety (so it contains no rational curves), then $\overline{M}_{0,n}(X,\beta) = \emptyset$ unless $\beta = 0$.

—$\overline{M}_{0,0}(\mathbb{P}^r, 1) = G(1,r)$, the Grassmannian of lines in $\mathbb{P}^r$. Indeed, there are no marked points, so no reducible curves can map stably in the above sense.

—$\overline{M}_{0,1}(\mathbb{P}^r, 1) =$ data of a line in $\mathbb{P}^r$, with a marked point. Again, no component may contract because there aren't enough marked points. So $\overline{M}_{0,1}(\mathbb{P}^r, 1)$ is the tautological line over $G(1,r)$.

—$\overline{M}_{0,0}(\mathbb{P}^2, 2)$ recovers the space of complete conics. Indeed, the general element is a smooth conic (up to automorphisms, a smooth conic has a unique parametrization); this can degenerate to a pair of lines, parametrized by a pair of intersecting lines in the obvious way; if the two components of the domain map to the same $\mathbb{P}^1$ in $\mathbb{P}^2$ we get a double line with a marked point (corresponding to the image of the intersection of the two components); and finally we find double lines with two marked points, arising from double covers (ramified at the two points).

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3. Gromov–Witten invariants—J. Thomsen, 9/24/96

We work over $\mathbb{C}$. For X a scheme, $\beta \in A_1X$, and $g, n \geq 0$, we have the functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$ from Algebraic Schemes to Sets, defined by

$$\overline{\mathcal{M}}_{g,n}(X,\beta)(S) = \left\{ \begin{array}{l} \text{stable families of maps over } S \\ \text{from } n\text{-pointed, genus-}g \text{ curves to } X, \\ \text{which represent } \beta \end{array} \right\}_{/\text{isomorphism}}$$

A stable family as above will be denoted $(\pi : \mathcal{C} \to S, \{p_i\}_{1\leq i \leq n}, \mu : \mathcal{C} \to X)$; here

(i) $\pi : \mathcal{C} \to S$ is flat, projective;
(ii) $p_i : S \to \mathcal{C}$ are sections, $1 \leq i \leq n$;
(iii) each geometric fiber of π, $\pi_s : \mathcal{C}_s \to \{s\}$ is a connected, nodal curve of arithmetic genus g and marked with distinct nonsingular points $p_1(s), \ldots, p_n(s)$;
(iv) $\forall s \in S$, $\mu_s : \mathcal{C}_s \to X$ is stable: contracted genus-0 components have at least 3 'special' (marked or singular) points; contracted genus-1 components have at least 1 special point;
(v) $\forall s \in S$, $\mu_{s*}[\mathcal{C}_s] = \beta$.

From now on, X is taken to be projective. We will assume the following results:

THEOREM 1. *There exists a projective coarse moduli space $\overline{M}_{g,n}(X,\beta)$ for the functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$.*

THEOREM 2. *Assume X is nonsingular and convex. Then $\overline{M}_{0,n}(X,\beta)$ is a fine moduli space for $\overline{\mathcal{M}}_{0,n}(X,\beta)$ away from curves with non-trivial automorphisms. More precisely:*

(i) *$\overline{M}_{0,n}(X,\beta)$ is a locally normal projective variety of pure dimension*

$$\dim X + \int_\beta c_1(T_X) + n - 3$$

(if nonempty);
(ii) *$\overline{M}_{0,n}(X,\beta)$ is locally a quotient of a nonsingular variety by a finite group;*
(iii) *The closed points in $\overline{M}_{0,n}(X,\beta)$ corresponding to irreducible curves form a dense open subscheme $M_{0,n}(X,\beta)$ of $\overline{M}_{0,n}(X,\beta)$;*
(iv) *There exists a nonsingular fine moduli space $\overline{M}^*_{0,n}(X,\beta)$ for automorphism-free curves;*
(v) *$\overline{M}^*_{0,n}(X,\beta)$ is an open subset of $\overline{M}_{0,n}(X,\beta)$ (dense if $n \geq 3$);*
(vi) *If $\overline{M}_{0,n-1}(X,\beta) \neq \emptyset$, then there exists a map $\overline{M}_{0,n}(X,\beta) \xrightarrow{\pi} \overline{M}_{0,n-1}(X,\beta)$ 'forgetting' the first point. For s general in $\overline{M}_{0,n-1}(X,\beta)$, $C_s = \pi^{-1}(s)$ is the curve corresponding to s, and the composition $C_s \to \overline{M}_{0,n}(X,\beta) \xrightarrow{\rho_1} X$ is the corresponding map to X, where ρ_1 denotes the 'evaluation map' at the first point.*

The evaluation maps ρ_i, $1 \leq i \leq n$, are defined as follows. To give maps $\rho_i : \overline{M}_{0,n}(X,\beta) \to X$ amounts to giving natural transformations

$$\theta_i : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \mathrm{Hom}(*, X) \quad :$$

at a set S, $\theta_i(S)$ sends a stable family over S:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mu} & X \\ \downarrow & & \\ S & & \end{array}$$

with sections $p_1, \ldots, p_n$, to the composition

$$S \xrightarrow{p_i} \mathcal{C} \xrightarrow{\mu} X$$

For $S = \mathrm{Spec}\,\mathbb{C}$, that is for a single stable curve C, this simply evaluates the map μ to X at $p_i \in C$.

For a permutation $\sigma \in S_n$, there is an automorphism of $\overline{M}_{0,n}(X,\beta)$ interchanging the points:

$$\overline{M}_{0,n}(X,\beta) \xrightarrow{\Sigma(\sigma)} \overline{M}_{0,n}(X,\beta)$$

This is induced by the natural transformation sending a family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mu} & X \\ \downarrow & & \\ S & & \end{array}$$

with sections $p_1, \ldots, p_n$, to the same diagram but with sections $p_{\sigma(1)}, \ldots, p_{\sigma(n)}$. It is clear that

$$\rho_{\sigma(i)} = \rho_i \circ \Sigma(\sigma)$$

From now on, assume X is projective, nonsingular and convex.

DEFINITION. (Gromov-Witten invariants.) Given $\beta \in A_1X$, and $\gamma_1, \ldots, \gamma_n \in A^*X$ (the *operational* Chow ring; $\int_A$ will denote evaluation at $A \in A_*$), define

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n)$$

REMARKS.
(1) If the γ_i are homogeneous, $I_\beta(\gamma_1 \cdots \gamma_n) = 0$ if $\sum_i \operatorname{codim} \gamma_i \neq \dim \overline{M}_{0,n}(X,\beta)$;
(2) Due to the presence of the S_n-action on $\overline{M}_{0,n}(X,\beta)$, $I_\beta(\gamma_1 \cdots \gamma_n)$ is invariant under permutation of the γ_i's.

LEMMA. *Assume X is homogeneous: $X = G/P$; and $\Gamma_1, \ldots, \Gamma_n \subset X$ $(n \geq 3)$ are subvarieties, such that $\sum \operatorname{codim}(\Gamma_i) = \dim \overline{M}_{0,n}(X,\beta)$. Then for a general $\sigma = (g_1, \ldots, g_n) \in G^n$, the scheme-theoretic intersection*

$$\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n)$$

consists of reduced points supported on the locus $M_{0,n}(X,\beta)$. Further, the number of points in this intersection equals $I_\beta(\check{\Gamma}_1\cdots\check{\Gamma}_n)$.

Proof. This follows from judicious use of Kleiman-Bertini [Kleiman]. Let G be a connected algebraic group, and X a homogeneous G-variety, Y, Z varieties mapping to X:

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

For $\sigma \in G$, denote by Y^σ the image of the composition $Y \xrightarrow{f} X \xrightarrow{\sigma} X$. Then

(1) There is a dense open subscheme G° of G such that, for $\sigma \in G^\circ$, $Y^\sigma \times_X Z$ is either empty, or of pure dimension

$$\dim Y + \dim Z - \dim X$$

(2) Further, if Y and Z are nonsingular, then G° can be found so that $Y^\sigma \times_X Z$ is nonsingular.

This is used four times in our proof of the Lemma. Notations: $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$; $\Gamma^\circ \subset \Gamma$ is the nonsingular locus; Γ_{sing} is the singular locus of Γ; $M_{0,n}(X,\beta)^c$ is the complement of $M_{0,n}(X,\beta)$ in $\overline{M}_{0,n}(X,\beta)$. Four applications of [Kleiman]:

$$\text{(I)}\qquad \begin{tikzcd} & \overline{M}_{0,n}(X,\beta) \arrow[d, "(\rho_1,\dots,\rho_n)"] \\ \Gamma \arrow[r] & X^n \end{tikzcd} \quad :$$

for a general $\sigma \in G^n$, $\Gamma^\sigma \times_{X^n} \overline{M}_{0,n}(X,\beta)$ is either empty, or of dimension 0;

$$\text{(II)}\qquad \begin{tikzcd} & \overline{M}_{0,n}(X,\beta) \arrow[d] \\ \Gamma_{\text{sing}} \arrow[r] & X^n \end{tikzcd} \quad :$$

for a general $\sigma \in G^n$, $\Gamma^\sigma_{\text{sing}} \times_{X^n} \overline{M}_{0,n}(X,\beta)$ is either empty, or of dimension < 0; hence, it is necessarily empty. That is, we may assume that $\Gamma = \Gamma^\circ$ is nonsingular;

$$\text{(III)}\qquad \begin{tikzcd} & M_{0,n}(X,\beta)^c \arrow[d] \\ \Gamma \arrow[r] & X^n \end{tikzcd} \quad :$$

for a general $\sigma \in G^n$, $\Gamma^\sigma \times_{X^n} M_{0,n}(X,\beta)^c$ is again empty, or of dimension < 0; hence, empty. So the intersection is supported on $M_{0,n}(X,\beta)$. Finally

$$\text{(IV)}\qquad \begin{tikzcd} & M_{0,n}(X,\beta) \arrow[d] \\ \Gamma^\circ \arrow[r] & X^n \end{tikzcd} \quad :$$

for a general $\sigma \in G^n$, $\Gamma^{\circ\sigma} \times_{X^n} M_{0,n}(X,\beta)$ is smooth, hence reduced, of dimension 0 (if nonempty).

Summarizing we have that, for a general $\sigma = (g_1,\dots,g_n) \in G$,

$$\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n) = \Gamma^\sigma \times_{X^n} \overline{M}_{0,n}(X,\beta)$$

is reduced, of dimension 0, and supported on the nonsingular subset $M_{0,n}(X,\beta)$. To see that the number of points in this intersection equals $I_\beta(\check{\Gamma}_1\cdots\check{\Gamma}_n)$, observe that we have the fiber square

$$\begin{tikzcd} \rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n) \arrow[r] \arrow[d] & \overline{M}_{0,n}(X,\beta) \times \Gamma^\sigma \arrow[d] \\ \overline{M}_{0,n}(X,\beta) \arrow[r, "i=(id,\rho)"] & \overline{M}_{0,n}(X,\beta) \times X^n \end{tikzcd}$$

with the bottom line a regular embedding, of codimension $n \dim X$. As the dimensions match, and as $\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n)$ is reduced,

$$[\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n)] = i^!(\overline{M}_{0,n}(X,\beta) \times \Gamma^\sigma)$$

where $i^!$ denotes the gysin map. On the other hand, $i^!(\overline{M}_{0,n}(X,\beta) \times \Gamma^\sigma) = \psi(g_1\Gamma_1 \times \cdots \times g_n\Gamma_n)$, where ψ is the composition

$$A_*(X)^{\otimes n} \to A_*(X^n) \xrightarrow{\cong} A^*(X^n) \xrightarrow{\rho^*} A^*(\overline{M}_{0,n}(X,\beta)) \xrightarrow{\cap[\overline{M}_{0,n}(X,\beta)]} A_*(\overline{M}_{0,n}(X,\beta))$$

Tracing the definition, $\psi(g_1\Gamma_1 \times \cdots \times g_n\Gamma_n)$ equals $I_\beta((g_1\Gamma_1)^\vee \cdots (g_n\Gamma_n)^\vee)$; finally, $[\Gamma_i] = [g_i\Gamma_i]$ and the result follows. $\square$

The Gromov-Witten invariants satisfy a number of general properties. For example: assume X is homogeneous, and $\gamma_1 \in A^1(X)$ and $n \geq 4$ (or $n \geq 1$ if $\beta \neq 0$). Then

$$I_\beta(\gamma_1\cdots\gamma_n) = \left(\int_\beta \gamma_1\right) I_\beta(\gamma_2\cdots\gamma_n)$$

Indeed, consider $\phi : \overline{M}_{0,n}(X,\beta) \xrightarrow{\rho_1\times\text{forget.}} X \times \overline{M}_{0,n-1}(X,\beta)$; then write

$$\phi_*[\overline{M}_{0,n}(X,\beta)] = \beta' \times [\overline{M}_{0,n-1}(X,\beta)] + \alpha$$

where α dominates a proper subset of $\overline{M}_{0,n-1}(X,\beta)$. (Here we use that X is homogeneous, so that $A_*(X \times \overline{M}_{0,n-1}(X,\beta)) = A_*(X) \otimes A_*(\overline{M}_{0,n-1}(X,\beta))$.) Pushing forward doesn't change degrees, so

$$\begin{aligned} I_\beta(\gamma_1\cdots\gamma_n) &= \int_{[\overline{M}_{0,n}(X,\beta)]} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) \\ &= \int_{\phi_*[\overline{M}_{0,n}(X,\beta)]} \gamma_1 \times (\rho_2^*(\gamma_2) \cup \cdots \cup \rho_n^*(\gamma_n)) \\ &= \int_{\beta'\times[\overline{M}_{0,n-1}(X,\beta)]+\alpha} \gamma_1 \times (\rho_2^*(\gamma_2) \cup \cdots \cup \rho_n^*(\gamma_n)) \\ &= \left(\int_{\beta'} \gamma_1\right) \int_{[\overline{M}_{0,n-1}(X,\beta)]} \rho_2^*(\gamma_2) \cup \cdots \cup \rho_n^*(\gamma_n) \end{aligned}$$

by the projection formula: α is killed after push-forward to $\overline{M}_{0,n-1}(X,\beta)$

$$= \left(\int_{\beta'} \gamma_1\right) I_\beta(\gamma_2 \cdots \gamma_n)$$

Now the claim is that $\beta' = \beta$. To see this, consider a generic point $pt = (C, p_2, \dots, p_n, \mu)$ of $\overline{M}_{0,n-1}(X,\beta)$. Using Theorem 2(vi) we get the fiber diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "{(\mu,pt)}"'] & \overline{M}_{0,n}(X,\beta) \arrow[d, "\phi"] \\
X \times \{pt\} \arrow[r] \arrow[d] & X \times M_{0,n-1}(X,\beta) \arrow[d, "p_2"] \\
\{pt\} \arrow[r, "j"] & \overline{M}_{0,n-1}(X,\beta)
\end{tikzcd}$$

For a generic pt, j is a regular embedding. Therefore

$$j^!\phi_*[\overline{M}_{0,n}(X,\beta)] = (\mu,pt)_* j^![\overline{M}_{0,n}(X,\beta)] = (\mu,pt)_*[C] = \beta \times \{pt\}$$

on the other hand,

$$j^!\phi_*[\overline{M}_{0,n}(X,\beta)] = j^!(\beta' \times [\overline{M}_{0,n-1}(X,\beta)] + \alpha) = j!(\beta' \times [\overline{M}_{0,n-1}(X,\beta)]) = \beta' \times \{pt\}$$

(no contribution from α, since it will miss a general pt). Hence $\beta \times \{pt\} = \beta' \times \{pt\}$, as needed.

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4. Associativity of the quantum product—K. Ranestad, 10/1/96

Usual notations: X is a variety, $\beta \in A_1(X)$, $\gamma_i \in A^*(X)$, μ is a map from $\mathbb{P}^1$ to X, or from a tree of $\mathbb{P}^1$'s; $p_1, \dots, p_n$ are marked points, $\rho_i(*) = \mu(p_i)$ for $* \in \overline{M}_{0,n}(X,\beta)$.

Conditions on X:

(1) X is smooth, projective, convex (that is: $\forall \mu$, $h^1(\mathbb{P}^1, \mu^* T_X) = 0$);
(2) the Chow and topological homology theories of X are isomorphic;
(3) the effective cone in $A_1(X)$ is $\{\sum_{i=1}^p a_i\beta_i : a_i \in \mathbb{Z}_{\geq 0}, \beta_i \text{ in the form } \mu_*[\mathbb{P}^1]\}$.

The Gromov-Witten invariants are the intersection numbers

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) \qquad \in \mathbb{Z}$$

We will need two properties of these numbers; a third one was stated and proved in Thomsen's lecture:

(1) if $\beta = 0$, so $\overline{M}_{0,n}(X,0) = \overline{M}_{0,n} \times X$, then $I_0(\gamma_1\gamma_2\gamma_3) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$, and $I_0(\gamma_1 \cdots \gamma_n) = 0$ for $n > 3$;

(2) if $\gamma_1 = 1$: $I_\beta(1, \gamma_2, \dots, \gamma_n) = \begin{cases} 0 & n > 3 \text{ or } \beta \neq 0 \\ \int_X \gamma_2 \cup \gamma_3 \text{ otherwise} \end{cases}$.

Both of these follow easily from the projection formula.

We can write the classical intersection product in a fancy way, using these notations. Pick generators $T_0 = 1, T_1, \dots, T_m$ for A^*X, set $g_{ij} = \int_X T_i \cup T_j$, and let g^{ij} be the inverse matrix of g_{ij} (the inverse exists by Poincaré duality, and since by assumption (2) we have no torsion). Via the isomorphism $A^*(X \times X) = A^*X \otimes A^*X$, the class of the diagonal $\Delta \subset X \times X$ is

$$[\Delta] = \sum g^{ij} T_i \otimes T_j$$

Denoting by p_1, p_2 the two projections, we have then

$$\begin{aligned}
T_i \cup T_j &= p_{2*}(p_1^*(T_i \cup T_j) \cup \Delta) \\
&= p_{2*}\left(\sum_{e,f} (T_i \cup T_j \cup T_e) \otimes g^{ef} T_f\right) \\
&= \sum_{e,f} \left(\int_X T_i \cup T_j \cup T_e\right) g^{ef} T_f \\
&= \sum_{e,f} I_0(T_i T_j T_e) g^{ef} T_f
\end{aligned}$$

With this understood, set for $\gamma \in A^*X$:

$$\phi(\gamma) = \sum_{n \geq 3} \frac{1}{n!} \sum_\beta I_\beta(\gamma^n)$$

Writing $\gamma = \sum_{i=0}^m y_i T_i$, this expands to

$$\phi(\gamma) = \sum_{n_0 + \cdots + n_m \geq 3} \sum_\beta I_\beta(T_0^{n_0} \cdots T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}$$

Lemma. *This is a power series. In fact, for given n, $I_\beta(\gamma^n)$ is nonzero only for finitely many β.*

Proof. This uses assumptions (1) and (3) on X. Since X is convex, we have $h^1(\mu^* T_X) = 0$ for all maps μ from $\mathbb{P}^1$ to X. Writing $\mu^* T_X = \oplus_{i=1}^{\dim X} \mathcal{O}(d_i)$ and perhaps composing first μ with a cover of $\mathbb{P}^1$, we may assume all $d_i \geq 0$. Also, for $\mu_*[\mathbb{P}^1] \neq 0$, the differential $d\mu : T_{\mathbb{P}^1} \to \mu^* T_X$ is injective, so at least one d_i is ≥ 2. Representing an effective β as a nonnegative combination of push-forwards of $\mathbb{P}^1$'s, we see that $\int_\beta c_1(T_X) \geq 2$. Also, as the effective cone is finitely generated we see that for a given N there are only a finite number of effective β for which $\int_\beta c_1(T_X) \leq N$. But if $I_\beta(\gamma^n)$ is nonzero, then $\dim X + \int_\beta c_1(T_X) + n - 3 = \dim \overline{M}_{0,n}(X,\beta) \leq n \cdot \operatorname{codim} \gamma \leq n \dim X$, and this bounds $\int_\beta c_1(T_X)$. $\square$

Now we define the quantum product $*$. Consider the third partial derivatives

$$\begin{aligned}
\phi_{ijk} &:= \frac{\partial^3 \phi}{\partial y_i \partial y_j \partial y_k} \\
&= \sum_{n_0 + \cdots + n_m \geq 0} \sum_\beta I_\beta(T_0^{n_0} \cdots T_m^{n_m} T_i T_j T_k) \frac{y_0^{n_0} \cdots y_m^{n_m}}{n_0! \cdots n_m!} \\
&= \sum_n \frac{1}{n!} \sum_\beta I_\beta(\gamma^n T_i T_j T_k) \quad ,
\end{aligned}$$

a power series in $y_0, \dots, y_m$; we set

$$T_i * T_j := \sum_{e,f} \phi_{ije} g^{ef} T_f \qquad \in A^*X \otimes \mathbb{Q}[[y_0, \dots, y_m]]$$

This product is clearly commutative, and has $T_0 = 1$ as unit by property (2) of the invariants:

$$\begin{aligned} T_0 * T_j = \sum_{e,f} \phi_{0je} g^{ef} T_f &= \sum_{e,f} \sum_{n \geq 0} \frac{1}{n!} \sum_\beta I_\beta(\gamma^n 1 T_j T_e) g^{ef} T_f \\ &= \sum_{e,f} I_0(1 T_j T_e) g^{ef} T_f = \sum_{ef} \left(\int_X T_j \cup T_e \right) g^{ef} T_f \\ &= \sum_{e,f} g_{je} g^{ef} T_f = T_j \end{aligned}$$

Associativity of $*$. The quantum product defined above is associative. To check this, first write out what it means:

$$(T_i * T_j) * T_k = \left(\sum_{e,f} \phi_{ije} g^{ef} T_f \right) * T_k = \sum_{e,f,c,d} \phi_{ije} g^{ef} \phi_{fkc} g^{cd} T_d$$

$$T_i * (T_j * T_k) = T_i * \left(\sum_{e,f} \phi_{jke} g^{ef} T_f \right) = \sum_{e,f,c,d} \phi_{jke} g^{ef} \phi_{ifc} g^{cd} T_d$$

That is, setting

$$F(ij|k\ell) = \sum_{e,f} \phi_{ije} g^{ef} \phi_{fk\ell} \quad ,$$

we need to show that

$$F(ij|k\ell) = F(jk|i\ell) \qquad \forall \ell$$

Expanding:

$$F(ij|k\ell) = \sum_{\beta_1, \beta_2; n_i \geq 0; 0 \leq e,f \leq m} \frac{1}{n_1! n_2!} I_{\beta_1}(\gamma^{n_1} T_i T_j T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} T_f T_k T_\ell)$$

Now, the boundary of $\overline{M} = \overline{M}_{0,n}(X, \beta)$ consists of irreducible divisors $D = D(A, B, \beta_1, \beta_2)$, where $A \amalg B = \{1, \dots, n\}$, and $\beta_1 + \beta_2 = \beta$. The general point of this divisor has a description similar to the case of $\overline{M}_{0,n}$ (see §3), by glueing stable curves at a point. More precisely,

$$D \cong \overline{M}_{0, A \cup \{\bullet\}}(X, \beta_1) \times_X \overline{M}_{0, B \cup \{\bullet\}}(X, \beta_2)$$

when $A \neq \emptyset$, $B \neq \emptyset$, with obvious notations. Setting $\overline{M}_A = \overline{M}_{0, A \cup \{\bullet\}}(X, \beta_1)$, etc. we have the diagram

$$\begin{tikzcd} \overline{M} \arrow[d, "\rho"'] & D \arrow[l, "\alpha"'] \arrow[r, "i"] \arrow[d, "\varphi"'] & \overline{M}_A \times_{\mathbb{C}} \overline{M}_B \arrow[d, "\rho'"] \\ X^n & X^{n+1} \arrow[l, "\alpha'"'] \arrow[r, "i'"] & X^{n+2} \end{tikzcd}$$

with the right square a fiber square. Chasing this diagram gives:

Lemma.

$$i_* \alpha^*(\rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n)) = \sum_{e,f} g^{ef} \left(\prod_{a \in A} \rho_a^*(\gamma_a) \rho_\bullet^*(T_e) \right) \left(\prod_{b \in B} \rho_b^*(\gamma_b) \rho_\bullet^*(T_f) \right)$$

Now fix $\beta \in A_1 X$, $\gamma_1, \dots, \gamma_n \in A^*X$, integers q, r, s, t in $\{1, \dots, n\}$, and put

$$G(qr|st) = \sum_{e,f; \beta_1 + \beta_2 = \beta; A \amalg B = \{1, \dots, n\}; q,r \in A; s,t \in B} I_{\beta_1}\Big(\prod_{a \in A} \gamma_a T_e\Big) g^{ef} I_{\beta_2}\Big(\prod_{b \in B} \gamma_b T_f\Big)$$

By the Lemma,

$$G(qr|st) = \sum \int_{D(A,B;\beta_1,\beta_2)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) = \int_{D(qr|st)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n)$$

where we denote by $D(qr|st)$ the sum of the relevant boundary divisors.

Now for the key observation: denoting by π the composition of the forgetful maps

$$\overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,n} \to \overline{M}_{0,\{q,r,s,t\}} \cong \mathbb{P}^1$$

one checks that $D(qr|st)$ is the inverse image via π of the divisor of $M_{0,\{q,r,s,t\}}$ corresponding to the partition $\{q, r\} \cup \{s, t\}$ (the convexity of X is used to show that all components in the preimage appear with multiplicity one). In particular,

$$D(qr|st) \cong D(rs|qt)$$

up to linear equivalence, since this equality holds in $M_{0,\{q,r,s,t\}} \cong \mathbb{P}^1$; and therefore

$$G(qr|st) = G(rs|qt)$$

for all $\gamma_1, \dots, \gamma_n$. Setting all but $\gamma_q, \gamma_r, \gamma_s, \gamma_t$ equal to γ, we have by definition

$$G(qr|st) = \sum_{e,f; n_1 + n_2 = n; n_i \geq 2} \binom{n-4}{n_1 - 2} I_{\beta_1}(\gamma^{n_1 - 2} \gamma_q \gamma_r T_e) g^{ef} I_{\beta_2}(\gamma^{n_2 - 2} \gamma_s \gamma_t T_f)$$

or, by a shift of the indices:

$$G(qr|st) = (n-4)! \sum_{e,f; n_1 + n_2 = n-4; n_i \geq 0} \frac{1}{n_1! n_2!} I_{\beta_1}(\gamma^{n_1} \gamma_q \gamma_r T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} \gamma_s \gamma_t T_f)$$

Therefore, the equality of G's says

$$\begin{aligned} &\sum_{e,f; n_1 + n_2 = n-4; n_i \geq 0} \frac{1}{n_1! n_2!} I_{\beta_1}(\gamma^{n_1} \gamma_q \gamma_r T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} \gamma_s \gamma_t T_f) \\ &\qquad = \sum_{e,f; n_1 + n_2 = n-4; n_i \geq 0} \frac{1}{n_1! n_2!} I_{\beta_1}(\gamma^{n_1} \gamma_r \gamma_s T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} \gamma_q \gamma_t T_f) \quad ; \end{aligned}$$

setting $\gamma_q = T_i$, $\gamma_r = T_j$, $\gamma_s = T_k$, $\gamma_t = T_\ell$ and adding over n:

$$\begin{aligned} &\sum_{e,f; n_1, n_2 \geq 0} \frac{1}{n_1! n_2!} I_{\beta_1}(\gamma^{n_1} T_i T_j T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} T_k T_\ell T_f) \\ &\qquad = \sum_{e,f; n_1, n_2 \geq 0} \frac{1}{n_1! n_2!} I_{\beta_1}(\gamma^{n_1} T_j T_k T_e) g^{ef} I_{\beta_2}(\gamma^{n_2} T_i T_\ell T_f) \quad ; \end{aligned}$$

and finally, adding over all $\beta_1 + \beta_2 = \beta$:

$$F(ij|k\ell) = F(jk|i\ell)$$

as needed. $\square$

5. Applications of QH^* to enumerative geometry—S.L. Kleiman, 10/8/96

Claude Itzykson, in memoriam

The main reference for the following material is [F-P], especially §9; a secondary reference is [DF-I]. The material is organized into these sections:

—§1. *Gromov–Witten invariants*
—§2. *The potential*
—§3. *The projective plane*
—§4. *Feynman diagrams*
—§5. *Surfaces in general*
—§6. *Del Pezzo and Hirzebruch surfaces*

§**1. Gromov–Witten invariants.** Let X be a smooth irreducible projective variety. If X is, in fact, the quotient G/P of a reductive group G by a parabolic subgroup P, then X is convex and its singular homology $H_*(X,\mathbb{Z})$ is algebraic. As we have seen, it follows that there are good moduli spaces $\overline{M} = \overline{M}_{0,n}(X,\beta)$ of marked Kontsevich-stable maps, and that X has an associative quantum cohomology ring $QH^*(X)$. Later (in §5) we'll see that conversely, at least for surfaces, if X is convex and $H_*(X,\mathbb{Z})$ is algebraic, then, for all practical purposes, $X = G/P$.

Let $\beta \in A_1X := H_2(X,\mathbb{Z})$. The moduli space $\overline{M}$ parameterizes maps of class β:

$$\overline{M} := \left\{ (C \xrightarrow{\mu} X; p_1,\dots,p_n) \;\middle|\; \begin{array}{l} C = \mathbb{P}^1 \text{ or a tree of } \mathbb{P}^1\text{s}, \\ \mu_*[C] = \beta,\ p_i \in C, \\ \text{and } \mu \text{ is Kontsevich-stable} \end{array} \right\} \Big/ \text{isom}$$

The *expected dimension* of $\overline{M}$ is

$$\operatorname{exp.dim}(\overline{M}) := \dim X + \int_\beta c_1(TX) + n - 3.$$

Moreover, there are *evaluation maps* $\rho_i\colon \overline{M} \to X$, which take an element of $\overline{M}$ as above to $\mu(p_i)$. Note that, if $\overline{M}$ is nonempty, then β is *effective*; that is, β is the class of the image of a tree C of $\mathbb{P}^1$s.

Let $\gamma_1,\dots,\gamma_n \in A^*X := H^*(X,\mathbb{Z})$. The corresponding *Gromov–Witten invariant* is

$$I_\beta(\gamma_1\cdots\gamma_n) := \begin{cases} 0, & \begin{array}{l}\text{unless } \beta \text{ is effective and}\\ \quad \sum \dim\gamma_i = \operatorname{exp.dim}(\overline{M});\end{array} \\ \int_\varphi \rho_1^*\gamma_1 \cup\cdots\cup \rho_n^*\gamma_n, & \text{if so.}\end{cases}$$

Here φ is the fundamental class of the moduli space: $\varphi := [\overline{M}]$ if X is convex; φ is the 'virtual fundamental class' otherwise [Li-Tian], [B-F] and [Behrend] (see §6 for some examples). This class may even be fractional.

At least in the case $X = G/P$, we have

$$I_\beta(\gamma_1\cdots\gamma_n) = \#(\rho_1^{-1}\Gamma_1 \cap\cdots\cap \rho_n^{-1}\Gamma_n),$$

where the Γ_i are representatives, in general position, of the Poincaré duals to the γ_i. In other words, I_β is the number of pointed maps μ such that $\mu_*[C] = \beta$ and $\mu p_i \in \Gamma_i$. Moreover, the intersection on the right consists of general points of $\overline{M}$. In particular, the corresponding C are each equal to $\mathbb{P}^1$. Now, it may be that the general map μ is not birational onto its image. For example, this is the case if $X = \mathbb{P}^1$ and $\beta = d[X]$ with $d \ge 2$, or if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\beta = d[\text{ruling}]$ with $d \ge 2$. However, if the general map is not birational, then the intersection would not be finite unless it's empty; hence, it's empty, and $I_\beta(\gamma_1\cdots\gamma_n)$ vanishes. Thus I_β is the number of irreducible rational curves in X $(= G/P)$ of class β meeting the Γ_i.

In general, the Gromov–Witten invariants possess the following three properties:

$$\text{(I)} \qquad I_0(\gamma_1\cdots\gamma_n) = \begin{cases} 0, & \text{if } n > 3; \\ \int_X \gamma_1\cup\gamma_2\cup\gamma_3, & \text{if } n = 3.\end{cases}$$

$$\text{(II)} \qquad I_\beta(1\cdot\gamma_2\cdots\gamma_n) = \begin{cases} 0, & \text{if } n > 3 \text{ or } \beta \ne 0; \\ \int_X \gamma_2\cup\gamma_3, & \text{if } n = 3 \text{ and } \beta = 0.\end{cases}$$

If γ_1 is a divisor class ($\gamma_1 \in A^1X$), then

$$\text{(III)} \qquad I_\beta(\gamma_1\gamma_2\cdots\gamma_n) = \Big(\int_\beta \gamma_1\Big) I_\beta(\gamma_2\cdots\gamma_n).$$

§**2. The potential.** The *potential* is defined by the formula,

$$\Phi(\gamma) = \sum_{n\ge 3}\sum_\beta \frac{1}{n!} I_\beta(\gamma^n).$$

Take a graded basis of A^*X with $T_0 = 1$, with $T_1,\dots,T_p \in A^1X$, and with $T_{p+1},\dots,T_m$ of higher degree. Write $\gamma = \sum y_iT_i$, and use the linearity of I_β to expand Φ:

$$\Phi(y_0,\dots,y_m) = \sum_{n_0+\cdots+n_m\ge 3}\sum_\beta I_\beta(T_0^{n_0}\cdots T_m^{n_m})\frac{y_0^{n_0}}{n_0!}\cdots\frac{y_m^{n_m}}{n_m!}.$$

The right-hand side is a well-defined formal power series in $\mathbb{Q}[[y]]$ if, given γ^n, we have $I_\beta(\gamma^n) = 0$ for almost all β. For example, this is the case if the effective classes lie in the cone generated by finitely many β such that $\int_\beta c_1TX > 0$ (which is the key ingredient in the first lemma in the preceding lecture of Ranestad's). This condition is satisfied if $X = G/P$, or more generally whenever $-K_X := c_1TX$ is ample. However, this issue becomes unimportant when Φ is modified as described next (see [G-P, §2]).

Break up the potential into two pieces, $\Phi = \Phi^{cl} + \Phi^{qtm}$, where Φ^{cl}, the *classical part*, is the contribution due to the condition $\beta = 0$, and where Φ^{qtm}, the *quantum correction*, is the contribution coming from all nonzero β. Property (I) of §1 yields

$$\Phi^{cl} = \frac{1}{3!}\int_X \gamma^3 = \sum_{n_0+\cdots+n_m=3}\int_X T_0^{n_0}\cup\cdots\cup T_m^{n_m}\frac{y_0^{n_0}}{n_0!}\cdots\frac{y_m^{n_m}}{n_m!}.$$

Since the quantum product $*$ (recalled below) involves only the third derivatives of Φ, we may modify Φ by terms of degree at most two. So (I)–(III) imply that we may replace Φ^{qtm} by

$$\Gamma := \sum_{n_{p+1}+\cdots+n_m\geq 3}\sum_{\beta\neq 0} I_\beta(T_{p+1}^{n_{p+1}}\cdots T_m^{n_m})\left(\prod_{i=1}^{p} e^{y_i\int_\beta T_i}\right)\frac{y_{p+1}^{n_{p+1}}}{n_{p+1}!}\cdots\frac{y_m^{n_m}}{n_m!},$$

which is a formal power series in $y_{p+1},\dots,y_m$ and in new variables $e^{y_1},\dots,e^{y_p}$ (with appropriate derivatives) and in their inverses. The product $\prod_{i=1}^p e^{\int_\beta y_iT_i}$ may be abbreviated to $e^{\int_\beta\gamma}$ as only the T_i in A^1X give nonzero contributions to the integral.

Set $\Phi_{ijk} := \dfrac{\partial^3\Phi}{\partial y_i\partial y_j\partial y_k}$, set $(g^{ef}) := (g_{ef})^{-1}$ for $g_{ef} = \int_X T_e\cup T_f$, and set

$$T_i * T_j := \sum_{e,f}\Phi_{ije}g^{ef}T_f.$$

It is immediate to check that this *quantum product* $*$ is commutative, with unit T_0. It's far less obvious that $*$ is associative; see the preceding lecture for a proof.

§**3. The projective plane.** As a first example, which is simple yet gives the general flavor, let's work out the above theory for $X = \mathbb{P}^2$. Let T_0, T_1, and T_2 be the classes of X, a line, and a point. Say $\beta = dT_1$ and set $\delta := \frac{1}{2}(d-1)(d-2)$. Then

$$\begin{aligned} I_\beta(T_2^{n_2}) &= \text{the number } N_d \text{ of } \delta\text{-nodal plane curves through } n_2 \text{ points}\\ &= 0,\ \text{unless } n_2 = 3d-1.\end{aligned}$$

For instance, N_1 is simply the number of lines through two points, namely, 1. Remarkably, all the other N_d follow formally from this one! (This fact was noted by Kontsevich first, according to [DF-I, p.82], and it has inspired the determination, via quantum cohomology, of a lot of new geometric numbers.)

With the above choice of the T_i, we find

$$\Phi^{cl} = \frac{1}{2}y_0^2y_2 + \frac{1}{2}y_0y_1^2,\ \text{and}\ \Gamma = \sum_{d\geq 1} N_d e^{dy_1}\frac{y_2^{3d-1}}{(3d-1)!}.$$

Moreover, we have

$$(g_{ef}) = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} = (g^{ef}).$$

Hence, $T_i * T_j = \Phi_{ij2}T_0 + \Phi_{ij1}T_1 + \Phi_{ij0}T_2$. Therefore,

$$\begin{aligned} T_1 * T_1 &= T_2 + \Gamma_{111}T_1 + \Gamma_{112}T_0,\\ T_1 * T_2 &= \qquad\ \Gamma_{121}T_1 + \Gamma_{122}T_0,\\ T_2 * T_2 &= \qquad\ \Gamma_{221}T_1 + \Gamma_{222}T_0.\end{aligned}$$

Straightforward computation now yields the formulas,

$$\begin{aligned} (T_1 * T_1) * T_2 &= (\Gamma_{222} + \Gamma_{111}\Gamma_{122})T_0 + \dots\\ T_1 * (T_1 * T_2) &= \qquad\qquad \Gamma_{112}\Gamma_{121}T_0 + \dots\end{aligned}$$

So associativity implies the equation,

$$\Gamma_{222} + \Gamma_{111}\Gamma_{122} = \Gamma_{112}\Gamma_{121}.$$

Differentiating Γ, we find

$$\Gamma_{222} = \sum_{d\geq 2} N_d e^{dy_1}\frac{y_2^{3d-4}}{(3d-4)!},\qquad \Gamma_{122} = \sum_{d\geq 1} dN_d e^{dy_1}\frac{y_2^{3d-3}}{(3d-3)!},$$
$$\Gamma_{112} = \sum_{d\geq 1} d^2N_d e^{dy_1}\frac{y_2^{3d-2}}{(3d-2)!},\qquad \Gamma_{111} = \sum_{d\geq 1} d^3N_d e^{dy_1}\frac{y_2^{3d-1}}{(3d-1)!}.$$

Multiplying, we get

$$\Gamma_{112}^2 = \sum_{d\geq 2}\sum_{d_1+d_2=d} d_1^2N_{d_1}d_2^2N_{d_2}e^{dy_1}\frac{y_2^{3d-4}}{(3d-2)!(3d-2)!},$$
$$\Gamma_{111}\Gamma_{122} = \sum_{d\geq 2}\sum_{d_1+d_2=d} d_1^3N_{d_1}d_2N_{d_2}e^{dy_1}\frac{y_2^{3d-4}}{(3d-1)!(3d-3)!}.$$

Finally, equating coefficients yields Kontsevich's celebrated recursion formula,

$$N_d = \sum_{d_1+d_2=d} N_{d_1}N_{d_2}\left[d_1^2d_2^2\frac{(3d-4)!}{(3d-2)!(3d-2)!} - d_1^3d_2\frac{(3d-4)!}{(3d-1)!(3d-3)!}\right].$$

The following table gives the first five values of the number N_d of rational plane curves of degree d (with δ nodes) through $3d-1$ points:

d	1	2	3	4	5
N_d	1	1	12	620	87304
δ	0	0	1	3	6
$3d-1$	2	5	8	11	14

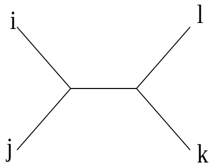
The first three values of N_d have been known for a long time. The fourth was found (after a lot of work) by Zeuthen in 1873. The fifth number was found as above by Kontsevich in December 1993 (it had already been found implicitly via more traditional means by Ran and by Vainsencher).

Along the same lines, we can work out the cases of $\mathbb{P}^3$ and of the quadric threefold Q^3 in $\mathbb{P}^4$ (the details are found in [F-P, §9]) and the case of the point-line incidence variety I^3 in $\mathbb{P}^2\times\mathbb{P}^2$ (the details are found in [DF-I, pp.125–29]). It is also possible to handle conditions of tangency by using the associativity of an appropriate generalized quantum cohomology ring. This was done to some extent by Di Francesco and Itzykson in [DF-I, pp.103–105]) and in full by Ernström and Kennedy in [E-K]. On the other hand, Pandharipande [P] worked with conditions of tangency by using intersection theory on $\overline{M}_{0,n}(\mathbb{P}^r)$ without the power of associativity.

§4. **Feynman diagrams.** Feynman diagrams can be used as an effective mnemonic device for writing down the associativity equations efficiently. The sum,

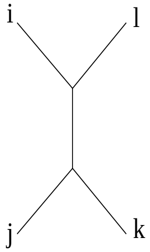
$$F(i,j|k,l) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl},$$

is represented by the diagram,



Heuristically, i and j are coupled on the left, and k and l are coupled on the right. Each couple can be in a number of intermediate "states," which are indexed by e and f, and quantified by Φ_{ije} and Φ_{fkl}. Each pair of states is correlated by a "propagator," which is represented by the horizontal link, and quantified by g^{ef}. The total "4-point correlation" is represented by the diagram, and quantified by the above sum $F(i,j|k,l)$.

The same 4-point correlation can be decomposed in a second way into a complete set of intermediate states. The corresponding diagram is



So the corresponding sum is $F(j,k|l,i)$. In physics, the duality relation of topological field theory is symbolized by an equation, with the first diagram above on the left and the second on the right. This duality relation corresponds to the following associativity equation:

$$A(i,j,k,l) : F(i,j|k,l) = F(j,k|l,i).$$

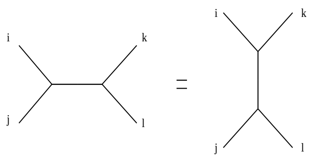
(This differential equation was called a *WDVV equation* after E. Witten, R. Dijkgraaf, H. Verlinde and E. Verlinde by B. Dubrovin.) Every associativity equation arises in this way for suitable values of i,j,k,l. However, many of the equations are equivalent, and others are trivially satisfied, as we'll now see.

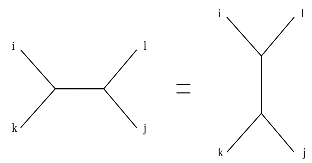
The Φ_{ije} are symmetric in i,j,e, and the g^{ef} are symmetric in e,f. So a simple formal calculation yields the following equivalences of equations:

$$A(i,j,k,l) \equiv A(j,k,l,i) \equiv A(k,l,i,j) \equiv A(l,i,j,k) \equiv$$
$$A(i,l,k,j) \equiv A(j,i,l,k) \equiv A(k,j,i,l) \equiv A(l,k,j,i).$$

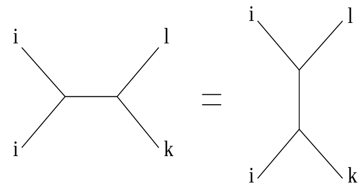
To obtain each of these eight equations, pick one of the four indices and read progressively around either one of the above diagrams either counterclockwise or clockwise.

Similarly, there are two more groups of eight equivalent equations, and they correspond to the following two duality relations:

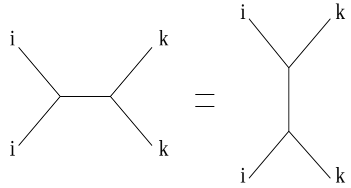




If all four of i,j,k,l are distinct, then there are twenty-four possible equations, and they divide into the above three groups of eight equivalent equations. If only three indices are distinct, say $i = j$, then there is, up to equivalence, only one nontrivial associativity equation $A(i,i,k,l)$, and it corresponds to the duality relation,



The equations of the other two groups are trivially satisfied because of the symmetry of the Φ_{ije}. If only two indices are distinct, say $i = j$ and $k = l$, then there is again, up to equivalence, only one nontrivial associativity equation $A(i,i,k,k)$, and it corresponds to the duality relation,



If three or four indices coincide, then the resulting equations are automatically satisfied. The same happens if one of the indices is 0. Indeed, $\Gamma_{0jk} = 0$; so null indices matter only for Φ^{cl}. However, the classical product is already associative!

Consequently, if the rank of $A^*(X)$ is $1+m$, then the total number of basic associativity equations is

$$3\binom{m}{4} + m\binom{m-1}{2} + \binom{m}{2} = \frac{m(m-1)(m^2-m+2)}{8}.$$

For example, for $m = 2,\ldots,7$, the numbers are 1, 6, 21, 55, 120, 231. Thus, if X is $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^3$, then m is 3, and the number of basic equations is 6.

Thus, given $A^*(X)$, including the anticanonical class $-K_X$, and a suitable number of initial conditions, it is a formal matter to set up the associativity equations and then to solve for the Gromov–Witten invariants. An algorithm to do so was recently developed and implemented as a (200K) C-program, `farsta`, by Andrew Kresch.

§**5. Surfaces in general.** Again, let X be a smooth irreducible projective variety of any dimension, and suppose that $H_*(X, \mathbb{Z})$ is algebraic. Then in particular $H_{2n+1}(X, \mathbb{Z}) = 0$. So the universal-coefficient theorem implies that $H^{2n+1}(X, \mathbb{Z}) = 0$ and that $H^{2n}(X, \mathbb{Z})$ is torsion-free.

By Hodge theory, $H^1(X, \mathbb{C}) = H^1(\mathcal{O}_X) \oplus H^0(\Omega^1_X)$; so $H^1(\mathcal{O}_X) = 0$. Further,

$$H^2(X, \mathbb{C}) = H^2(\mathcal{O}_X) \oplus H^1(\Omega^1_X) \oplus H^0(\Omega^2_X),$$

and the algebraic cycles map into $H^1(\Omega^1_X)$. They span all of $H^2(X, \mathbb{C})$ by assumption. Hence $H^2(\mathcal{O}_X) = 0$.

Consider the exponential sequence:

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0.$$

It yields the long exact sequence:

$$H^1(\mathcal{O}_X) \to H^1(\mathcal{O}^*_X) \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X).$$

The extreme terms vanish. So we get

$$\mathrm{Pic}(X) = H^1(\mathcal{O}^*_X) \xrightarrow{\sim} H^2(X, \mathbb{Z}).$$

Thus $\mathrm{Pic}(X)$ is discrete and torsion-free. In particular, if X is a *surface,* and if β is given in $A_1X := H_2(X, \mathbb{Z})$, then the divisors D of class (Poincaré dual to) β are linearly equivalent.

Let $\mu\colon \mathbb{P}^1 \to X$ be a map that's birational onto its image. Form the sequence,

$$0 \to \mathcal{T}_{\mathbb{P}^1} \to \mu^*\mathcal{T}_X \to \mathcal{N}_\mu,$$

(where $\mathcal{N}_\mu$ is the dual of $\mathrm{Ker}(\mu^*\Omega^1_X \to \Omega^1_{\mathbb{P}^1})$). This sequence must be of the form,

$$0 \to \mathcal{O}(2) \xrightarrow{u} \mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathcal{O}(c),$$

say with $a \geq b$. Since the map u is nonzero, so is its composition with the projection to $\mathcal{O}(a)$ or else with that to $\mathcal{O}(b)$; hence, $a \geq 2$ or $b \geq 2$. Therefore, $a \geq 2$ since $a \geq b$. Similarly, $c \geq a$ or $c \geq b$; hence $c \geq b$. Further, if X is convex, then $b \geq 0$ and so also $c \geq 0$.

Suppose that X is a convex surface. Set $D := \mu_*\mathbb{P}^1$. Then $\mathcal{N}_\mu = \mathcal{O}(\mu^*D)$. So $D^2 \geq 0$ as $c \geq 0$. Furthermore, $\langle -K_X \cdot D\rangle \geq 2$ with $-K_X = c_1T_X$ as $a \geq 2$ and $b \geq 0$. In particular, there are no (-1)-curves. Hence X is relatively minimal. Therefore, if X is rational, then $X = \mathbb{P}^2$ or $X = \mathbb{F}_e$ (the Hirzebruch surface). In the latter case, $X = \mathbb{P}^1 \times \mathbb{P}^1$ necessarily, because $\mathbb{F}_e$ has a section of square $-e$. If X is irrational, then K_X is nef and so, since $\langle -K_X \cdot D\rangle \geq 2$, there are no rational curves on X.

In sum, if X is a convex surface with $H_*(X, \mathbb{Z})$ algebraic, then either $X = \mathbb{P}^2$, or $X = \mathbb{P}^1 \times \mathbb{P}^1$ (and in both these cases $X = G/P$), or else X has no rational curves! In other words, for surfaces, the requirement of convexity is rather restrictive for the applications of quantum cohomology to enumerative geometry.

From now on, X is a surface with $H_*(X, \mathbb{Z})$ algebraic, but X is not necessarily convex. Given $\beta \in A_1X$, set

$$k(\beta) := \langle -K_X \cdot \beta\rangle = \int_\beta c_1(TX).$$

Let π be the class of a point. If $I_\beta(\pi^n) \neq 0$, then the equations,

$$n \cdot \mathrm{codim}\, \pi = \mathrm{exp.\,dim}(\overline{M}) = \dim X + k(\beta) + n - 3,$$

give the formula,

$$n = k(\beta) - 1,$$

because $\mathrm{codim}\, \pi = 2$ and $\dim X = 2$. Set therefore

$$N_\beta := I_\beta(\pi^{k(\beta)-1}).$$

Normally, N_β is the number of curves of class β with $p_a(\beta)$ nodes (so they're immersed $\mathbb{P}^1$s) that pass through $k(\beta) - 1$ points. Recall that, if β is not effective (that is, not the class of the image of a tree of $\mathbb{P}^1$s), then $N_\beta = 0$; it is reasonable to conjecture that, if the arithmetic genus $p_a(\beta)$ is strictly negative, then again $N_\beta = 0$.

As in §2, replace the potential Φ by the sum $\Phi^{cl} + \Gamma$ where

$$\Gamma(\gamma) := \sum_{\beta \neq 0} N_\beta e^{\int_\beta \gamma} \frac{y_m^{k(\beta)-1}}{(k(\beta)-1)!}.$$

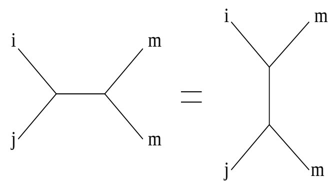
Write $\gamma = \gamma_0 + \gamma_1 + \gamma_2$ with $\gamma_i \in A_i(X)$. Then

$$\Phi^{cl} = \sum_{i+j+k=3} \frac{1}{i!j!k!} \int_X \gamma_0^i \gamma_1^j \gamma_2^k.$$

For the integral to be nonzero, necessarily $j + 2k = 2$, and so either $k = 0$, $j = 2$, $i = 1$ or $k = 1$, $j = 0$, $i = 2$. So

$$\Phi^{cl} = \frac{1}{2}\int \gamma_0\gamma_1^2 + \frac{1}{2}\int \gamma_0^2\gamma_2.$$

For $0 < i, j < m$, the duality relation



corresponds to the associativity equation,

$$A(i,j,m,m): \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fmm} = \sum_{e,f} \Phi_{jme} g^{ef} \Phi_{fmi}.$$

On the left, consider the terms with $f = m$. First, $\Phi^{cl}_{mmm} = 0$; so $\Phi_{mmm} = \Gamma_{mmm}$. Next, $g^{em} = 0$ if $e \neq 0$, and $g^{0m} = 1$. Finally, $\Phi_{ij0} = \Phi^{cl}_{ij0} = g_{ij}$. Hence there's only one nonzero term with $f = m$, and it's equal to $g_{ij}\Gamma_{mmm}$. On the other hand, $\Phi^{cl}_{ije} = 0$ unless $e = 0$, and $\Phi^{cl}_{efm} = 0$ unless $e = 0$ and $f = 0$. Therefore, the associativity equation yields a formula of the following form:

$$g_{ij}\Gamma_{mmm} = \text{ a certain quadratic polynomial in the } \Gamma_{efm}.$$

As in the case of $\mathbb{P}^2$, this formula yields a recurrence relation of the following form:

$$g_{ij}N_\beta = \sum_{\beta_1+\beta_2=\beta} N_{\beta_1}N_{\beta_2} * \{ * - * \}.$$

Only one of these recurrence relations is needed to solve for the N_β; however, the others serve to reduce the number of necessary initial conditions, although redundantly. (For a further discussion of this matter whenever A^*X is generated by A^1X, see Kresch's paper [Kresch].)

§**6. Del Pezzo and Hirzebruch surfaces.** In this final section, we'll consider two examples: the Del Pezzo surfaces, and the Hirzebruch surfaces.

A *Del Pezzo surface* is a smooth irreducible projective surface X such that $-K_X$ is ample. One such X is $\mathbb{P}^1 \times \mathbb{P}^1$; it is also a Hirzebruch surface, and will be considered below. Otherwise, X is obtained by blowing up the plane at r points in general position, where $0 \leq r \leq 8$. Note that X is not convex for $r > 0$. A natural basis for A^1X is $\{h, e_1, \ldots, e_r\}$ where h is the pullback of the class of a line, and e_i is the class of the ith exceptional divisor. The potential Φ is a well-defined power series because $-K_X$ is ample. So the associativity equation $A(1,1,m,m)$ yields the following recursion formula:

$$N_\beta = \sum N_{\beta_1}N_{\beta_2}\langle\beta_1\cdot\beta_2\rangle\langle h\cdot\beta_1\rangle\left[\langle h\cdot\beta_2\rangle\binom{k(\beta)-4}{k(\beta_1)-2} - \langle h\cdot\beta_1\rangle\binom{k(\beta)-4}{k(\beta_2)-1}\right].$$

Needless to say, for $r = 0$, we recover the formula of §3 for the plane. The case $r = 6$, where X is equal to a cubic surface in $\mathbb{P}^3$, was worked out in detail by Di Francesco and Itzykson in [DF-I, §3.3]; the general case was treated briefly by Kontsevich and Manin in [K-M, §5.2.3]. The case of arbitrary r (possibly greater than 8 where $-K_X$ is *not* ample) was treated in depth by Göttsche and Pandharipande in [G-P]; they also considered the enumerative significance of the N_β.

The *Hirzebruch surface* of index e is the rational ruled surface,

$$\mathbb{F}_e := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)\big).$$

It has a unique section E over $\mathbb{P}^1$ such that $E^2 = -e$ if $e \geq 1$. However, $\mathbb{F}_0$ is $\mathbb{P}^1 \times \mathbb{P}^1$; in this case, let E be any section. Of course, $\mathbb{F}_0$ can be embedded as the

quadric surface in $\mathbb{P}^3$, and the latter was studied in some detail by Di Francesco and Itzykson in [DF-I, §3.2]. The next surface $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ at a single point, a Del Pezzo again.

For any e, a natural basis for A^1X is given by the class $[E]$ and that $[F]$ of a fiber. For $e \geq 2$, there are infinitely many classes $\beta = a[E] + b[F]$ having $a, b \geq 0$ and given $k(\beta)$. However, only finitely many have $p_a(\beta) \geq 0$. Indeed,

$$k(\beta) = (2-e)a + 2b \text{ and } p_a(\beta) = (a-1)(2b-2-ae)/2.$$

So, if $p_a(\beta) \geq 0$ and $a \geq 2$, then $2b - ae \geq 2$; hence, if also $k(\beta)$ is given, then a is bounded, and so b is bounded too. So the conjecture of the preceding section would imply that the potential Φ is a well-defined power series, but this question is unimportant when Φ is modified as explained in §2. The conjecture would also provide some useful initial conditions, but these conditions can also be obtained by using a number of associativity equations.

The associativity equation $A(1,2,3,3)$ yields the recursion formula,

$$N_\beta = \sum N_{\beta_1}N_{\beta_2}\langle\beta_1\cdot\beta_2\rangle\langle E\cdot\beta_1\rangle\left[\langle F\cdot\beta_2\rangle\binom{k(\beta)-4}{k(\beta_1)-2} - \langle F\cdot\beta_1\rangle\binom{k(\beta)-4}{k(\beta_2)-1}\right].$$

(This formula was worked out for the first time by Ragni Piene and the lecturer in March of 1994.) It turns out experimentally that, on writing $N(a,b;e)$ for N_β where $\beta = a[E] + b[F]$ on $\mathbb{F}_e$, we find the relation,

$$N(a,b;e) = N(a, b+a; e+2).$$

A conceptual explanation for it (explained to the lecturer by Sheldon Katz in September 1994) is this: $\mathbb{F}_e$ degenerates into $\mathbb{F}_{e+2}$, transforming F to F and E to $E + F$, while leaving the quantum cohomology invariant.

The enumerative significance of the N_β is, of course, nontrivial to establish. On the other hand, there are more geometric computations of related numbers, which do not make use of quantum cohomology: Caporaso and Harris have obtained numbers of *irreducible* rational curves of class β on $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$ and of a special β on an arbitrary $\mathbb{F}_e$. Following in their footsteps, Vakil has obtained all the numbers of both the reducible and irreducible curves on an arbitrary $\mathbb{F}_e$ (and the corresponding numbers in arbitrary genus as well!). Abramovich and Bertram, in work in progress, have been obtaining numbers by carefully studying the degeneration of $\mathbb{F}_e$ into $\mathbb{F}_{e+2}$.

Here is a concrete example, which illustrates some of the subtleties involved in the enumeration. (This example was explained to the lecturer by Dan Abramovich in November 1995.) Consider $\mathbb{F}_2$. Note that $k(a[E]+b[F]) = 2b$, which is independent of a! Now, take $\beta := 3[E] + 6[F]$. Then $k(\beta) - 1 = 11$ and

$$N_\beta = 3510 = 2232 + 2\cdot 636 + 6\cdot 1 + 0$$

where the four terms on the right arise as follows:

2232 is the contribution of the irreducible curves of class β through 11 general points.

$2\cdot 636$ is the contribution of the curves breaking up as E union an irreducible curve of class γ, where $\gamma := 2[E] + 6[F]$, through the 11 points. The latter curve meets E twice: $\langle (2E+6F)\cdot E\rangle = -4+6 = 2$. So there are two ways to partially normalize the curve into a tree of $\mathbb{P}^1$s. There are 636 such curves of class γ, because $N_\gamma = 640$ and $N_\gamma = 636 + 4\cdot 1$, where $4\cdot 1$ is the contribution (to N_γ now!) of curves breaking up as E union a curve of class $\delta := [E]+6[F]$. Indeed, $\langle \delta\cdot [E]\rangle = -2+6 = 4$, and therefore there are 4 ways to attach a curve of class δ to E. Finally, there are N_δ curves of class δ through the 11 points, and $N_\delta = 1$ because $p_a(\delta) = 0$.

$6\cdot 1$ is the contribution of curves splitting as $2[E]+\delta$ where $\delta := [E]+6[F]$ as before. The two reduced components again meet in 4 points as the intersection number $\langle (E+6F)\cdot E\rangle$ is 4. There are 6 ways to map a tree of 3 $\mathbb{P}^1$s with 2 nodes onto each curve, mapping the two extreme $\mathbb{P}^1$s to E, the connecting $\mathbb{P}^1$ to the other component, and the 2 nodes to 2 of the 4 points. Finally, as before, there is 1 curve of class δ through the 11 points.

0 is the contribution of the same curves as in the previous case, but with a tree of 2 $\mathbb{P}^1$s with 1 node and 1 $\mathbb{P}^1$ mapping 2-to-1 to E. It can be shown that these form a component of $\overline{M}$ of the wrong dimension and contribution 0, via an explicit analysis of the degeneration of $\mathbb{F}_e$ into $\mathbb{F}_{e+2}$.

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6. About $\overline{M}(X,\beta)$—I. Ciocan-Fontanine, 10/22/96

We go back now to the proof of the properties of $\overline{M}_{0,n}(X,\beta)$ that we used in the enumerative geometry computations and the construction of QH^*. The plan for this lecture is:
—*§1. Overview of the construction of $\overline{M}_{g,n}(X,\beta)$ and of the proofs*
—*§2. The idea behind the construction for $X = \mathbb{P}^r$*
—*§3. Begin the formal proofs*

§1. Overview. Let X be a projective variety over $\mathbb{C}$, $\beta \in A_1 X$. We have a functor

$$\overline{\mathcal{M}}_{g,n}(X,\beta):\quad \{\text{schemes}/\mathbb{C}\} \to \{\text{Sets}\}$$

defined by

$$\mathcal{M}_{g,n}(X,\beta)(S) = \{\text{isom. classes of stable maps of genus } g,\ n\text{-pointed curves, etc.}\}$$

We first list the results we need.

THEOREM 1. *There exists a projective coarse moduli space $\overline{M}_{g,n}(X,\beta)$ for this functor.*

The technical formulation of this statement is: there exists a projective scheme $\overline{M}_{g,n}(X,\beta)$ together with a natural transformation of functors

$$\phi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \operatorname{Hom}(\cdot, \overline{M}_{g,n}(X,\beta))$$

satisfying

(1) $\phi(\operatorname{Spec}(\mathbb{C}))$ is a bijection of sets;
(2) If there is a $\psi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \operatorname{Hom}(\cdot, Z)$, then there is a unique morphism $\overline{M}_{g,n}(X,\beta) \xrightarrow{\gamma} Z$ such that $\psi = \tilde{\gamma}\circ\phi$, with $\tilde{\gamma} = \operatorname{Hom}(\cdot,\gamma)$).

If we assume in addition that X is nonsingular and convex, and that $g = 0$, then we can say more: we have a local description of $\overline{M}_{0,n}(X,\beta)$:

THEOREM 2.
(i) *$\overline{M}_{0,n}(X,\beta)$ is a locally normal projective variety of pure dimension*

$$\dim X + \int_\beta c_1(T_X) + n - 3$$

(ii) *$\overline{M}_{0,n}(X,\beta)$ is locally a quotient of a smooth quasi-projective variety by a finite group;*
(iii) *$\overline{M}^*_{0,n}(X,\beta)$ (=automorphism-free locus) is smooth, with a universal family.*

The *boundary* of $\overline{M}_{0,n}(X,\beta)$ is the complement of the subset parametrizing irreducible curves; that is, $\overline{M}_{0,n}(X,\beta) - M_{0,n}(X,\beta)$.

THEOREM 3. *The boundary of $\overline{M}_{0,n}(X,\beta)$ is a normal crossing divisor, up to a quotient by a finite group.*

Outline of the proofs:

(1) *Existence:* First construct $\overline{M}_{g,n}(\mathbb{P}^r,d)$ and show that this is a projective scheme. Next, for X projective choose an embedding $X \xrightarrow{i} \mathbb{P}^r$, and show that there exists a natural closed subscheme $\overline{M}_{g,n}(X,d) \subset \overline{M}_{g,n}(\mathbb{P}^r,d)$; then $\overline{M}_{g,n}(X,d) = \amalg_{i_*\beta = d(\text{line})}\overline{M}_{g,n}(X,\beta)$. The universality property of the coarse moduli space will imply the independence of the space from the chosen embedding.
(2) *Local structure:* Comes for free from the construction when $X = \mathbb{P}^r$.
(3) *Boundary:* Ditto.

§2. Outline for $X = \mathbb{P}^r$. Let $d, r > 0$, and let $(C;\{p_i\},\mu)$ be a stable n-pointed map to $\mathbb{P}^r$, with image of degree d.

Choose coordinates $(x_0 : \cdots : x_r)$ for $\mathbb{P}^r$. The map $(C;\{p_i\},\mu)$ determines and is determined by the data of
$(C,\{p_i\})$, n-pointed quasi-stable;
a line bundle $\mathcal{L}$ on C, that is $\mathcal{L} = \mu^*\mathcal{O}_{\mathbb{P}^r}(1)$;
$r+1$ general sections $s_i = \mu^*(x_i)$.
This is what we would like to parametrize.

A generic map (if there is one) will have transversal intersection with the coordinate hyperplanes $\{x_i = 0\}$, away from $\{p_i\}$ and from the nodes. The divisor of s_i consists then of distinct points $\{q_{i1},\ldots,q_{id}\}$: we get additional $d(r+1)$ marked points on C. Assume C is generic:

CLAIM. *($(r,d) \neq (1,1)$) μ is stable if and only if $(C;\{p_i\},\{q_{ij}\})$ is Deligne-Mumford-Knudsen stable.*

PROOF. $\Longrightarrow$: Say that $E \subset C$, $E \cong \mathbb{P}^1$; if $\mu(E)$ is a point, we are done by the stability of μ. Otherwise, there are two cases: if $E = C$, all $d(r+1) \geq 3$ extra marked points are on C; and if $E \neq C$, we have at least one node, and $(r+1) \geq 2$ of the new markings on E. So every contracted component has at least 3 special points, as needed.

The other direction is easier. $\square$

Summing up, the generic $(C, \{p_i\}, \mu)$ determines a point in $\overline{M}_{g,m}$, with $m = n + d(r+1)$. Note that there is a $\mathbb{C}^*$ ambiguity in the choices of each of the s_i, modulo a $\mathbb{C}^*$ by homogeneity. That is, given the points corresponding to the sections (the information in the q_{ij}'s) there is still a $(\mathbb{C}^*)^r$ worth of additional choices to be made.

Further, we have to take account of the permutations of the points in each hyperplane section, that is, of the action of $G = S_d \times \cdots \times S_d = (S_d)^{r+1}$, where the i-th factor permutes $\{q_{i1}, \ldots, q_{id}\}$.

What about nongeneric curves? All curves are generic for some choice of the coordinates; a given choice gives a coordinate patch for an open in a cover of $\overline{M}$. Then we need to show that these coordinate patches do patch, and a boundedness result to show that the scheme is of finite type. Then we need to show that the scheme is proper; and finally that it is projective (this is technically harder).

Summarizing: we have to consider a (quotient of a) torus bundle over a subset of $\overline{M}_{g,m}$. Which subset? (In the genus-0 case, we will end up with an open subset of $\overline{M}_{0,m}$.) Let $B \subset \overline{M}_{g,m}$ be the subset determined as above. Remark: on C, the r divisors $\operatorname{div} s_i$ are linearly equivalent; this puts a condition on $\overline{M}_{g,m}$, as we need the divisors $\{q_{01}, \ldots, q_{0d}\}$, $\ldots$, $\{q_{r1}, \ldots, q_{rd}\}$ on $C \in B$ to be linearly equivalent divisors (and very ample). In particular, the degrees of these divisors on a given component must be equal; in genus 0, this suffices essentially to determine everything, as the number of points in a divisor determines its class. In genus>0, the ambiguity is measured by the Jacobian; in practice, we can't get our hands on the local structure of the resulting spaces.

§**3. Formal proofs.** Say $\mathbb{P}^r = \mathbb{P}V$, where V is an $(r+1)$-dimensional complex vector space; $H^0(\mathcal{O}(1)) = V^*$ Fix $\bar{t} = \{t_0, \ldots, t_r\}$, a basis of V^*.

Definition 1. A $\bar{t}$-stable family of degree-d maps from n-pointed curves to $\mathbb{P}^r$ consists of

$$(\pi : \mathcal{C} \to S; \{p_i\}_{i=1,\ldots,n}, \{q_{ij}\}_{0\le i\le r, 1\le j\le d}, \mu)$$

such that

(i) $(\pi : \mathcal{C} \to S, \{p_i\}, \mu)$ is a stable family of degree-d maps from n-pointed curves to $\mathbb{P}^r$;
(ii) $(\pi : \mathcal{C} \to S; \{p_i\}, \{q_{ij}\})$ is a stable $m = n + d(r+1)$-pointed curve;
(iii) $\mu^*(t_i) = q_{i1} + \cdots + q_{id}$ as effective Cartier divisors.

Definition 2. $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d, \bar{t})$ is the functor of $\bar{t}$-rigid stable maps: that is, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d, \bar{t})(S) =$\{isomorphism classes of families over S as in Definition 1\}.

Proposition 1. *There exists a quasi-projective moduli space $\overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$, which is coarse for $g > 0$ and fine and nonsingular for $g = 0$.*

Proof. (Only for $g = 0$.) Let $\overline{M} = \overline{M}_{0,m}$; $\pi : \overline{U} \to \overline{M}$ the universal family; p_i, q_{ij} are sections of $\overline{U} \to \overline{M}$. As $\overline{U}$ is nonsingular, $q_{i1} + \cdots + q_{id}$ determines a line bundle $\mathcal{H}_i = \mathcal{O}_{\overline{U}}(q_{i1} + \cdots + q_{id})$. Denote by s_i a corresponding section, that is a 'global equation' for $q_{i1} + \cdots + q_{id}$.

Denote by $B \subset \overline{M}$ the open subscheme determined by the property that for $b \in B$, the geometric fiber $C_b = \overline{U}_b$ will have $\deg \mathcal{H}_i = \deg \mathcal{H}_j$ on every component

of C_b.

$$\begin{array}{ccc}
\overline{U}_B & \xrightarrow{\ \bar{i}\ } & \overline{U} \\
{\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi} \\
B & \xrightarrow{\ i\ } & M
\end{array}$$

(Note: for $g > 0$, we would require instead that $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ 'comes from the base' for all i.) Now $(\pi_B)_* \bar{i}^* (\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is a line bundle $\mathcal{G}_i$ on B, and has an associated $\mathbb{C}^*$-bundle $\tau_i : Y_i \to B$ obtained by deleting the zero-section.

Let then $Y = Y_1 \times_B \cdots \times_B Y_r$, coming with projections $\rho_i : Y \to Y_i$, $\tau : Y \to B$. Note that $\tau_i^* \mathcal{G}_i$ is canonically trivial.

$$\begin{array}{ccccc}
\overline{U}_Y & \xrightarrow{\ \bar{\tau}\ } & \overline{U}_B & \xrightarrow{\ \bar{i}\ } & \overline{U} \\
{\scriptstyle \pi_Y}\downarrow & & {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\ \tau\ } & B & \xrightarrow{\ i\ } & \overline{M}
\end{array}$$

Claim. *On $\overline{U}_Y$ there is a canonical isomorphism $\bar{\tau}^* i^* \mathcal{H}_i \cong \bar{\tau}^* i^* \mathcal{H}_0 =: \mathcal{L}$.*

Pf. $\bar{\tau}^* \bar{i}^* \mathcal{H}_i \otimes \mathcal{H}_0^{-1} = \bar{\tau}^* \pi_B^* \pi_{B*} \bar{i}^* \mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ (since $\bar{i}^* \mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ comes from B, $\pi_B^* \pi_{B*}$ leaves it alone), which equals $\pi_Y^* \tau^* \mathcal{G}_i = \pi_Y^* \rho_i^* \tau_i^* \mathcal{G}_i$, and $\tau_i^* \mathcal{G}_i$ is canonically trivial. □

Now $\bar{\tau}^* s_i$ generates $\bar{\tau}^* \mathcal{H}_i$, for $i = 0, \ldots, r$. There exists a unique $\mu : \overline{U}_Y \to \mathbb{P}^r$ such that $\mu^* \mathcal{O}(1) = \mathcal{L}$ and $\mu^* t_i = \bar{\tau}^* s_i$.

Claim. *$(\pi_Y : \overline{U}_Y \to Y, \{p_i\}, \{q_{ij}\}, \mu)$ is a universal family over Y.*

That is, Y represents the functor of $\bar{t}$-rigid stable maps.

Pf. By construction it is a $\bar{t}$-rigid stable family. To show it is universal: pick another $(\pi_S : \mathcal{C}_S \to S, \{p_i\}, \{q_{ij}\}, \nu)$, and show that there is a unique morphism $S \to Y$ such that this family is canonically isomorphic to the pull-back of $\overline{U}_Y$. For this, $\pi_S : \mathcal{C}_S \to S$ is a family of m-pointed curves, in particular, so there is a unique $\lambda : S \to \overline{M}$:

$$\begin{array}{ccc}
\mathcal{C}_S & \xrightarrow{\ \bar{\lambda}\ } & \overline{U} \\
{\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi} \\
S & \xrightarrow{\ \lambda\ } & \overline{M}
\end{array}$$

$\lambda^* \mathcal{H}_i \cong \nu^* \mathcal{O}_{\mathbb{P}^r}(1)$, so $S \xrightarrow{\lambda} \overline{M}$ factors $S \xrightarrow{\lambda} B \to \overline{M}$. There is a canonical isomorphism:

$$\mathcal{O}_S \cong \pi_{S*} \bar{\lambda}^* \mathcal{H}_i \otimes \mathcal{H}_0^{-1} \cong \lambda^* \mathcal{G}_i \quad ;$$

because of this, there is a canonical isomorphism $\pi_{S*} \pi_S^* \mathcal{N} \cong \mathcal{N}$ for all line bundles $\mathcal{N}$ on S; in particular, there are canonical isomorphisms

$$\lambda^* \mathcal{G}_i \cong \pi_{S*} \pi_S^* \lambda^* \mathcal{G}_i = \pi_{S*} \bar{\lambda}^* \pi_S^* \mathcal{G}_i = \pi_{S*} \bar{\lambda}^* \pi_B^* \pi_{B*} \bar{i}^* (\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \cong \pi_{S*} \bar{\lambda}^* (\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$$

Therefore there is a canonical $S \to Y$ commuting in

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, "\lambda"'] & & Y \arrow[dl, "\tau"] \\ & B & \end{tikzcd}$$

The pull-back of the universal family is the given family on S because at each step all choices were canonical. $\square$

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7. The construction of $\overline{M}_{0,n}(\mathbb{P}^r, d)$—J. Thomsen, 11/5/96

We fix n, r, d, and only deal with the genus$=0$ case. Let $V^* = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$, and let $\overline{t} = \{t_0, \dots, t_r\}$ be a basis of V^*. Consider the functor

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d, \overline{t}) : \quad \text{Alg. schemes over } \mathbb{C} \to \text{Sets}$$

sending S to the set of isomorphism classes of $\overline{t}$-rigid families over S. Here a *$\overline{t}$-rigid family* over S is an object

$$(\pi : C \to S; \{p_i\}_{i=1,\dots,n}, \{q_{ij}\}_{0\le i\le r, 1\le j\le d}, \mu : C \to \mathbb{P}^r)$$

where

(i) $(\pi : C \to S, \{p_i\}, \mu)$ is a stable family of degree-d maps from n-pointed genus-0 curves to $\mathbb{P}^r$;
(ii) $(\pi : C \to S, \{p_i\}, \{q_{ij}\})$ is a stable family of $m = (n + d(r+1))$-pointed genus-0 Deligne-Mumford stable curves;
(iii) $\mu^*(t_i) = q_{i1} + \cdots + q_{id}$, $i = 0, \dots, r$.

Recall from Ciocan-Fontanine's lecture:

THEOREM. *There exists a quasiprojective nonsingular variety $\overline{M}(\overline{t})$, which is a fine moduli space with respect to $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d, \overline{t})$.*

Our task is to patch these moduli spaces together into a space $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$.

Suppose that we already have a fine moduli space $\overline{M}$ for $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. For a $\overline{t}$-rigid family $(\pi : C \to S, \{p_i\}, \{q_{ij}\}, \mu)$, we consider the diagram

$$\begin{tikzcd} D_i \arrow[rr, dashed] \arrow[d] & & D_i \arrow[r] \arrow[d] & D_i \arrow[r] \arrow[d] & D_i = Z(\mu^* t_i) \arrow[d] & \\ C \arrow[r] \arrow[d] & \mathcal{U}_{\mathcal{D}} \arrow[r] \arrow[d] & \mathcal{U}_{\overline{t}} \arrow[r] \arrow[d] & \mathcal{U}_{\mathrm{tr}} \arrow[r] \arrow[d] & \mathcal{U} \arrow[r, "\mu"] \arrow[d] & \mathbb{P}^r \\ S \arrow[r] & \mathcal{D} \arrow[r, "\varphi"] \arrow[uur, dashed] & \overline{M}_{\overline{t}} \arrow[r] & \overline{M}_{\mathrm{tr}} \arrow[r] & \overline{M} & \end{tikzcd}$$

The map $S \to \overline{M}$ realizing the family via pull-back is not surjective, as the curves over S intersect the D_i transversally (by rigidity): so the map must factor through the open subset

$$\overline{M}_{\mathrm{tr}} = \{\text{curves intersecting the } D_i\text{'s transversally}\}$$

Similarly, the map factors through the smaller open $\overline{M}_{\overline{t}}$ where the p_i's are distinct from the D_i's, and the D_i's intersect trivially.

Next, the q_{ij} define a map from S to

$$\mathcal{D} = \prod_i \left[\left(\underbrace{D_i \times_{\overline{M}_{\overline{t}}} \cdots \times_{\overline{M}_{\overline{t}}} D_i}_{d} \right) - \text{ big diagonals} \right]$$

through which the map to $\overline{M}_{\overline{t}}$ also factors. As $\mathcal{D}$ is independent of the $\overline{t}$-rigid family we started with, it is a concrete realization of $\overline{M}(\overline{t})$.

Also, $G_{d,r} = \underbrace{S_d \times \cdots \times S_d}_{r+1}$ acts on $\mathcal{D}$ over $\overline{M}_{\overline{t}}$; and we claim that

$$\mathcal{D}/G_{d,r} \xrightarrow{\phi/G_{d,r}} \overline{M}_{\overline{t}}$$

is an isomorphism. To see this (at least on closed fibers), look at the fiber over $s \in S \to \overline{M}_{\overline{t}}$:

$$\begin{tikzcd} \pi_i^{-1}(s) \arrow[r] \arrow[d] & D_i \arrow[d, "\pi_i"] \\ \{s\} \arrow[r] & \overline{M}_{\overline{t}} \end{tikzcd} \qquad \begin{tikzcd} \varphi^{-1}(s) \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "\varphi"] \\ \{s\} \arrow[r] & \overline{M}_{\overline{t}} \end{tikzcd}$$

Clearly $\pi_i^{-1}(s) = d$ points of intersection of t_i with (the curve represented by) s; and

$$\varphi^{-1}(s) = \left\{ (x_{01}, \dots, x_{0d}, \dots, x_{r1}, \dots, x_{rd}) \,\middle|\, x_{ij} \text{ distinct, and } \{x_{ij}\} = \pi_i^{-1}(s) \right\}$$

Summarizing, the open $\overline{M}_{\overline{t}}$ of $\overline{M}$ (if the latter exists) must be isomorphic to $\mathcal{D}/G_{d,r} = \overline{M}(\overline{t})/G_{d,r}$. So we should be able to construct $\overline{M}_{0,n}(\mathbb{P}^r, d)$ by gluing together $\overline{M}(\overline{t})/G_{d,r}$, where $\overline{t}$ runs through all bases of V^*.

Now

$$\overline{M}(\overline{t})/G_{d,r} \leftrightarrow \begin{array}{l} \text{curves (i.e., points on } \overline{M}_{0,n}(\mathbb{P}^r, d)) \text{ which the hyperplanes } t_i \\ \text{intersect transversally in distinct points and away from the} \\ \text{marked points} \end{array}$$

If $\overline{t}'$ is another basis,

$$\text{“}\overline{M}(\overline{t})/G_{d,r} \cap \overline{M}(\overline{t}')/G_{d,r}\text{”} \leftrightarrow \begin{array}{l} \text{curves which the hyperplanes } t_i \text{ and } t_i' \\ \text{intersect transversally in distinct points (each)} \\ \text{and away from marked points} \end{array}$$

In order to control the patching, we have to construct this intersection explicitly (as we did for $\overline{M}(\overline{t})/G_{d,r}$) a moment ago). This will be obtained as the quotient by $G_{d,r} \times G_{d,r}$ of a suitable $\overline{M}(\overline{t}, \overline{t}')$. In fact it is useful and not harder to consider arbitrary (finite) sets of bases at once:

DEFINITION. Let $\{\bar{t}_\ell\}_{1\le\ell\le h}$ be a set of bases of V^*. Then

$$(\pi : C \to S, \{p_i\}_{1\le i\le n}, \{q_{ij\ell}\}_{0\le i\le r, 1\le j\le d, 1\le \ell\le h}, \mu)$$

is called a $\{\bar{t}_\ell\}_{1\le\ell\le h}$*-rigid family* if for all ℓ,

$$(\pi : C \to S, \{p_i\}_{1\le i\le n}, \{q_{ij\ell}\}_{0\le i\le r, 1\le j\le d}, \mu)$$

is $\bar{t}_\ell$-rigid.

DEFINITION. Define a functor $\overline{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_h)$ from Algebraic Schemes to Sets sending S to the set of isomorphism of $\{\bar{t}_\ell\}_{1\le\ell\le h}$-rigid families over S (where isomorphisms of families, etc. are defined in the usual way).

THEOREM. *There is a fine moduli space $\overline{M}(\bar{t}_1, \dots, \bar{t}_h)$ for this functor, carrying a $G_{d,r} \times \cdots \times G_{d,r}$-action.*

The idea of course is that the intersection "$\cap_{\ell=1}^h \overline{M}(\bar{t}_\ell)/G_{d,r}$" should correspond to $\overline{M}(\bar{t}_1, \dots, \bar{t}_h)/G_{d,r} \times \cdots \times G_{d,r}$.

PROOF OF THE THEOREM. This is done by induction on h; the case $h = 1$ has already been done. For $h > 1$, assume $\overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})$ has been constructed already, let $\bar{t}_h = (t_{h0}, \dots, t_{hr})$ and consider

$$\begin{tikzcd}
D_i \arrow[r] \arrow[d] & D_i \arrow[r] \arrow[d] & D_i \arrow[r] \arrow[d] & D_i = Z(\mu^* t_{hi}) \arrow[d] & \\
\mathcal{U}_{\mathcal{D}} \arrow[r] \arrow[d] & \mathcal{U}(\bar{t}_h) \arrow[r] \arrow[d] & \mathcal{U}_{\mathrm{tr}} \arrow[r] \arrow[d] & \mathcal{U} \arrow[r, "\mu"] \arrow[d] & \mathbb{P}^r \\
\mathcal{D} \arrow[r] \arrow[uur, dashed] & \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})(\bar{t}_h) \arrow[r] & \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})_{\mathrm{tr}} \arrow[r] & \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1}) &
\end{tikzcd}$$

(i) $\overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})_{\mathrm{tr}} \subset \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})$ is the maximal open subscheme where the sheaf of relative differentials $\Omega_{D_i/\overline{M}(\bar{t}_1,\dots,\bar{t}_{h-1})} = 0$;
(ii) $\overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})(\bar{t}_h) \subset \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})_{\mathrm{tr}}$ is the maximal open subscheme where the p_i and the D_i are distinct;
(iii) $\mathcal{D} = \prod_i ((D_i \times_M \cdots \times_M D_i) -$ big diag.) where $M = \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})(\bar{t}_h)$.

The natural projections $\mathcal{D} \to D_i$ give the new sections q_{ijh}; $\overline{M}(\bar{t}_1, \dots, \bar{t}_h) := \mathcal{D}$ is the fine moduli space for $\overline{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_h)$. The action of the group is the natural one. $\square$

Note that there will be natural maps

$$\overline{M}(\bar{t}_1, \dots, \bar{t}_h) \xrightarrow{\varphi_{\bar{t}_1,\dots,\bar{t}_{h-1},\bar{t}_h}} \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})(\bar{t}_h) \overset{\text{open}}{\hookrightarrow} \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})$$

and

(1) $\varphi_{\bar{t}_1,\dots,\bar{t}_{h-1},\bar{t}_h}$ is $(G_{d,r})_1 \times \cdot \times (G_{d,r})_{h-1}$-equivariant;
(2) $\overline{M}(\bar{t}_1, \dots, \bar{t}_h)/(G_{d,r})_h \cong \overline{M}(\bar{t}_1, \dots, \bar{t}_{h-1})(\bar{t}_h)$

Now we can glue the spaces together. The data of the gluing:
- $\overline{M}_{\bar{t}} = \overline{M}(\bar{t})/G_{d,r}$, $\bar{t}$ running through all bases of V^*;
- $\overline{M}_{\bar{t}\bar{t}'}$, defined as $\overline{M}(\bar{t})(\bar{t}')/G_{d,r}$;
- $\varphi_{\bar{t}\bar{t}'} : \overline{M}_{\bar{t}\bar{t}'} \to \overline{M}_{\bar{t}'\bar{t}}$, isomorphisms induced from

$$\begin{tikzcd}
\overline{M}(\bar{t}, \bar{t}')/G_{d,r} \times G_{d,r} \arrow[r, "\sim"] \arrow[d] & \overline{M}(\bar{t}', \bar{t})/G_{d,r} \times G_{d,r} \arrow[d] \\
\overline{M}(\bar{t})(\bar{t}')/G_{d,r} = \overline{M}_{\bar{t}\bar{t}'} \arrow[r, "\varphi_{\bar{t}\bar{t}'}"] & \overline{M}_{\bar{t}'\bar{t}} = \overline{M}(\bar{t}')(\bar{t})/G_{d,r}
\end{tikzcd}$$

Compatibility is checked on triple intersection. The resulting scheme is $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

Next, we will check that $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is of finite type over $\mathbb{C}$.

Let $S \in \overline{M}_{0,n}(\mathbb{P}^r, d)$, and let $(C; \{p_i\}, \mu)$ be a corresponding curve. Define

$$\mathcal{L} = \omega_C(p_1 + \cdots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3))$$

CLAIM. *$\mathcal{L}$ is ample.*

Indeed, let $E \subset C$ be a component. Then

(i) $(\omega_C)_{|E} = \omega_E \otimes \mathcal{O}(\text{nodes of } C \text{ along } E)$;
(ii) $\deg(\omega_C(p_1 + \cdots + p_n)_{|E}) = -2 + \#$ special points on E;
(iii) $\deg(\mathcal{L}_{|E}) = -2 + \#$ special points on $E + 3\,d_E > 0$ (where d_E =degree of $\mu_*[E]$)

as needed. One can in fact check that

CLAIM. *$\mathcal{L}^2$ is very ample, and $h^1(C, \mathcal{L}^2) = 0$.*

Define

$$e := \deg \mathcal{L}^2 = 2 \deg \mathcal{L} = 2(-2 + n + 3d)$$

then it follows from Riemann-Roch that $h^0(C, \mathcal{L}^2) = e + 1$. Note that e is independent of C.

Denote by $i : C \hookrightarrow \mathbb{P}^e$ the embedding induced by $\mathcal{L}^2$, and let $\gamma = (i, \mu) : C \hookrightarrow \mathbb{P}^e \times \mathbb{P}^r$. The image of C in $\mathbb{P}^e \times \mathbb{P}^r$ has bidegree (e, d).

Look then at the Hilbert scheme H of genus-0 curves in $\mathbb{P}^e \times \mathbb{P}^r$, of bidegree (e, d); this comes with a universal family $W \to H$. Denote by H_n the space $\underbrace{W \times_H \cdots \times_H W}_{n}$; H_n is a fine moduli space for families of curves of genus 0 and bidegree (e, d), together with n sections. It has a universal family $\mathcal{U}_n$:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & \mathcal{U}_n \arrow[r] \arrow[d] & \mathbb{P}^e \times \mathbb{P}^r \times H_n \arrow[r] & \mathbb{P}^r \\
\mathrm{Spec}\,\mathbb{C} \arrow[r] & H_n & &
\end{tikzcd}$$

The space H_n is of finite type over $\mathcal{C}$; we want to deduce that $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is then also of finite type.

Given a basis $\overline{t}$ of V^*, we let $H_{n,\overline{t}} \subset H_n$ be the open subscheme where the elements of the basis intersect the curve transversally and in distinct points. Let $H_b = \cup_{\overline{t}\text{ basis of } V^*} H_{n,\overline{t}}$.

(i) H_b is of finite type, so H_b can be covered by a finite number of $H_{n,\overline{t}}$;
(ii) by Bertini's theorem, every stable map from n-pointed genus-0 curves to $\mathbb{P}^r$ is induced from H_b.

The conclusion is that there exist a finite number $\overline{t}_1, \dots, \overline{t}_h$ of bases of V^* such that

$$\overline{M}_{0,n}(\mathbb{P}^r, d) = \cup_{i=1}^h \overline{M}(\overline{t}_i)$$

The space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is then of of finite type, since it is covered by finitely many schemes of finite type.

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8. $\overline{M}_{0,n}(X,\beta)$—E. Tjøtta, 11/12/96

Our goal:

THEOREM 1. *If X is projective, there exists a coarse moduli space $\overline{M}_{0,n}(X,\beta)$.*

THEOREM 2. *For X nonsingular and convex:*

i) $\dim \overline{M}_{0,n}(X,\beta) = \dim X + \int_\beta c_1(TX) + n - 3$;
ii) *$\overline{M}_{0,n}(X,\beta)$ is locally a quotient of a nonsingular variety by a finite group.*

Recall how this was done for $X = \mathbb{P}^r$: we constructed a fine moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d, \overline{t})$ for $\overline{t}$-rigid maps, where $\overline{t} = \{t_0, \dots, t_r\}$ is a basis for $H^0(\mathbb{P}^r, \mathcal{O}(1))$. This space $\overline{M}_{0,n}(\mathbb{P}^r, d, \overline{t})$ is a $(\mathbb{C}^*)^r$-bundle over an open B in $\overline{M}_{0,m}$, with $m = n + d(r+1)$. The group $G = (S_d)^{r+1}$ acts on $\overline{M}_{0,n}(\mathbb{P}^r, d, \overline{t})$, and the quotients as $\overline{t}$ varies glue together, giving $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

A) $\overline{M}_{0,n}(X,\beta)$. Let X be a subvariety of $\mathbb{P}^r$: $X \overset{i}{\hookrightarrow} \mathbb{P}^r$; let $i_*\beta = d[\text{line}]$. There exists a closed subscheme $\overline{M}_{0,n}(X,\beta,\overline{t}) \subset \overline{M}_{0,n}(\mathbb{P}^r, d, \overline{t})$ such that for all families of $\overline{t}$-rigid stable maps $\mu : \mathcal{C} \to \mathbb{P}^r$ over a base S, $\exists\tilde{\mu} \iff \exists\tilde{\nu}$ in the commutative diagram:

$$\begin{tikzcd}
 & & X \arrow[d] \\
\mathcal{C} \arrow[r] \arrow[d, "\pi"'] \arrow[urr, dashed, "\tilde{\mu}"] & \mathcal{U} \arrow[r] \arrow[d] & \mathbb{P}^r \\
S \arrow[r, "\nu"] \arrow[dr, dashed, "\tilde{\nu}"'] & \overline{M}_{0,n}(\mathbb{P}^r, d, \overline{t}) & \\
 & \overline{M}_{0,n}(X,\beta,\overline{t}) \arrow[u] &
\end{tikzcd}$$

(with $\mathcal{U}$ the universal family, and sections $S \to \mathcal{C}$, etc., as usual).

PROOF. For $\mu : \mathcal{C} \to X$ and $k > 0$, $H^1(\mathcal{C}, \mu^*\mathcal{O}(k)) = 0$; by base change, $\pi_*\mu^*\mathcal{O}(k)$ is a vector bundle with fibers $\pi_*\mu^*\mathcal{O}(k)_s = H^0(\mathcal{C}_s, \mu^*\mathcal{O}(k))$. Let $\ell > 0$ such that $\mathcal{I}_X(\ell)$ is generated by global sections. With $M = \overline{M}_{0,n}(\mathbb{P}^r, d, \overline{t})$ in

$$\begin{tikzcd}
H^0\mathcal{I}_X(\ell)_M \arrow[d] \arrow[dr, "\rho"] & \\
H^0\mathcal{O}_{\mathbb{P}^r}(\ell)_M \arrow[r] & \pi_*\mu^*\mathcal{O}(\ell)
\end{tikzcd}$$

we identify $\rho(F)$ over $s \in S$ with $\mu^*F_{|\mathcal{C}_s}$. Then $\mu(\mathcal{C}_s) \subset X \iff s \in Z =$ the zero scheme of $\{\rho(F) : F \in H^0\mathcal{I}_X(\ell)\}$.

Let then $\overline{M}_{0,n}(X,\beta,\overline{t}) = \{s \in Z : [\mu(C_s)] = \beta\}$ be the component of Z determined by β. The group G acts on $\overline{M}_{0,n}(X,\beta,\overline{t})$; we construct $\overline{M}_{0,n}(X,\beta)$ by gluing the various quotients. $\square$

Note: this works also for general genus.

B) Assume X is nonsingular and convex. Recall that X convex
$\iff \forall\varphi : \mathbb{P}^1 \to X$, $H^1(\mathbb{P}^1, \varphi^*TX) = 0$, that is
$\iff \forall\varphi : \mathbb{P}^1 \to X$, $\varphi^*TX = \oplus\mathcal{O}(n_i), n_i \geq 0$.

Claim: this is equivalent to $\forall\varphi : C \to X$, C at worst nodal, of arithmetic genus 0, $H^1(C, \varphi^*TX) = 0$.

The proof is by induction on the number of components. If $C = C' \cup L$, and $C' \cap L = \{p\}$, we have the exact sequence on C

$$0 \to \varphi^*TX \to \varphi^*TX_{|C'} \oplus \varphi^*TX_{|L} \to \varphi^*TX_p \to 0 \quad ,$$

whose long exact sequence yields the vanishing of $H^1(\varphi^*TX)$.

C) Local study of $M = \overline{M}_{0,n}(X,\beta,\overline{t})$. Let $[\mu] \in M$, that is, $[\mu]$ is an object $(\mu : C \to X; p_i, q_{ij})$. For $D = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and $d_0 =$ the closed point of D,

$$\begin{aligned}
TM_{[\mu]} &= \{\varphi : D \to M | \varphi(d_0) = [\mu]\} \\
&= \text{space of first order deformations of } (\mu : C \to X; p_i, q_{ij}) \\
&= \text{space of first order deformations of } (\mu : C \to X; p_i) =: \operatorname{Def}(\mu)
\end{aligned}$$

Let $\operatorname{Def}_G(\mu)$ be the space of first order deformations of $(\mu : C \to X; p_i)$, preserving the combinatorial type G of C. Then $\operatorname{Def}_G(\mu) \subset \operatorname{Def}(\mu)$, and its codimension is $\leq q =$ the number of nodes of C. We will show that this is in fact an equality.

Consider $\operatorname{Hom}(C, X) \times (C^n \setminus \text{diag.})$, containing the open subset $H^{st}_\beta(C)$ defined by $\{\text{stable } (\varphi; p_1, \dots, p_n) : [\operatorname{im}\varphi] = \beta\}$, mapping to $\overline{M}_{0,n}(X,\beta)$. Act on $H^{st}_\beta(C)$ with $\operatorname{Aut}(C)$:

$$\begin{gathered}
\operatorname{Aut}(C) \times H^{st}_\beta(C) \to H^{st}_\beta(C) \\
(\psi, (\varphi; p_1, \dots, p_n)) \mapsto (\varphi \circ \psi; \psi(p_1), \dots, \psi(p_n))
\end{gathered}$$

This action is not free, but has finite stabilizers, giving étale maps $\operatorname{Aut}(C) \to$ orbit of $(\varphi; p_1, \dots, p_n)$.

Relativize this construction: for a flat family $\mathcal{C} \to S$ of $g = 0$ nodal curves, consider $\mathcal{C}^n = \underbrace{\mathcal{C} \times_S \cdots \times_S \mathcal{C}}_{n}$, $\mathrm{Hom}_S(\mathcal{C}, X \times S) \times_S (\mathcal{C}^n \setminus \mathrm{diag})$ and its open subset $H^{st}_\beta(\mathcal{C})$, mapping (over S) to $M_{0,n}(X, \beta, \bar{t})$. $\mathrm{Aut}_S(\mathcal{C})$ acts fiberwise with finite stabilizer on $H^{st}_\beta(\mathcal{C})$.

Now $\mathrm{Hom}_S(\mathcal{C}, X \times S)$ is an open subset of the Hilbert scheme of graphs. This allows us to compute its tangent space and dimension: for $\mu : \mathcal{C}_s \to X$,

—$T\mathrm{Hom}(\mathcal{C}_s, X)_{[\mu]} = H^0(\mathcal{C}_s, \mu^* TX)$;

—for X nonsingular, the dimension of all components of $\mathrm{Hom}_S(\mathcal{C}, X \times S)$ at $[\mu]$ is at least the expected one, that is $\dim H^0 \mu^* TX - \dim H^1 \mu^* TX + \dim S$;

—by convexity, $H^1 = 0$. The dimension equals the expected one.

The fibers of $\mathrm{Hom}_S(\mathcal{C}, X \times S) \to S$ have dimension $H^0 \mu^* TX$; if S is smooth, $\mathrm{Hom}_S(\mathcal{C}, X \times S) \to S$ is smooth at $[\mu]$. It follows that $H^{st}_\beta(\mathcal{C}) \to S$ is smooth at $[\mu]$, of relative dimension $\dim H^0 \mu^* TX + n$ (*).

Again let $\mathrm{Def}_G(C)$ be the space of first order deformations of C preserving the combinatorial type G, and consider the universal base space B of deformations of C preserving G. The space B can be seen to be smooth as follows. Stabilize C by replacing unstable components with marked points to get a marked curve C^{st} in some $\overline{M}_{0,m}$. The locus of curves in $\overline{M}_{0,m}$ determined by the combinatorial type of C^{st} is smooth (see Belorousski's talk on $\overline{M}_{0,n}$). Then B is a ball around C^{st} in this locus. Taking $S = B$ in the above, and $\mathcal{U} \to B$ the universal curve, we obtain $H^{st}_\beta(\mathcal{U}) \to \overline{M}_{0,n}(X, \beta)$ over B. Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H^0(C, TC) \arrow[r, equal] \arrow[d] & T\mathrm{Aut}(C)_{id} \arrow[d] & \mathrm{Def}_G(C) \arrow[d, equal] & \\
0 \arrow[r] & H^0 \mu^* TX \oplus \mathbb{C}^n \arrow[r] \arrow[d] & TH^{st}_\beta(\mathcal{U})_{[\mu]} \arrow[r] \arrow[d] & TB_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Coker} \arrow[r] \arrow[d] & \mathrm{Def}_G(\mu) \arrow[r] \arrow[d] & \mathrm{Def}_G(C) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where the middle vertical map is the differential of $H^{st}_\beta(\mathcal{U}) \to \overline{M}_{0,n}(X, \beta)$. By Riemann-Roch and (*) we get:

$$\begin{aligned}
\dim \mathrm{Def}_G(\mu) &= \dim \mathrm{Def}_G(C) + \dim H^0 \mu^* TX + n - \dim H^0(C, TC) \\
&= \Big(\sum_{|v| \geq 4} (|v| - 3)\Big) + \Big(\dim X + \int_\beta c_1 TX\Big) + n - \Big(\sum_{|v| \leq 3} (3 - |v|)\Big)
\end{aligned}$$

where the $\sum$ is over the vertices v of G, and $|v|$ denotes valence. This gives

$$\dim \mathrm{Def}_G(\mu) = \dim X + \int_\beta c_1 TX + n - 3 - q$$

with $q = \#$ nodes of C.

Let $\mathcal{C} \to S$ be a smoothing of C, with S smooth. By (*) we have that $M_{0,n}(X, \beta)$ is dense in $\overline{M}_{0,n}(X, \beta)$, hence $\dim M_{0,n}(X, \beta) \leq \dim \mathrm{Def}(\mu)$. Since $H^{st}_\beta(\mathbb{P}^1) \to M_{0,n}(X, \beta)$ is surjective, $\dim M_{0,n}(X, \beta) = \dim X + \int_\beta c_1(TX) + n - 3$. Putting all together,

$$\dim X + \int_\beta c_1(TX) + n - 3 \leq \dim \mathrm{Def}(\mu) \leq \dim \mathrm{Def}_G(\mu) + q = \dim X + \int_\beta c_1(TX) + n - 3 \quad ,$$

giving equalities through. Therefore

$$\dim T\overline{M}_{0,n}(X, \beta, \bar{t})_{[\mu]} = \dim \overline{M}_{0,n}(X, \beta, \bar{t})$$

and the space is smooth, as needed for Theorem 2.

D) Boundary. The boundary is studied by using the fact that $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$ is a $(\mathbb{C})^r$-bundle over an open subset of $\overline{M}_{0,m}$ (where $m = n + d(r+1)$). The conclusion is that the boundary of $\overline{M}_{0,n}(X, \beta, \bar{t})$ is a divisor with normal crossings.

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9. The boundary of $\overline{M}_{0,n}(X,\beta)$—E. Rødland, 11/19/96

Reminder of the construction: $\overline{M}_{0,n}(X, \beta)$ is obtained by gluing together quotients $\overline{M}_{0,n}(X, \beta, \bar{t})/G_{d,r}$, where $\overline{M}_{0,n}(X, \beta, \bar{t})$ are closed subschemes of schemes $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$, in turn obtained as $(\mathbb{C}^*)^r$-covers of a Zariski-open $\mathcal{B}$ of $\overline{M}_{0,m}$, where $m = n + d(r+1)$. The boundary $\partial \overline{M}_{0,m}$ of $\overline{M}_{0,n}(X, \beta)$, that is the subset corresponding to maps from reducible curves, will be constructed from the boundary of $\overline{M}_{0,m}$.

Recall that $\partial \overline{M}_{0,m}$ is a normal-crossing divisor, with components $D(A|B)$, where $A \cup B = \{1, \ldots, m\}$: points of $D(A|B)$ correspond to curves $C = C_A \cup C_B$ with $C_A \cap C_B$ =point, and C_A, C_B resp. containing points marked from resp. A, B.

As $\mathcal{B}$ is Zariski-open in $\overline{M}_{0,m}$, the intersections $D(A|B) \cap \mathcal{B}$ will also be divisors crossing normally, and so will the pull-backs to $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$. Restricting to $\overline{M}_{0,n}(X, \beta, \bar{t})$, the intersection is transversal enough so that the restricted boundary divisors still intersect with normal crossings (dimension count).

On the quotient $\overline{M}_{0,n}(X, \beta, \bar{t})/G_{d,r}$, the boundary will be a normal crossing divisor up to a finite group.

Let us describe the components of the boundary.

For $n = 0$: $\partial \overline{M}_{0,0}(X, \beta) = \cup_{\beta = \beta_A + \beta_B} D(\beta_A, \beta_B)$, where β_A, β_B are effective, and $D(\beta_A, \beta_B)$ consists of maps μ from $C = C_A \cup C_B$ with $C_A \cap C_B$ =point, and $\mu|_{C_A}, \mu|_{C_B}$ resp. represent β_A, β_B.

For $n > 0$, the situation is slightly more complicated. $\partial \overline{M}_{0,0}(X, \beta)$ is the union of components $D(A, B, \beta_A, \beta_B)$ where $A \cup B = \{1, \ldots, n\}$ and $A \cap B = \emptyset$; β_A, β_B are effective and adding up to β; if $\beta_A = 0$ then $|A| \geq 2$ and similarly for B (stability condition); and $D(A, B, \beta_A, \beta_B)$ consists of maps $C = C_A \cup C_B \xrightarrow{\mu} X$ where

a) $C_A \cap C_B$ =point, C_A, C_B genus-0 quasi-stable curves;
b) the markings from A (resp., B) are on C_A (resp., C_B);
c) $\mu_A = \mu|_{C_A}$, $\mu_B = \mu|_{C_B}$ represent β_A, β_B.

By the dimension computations from the results on deformations, one sees that the set of curves with $C_A \cong C_B \cong \mathbb{P}^1$ is dense in $D(A, B, \beta_A, \beta_B)$.

Denote by K the divisor $D(A, B, \beta_A, \beta_B)$, and by M_A the space $\overline{M}_{0,A\cup\{\bullet\}}(X, \beta)$, mapping to X by $\rho_A(C)$ =image of $\bullet$ (and similarly M_B =etc.). Then we claim that K is 'almost' the product $M_A \times M_B$.

To be precise, consider

$$\widetilde{K} = M_A \times_X M_B = (\rho_A \times \rho_B)^{-1}(\Delta_X) \quad .$$

Letting similarly

$$\widetilde{K}(X, \overline{t}_A, \overline{t}_B) = M_A(X, \overline{t}_A) \times_X M_B(X, \overline{t}_B)$$

with the evident groups G_A, G_B acting on the factors, the map $\widetilde{K}(X, \overline{t}_A, \overline{t}_B)/G_A \times G_B \to K$ induces $\psi : \widetilde{K} \to K$. If $C_A = \mathbb{P}^1$, $[\mu_A] \in \overline{M}_A(X, \overline{t}_A)$, we have

$$T_{M_A}([\mu_A]) \cong \mathrm{Def}(\mu_A) \twoheadrightarrow T_{M_{\{\bullet\}}}([\mu_A]) \cong H^0(\mu_A^* T_X / T_C(-p_0)) \to T_X(\mu_A(p_0))$$

Because $\mu_A^* T_X$ is generated by global sections, the second map too is surjective. Hence the composition, that is the differential of ρ_A (resp., ρ_B), is surjective, making $\widetilde{K}(X, \overline{t}_A, \overline{t}_B)$ smooth. It follows that $\widetilde{K}$ is locally normal, with finite quotient singularities.

To understand ψ, consider $A, B \neq \emptyset$ and $[\mu] \in K$, corresponding to a reducible curve $C = \cup C_i$. For $q_A \in A$ and $q_B \in B$, there is a unique path of components of C from q_A to q_B (since the components of C form a tree). Find this path $\{C_i\}_{i=1,\ldots,e}$ and order the components so that $q_A \in C_1$, $q_B \in C_e$. Denote by x_i the intersection $C_i \cap C_{i+1}$; and let $C_{A,i}$ be the closure of the connected component of $C \setminus \{x_i\}$ containing q_A (and define $C_{B,i}$ similarly). This yields a sequence of splittings of C into two components. For some i we will have that $\mu_*[C_{A,i}] = \beta_A$, and necessarily $\mu_*[C_{B,i}] = \beta_B$. Take the smallest such i. (Note: if $C_{A,j}$, $C_{A,j+1}$ realize the same class, then there must be at least one extra marking on C_j, by stability.)

This shows how to decompose $C = C_A \cup C_B$ uniquely, and in short that the map $\psi : \widetilde{K} \to K$ must be a bijection. Moreover K normal, ψ bijection $\implies$ ψ isomorphism, which is what we claimed.

For $A \neq \emptyset$ or $B \neq \emptyset$, or $\beta_A \neq \beta_B$ a similar discussion yields that ψ is bijective almost everywhere, hence birational.

For $A = B = \emptyset$ (so $n = 0$) and $\beta_A = \beta_B$, the set of maps from curves with two components is still dense in the corresponding $D(\ldots)$, but swapping the components will make ψ generically $2:1$ onto its image.