By the dimension computations from the results on deformations, one sees that the set of curves with $C_A \cong C_B \cong \mathbb{P}^1$ is dense in $D(A, B, \beta_A, \beta_B)$.

Denote by K the divisor $D(A, B, \beta_A, \beta_B)$, and by M_A the space $\overline{M}_{0,A\cup\{\bullet\}}(X, \beta)$, mapping to X by $\rho_A(C)$ =image of $\bullet$ (and similarly M_B =etc.). Then we claim that K is 'almost' the product $M_A \times M_B$.

To be precise, consider

$$\widetilde{K} = M_A \times_X M_B = (\rho_A \times \rho_B)^{-1}(\Delta_X) \quad .$$

Letting similarly

$$\widetilde{K}(X, \overline{t}_A, \overline{t}_B) = M_A(X, \overline{t}_A) \times_X M_B(X, \overline{t}_B)$$

with the evident groups G_A, G_B acting on the factors, the map $\widetilde{K}(X, \overline{t}_A, \overline{t}_B)/G_A \times G_B \to K$ induces $\psi : \widetilde{K} \to K$. If $C_A = \mathbb{P}^1$, $[\mu_A] \in \overline{M}_A(X, \overline{t}_A)$, we have

$$T_{M_A}([\mu_A]) \cong \mathrm{Def}(\mu_A) \twoheadrightarrow T_{M_{\{\bullet\}}}([\mu_A]) \cong H^0(\mu_A^* T_X / T_C(-p_0)) \to T_X(\mu_A(p_0))$$

Because $\mu_A^* T_X$ is generated by global sections, the second map too is surjective. Hence the composition, that is the differential of ρ_A (resp., ρ_B), is surjective, making $\widetilde{K}(X, \overline{t}_A, \overline{t}_B)$ smooth. It follows that $\widetilde{K}$ is locally normal, with finite quotient singularities.

To understand ψ, consider $A, B \neq \emptyset$ and $[\mu] \in K$, corresponding to a reducible curve $C = \cup C_i$. For $q_A \in A$ and $q_B \in B$, there is a unique path of components of C from q_A to q_B (since the components of C form a tree). Find this path $\{C_i\}_{i=1,\dots,e}$ and order the components so that $q_A \in C_1$, $q_B \in C_e$. Denote by x_i the intersection $C_i \cap C_{i+1}$; and let $C_{A,i}$ be the closure of the connected component of $C \setminus \{x_i\}$ containing q_A (and define $C_{B,i}$ similarly). This yields a sequence of splittings of C into two components. For some i we will have that $\mu_*[C_{A,i}] = \beta_A$, and necessarily $\mu_*[C_{B,i}] = \beta_B$. Take the smallest such i. (Note: if $C_{A,j}$, $C_{A,j+1}$ realize the same class, then there must be at least one extra marking on C_j, by stability.)

This shows how to decompose $C = C_A \cup C_B$ uniquely, and in short that the map $\psi : \widetilde{K} \to K$ must be a bijection. Moreover K normal, ψ bijection $\implies$ ψ isomorphism, which is what we claimed.

For $A \neq \emptyset$ or $B \neq \emptyset$, or $\beta_A \neq \beta_B$ a similar discussion yields that ψ is bijective almost everywhere, hence birational.

For $A = B = \emptyset$ (so $n = 0$) and $\beta_A = \beta_B$, the set of maps from curves with two components is still dense in the corresponding $D(\dots)$, but swapping the components will make ψ generically $2:1$ onto its image.

Part II
Topics in Quantum Cohomology

1. $QH^*(flag)$—W. Fulton, 9/5/96

Two different things can be called the *quantum cohomology* of a flag manifold: the large ring, and the small ring.

Large: involves the numbers of maps $\mathbb{P}^1 \xrightarrow{f} X$, with prescribed $f_*[\mathbb{P}^1]$, that meet given general Schubert varieties $\Omega_1, \dots, \Omega_t$;

Small: as above, but $t = 3$.

General story. Take a basis of the cohomology $H^*X = A^*X$, say $T_0 = 1$; $T_1, \dots, T_p$ for A^1; and $T_{p+1}, \dots, T_m$ for the rest. For X a flag manifold, the basis of classes of Schubert varieties is particularly effective: the product of two of these classes is a positive combination of these classes.

Define a power series:

$$\phi(y_0, \dots, y_m) = \sum_{n_0+\cdots+n_m \geq 3} \sum_{\beta \in A_1 X} I_\beta(T_0^{n_0} \cdots T_m^{n_m}) \cdot \prod \frac{y_i^{n_i}}{n_i!}$$

where $I_\beta(T_0^{n_0} \cdots T_m^{n_m})$ counts the number of maps f as above, touching n_i T_i's and with $\beta = f_*[\mathbb{P}^1]$. Note: T_0 and the divisor classes will 'factor out' easily in this definition, as curves meet divisors predictably.

Make $H^*X \otimes \mathbb{Q}[[y_0, \dots, y_m]]$ into a $\mathbb{Q}[[y]]$-*algebra* by setting

$$T_i * T_j = \sum \phi_{ijk} g^{k\ell} T_\ell$$

where $\phi_{ijk} = \frac{\partial^3 \phi}{\partial y_i \partial y_j \partial y_k}$, and $g^{k\ell}$ is the inverse of the matrix $g_{k\ell}$ given by the ordinary intersection product on X: $g_{k\ell} = \int_X T_k \cdot T_\ell$. Note: for flag manifolds, the Schubert basis diagonalizes the intersection product, so this g is very simple.

THEOREM. *The product $*$ defined above is associative, with T_0 as unit element.*

The associativity of $*$ yields many relations between the I_β's: $\frac{m(m-1)(m^2-m+2)}{8}$, in fact. Often these relations alone and essentially trivial enumerative results suffice to determine all the I_β's.

Example. For $X = F\ell(\mathbb{C}^6)$, $H^*X = \mathbb{Z}[x_1, \dots, x_6]/(e_1, \dots, e_6)$ (e_i: elementary symmetric polynomial of degree i); a $\mathbb{Z}$-basis for the cohomology is given by the *Schubert polynomials* S_w, $w \in S_6$. In principle one could write down the 35 billion (!) equations arising from the above, and derive the numbers; in practice, this is essentially undoable.

The *small* QH^* is defined similarly, but using

$$\overline{\phi}_{ijk}(y_0, \dots, y_p) = \phi_{ijk}(y_0, \dots, y_p, 0, \dots, 0)$$

This $\overline{\phi}$ is easier to describe: $\overline{\phi}_{ijk}$ equals $\int_X T_iT_jT_k + \sum_{\beta\neq 0} I_\beta(T_iT_jT_k)\prod_{i=1}^p q_i^{\int_\beta T_i}$, with $q_i = e^{y_i}$.

The corresponding $*$ makes $QH^*X = H^*X \otimes \mathbb{Z}[q_1, \dots, q_p]$ into a $\mathbb{Z}[q]$-algebra.

Example. For $X = Gr(\ell, \mathbb{C}^n)$, $X = F\ell(\mathbb{C}^n)$, it is not too difficult to obtain $QH^*X = \mathbb{Z}[\dots, q_1, \dots, q_p]/(\text{explicit ideal})$. But note: this does *not* compute even the 3-point numbers $I_\beta(T_iT_jT_k)$: one needs "Quantum-Giambelli" formulas for the classes of T_i in QH^*.

More explicitly, take $X = Gr(\ell, \mathbb{C}^{n=k+\ell})$:

$$H^*X = \mathbb{Z}[\sigma_1, \dots, \sigma_k]/(Y_{\ell+1}, \dots, Y_n)$$

with $\sigma_i = c_i(\text{quotient bundle})$, and S_i = the determinant of the $i \times i$ matrix $\begin{pmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_i \\ 1 & \sigma_1 & \dots & \\ & & \ddots & \\ & & & \sigma_1 \end{pmatrix}$. Then:

$$QH^*X = \mathbb{Z}[\sigma_1, \dots, \sigma_k, q]/(S_{\ell+1}, \dots, S_{n-1}, S_n + (-1)^k q)$$

But again, this alone does not give the numbers! What one still needs is a quantum Giambelli. Surprisingly, this turns out to be the same recipe (with Young diagrams) as for the ordinary Giambelli: no 'quantum correction' is necessary.

Everything else is formal from here, and one can derive quantum-Pieri, quantum Littlewood-Richardson rules, etc.

For flag manifolds, $H^*F\ell(\mathbb{C}^n) = \mathbb{Z}[x_1, \dots, x_n]/(e_1, \dots, e_n)$ with e_i = elementary symmetric polynomials. A basis consists of Schubert varieties $\mathfrak{S}_w$, with $w \in S_n$, corresponding to Schubert polynomials. So there must exist integer coefficients c_{uv}^w such that $\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum c_{uv}^w \mathfrak{S}_w$; note: there is no known formula for the c_{uv}^w!

Fact: take any homogeneous presentation of H^*X, and find any q-deformation of the relations that hold in QH^*. Then the 'deformed' presentation computes QH^* (Fulton-Pandharipande).

For example, take the above presentation for H^*X, $X = F\ell(\mathbb{C}^n)$; deform the symmetric e_i to their quantum counterpart E_i; then

$$QH^*X = \mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}]/(E_1, \dots, E_n)$$

Here is an explicit description of the E_i. For the e_i, consider n marked dots:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & \dots & x_n \\ \bullet & \bullet & \bullet & \dots & \bullet \end{array}$$

then the ordinary symmetric polynomials e_i are the sum of all monomials in the x_j's obtained by 'covering' i dots. For the E_i, one labels pairs of adjacent dots by q_i's:

$$\begin{array}{ccccccccc} x_1 & & x_2 & & x_3 & & \dots & & x_n \\ \bullet & q_1 & \bullet & q_2 & \bullet & q_3 & \dots & q_{n-1} & \bullet \end{array}$$

and again one can write E_i as a sum of monomials corresponding to ways to 'cover' i dots; the q_i's (pairs of adjacent dots) must be disjoint, and will have degree 2 in the monomials. So for example $x_1q_2q_4$ will be a monomial in E_5 ($n \geq 5$).

There are quantum-Giambelli formula for quantum Schubert polynomials $\mathfrak{S}_w^q$ in $QH^*F\ell(\mathbb{C}^n)$, and quantum-Monk formulas. One remarkable thing about $\mathfrak{S}_w^q$: almost all (but not all as in the Grassmannian case) of them have no quantum correction.

2. The small quantum cohomology of the Grassmannian—R. Pandharipande, 9/19/96

Reference: A. Bertram, [Bertram].

$G = G(k, n)$ will denote the Grassmannian of k-spaces in $\mathbb{C}^n$, with tautological sequence

$$0 \to S \to \mathbb{C}^n \to Q \to 0$$

First review the classical story.

(I) Additive structure of $H^*(G, \mathbb{Z})$. Fix a flag

$$0 = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n$$

We have one *Schubert cell* for each non-increasing partition $\lambda = (\alpha_1, \dots, \alpha_k)$: the corresponding closed Schubert cell is

$$\Omega_\lambda = \{V : \dim(V \cap F_{n-k+i-\alpha_i}) \geq i\}$$

and its class $[\Omega_\lambda]$ will be denoted by W_λ.

Fact. *$\{W_\lambda\}$ is a $\mathbb{Z}$-basis of $H^*(G, \mathbb{Z})$.*

(II) Multiplicative structure of $H^*(G, \mathbb{Z})$. Let σ_i be $c_i(Q)$, $1 \leq i \leq n-k$, and define formally polynomials $S_j(\sigma_1, \dots, \sigma_{n-k})$ by

$$\frac{1}{1 + \sigma_1 t + \sigma_2 t^2 + \dots} = \sum_{j=0}^{\infty} (-1)^j S_j t^j$$

Notice that $S_j(\sigma) = c_j(\check{S})$; so necessarily $S_j(\sigma) = 0$ for $k+1 \leq j \leq n$ in $H^*(G)$: we get a set of relations between the σ's as elements of $H^*(G)$.

FACT. *This is a complete set of relations; that is,*

$$H^*(G,\mathbb{Z}) \cong \mathbb{Z}[\sigma_1,\dots,\sigma_{n-k}]/(S_{k+1},\dots,S_n)$$

(III) Expressing W_λ as polynomials in the σ's. Giambelli's formula: with $\lambda = (\alpha_1,\dots,\alpha_k)$ as before:

$$W_\lambda = |\sigma_{\alpha_i+i-j}| = \begin{vmatrix} \sigma_{\alpha_1} & \sigma_{\alpha_1+1} & \cdots & \\ \sigma_{\alpha_2-1} & \sigma_{\alpha_2} & & \\ \vdots & & \ddots & \\ \sigma_{\alpha_k-k+1} & & & \sigma_{\alpha_k} \end{vmatrix}$$

Also, since the W_λ's give an integral basis for $H^*(G)$, there must be integers $c^\gamma_{\lambda\mu}$ such that

$$W_\lambda \cdot W_\mu = \sum c^\gamma_{\lambda\mu} W_\gamma$$

There is in fact an explicit formula for the $c^\gamma_{\lambda\mu}$, given by the *Littlewood-Richardson rule.* A simpler case is for $W_{(m,0,\dots,0)} = \sigma_m$, for which *Pieri's formula* says that

$$W_\lambda \cdot \sigma_m = \sum c^\gamma_{\lambda m} W_\gamma$$

with the $c^\gamma_{\lambda m}$ all zero, except if γ can be obtained from λ by adding m boxes to its Young diagram, but no two on the same column (see e.g. [Fulton], p. 264). In the latter, case the coefficient is 1.

Small quantum cohomology. Move now to the Quantum setting. First, define the 3-point Gromov-Witten invariants: the effective generator of $A_1G(k,n) = \mathbb{Z}$ determines an identification of $A_1G(k,n)$ with $\mathbb{Z}$; let

$$I_d(W_{\lambda_1}, W_{\lambda_2}, W_{\lambda_3}) = \begin{cases} \#\text{ of rational curves in } G(k,n) \text{ of class } d, \\ \text{meeting general translates of representatives of} \\ \Omega_{\lambda_1}, \Omega_{\lambda_2}, \Omega_{\lambda_3} \text{ (if this number is finite; 0 otherwise)} \end{cases}$$

We will describe the quantum cohomology ring $QH^*(G) = QH^*(G(k,n))$ by retracing the classical steps.

(I) Additive structure. Additively, $QH^*(G(k,n))$ is the free $\mathbb{Z}[q]$-module $H^*(G) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. We have the obvious inclusion

$$\begin{aligned} H^*(G) &\to QH^*(G) \\ W_\lambda &\mapsto W_\lambda \otimes 1 \end{aligned}$$

Via this inclusion, the Schubert classes span a free basis of the quantum cohomology as a $\mathbb{Z}[q]$-module.

(II) Quantum product. There is a $*$-product which makes $QH^*(G)$ an associative, commutative $\mathbb{Z}[q]$-algebra with unit:

$$W_{\lambda_1} * W_{\lambda_2} = \sum_{d\geq 0} q^d \sum_\mu I_d(W_{\lambda_1}, W_{\lambda_2}, W_\mu) W_{\check\mu}$$

where $W_{\check\mu}$ is Poincaré dual to W_μ. For short, we will write $\langle W_{\lambda_1} W_{\lambda_2} W_\mu \rangle_d$ for $I_d(W_{\lambda_1}, W_{\lambda_2}, W_\mu)$.

REMARKS. (i) $\dim \overline{M}_{0,3}(G,d) = dn + \dim G$, so in order for a term in the sum not to be 0 it is necessary that

$$\operatorname{codim} W_{\lambda_1} + \operatorname{codim} W_{\lambda_2} + \operatorname{codim} W_\mu = dn + \dim G$$

In other words,

$$\operatorname{codim} W_{\lambda_1} + \operatorname{codim} W_{\lambda_2} = dn + \operatorname{codim} W_{\check\mu}$$

This shows that we can give a grading on $QH^*(G)$ compatible with the codimension grading on H^* and with $*$, if we take q to have degree n. $QH^*(G)$ is then a graded ring.

(ii) Since QH^* is graded, it is clear that only finitely many d's contribute to the $\sum_{d\geq 0}$ defining $*$.

(iii) $QH^*(G) \otimes (\mathbb{Z}[q]/(q)) \cong H^*(G)$. That is, the $d = 0$ contribution to $W_{\lambda_1} * W_{\lambda_2}$ is the usual intersection product $W_{\lambda_1} \cdot W_{\lambda_2}$ in the Grassmannian. $[G] = \sigma_0$ is the unit element for both $\cdot$ and $*$.

EXAMPLE. $G(1,n) = \mathbb{P}^{n-1}$. Here σ_1 is the class of a hyperplane, σ_{n-1} is the class of a point. What is $\sigma_1 * \sigma_{n-1}$?

—As point and hyperplane do not meet in $\mathbb{P}^n$, the contribution in degree 0 is 0;

—in degree 1: $\langle \sigma_1 \sigma_{n-1} \sigma_i \rangle_1$ is nonzero only for $k = n-1$; for $i = n-1$ it is the number of lines through two points and meeting a hyperplane, i.e., $\langle \sigma_1 \sigma_{n-1} \sigma_{n-1} \rangle_1 = 1$;

—in degree $d \geq 1$: $\langle \sigma_1 \sigma_{n-1} \sigma_i \rangle_d = 0$ by grading considerations, as q^d then has degree $dn > n - 1 + 1$.

Therefore $\sigma_1 * \sigma_{n-1} = q\sigma_0 = q$.

PROPOSITION. *The classes $\sigma_1,\dots,\sigma_{n-k} \in QH^*(G(k,n))$ generate it as an algebra over $\mathbb{Z}[q]$.*

PROOF. Induction. It suffices to prove that $H^*(G) \otimes 1$ is contained in the subalgebra generated by $\sigma_1,\dots,\sigma_{n-k}$. Consider then $\xi \in H^*(G)$; if $\operatorname{codim} \xi = 0$, ξ is trivially in this subalgebra. For ξ of nonzero codimension ℓ, write $\xi = f(\sigma_1,\dots,\sigma_{n-k})$ in H^* (that is, 'classically'); then we see that if computed in QH^* (with $*$ replacing $\cdots$)

$$f(\sigma_1,\dots,\sigma_{n-k}) = \xi + qA_1 + q^2A_2 + \cdots + q^fA_f \tag{1}$$

with $A_i \in H^*(G)$ of lower codimension (by the grading). By induction the A_i's are in the subalgebra, and hence so is ξ by (1). $\square$

Hence we have a surjection from the polynomial ring

$$\mathbb{Z}[q, \sigma_1, \ldots, \sigma_{n-k}] \to QH^*(G)$$

and we seek generators for the kernel. Recall that the classical relations are the $S_{k+1}, \ldots, S_n$; S_j has codimension j.

Remark. If $\operatorname{codim} W_{\lambda_1} + \operatorname{codim} W_{\lambda_2} \le n-1$, then $W_{\lambda_1} * W_{\lambda_2} = W_{\lambda_1} \cdot W_{\lambda_2}$. This is again immediate from grading considerations, as $\deg q = n$. Therefore, the classical relations $S_{k+1}, \ldots, S_{n-1}$ must still hold in QH^*, with $*$ replacing $\cdot$.

What happens to S_n?

The definition of the S_i's yields the formal polynomial identity

$$S_n(\sigma) - \sigma_1 S_{n-1}(\sigma) + \sigma_2 S_{n-2}(\sigma) + \cdots + (-1)^{n-k}\sigma_{n-k}S_k = 0$$

This holds in the polynomial ring, so it must hold in QH^*. However, we have just seen that $S_{k+1} = \cdots = S_{n-1} = 0$ in QH^*, therefore

$$S_n(\sigma) + (-1)^{n-k}\sigma_{n-k} * S_k = 0 \quad \text{in } QH^*(G).$$

Now,

σ_{n-k} = class of k-spaces containing a fixed line $\subset \mathbb{C}^n$;

$S_k(\sigma) = W_{(\underbrace{1,\ldots,1}_{k})}$ = class of k-spaces contained in a fixed $(n-1)$-space $\subset \mathbb{C}^n$

(note that the classical S_k coincides with the quantum S_k by the above remark).

It is easy to conclude from this that the only nonzero contribution to $\sigma_{n-k} * S_k$ is

$$\langle \sigma_{n-k}\, W_{(\underbrace{1,\ldots,1}_{k})} W_{(\underbrace{1,\ldots,1}_{n-k})} \rangle_1 = 1$$

and the conclusion is that $\sigma_{n-k} * S_k = q$. Therefore, the relation involving $S_n(\sigma)$ in QH^* is

$$S_n(\sigma) + (-1)^{n-k} q = 0$$

Rank considerations show that these relations are all there is, and therefore

$$QH^*(G) = \mathbb{Z}[q, \sigma_1, \ldots, \sigma_{n-k}]/(S_{k+1}, \ldots, S_{n-1}, S_n + (-1)^{n-k}q)$$

This was first obtained by Witten, and Siebert-Tian.

(III). Surprisingly, Giambelli's formula holds in QH^*: that is, for every partition $\lambda = (\alpha_1, \ldots, \alpha_k)$

$$W_\lambda = \begin{vmatrix} \sigma_{\alpha_1} & \sigma_{\alpha_1+1} & \cdots & \\ \sigma_{\alpha_2-1} & \sigma_{\alpha_2} & & \\ \vdots & & \ddots & \\ \sigma_{\alpha_k-k+1} & & & \sigma_{\alpha_k} \end{vmatrix} \in QH^*(G(k,n))$$

where of course $*$ replaces $\cdot$ in computing the determinant. This is due to Bertram, and we will sketch a proof of this fact here.

First, we need an expression for products $W_{\lambda_1} * W_{\lambda_2} * \cdots * W_{\lambda_m}$. Define another Gromov-Witten-type invariant, by setting

$$\langle W_{\lambda_1} \ldots W_{\lambda_m} W_\mu \rangle_d = \begin{cases} \#\text{ of solutions (if finite) of the following} \\ \text{enumerative problem: fix general } p_1, \ldots, p_{m+1} \in \mathbb{P}^1, \\ \text{and general translates of } \Omega_\bullet, \text{ then count} \\ \text{the number of maps } f : \mathbb{P}^1 \to G \text{ with } [f(\mathbb{P}^1)] = d, \\ \text{and mapping } p_1 \text{ to } \Omega_{\lambda_1}, \ldots, p_m \text{ to } \Omega_{\lambda_m}, \\ \text{and } p_{m+1} \text{ to } \Omega_\mu. \end{cases}$$

(Note: these differ from the usual invariants in that we are fixing the $p_i \in \mathbb{P}^1$.)

Proposition. $W_{\lambda_1} * \cdots * W_{\lambda_m} = \sum_{d \ge 0} q^d \sum_\mu \langle W_{\lambda_1} \ldots W_{\lambda_m} W_\mu \rangle_d W_{\check\mu}$.

Proof. Let e_i $i = 1, \ldots, m+1$ be the evaluations maps from $\overline{M}_{0,m+1}(G,d)$ to $G(k,n)$, and let π be the forgetful map $M_{0,m+1}(G,d) \to \overline{M}_{0,m+1}$. Rephrasing the definition of $\langle W_{\lambda_1} \ldots W_{\lambda_m} W_\mu \rangle_d$, we have

$$\pi_*(e_1^* W_{\lambda_1} \cdots e_{m+1}^* W_\mu) = \langle W_{\lambda_1} \ldots W_{\lambda_m} W_\mu \rangle_d [\overline{M}_{0,m+1}]$$

or

$$\langle W_{\lambda_1} \ldots W_{\lambda_m} W_\mu \rangle_d = e_1^* W_{\lambda_1} \cdots e_{m+1}^* W_\mu \cdot [\pi^{-1}(p)]$$

for arbitrary $p \in \overline{M}_{0,m+1}$.

Claim.

$$\langle W_{\lambda_1} \ldots W_{\lambda_m} W_\mu \rangle_d = \sum_{d_1+d_2=d} \sum_\nu \langle W_{\lambda_1} \ldots W_{\lambda_{m-1}} W_\nu \rangle_{d_1} \langle W_{\check\nu} W_{\lambda_m} W_\mu \rangle_{d_2}$$

This follows by choosing a general $p \in D(12\ldots(m-1)|m(m+1))$.

The Claim implies the proposition, via an easy induction. □

The 'Quantum-Giambelli' formula will follow in the end by applying Kempf-Laksov's formula, [K-L], which we recall here. Let M be a nonsingular variety, and consider a bundle map $\mathbb{C}^n \to E^{n-k}$ (note: not necessarily surjective). Also, fix a flag $0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^n$. Let D_{i,α_i} be the scheme-theoretic locus where $F_{n-k+i-\alpha_i} \to E$ has kernel of dimension $\ge i$. Finally, for $\lambda = (\alpha_1, \ldots, \alpha_k)$ let $C_\lambda = D_{1,\alpha 1} \cap \cdots \cap D_{k,\alpha_k}$. The formula states that if C_λ is pure of expected dimension, then $[C_\lambda]$ is given by the Giambelli determinant:

$$[C_\lambda] = \begin{vmatrix} \sigma_{\alpha_1} & \sigma_{\alpha_1+1} & \cdots & \\ \sigma_{\alpha_2-1} & \sigma_{\alpha_2} & & \\ \vdots & & \ddots & \\ \sigma_{\alpha_k-k+1} & & & \sigma_{\alpha_k} \end{vmatrix}$$

with $\sigma_j = c_j(E)$.

Also, we will need to use the *Quot scheme* (see for example [Strømme]) $Q_d = \mathrm{Quot}_{k,d}(\mathbb{C}^{n\,*}/\mathbb{P}^1)$ parametrizes (flatly) exact sequences

$$0 \to S \to \mathbb{C}^{n\,*} \to T \to 0$$

of quotient sheaves on $\mathbb{P}^1$, with $\mathrm{rk}(T) = k$, $\deg(T) = d$. There is a universal sequence of sheaves on $Q_d \times \mathbb{P}^1$:

$$0 \to \mathcal{S} \to \mathbb{C}^{n\,*} \to \mathcal{T} \to 0 \quad ,$$

and we let $M_d \subset Q_d$ be the largest open subset such that the restriction of this universal sequence to $M_d \times \mathbb{P}^1$ is in fact a sequence of vector bundles. For $x \in M_d$ we have the exact sequence

$$0 \to \mathcal{S}_x \to {\mathbb{C}^n}^*_x \to \mathcal{T}_x \to 0 \quad ,$$

of vector bundles on $\mathbb{P}^1$; dualize:

$$0 \to \mathcal{T}_x^* \to {\mathbb{C}^n}_x \to \mathcal{S}_x^* \to 0$$

Thus we obtain from each $x \in M_d$ a rank-k subbundle of the trivial n-bundle over $\mathbb{P}^1$, and hence a degree-d map $\mathbb{P}^1 \to G(n,k)$; and conversely. In other words, we can think of Quot as a compactification of $M_{0,3}(G,d)$.

Quot is a nonsingular variety of dimension $dn + \dim G(k,n)$.

Note: in the above universal sequence

$$0 \to \mathcal{S} \to \mathbb{C}^{n\,*} \to \mathcal{T} \to 0 \quad ,$$

$\mathcal{S}$ is in fact locally free (while $\mathcal{T}$ is not).

Now consider the map

$$\mathbb{C}^n \to \mathcal{S}^*$$

on $Q_d \times \mathbb{P}^1$. We will apply [K-L] to this map. First, define 'Schubert cycles' on Q_d. Note: as seen above, we have a map $M_d \times \mathbb{P}^1 \to G(k,n)$; morally we would like to choose $p \in \mathbb{P}^1$ and pull-back the usual Schubert cycles via

$$Q_d \times \{p\} \dashrightarrow M_d \times \{p\} \xrightarrow{e_p} G(k,n)$$

but we have to be careful as we go through the rational map. Schubert cycles can be defined in $M_d \times \{p\}$ by just setting $W_\lambda(p) = e_p^{-1}(\Omega_\lambda)$; define then

$$\overline{W}_\lambda(p) = \text{degeneracy locus in } Q_d \text{ of the corresponding bundle map } \mathbb{C}^n_p \to \mathcal{S}^*_p.$$

REMARKS. (i) For any points $p_1, \dots, p_N \in \mathbb{P}^1$, and general translates of Ω_{λ_1}, $\dots$, Ω_{λ_N},

$$W_{\lambda_1}(p_1) \cap \dots \cap W_{\lambda_N}(p_N) \subset M_d$$

is smooth of pure expected dimension, by Kleiman-Bertini.

(ii) If $p_1, \dots, p_N$ are distinct points in $\mathbb{P}^1$, choose N general flags in $\mathbb{C}^n$ in defining the Ω_λ's; then

$$\overline{W}_{\lambda_1}(p_1) \cap \dots \cap \overline{W}_{\lambda_N}(p_N) \subset Q_d$$

has pure expected dimension, and

$$W_{\lambda_1}(p_1) \cap \dots \cap W_{\lambda_N}(p_N) \subset M_d$$

is Zariski dense in it. This is the main 'moving lemma' in Bertram's paper.

(iii) Therefore, we have an alternative definition for the new Gromov-Witten invariants:

$$\langle W_{\lambda_1} \dots W_{\lambda_m} \rangle_d = \#(\overline{W}_{\lambda_1}(p_1) \cap \dots \cap \overline{W}_{\lambda_N}(p_N))$$

if the latter is finite: indeed, both Quot and $\overline{M}_{0,3}(G,d)$ are compactifications of M_d; by (ii), $\cap W = \cap \overline{W}$ if finite; so we may measure the intersection number in $\overline{M}_{0,3}(G,d)$, which is the original definition of the invariants, by computing in Quot.

And now we get from [K-L] that (iv) $\overline{\sigma}_i(p) := c_i(\mathcal{S}_p^*)|_{Q_d}$ is independent of p; and that

$$\text{(v)} \qquad [\overline{W}_\lambda(p)] = \begin{vmatrix} \overline{\sigma}_{\alpha_1}(p) & \overline{\sigma}_{\alpha_1+1}(p) & \dots & \\ \overline{\sigma}_{\alpha_2-1}(p) & \overline{\sigma}_{\alpha_2}(p) & & \\ \vdots & & \ddots & \\ \overline{\sigma}_{\alpha_k-k+1}(p) & & & \overline{\sigma}_{\alpha_k}(p) \end{vmatrix} \quad .$$

Now we are ready to prove Quantum-Giambelli. Let $\Delta_\lambda(\sigma) \in QH^*(G)$ be Giambelli's determinant (with $*$-product). Extending $\langle \cdot \rangle_d$ by linearity to $\langle P(\sigma), W_\mu \rangle_d$ for all homogeneous polynomials P, we have

$$\Delta_\lambda(\sigma) = \sum_{d \ge 0} q^d \sum_\mu \langle \Delta_\lambda(\sigma), W_\mu \rangle_d \, W_{\check{\mu}}$$

(using the last proposition). Now by the classical Giambelli the $d = 0$ contribution in the sum equals W_λ; so it is enough to show that

$$\langle \Delta_\lambda(\sigma), W_\mu \rangle_d = 0 \quad \text{for } d \ge 1.$$

Computing on Q_d as in (iii):

$$\langle \Delta_\lambda(\sigma), W_\mu \rangle_d = \begin{vmatrix} \overline{\sigma}_{\alpha_1}(p_1) & \overline{\sigma}_{\alpha_1+1}(p_1) & \dots & \\ \overline{\sigma}_{\alpha_2-1}(p_2) & \overline{\sigma}_{\alpha_2}(p_2) & & \\ \vdots & & \ddots & \\ \overline{\sigma}_{\alpha_k-k+1}(p_k) & & & \overline{\sigma}_{\alpha_k}(p_k) \end{vmatrix} \cdot \overline{W}_\mu(p_{k+1}) \quad ;$$

the $\overline{\sigma}_i(p)$ are independent of p by (iv), so we may choose $p_1 = \dots = p_k$; by (v), the determinant then evaluates a cycle $\overline{W}_\lambda$:

$$\begin{aligned} \langle \Delta_\lambda(\sigma), W_\mu \rangle_d &= \overline{W}_\lambda(p_1) \cdot \overline{W}_\mu(p_{k+1}) \\ &= \# \text{ maps } \mathbb{P}^1 \to G \text{ with } [f(\mathbb{P}^1)] = d, \text{ and } f(p_1) \in \Omega_\lambda, f(p_{k+1}) \in \Omega_\mu \end{aligned}$$

The above intersection has pure expected dimension 0. However, if it is non-empty, it must have dimension at least 1 (since the automorphism of $\mathbb{P}^1$ with two markings acts). Hence, the intersection must be empty.

This proves that the $d \ge 1$ contributions vanish, and concludes the proof of the Quantum-Giambelli formula.

3. Rational curves on complete intersections in toric varieties (after Givental)—V. Batyrev, 9/26/96

The starting point is the famous paper by Candelas, de la Ossa et al.[CDGP]: consider a hypersurface $V_5 \subset \mathbb{P}^4$ of degree 5; $c_1(V_5) = 0$ (that is, V_5 is a Calabi-Yau manifold). For each degree d, one expects a finite number n_d of rational curves of degree d (although this number was proved to be finite only for relatively small d). The mathematicians had shown that $n_1 = 2875$; $n_2 = 609250$, and they were in the process of computing n_3. [CDGP] claimed that they could compute the power series

$$K(q) = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}$$

How? Let $\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$; and $\theta = z\frac{\partial}{\partial z}$; also, consider the operator

$$\mathcal{D} = \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$$

Then $\phi_0(z) = 0$ is the only regular solution at $z = 0$; there are three other solutions, with logarithmic singularities at $z = 0$. Set

$$\phi_0(z,\epsilon) = \sum_{n\geq 0} \frac{\Gamma(5(n+\epsilon)+1)}{\Gamma(n+\epsilon+1)^5} z^{n+\epsilon}$$

and differentiate formally with respect to ϵ:

$$\frac{\partial}{\partial\epsilon}\phi_0(z,\epsilon)|_{\epsilon=0} = \phi_1(z) = (\log z)\phi_0(z) + \psi(z) \quad , \quad \psi(o) = 0$$

Then

$$(*) \qquad K(q)\left(\frac{dq}{q}\right)^{\otimes 3} = \frac{5}{(1-5^5 z)\phi_0^2(z)}\left(\frac{dz}{z}\right)^{\otimes 3}$$

with $q = \exp\frac{\phi_1(z)}{\phi_0(z)}$. This determines $K(q)$, giving a 'prediction' for the number n_d.

Later, Kontsevich computed n_4, showing it agrees with this prediction. Givental [Givental] found ways to go further, and managed to prove (*) rigorously. Note: it is very nontrivial that the n_d found this way should be nonnegative integers—the only known proof is via Givental's work.

Givental's framework covers many other cases. Denote by $V_{\ell_1,\dots,\ell_r} \subset \mathbb{P}^n$ a complete intersection of hypersurfaces of degrees $\ell_1,\dots,\ell_r$. Assume $\ell_1+\dots+\ell_r \leq n+1$. Three cases are distinguished:

(1) $\ell_1+\dots+\ell_r \leq n-1$
(2) $\ell_1+\dots+\ell_r = n$
(3) $\ell_1+\dots+\ell_r = n+1$ (Calabi-Yau).

Denote by R the pull-back:

$$R : H^*(\mathbb{P}^n) \to H^*(V_{\ell_1,\dots,\ell_r}) \quad ;$$

for $H \in H^2(\mathbb{P}^n)$ the hyperplane class, $R(H)$ usually generates Pic$X$.

In case (1): in the quantum cohomology ring $QH^*(V_{\ell_1,\dots,\ell_r})$, we have the following relation:

$$X^{n+1-r} = \ell_1^{\ell_1}\cdots\ell_r^{\ell_r}\cdot q\cdot X^{\sum \ell_i - r}$$

(so that the ring is graded, with $\deg q = n+1-\sum \ell_i$).

Examples. (i) $\mathbb{P}^k \subset \mathbb{P}^n$, $n = r-k$, $\ell_1 = \dots = \ell_r = 1$, so this says $x^{k+1} = q$.
(ii) $G(2,4) \subset \mathbb{P}^5$: get $X^5 - 2^2 qX = 0$.
(iii) $\mathbb{P}^1\times\mathbb{P}^1 \subset \mathbb{P}^3$: consider X only, although here Pic$=\mathbb{Z}\oplus\mathbb{Z}$; then $X^3 - 4qX = 0$. (For X_1, X_2 generators of Pic, so that $X = X_1 + X_2$, $X_i^2 = q$.)

In case (2), one finds the slightly more complicated

$$\left(X + \prod_{i=1}^r \ell_i! q\right)^{n+1-r} = \prod_{i=1}^r \ell_i^{\ell_i} q \left(X + (\prod_{i=1}^r \ell_i!)q\right)^{n-r}$$

Example. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, embedded as a conic; so $H \mapsto \mathcal{O}(2)$. Then this says

$$(X+2q)^2 = 2^2 q(X+2q)$$

that is: $X^2 = 4q^2$.

Givental's idea: define and use 'equivariant quantum cohomology'.

Reminder of usual equivariant cohomology. Given a topological space X and a Lie group G acting on X (typically $G \cong (S^1)^r$ or $(\mathbb{C}^*)^r$), define an equivariant cohomology ring $H_G^*(X)$ as follows: find a space EG which is contractible and on which G acts freely; set $BG = EG/G$; and define

$$H_G^*(X) = H^*((X\times EG)/G)$$

In particular, $H_G^*(pt) = H^*(BG)$.

Example. $G = S^1$. Then $EG = S^\infty$ (the 'Hilbert sphere') $= \{(x_i) : \sum |x_i^2| = 1$, almost all $x_i = 0\}$. Then $H_G^*(pt) = H^*(EG/G) = H^*(\mathbb{CP}^\infty) = \mathbb{C}[x]$.

Properties of equivariant cohomology.

(1) A G-morphism $f : X_1 \to X_2$ induces a pull-back $f^* : H_G^*(X_2) \to H_G^*(X_1)$, a ring homomorphism, homogeneous of degree 0.

(2) A proper G-morphism $f : X_1 \to X_2$ induces a push-forward $f^* : H_G^*(X_1) \to H_G^*(X_2)$, homogeneous of degree $\dim X_2 - \dim X_1$.

In particular, the map $X \to pt$ defines an $H_G^*(pt) = H^*(BG)$-module structure on any $H_G^*(X)$; if X is proper, we also have a map $H_G^*(X) \to H_G^*(pt)$ (analogous to $\int$).

Example. For $G = (S^1)^r$ (or $(\mathbb{C}^*)^r$), $H_G^*(pt) = \mathbb{C}[x_1,\dots,x_r]$. The $\int$ of classes in $H_G^*(X)$ are polynomials in r variables for $G = S^1$.

Example. $X = \mathbb{CP}^1$, $G = S^1$ or $\mathbb{C}$. Fix two points $0,\infty$, and corresponding inclusions i_0, i_∞ to $\mathbb{CP}^1$. Set $X_0 = i_{0*}[1]$, $X_\infty = i_{\infty*}[1] \in H_G^*(\mathbb{CP}^1)$; then $H_G^*(\mathbb{CP}^1) = \mathbb{C}[X_0, X_\infty]/(X_0 X_\infty)$. Here $H_G^*(pt) = \mathbb{C}[\hbar]$; the indeterminate $\hbar$ acts by multiplication by $X_0 - X_\infty$.

Back to Givental's work. Consider a map $\mathbb{P}^1 \to \mathbb{P}^n$, given by $(f_0 : \dots : f_n)$ with f_i homogeneous in u, v, of degree d (so that the image is a degree-d curve). Also, let V_k be a hypersurface of degree k. For $n = 4$, a map as above is specified by $5(d+1)$ homogeneous coefficient, so we could take

$$L'_d = \mathbb{P}^{5(d+1)-1}$$

as a naive moduli space of mappings. Imposing that the image lies in a V_k for $k = 5$ amounts to $(5d+1)$ conditions, giving a virtual dimension of 3 for the subscheme in $\mathbb{P}^{5(d+1)-1}$ of such rational curves. This accounts of course for the dim-3 group of automorphisms of $\mathbb{P}^1$; we can let n_d = the length of the scheme defined by these conditions. For situation (1) ($\sum \ell_i \le n-1$), this naive method in fact works fine.

Main Lemma. For L_d = space of stable maps of degree $(d,1)$ $C \to \mathbb{P}^4 \times \mathbb{P}^1$, there is a morphism $\mu : L_d \to L'_d$. For

$$C = C_0 \cup C_1 \cup \cdots \cup C_k \qquad C_i \cong \mathbb{P}^1$$

say C_0 has degree $(d', 1)$, C_i map to $\mathbb{P}^4 \times \{x_i\}$ and have degree $(d_i, 0)$ with $\sum d_i = d - d'$. $\mu(C)$ is given by $(g_0 f_0, \ldots, g_0 f_4)$, where (f_i) define $C_0 \to \mathbb{P}^4$ and $g_0 = \prod(x - x_i)$.

This can be done in every n. $\mathbb{P}^{(n+1)d-1} \times \mathbb{P}^1 \subset \mathbb{P}^{(n+1)(d+1)-1}$, where we think of the left-hand term as parametrizing degree-d rational curves, and the right-hand term as parametrizing degree-$(d-1)$ curves with a $\mathbb{P}^1$ tail. Imposing the curve to be on a degree-k V_k amounts to conditions bringing the dimension down to

$$1 + (n+1)d - 1 - (k(d-1)+1)$$

If $k \le n-1$, this (the dimension of the moduli) is less than $(n+1-k)d_k - 1$; for $k = n$, the numbers will be the same and there will be a contribution from these curves.

Bott residue formula. Assume a compact X has an S^1(for example)-action with finitely many fixed points. Also, let E be an S^1-equivariant vector bundle, of rank $= \dim X = n$. Then

$$\int_X c_n(E) = \sum_{x \in X^{S^1}} \frac{\prod b_i}{\prod a_i}$$

where $a_i, b_i \in \mathbb{Z}$ are the weights of the S^1-action on $T_x X$ and E_x respectively, for $x \in X^{S^1}$.

In the context of enumerative geometry, this was first used in [E-S]. It motivated Kontsevich's work.

Now $(\mathbb{C}^*)^{n-1}$ acts on $\mathbb{P}^n$ (as the maximal torus of $\mathrm{PGL}(n+1)$), and on L_d. The map $L_d \to L'_d$ is equivariant. Now we have to choose a good basis for the cohomology; unfortunately, there is no natural choice; but this can be done in equivariant cohomology. Recall that the pull-back gives maps $H^*_G(X) \to H^*_G(p)$ for all p in X, and for $p_i \in X^G$ in particular. So we have a map

$$H^*_G(X) \to \sum_i H^*_G(p_i)$$

Theorem. *(Atiyah-Bott, "Localization theorem") This map is an isomorphism after inversion of some element in $H^*_G(pt)$.*

Example. Two fixed points: $0, \infty$ for the action of S^1 on $\mathbb{CP}^1$. This gives the obvious map

$$H^*_{S^1}(\mathbb{CP}^1) = \mathbb{C}[X_0, X_\infty]/(X_0 X_\infty) \to \mathbb{C}[X_0] \oplus \mathbb{C}[X_\infty]$$

This is an isomorphism after inverting $\hbar = X_0 - X_\infty$.

This is the start of the rather tricky computation in Givental's paper.

Givental's method can be generalized to all Calabi-Yau complete intersections in toric varieties. Note: for Σ a fan, $\mathbb{P}\Sigma$ is not necessarily convex; for example, $\mathbb{P}^2$ blown-up at a point already is not.

There are predictions for the number of rational curves on Calabi-Yau complete intersection. For example, $V_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$: the $N_{0,d}$ have some periodic properties. It's not clear however what the physicists are counting.

Also, the relation between QH^* and Hodge theory is still mysterious.

Another possible generalization should be complete intersection in Grassmannians, for which the moduli space of stable maps should have all the necessary information (no mirror symmetry needed here).

Final comment: Givental does prove that $n_d \in \mathbb{Z}$; so far, however, it is only a 'virtual' number. To show that it is the actual number of rational curves may be very difficult; for example, this is not the case already for $V_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$.

4. Equivariant QH^* (after Givental)—B. Kim, 10/3/96

Goal: to begin a study of the equivariant quantum cohomology ring QH^*_G, aiming to understand Givental's work.

First, recall the definition and basic properties of the classical equivariant cohomology ring H^*_G. Here

G is a connected compact Lie group;

X is an oriented smooth manifold or orbifold, acted upon by G;

X_G denotes the homotopic quotient, defined by $X \times_G EG$, where $EG \to BG$ is the universal G-bundle ($X \times_G EG = X \times EG/_\sim$, where $(x, yg) \sim (gx, y)$ for $x \in X$, $y \in EG$, $g \in G$).

Definition. $H^*_G(X) := H^*(X_G, \mathbb{C})$.

Remark. $X \times_G EG \xrightarrow{\pi} BG$ is a fiber bundle, with fiber X, so π^* gives $H^*_G(X)$ an $H^*(BG)$-module structure. Suppose the Leray spectral sequence of $X_G \to BG$ degenerates at $E_2 = H^*(X) \otimes H^*(BG)$, so that $H^*_G(X) \cong H^*(X) \otimes H^*(BG)$ as $H^*(BG)$-modules (not as rings); and we have the exact sequence

$$0 \to I \cdot H^*_G(X) \to H^*_G(X) \to H^*(X) \to 0$$

with $I = \widetilde{H^*(BG)} = \ker(H^*(BG) \to H^*(pt))$ (the augmented cohomology group). (Note: the spectral sequence does degenerate in most algebro-geometric applications.) Then

(1) $H^*_G(X)$ is free over $H^*(BG)$; choose a basis $\{h_i\}$

(2) the composition $< \cdot, \cdot >$: $H^*_G(X) \otimes H^*_G(X) \xrightarrow{\cup} H^*_G(X) \xrightarrow{\pi_*} H^*(BG)$ is nondegenerate, and $\det(g_{ij} =< h_i, h_j >) \in \mathbb{C}$ (a priori $\in H^*(BG)$ only).

Equivariant Gromov-Witten invariants. Let X be a convex variety, acted upon by G. The evaluation/contraction diagrams

$$\begin{array}{ccc} \overline{M}_{0,n}(X,d) & \xrightarrow{ev_i} & X \\ \downarrow{\scriptstyle\pi} & & \\ \overline{M}_{0,n} & & \end{array}$$

pass to the homotopic quotients:

$$\begin{array}{ccc} \overline{M}_{0,n}(X,d)_G & \xrightarrow{ev_i} & X_G \\ \downarrow & & \\ \overline{M}_{0,n} \times BG & & \end{array}$$

(note $\overline{M}_{0,n} \times BG = (\overline{M}_{0,n})_G$: the action of G is trivial here).

Define $< \cdots >_{n,d}$: $H_G^{\otimes n}(X) \to H^*(BG)$ by

$$< \gamma_1 \cdots \gamma_n >_{n,d} = \int ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \in H^*(BG)$$

where $\int$ is the push-forward via $\overline{M}_{0,n}(X,d)_G \to BG$. Next, define the potential (up to quadratic terms)

$$\phi(\gamma) = \sum_{n,d} \frac{1}{n!} < \gamma^{\otimes n} >_{n,d} \in H^*(BG) \quad .$$

Choosing a basis $\{T_i\}$ of $H^*_G(X)$ over $H^*(BG)$, $\phi(\gamma^n)$ can be expanded as a formal power series:

$$(\phi_G =)\phi(\sum y_i T_i) = \sum (\cdots) \frac{y_1^{n_1} \cdots y_r^{n_r}}{n_1! \cdots n_r!}$$

(note: here $y_i \in H^*(BG)$.) For $x, y \in H^*(X_G)$, define $x \circ y$ by

$$< x \circ y, z > = \text{ partial derivative of } \phi \text{ in the directions } x, y, z$$

(which specifies it uniquely). One can prove that this $\circ$ is (super-)commutative, has a unit, is associative, and more.

Exercise. Let $S^1 \times S^1$ act on $\mathbb{P}^1$ by $(z_1 : z_2) \mapsto (e^{2\pi i s} z_1 : e^{2\pi i t} z_2)$. Find ϕ_G of the quotient. (Answer: $QH^*_G(\mathbb{P}^1) = \mathbb{C}[u, v, \hbar_1, \hbar_2, q]/(u + v = \hbar_1 + \hbar_2, uv = \hbar_1\hbar_2 + q)$.)

Applications. Computation of the small $QH^*(F)$, for F a partial flag manifold; and Givental's proof of the mirror conjecture in Calabi-Yau and Fano complete intersections in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

For $\mathbb{CP}^n$ we can state a 'mirror theorem'. Consider a degree-ℓ hypersurface; the cases to be considered will be $\ell < n$, $\ell = n$, $\ell = n + 1$.

Theorem (Givental). *The quantum differential equation on $\mathbb{CP}^n$ is hypergeometric.*

To clarify this statement, let $p = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n)$, and consider a $H^*(\mathbb{P}^n)$-valued formal function f:

$$f(t) = a_0(t)p^n + a_1(t)p^{n-1} + \cdots + a_n(t) \quad ;$$

here $e^t = q$, where q is the quantum correction in $QH^*(\mathbb{P}^n) = \mathbb{C}[p, q]/(p^{n+1} - q)$. The 'quantum differential equation' is

$$\frac{d}{dt} f(t) - p * f(t) = 0$$

where $*$ denotes the quantum product. Note that the equation implies in particular that $(\frac{d}{dt})^{n+1} a_0(t) = q a_0(t)$. The statement is that solutions to this differential equation have integral representations:

$$S(q) = \int_{\Gamma \subset Y_q} e^{u_0 + \cdots + u_n} \frac{du_0 \cdots du_n}{d(u_0 \cdots u_n)}$$

where $\omega = \frac{du_0 \cdots du_n}{d(u_0 \cdots u_n)}$ is the n-form defined by $du_0 \cdots du_n = \omega \wedge d(u_0 \cdots u_n)$ (so, $\omega = -\frac{1}{(n+1)q} \sum (-1)^i u_i du_0 \wedge \ldots \hat{du_i} \cdots \wedge du_n$). Also, $Y_q = \pi^{-1}(q)$ with $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^*$, $(u_0, \ldots, u_n) \mapsto u_0 \cdots u_n$, and Γ is some real n-dimensional cycle in Y_q.

Choose coordinates in Y_q: $u_1, \ldots, u_n$, so that $u_0 = \frac{q}{u_1 \cdots u_n}$; $Y_q = \mathbb{C}^*_{u_1} \times \cdots \times \mathbb{C}^*_{u_n}$. Choosing $\Gamma = \prod$ unit circles, the integral above is

$$\int_{|u_i|=1, i=1,\ldots,n} e^{u_1 + \cdots + u_n + \frac{q}{u_1 \cdots u_n}} \frac{du_1 \cdots du_n}{u_1 \cdots u_n} = \sum \frac{q^d}{(d!)^{n+1}}$$

up to a normalization factor. This satisfies $(\frac{d}{dt})^{n+1} - q = 0$. The other linearly independent *multi-valued* solutions are obtained by different choices of Γ.

From the point of view of Morse theory: $g = -\mathrm{Re}(F = u_0 + \cdots + u_n)$; $\Gamma =$ the unstable submanifold of ∇g in a Riemannian metric in Y_q. One can show that there are $(n + 1)$ real n-dimensional unstable submanifolds Γ, and check that the corresponding $S(q)$ give a complete set of solutions for the quantum differential equation.

Next, consider a general quintic $X_q \subset \mathbb{P}^4$. The corresponding series is

$$\sum_d \frac{(5d)!}{(d!)^5} z^d \quad .$$

For a degree-ℓ hypersurface in $\mathbb{P}^N$, with $0 < \ell \le N + 1$, this would be $\sum \frac{(\ell d)!}{(d!)^{N+1}} z^d$. The above $S(q)$ should play the role of the $\ell = 0$-case of this expression.

Let $H \subset \mathbb{P}^N$ be a hypersurface of degree $\ell \le N + 1$. Givental's proof of the mirror theorem for H amounts to the statement that the quantum differential equation, $D_1(U(q)) = 0$, will essentially coincide with the differential equation,

$D_2(V(z)) = 0$, satisfied by $\sum \frac{(\ell d)!}{(d!)^{N+1}} z^d$. Here 'essentially' means *up to a change of coordinates* and more: for $\ell < N$, $z = q$; for $\ell = N$, $z^d = e^{-\ell! q} q^d$; for the Calabi-Yau case, z =complicated.

A problem to overcome in this proof: while for $\mathbb{P}^n$ we knew the QH^*, we do not have as much for the generic hypersurface; hence, we cannot explicitly produce the quantum differential equation from the start. There are however complete solutions to the quantum differential equation in terms of intersections in $\overline{M}_{0,n}(X,d)$. The intersection theory so far was non-equivariant; equivariant intersection theory (equivariant under the action of the torus on $\mathbb{P}^n$) can be used to compute these solutions. So in a sense, although we do not have the quantum differential equation, we have its solutions. This reduces the computation to a suitable (and complicated) $\sum$ over trees. In fact, for $\ell < N$ the summation is simply a summation over chains, and yields recursion relations, from which the quantum differential equation can be recovered (and checked to agree with the non-Calabi-Yau mirror due to Givental's theory).

For $\ell = N$, $\sum$trees is a sum over chains, plus a correction; this correction can be evaluated with relative ease.

For $\ell = N + 1$, $\sum$trees is a sum over chains, plus several correction terms. Evaluating these is substantially harder.

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5. QH^* of blow-ups of $\mathbb{P}^2$—L. Göttsche and R. Pandharipande, 10/17/96

Part I (L. Göttsche). Let X_r denote the blow-up of $\mathbb{P}^2$ at r general points. The aim is to compute Gromov-Witten invariants of X_r, and show their enumerative significance in some cases.

Reminder on Gromov-Witten invariants. For X smooth and projective, and effective $\beta \in A_1 X$, there is a space $\overline{M}_{0,n}(X,\beta) = \{(\mu : C \to X; p_1,\dots,p_n)\}$ with evaluation maps ρ_i to X, $i = 1,\dots,n$. Then

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{[\overline{M}_{0,n}(X,\beta)]} \rho_1^*\gamma_1 \cup \cdots \cup \rho_n^*\gamma_n$$

Note that we are not assuming that X is convex. $[\overline{M}_{0,n}(X,\beta)]$ is a natural fundamental class: the obvious one if X is convex; or a clever one otherwise, for example defined by means of the work of Behrend–Fantechi or Li–Tian.

More notations:

—the (pull-back of the) hyperplane class in X_r will be denoted H;
—the classes of the exceptional divisors will be E_1 through E_r;
—for $\alpha = (a_1,\dots,a_r)$, (d,α) will be the divisor $dH - \sum a_i E_i$;
—$n_{d,\alpha}$ will be the expected dimension of $\overline{M}_{0,0}(X,(d,\alpha))$, that is $3d - 1 - \sum a_i$.

Write

$$N_{d,\alpha} = I_{(d,\alpha)}((\mathrm{pt})^{n_{d,\alpha}}) \quad :$$

since divisors factor out, these are the only 'interesting' Gromov-Witten numbers.

Intuitively, $N_{d,\alpha}$ is the number of rational curves in X_r of class (d,α) through $n_{d,\alpha}$ general points; that is, the number of rational curves in $\mathbb{P}^2$ of degree d, with

points of multiplicity a_i at r given general points, and passing through $n_{d,\alpha}$ more general points.

These numbers ought to satisfy the following properties:

(P1) $N_{0,\alpha} = 0$ unless $\alpha = (0,0,\dots,\overset{i}{-1},\dots,0) =: -[i]$;
(P2) $N_{d,\alpha} = 0$ if $d > 0$ and any $a_i < 0$;
(P3) $N_{d,\alpha} = N_{d,\alpha_\sigma}$ for any permutation σ;
(P4) $N_{d,\alpha} = N_{d,(\alpha,0)}$;
(P5) If $n_{d,\alpha} > 0$, then $N_{d,\alpha} = N_{d,(\alpha,1)}$;
(P6) $N_{d,\alpha} = N_{d',\alpha'}$ if (d',α') is obtained from (d,α) by a Cremona transformation, that is, if

$$d' = 2d - a_1 - a_2 - a_3 \quad \text{and}$$
$$\alpha' = (d - a_2 - a_3, d - a_1 - a_3, d - a_1 - a_2, a_4, \dots, a_r)$$

Further, one number is 'enumerative' (that is, it *does* count the appropriate number of rational curves) if so is the other.

Proof of (P6). Blow-up p_1, p_2 and p_3, obtaining exceptional divisors E_i, and proper transforms F_i of the lines L_i through them (L_1 through p_2 and p_3, etc.). Blow-down the F_i's to points q_i, and let $\overline{H}$ be the pull-back of the hyperplane from the blow-down. On the blow-up S there are two natural bases for Pic: $\{H, E_1, E_2, E_3\}$ and $\{\overline{H}, F_1, F_2, F_3\}$, with obvious relations

$$\overline{H} = 2H - E_1 - E_2 - E_3 \quad , \quad F_1 = H - E_2 - E_3 \quad , \quad \text{etc.}$$

For $x_4,\dots,x_r$ additional general points on S, we may see the blow-up of S at $x_4,\dots,x_r$ both as the blow-up of $\mathbb{P}^2$ at p_i, x_j and as the blow-up of (the other) $\mathbb{P}^2$ at q_i, x_j. From

$$dH - a_1E_1 - \dots - a_rE_r = (2d - a_1 - a_2 - a_3)\overline{H} - (d - a_2 - a_3)F_1 - \dots$$

one gets $\overline{M}_{0,0}(X_r,(d,\alpha)) \cong \overline{M}_{0,0}(X_r,(d',\alpha'))$, from which (P6) follows. □

Now the results are:

Theorem 1. *The $N_{d,\alpha}$ are determined by simple recursion formulas.*

Theorem 2. *The number of genus-0 stable maps with image (d,α) through $n_{d,\alpha}$ general points is finite. All of these maps are birational maps of $\mathbb{P}^1$ onto the image. Possibly counting multiplicities, $N_{d,\alpha}$ is this number.*

Theorem 3. *Assume one of the following:*

(1) *$n_{d,\alpha} > 0$;*
(2) *There is an i for which $i \in \{1,2\}$;*
(3) *$r \leq 8$.*

Then all curves count with multiplicity 1, and each map is an immersion.

COROLLARY. *If $d \le 10$, then $N_{d,\alpha}$ is enumerative.*

Roughly speaking, the proof is in two parts. First, the associativity of the quantum product determines the numbers. Second, assume $n_{d,\alpha} = 0$; then the space $\overline{M}_{0,0}(X,(d,\alpha))$ has dimension 0 and consists only of curves as in Theorem 2. Under the conditions of Theorem 3, $\overline{M}_{0,0}(X,(d,\alpha))$ is *smooth* of dimension 0.

Digression on quantum cohomology for nonconvex varieties. Let X be a smooth projective variety, and $\mathcal{B} \subset H_2(X,\mathbb{Z})$ the cone of effective classes. Choose a $\mathbb{Z}$-basis $T_0, \underbrace{T_1, \dots, T_p}_{\text{divisors}}, T_{p+1}, \dots, T_m = \{pt\}$ for $H^*(X,\mathbb{Z})$, and let $\{T_i^\vee\}$ be the dual basis, so that $T_i \cdot T_j^\vee = \delta_{ij}$.

For variables $q_1, \dots, q_p$, $y_{p+1}, \dots, y_m$ set

$$\Gamma(q,y) = \sum_{n_{p+1}+\cdots+n_m \ge 0} \sum_{\beta \in \mathcal{B}-\{0\}} I_\beta(T_{p+1}^{n_{p+1}} \cdots T_m^{n_m}) q_1^{\int_\beta T_1} \cdots q_p^{\int_\beta T_p} \frac{y_{p+1}^{n_{p+1}} \cdots y_m^{n_m}}{n_{p+1}! \cdots n_m!}$$

Note: we are 'separating the β's', so that there is no question of convergence (β is reconstructed from $\int_\beta T_1, \dots, \int_\beta T_p$ by duality).

$$\text{Thinking } q_i = e^{y_i}, \text{ set } \partial_i := \begin{cases} q_i \dfrac{\partial}{\partial q_i} & i = 1, \dots, p \\ \dfrac{\partial}{\partial y_i} & i = p+1, \dots, m \end{cases}, \text{ and } \Gamma_{abc} = \partial_a \partial_b \partial_c \Gamma.$$

We get a $\mathbb{Q}[[q,q^{-1},y]]$-algebra structure on the free $\mathbb{Q}[[q,q^{-1},y]]$-module generated by the T_i, by defining

$$T_i * T_j = (T_i \cdot T_j) + \sum_{r=1}^{m} \Gamma_{ijr} T_r^\vee$$

Fact: this is associative. This is shown by combining the properties of the virtual fundamental classes with the arguments for associativity from the convex case.

Back to X_r now. Here choose $T_0 = 1$, $T_1 = H$, $T_{i+1} = E_i$, $T_m = \{pt\}$. So

$$\Gamma(q,y) = \sum_{d,\alpha} N_{d,\alpha} q_1^d q_2^{a_1} \cdots q_{r+1}^{a_r} \frac{y_m^{n_{d,\alpha}}}{n_{d,\alpha}!}$$

$$T_i * T_j = (T_i \cdot T_j) T_m + \sum_{s=1}^{m} \epsilon_s \Gamma_{ijs} T_s + \Gamma_{ijm} T_0$$

with $\epsilon_s = 1$ if $s = 1$, -1 if $s > 1$.

LEMMA.

$$(g(m)) \qquad \Gamma_{mmm} = \sum_{s=1}^{m-1} \epsilon_s \left(\Gamma_{1sm}^2 - \Gamma_{11s}\Gamma_{smm}\right)$$

and, for $i = 2, \dots, r+1 = m$,

$$(g(i-1)) \qquad \Gamma_{iim} - \Gamma_{11m} = \sum_{s=1}^{m-1} \epsilon_s \left(\Gamma_{1is}^2 - \Gamma_{11s}\Gamma_{iis}\right)$$

PROOF. For $g(m)$, look at the coefficient of T_0 in

$$(T_1 * T_1) * T_m - T_1 * (T_1 * T_m) \quad ;$$

for $g(i-1)$, look at the coefficient of T_1 in

$$(T_1 * T_1) * T_i - T_1 * (T_1 * T_i) \quad . \quad \square$$

Notation:

$\vdash (d,\alpha) := \{((d_1,\beta),(d_2,\gamma)) : \text{both are} \ne 0;$
$\qquad \text{the sum is } (d,\alpha);\ n_{d_1,\beta} \ge 0;\ n_{d_2,\gamma} \ge 0;\ b_i \le d_1,\ c_i \le d_2\}$

(where the b_i are the components of β, and the c_i are components of γ. With this, the basic recursion is given by

THEOREM. *The $N_{d,\alpha}$ are determined by*

$$N_{1,(0,\dots,0)} = 1, N_{0,-[i]} = 1$$

and: if $n_{d,\alpha} \ge 3$,

$$N_{d,\alpha} = \sum_{\vdash(d,\alpha), d_i > 0} N_{d_1,\beta} N_{d_2,\gamma} \left(d_1 d_2 - \sum b_i c_i\right) \left(d_1 d_2 \binom{n_{d,\alpha}-3}{n_{d_1,\beta}-1} - d_1^2 \binom{n_{d,\alpha}-3}{n_{d_1,\beta}}\right)$$

if $n_{d,\alpha} \ge 0$,

$$\begin{aligned} d^2 a_i^2 N_{d,\alpha} &= (d^2 - (a_i^2 - 1)^2) N_{d,\alpha-[i]} \\ &\quad + \sum_{\vdash(d,\alpha-[i]), d_i > 0} N_{d_1,\beta} N_{d_2,\gamma} \left(d_1 d_2 - \sum b_i c_i\right) \left(d_1 d_2 b_i c_i - d_1^2 c_i^2\right) \binom{n_{d,\alpha}}{n_{d_1,\beta}} \end{aligned}$$

PROOF OUTLINE. $g(i)$ translates into something similar to $R(i)$; use this to prove (P1). After (P1) is proved, $g(i)$ does give $R(i)$, and $g(m)$ gives $R(m)$. (P2)—(P5) are proved by induction, using the recursions. $\square$

REMARK. Feeding the recursions into a computer, one can crunch out many examples. For all cases worked out so far, it so happens that two numbers are equal if and only if they must be equal by (P1)—(P6).

Part II (R. Pandharipande). Notations as in Göttsche's talk: we have $n_{d,\alpha} \ge 0$, $d > 0$; $\alpha = (\alpha_1, \dots, \alpha_r)$, with $\alpha_i \ge 0$; X_r; etc. The enumerative geometry problem we consider is:

count the number of maps $[v] \in \overline{M}_{0,0}(X_r,(d,\alpha))$ incident to $n_{d,\alpha}$ general points.

In this lecture we address the following results:

(I) The number of solutions is finite; all are birational maps from $\mathbb{P}^1$ to the image in X_r; $N_{d,\alpha} \ge$ the number of solutions ≥ 0.

(II) Let at least one of the following two conditions hold:

(i) $n_{d,\alpha} > 0$;
(ii) $\alpha_i \in \{1,2\}$ for some i.

Then $N_{d,\alpha}$ is the number of solutions. Moreover, they are immersions of $\mathbb{P}^1$ to X_r (questions: are they immersions to $\mathbb{P}^2$?)

Step 1. The trick we use is the "Elliptic herding" method (Harris–Caporaso, Kollár).

Let $(d,\alpha)\neq 0$, satisfying $n_{d,\alpha}<0$. Then $\overline{M}_{0,0}(X_r,(d,\alpha))=\emptyset$.

PROOF. Assume $d>0$, and let $B_r=\underbrace{\mathbb{P}^2\times\cdots\times\mathbb{P}^2}_{r}-$ diagonals. Consider the 'universal blow-up' $\mathcal{X}_r\xrightarrow{\pi}B_r$, whose fiber over $(b_1,\dots,b_r)$ is the blow-up X_b of $\mathbb{P}^2$ at $b=(b_1,\dots,b_r)$.

Let $\overline{M}_{0,0}(\mathcal{X}_r,(d,\alpha))$ be the π-relative space of maps. Let $\tau:\overline{M}_{0,0}(\pi,(d,\alpha))\to B_r$ be the natural map. Assume τ is generically surjective. Since τ is proper, it must then be surjective. That is, specializing $(b_1,\dots,b_r)$ we get stable curves in the limit.

Now 'herd' $b_1,\dots,b_r$ to lie on a smooth elliptic curve: we get a stable map $\mu:C\to X_b$, with $b_i\in$smooth elliptic curve $\mathcal{E}\subset\mathbb{P}^2$.

Simple numerology gives $C\cdot\mu^*(c_1(T_{X_b})=3d-\sum\alpha_i=n_{d,\alpha}+1\leq 0$.

Now the proper transform E of $\mathcal{E}$ is a section of T_{X_b} and the stable curve cannot have components on E. If $C=\cup C_i$, $C_i\cdot\mu^*E\geq 0$; so necessarily $C\cdot\mu^*E=0$. Pushing forward, we get a stable curve which intersects $\mathcal{E}$ only along the blow-up points. This leads to a contradiction since the points are general. □

Step 2. Suppose (d,α) satisfies $n_{d,\alpha}\geq 0$. Every map $[\mu]\in\overline{M}_{0,0}(X_r,(d,\alpha))$ incident to $n_{d,\alpha}$ general points in X_r is a birational map $\mathbb{P}^1\to X_r$; moreover, it is 'simply incident' to the points.

PROOF. This is easy using Step 1. We need to show that: the domain of μ is irreducible; μ is birational onto its image; and it is simply incident to the points. These are all shown by proving that failing any of these would contradict Step 1. For example, suppose that such a solution map has *reducible* source: $\mu:C=\cup C_i\to X_r$. There must be at least two components mapping nontrivially by μ (no marked points, so the tails must survive by stability). Let then $C_1(d_1,\alpha_1),\dots,C_s(d_s,\alpha_s)$ be the components that are not collapsed, with $s\geq 2$. Note that $n_{d,\alpha}=s-1+\sum_{i=1}^s n_{d_i,\alpha_i}>\sum_{i=1}^s n_{d_i,\alpha_i}$. Also, C goes through the $n_{d,\alpha}$ points: let $p_i=$number of points contained in $\mu(C_i)$. Then $\sum_{i=1}^s p_i\geq n_{d,\alpha}$ (every point is in one of $C_1,\dots,C_s$) $>\sum_{i=1}^s n_{d_i,\alpha_i}$. Then there must be a j such that $p_j>n_{d_j,\alpha_j}$; and this contradicts Step 1. Indeed, consider the map μ_j' from C_j to the blow-up of X_r at the p_j points in $\mu(C_j)$. The image has class $(d_j,(\alpha_j,m_1,\dots,m_{p_j}))$; the virtual dimension would be

$$n_{d_j,\alpha_j}-\sum m_i\leq n_{d_j,\alpha_j}-p_j<0$$

and the moduli space for this problem is empty by Step 1.

The other two parts to the argument are similar: the extra condition leads to negative virtual dimension. □

Step 3. Reduce to expected dimension 0. For this, blow-up the $n_{d,\alpha}$ (general) points producing a $X_{r+n_{d,\alpha}}$. The solutions to the original problem come from $\overline{M}_{0,0}(X_{r+n_{d,\alpha}},(d,(\alpha,1,\dots,1))$. The recursions from Göttsche's section show that $N_{d,\alpha}=N_{d,(\alpha,1,\dots,1)}$; so we may reduce to the case in which the expected dimension is 0 (note: the $X_{r+n_{d,\alpha}}$'s are even less convex than X_r!).

Step 4. Assume $n_{d,\alpha}=0$. Then $\overline{M}_{0,0}(X_r,(d,\alpha))$ is pure of dimension 0 (not claiming here nonsingular or nonempty).

PROOF. By Step 2, we know $\overline{M}_{0,0}(X_r,(d,\alpha))$ equals the locus $\overline{M}^{\#}_{0,0}(X_r,(d,\alpha))$ of birational maps $\mathbb{P}^1\to X_r$. Consider the normal sheaf sequence

$$0\to T_{\mathbb{P}^1}\to\mu^*T_{X_r}\to N_{X_r|\mathbb{P}^1}\to 0$$

The Zariski tangent space to $[\mu]$ in $\overline{M}_{0,0}(X_r,(d,\alpha))$ is $H^0(\mathbb{P}^1,N_{X_r|\mathbb{P}^1})$; the degree of the normal bundle is $3d-\sum\alpha_i-2=n_{d,\alpha}-1<0$.

This does not mean that $N_{X_r,\mathbb{P}^1}$ has no sections; but we have the torsion sequence

$$0\to\text{Torsion}\to N_{X_r\mathbb{P}^1}\to\text{Free}\to 0\quad :$$

the free part must have negative, and the torsion part positive, degree. The tangent space $H^0(\mathbb{P}^1,N_{X_r|\mathbb{P}^1})$ is equal to $H^0(\mathbb{P}^1,\text{torsion})$. It follows that the moduli space is of pure dimension 0. □

This concludes the proof of part I. We can only outline the proof to part II: if $n_{d,\alpha}=0$, and $\alpha_i\in\{1,2\}$ for some i, then $\overline{M}_{0,0}(X_r,(d,\alpha))$ is nonsingular, and each curve is immersed.

What is easy to see is that the nonsingularity of $\overline{M}_{0,0}(X_r,(d,\alpha))$ is equivalent to each curve being immersed. Indeed, refining the argument above, the degree of $N_{X_r,\mathbb{P}^1}$ must be -1. So $H^0(N_{X_r,\mathbb{P}^1})=0$ if and only if $N_{X_r,\mathbb{P}^1}$ is locally free, if and only if the curve is immersed.

By considering deformations of exceptional points $b_1,\dots,b_r$, cohomological conditions on relevant normal sequences are obtained. These are enough to force the immersion condition. However, the relevant deformation argument works through only if there is at least one nonsingular or double point in the list, and this translates into the condition on α. □

6. Quantum Schubert polynomials—W. Fulton, 11/7/96

This is in a sense a continuation of the first talk in this series, on the (small) quantum cohomology ring of flag varieties. The starting remark is that beyond an abstract description of QH^*, a *Giambelli formula* is needed to perform any computation, both in the classical and in the quantum case.

Review of the classical case. Notations: $X=F\ell(\mathbb{C}^n)=\{L_\bullet:L_1\subset L_2\subset\cdots\subset L_n=\mathbb{C}^n\}$ denotes a flag manifold; we have the universal flag of bundles

$$U_1\subset U_2\subset\cdots\subset U_n=V_X$$

over X, with $V=\mathbb{C}^n$; let $x_i=-c_1(U_i/U_{i-1})$; then

$$H^*X\cong\mathbb{Z}[x_1,\dots,x_n]/(e_1^n,\dots,e_n^n)$$

where $e_i^k=i$-th elementary symmetric polynomial in $x_1,\dots,x_k$. The reason for this is that we have a map $\mathbb{Z}[x_1,\dots,x_n]/(e_1^n,\dots,e_n^n)\to H^*X$ since $e_i^n=(-1)^ic_i(V_X)=$

0 in H^*X; to see $\cong$, realizing X as a sequence of $\mathbb{P}^r$-bundles shows that $\operatorname{rk} H^*X = n!$ as needed.

Bases for H^*X over $\mathbb{Z}$:

(i) $x^I = x_1^{i_1} \cdots x_n^{i_n}$, with $i_j \leq n - j$ (so that $i_n = 0$ necessarily). This basis is preferred by algebraists.

(ii) $e_J = e_{j_1}^1 \cdots e_{j_{n-1}}^{n-1}$, with $0 \leq j_p \leq p$.

(iii) Classes of Schubert varieties: $[\Omega_w]$, with $w \in S_n$; to describe Ω_w, fix a flag $V_\bullet$ and set

$$\Omega_w = \{L_\bullet \mid \dim L_p \cap V_{n+1-q} \geq \#\{i \leq p \mid w(i) \leq q\} \forall p, q\}$$

Ω_w is an irreducible subvariety of codimension $= \ell(w) = \#\{i < j \mid w(i) > w(j)\}$. Of course this basis is preferred by geometers.

Giambelli problem: write $[\Omega_w]$ in $\mathbb{Z}[x]/(\quad)$.

Solution: (Bernstein-Gelfand-Gelfand, Demazure, etc.)

(1) For $w = n\, n-1 \ldots 2\, 1$, $[\Omega_w]$ =the class of a point $= x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$;

(2) Suppose $w(i) > w(i+1)$ for some i. Let $w' = w \cdot s_i$ (where s_i =transposition $(i\; i+1)$), that is, interchange the values of i and $i+1$. Then

$$[\Omega_{w'}] = \partial_i [\Omega_w]$$

where ∂_i is a difference operator,

$$\partial_i P = \frac{P - s_i(P)}{x_i - x_{i+1}} \quad (\text{where } s_i(P) = P(\cdots, x_{i+1}, x_i, \cdots))$$

This recipe gives *polynomials,* denoted $\mathfrak{S}_w(x)$, representing $[\Omega_w]$. They have been defined, studied, and called *Schubert polynomials,* by Lascoux and Schützenberger.

REMARK. For any $u \in S_n$, we have an operator ∂_u: write $u = s_{i_1} \cdots s_{i_\ell}$ in the shortest possibly way; then $\partial_u = \partial_{i_1} \circ \cdots \circ \partial_{i_\ell}$. These operators are independent of the decomposition chosen for u, provided this has the shortest possible length. The Schubert polynomials $\mathfrak{S}_w$ can be written in terms of these operators.

EXAMPLES. $\mathfrak{S}_{12\ldots n} = 1$

$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$ ($\partial_i[\Omega_w] = 0$ if $w(i) < w(i+1)$; then $\mathfrak{S}_{s_i}$ cannot include any x_j for $j > i$, must be linear, etc.)

$n = 3:$

$$\begin{tikzcd}
 & \underset{x_1^2 x_2}{321} \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\underset{x_1 x_2}{231} \arrow[d, "\partial_2"'] & & \underset{x_1^2}{312} \arrow[d, "\partial_1"] \\
\underset{x_1}{213} \arrow[dr, "\partial_1"'] & & \underset{x_1 + x_2}{132} \arrow[dl, "\partial_2"] \\
 & \underset{1}{123} &
\end{tikzcd}$$

PROOF. (Of (2) from the Solution.) Let $X(i)$ = partial flag manifold (forget L_i). The natural map $X \to X(i)$ is a $\mathbb{P}^1$-bundle, $\mathbb{P}(U_{i+1}/U_{i-1})$ (abusing notations).

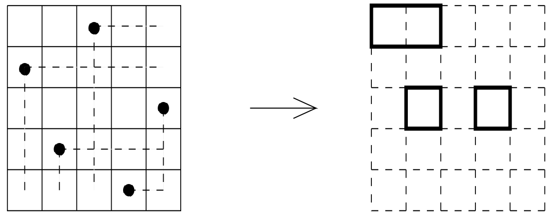
$$\begin{tikzcd}
 & X \times_{X(i)} X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr] & & X \arrow[dl] \\
 & X(i) &
\end{tikzcd}$$

(1) $(p_1)_* \circ (p_2)^* : H^*X \to H^*X$ is ∂_i;

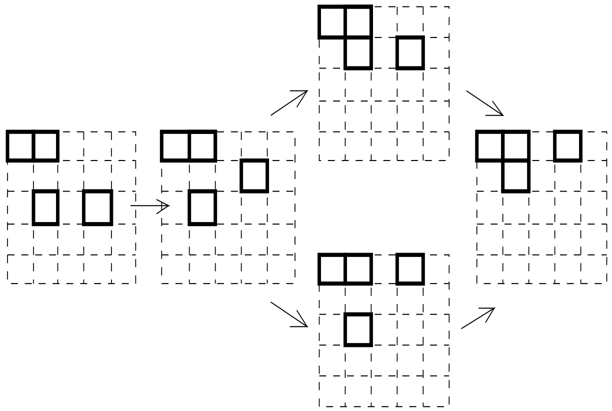
(2) p_1 maps $p_2^{-1}(\Omega_w)$ birationally onto $\Omega_{w'}$ (with notations as above).

This implies $[\Omega_{w'}] = \partial_i [\Omega_w]$, as needed. $\square$

DIGRESSION ON SCHUBERT POLYNOMIALS. It is clear that $\mathfrak{S}_w = \sum a_I x^I$, with $a_I \in \mathbb{Z}$. In fact $a_I \geq 0$, but we do not know a 'geometric' reason for it. Kohnert (in his thesis) conjectured an intriguing formula for a_I. The formula is best illustrated in a simple example: take $w = 31524$; its *diagram* $D(w)$ is obtained like this:



A *move* for such an arrangement consists of taking the box which is right-most in its row, and moving it up to the next available spot. Now play the following game: start with $D(w)$, and make all possible legal moves



List all results D, each once, and let $x^D = \prod x_i^{\#\text{boxes in } i^{\text{th}} \text{ row of } D}$. Then the claim is that

$$\mathfrak{S}_w(x) = \sum x^D \quad .$$

In the example, this gives

$$\mathfrak{S}_{31524} = x_1^2 x_3^2 + x_1^2 x_2 x_3 + x_1^2 x_2^2 + x_1^3 x_3 + x_1^3 x_2$$

Although formulas for the a_I's have been proved, Kohnert's conjecture remains open.
(End of the digression)

Pieri problem: write the product of a general Schubert variety by a special one: $\mathfrak{S}_{s_i} \cdot \mathfrak{S}_w = ?$

Solution: (Monk, Chevalley) This is $\sum \mathfrak{S}_{w'} = \sum \mathfrak{S}_{wt_{ab}}$, where t_{ab} is the transposition $a \leftrightarrow b$; the $\sum$ is over $a \le i < b$ such that $w(a) < w(b)$ and $w(j)$ is *not* between $w(a)$ and $w(b)$ for all j between a and b.

EXAMPLE. $\mathfrak{S}_{s_3} \cdot \mathfrak{S}_{31524} = \mathfrak{S}_{32514} + \mathfrak{S}_{41523}$
$\mathfrak{S}_{s_2} \cdot \mathfrak{S}_{31524} = \mathfrak{S}_{32514} + \mathfrak{S}_{51324} + \mathfrak{S}_{41523}$

REMARK. There must exist coefficients c_{uv}^w such that $\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum c_{uv}^w \mathfrak{S}_w$. By geometry, $c_{uv}^w \in \mathbb{Z}_{\ge 0}$. No formula for all c_{uv}^w is even guessed!

Finally, the basis of Schubert varieties behaves well with respect to the intersection pairing:

$$\int_X [\Omega_u] \cdot [\Omega_{w_0 v}] = \delta_{uv} \quad ,$$

where $w_0 = n\, n-1 \ldots 2\, 1$.

Quantum version. Take variables $q_1, \ldots, q_{n-1}$, corresponding to $[\Omega_{s_1}], \ldots, [\Omega_{s_{n-1}}]$; and duals $Y_i = [\Omega_{w_0 s_i}]$. Let $K = \mathbb{Z}[q_1, \ldots, q_{n-1}]$ ($\deg q_i = \int_{Y_i} c_1(TX) = 2$).

DEFINITION. $QH^*X = H^*X \otimes_{\mathbb{Z}} K$ as K-module; it is a K-algebra under

$$[V] * [W] = [V] \cup [W] + \sum_{d \ne 0} q^d I_d(V \cdot W \cdot \Omega_u)[\Omega_{w_0 u}]$$

with appropriate positions ($q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$, etc.)

Problem: Present QH^*: $QH^*X = K[x_1, \ldots, x_n]/(\ldots ? \ldots)$

For the relations, it suffices to find any deformations of $e_1^n, \ldots, e_n^n$ which hold in QH^*X.

THEOREM 1. *(Givental-Kim; Ciocan-Fontanine)*

$$QH^X = K[x_1, \ldots, x_n]/(E_1^n, \ldots, E_n^n)$$

where $E_i^k = i^{\text{th}}$ *'elementary quantum polynomial' in k variables.*

These quantum polynomials are

$$E_i^k = \sum_{2|I|+|J|=i,\ I,J \text{ 'disjoint'}} q_I x_J \quad :$$

where q_i 'covers' $i, i+1$ and x_j covers j, and I, J are disjoint in the sense of covering disjoint subsets of $\{1, \ldots, k\}$. For example,

$$\begin{array}{ccccccccccccccccc}
x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 & & x_8 & & x_9 \\
* & q_1 & \bullet & q_2 & \bullet & q_3 & \circ & q_4 & * & q_5 & * & q_6 & \circ & q_7 & \bullet & q_8 & \bullet
\end{array}$$

$q_2 q_8 x_1 x_5 x_6$ is a summand in E_7^9.

The original description of these polynomials was as follows: let A be the matrix

$$\begin{pmatrix}
x_1 & q_1 & 0 & \ldots & 0 \\
-1 & x_2 & q_2 & \ldots & 0 \\
 & & \ddots & & \\
 & & \ldots & x_{k-1} & q_{k-1} \\
0 & 0 & 0 & -1 & x_k
\end{pmatrix}$$

Then $\det(1 + \lambda A) = \sum E_i^k \lambda^i$. The two descriptions agree, as they both satisfy the recursion

$$E_i^k = E_i^{k-1} + x_k E_{i-1}^{k-1} + q_{k-1} E_{i-2}^{k-2} \quad .$$

Next, $c \mapsto c \otimes 1$ gives an inclusion

$$H^*X \hookrightarrow QH^*X = H^*X \otimes K$$

In the Grassmannian case, Giambelli's formula moves unchanged from the classical to the quantum ring (see Pandharipande's lecture on the Grassmannian).

THEOREM 2, 'QUANTUM GIAMBELLI'. *(Fomin-Gelfand-Postnikov, based on a result of Ciocan-Fontanine.) Write* $\mathfrak{S}_w = \sum n_{Jw} e_J$, *with* $e_J = e_{j_1}^1 \cdots e_{j_{n-1}}^{n-1}$ *and* $n_{jw} \in \mathbb{Z}$ (note: the n_{Jw} are not necessarily positive). *Then*

$$[\Omega_w] = \sum n_{Jw} E_J\,, \ \textit{with}\ E_J = E_{j_1}^1 \cdots E_{j_{n-1}}^{n-1}$$

THEOREM 3, 'QUANTUM MONK'. *(same people; also D. Petersen.) With* $\mathfrak{S}_w^q := \sum n_{Jw} E_J$,

$$\mathfrak{S}_{s_i}^q \cdot \mathfrak{S}_w^q = \sum_{\text{classical}} \mathfrak{S}_{wt_{ab}}^q + \sum_{\text{quantum}} q_{cd} \mathfrak{S}_{wt_{cd}}^q \quad ,$$

the second $\sum$ *over all* $c \le i < d$ *such that* $w(c) > w(j) > w(d)$ *for all* j *between* c *and* d; *and where* $q_{cd} = q_c q_{c+1} \cdots q_{d-1}$.

PROOF. Here are four bases of QH^*X over K:

(i) x^I;
(ii) E_J;
(iii) $[\Omega_w]$;
(iv) $\mathfrak{S}_w^q$.

We want to show that (iii) and (iv) are the same. Let K_+ be the cone spanned by all q^M, and let QH_+ be the cone spanned by all $q^M \cdot [\Omega_w]$, that is $\oplus K_+ \cdot [\Omega_w]$. Three facts:

(a) QH_+ is closed under multiplication;

(b) Each E_J^p is in QH_+, so each E_J is (this relies on Ciocan-Fontanine's result: $E_J^p = [\Omega_u]$ for $u = (p-j+1, p-j+2, \ldots, p+1)$;

(c) This needs more notations: for $F \in QH^*X$, define $< F >$ by any of the following recipes:

expand in terms of basis (i), take coefficient of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$; or
expand in terms of basis (iii), take coefficient of $[\Omega_{w_0}]$; or
expand in terms of basis (iv), take coefficient of $\mathfrak{S}^q_{w_0}$.

(It is easy to see that these coincide.) Remark: if $F \in QH_+$, then $< F > \in K_+$. The statement is then that

$$< \mathfrak{S}^q_u \cdot \mathfrak{S}^q_{w_0 v} >= \delta_{uv}$$

for all u, v. This is harder than it looks.

Also,

Lemma. *Fix $k \le \binom{n}{2}$. For all $w \in S_n$ of length $\ell(w) = k$, there exist $a_w > 0$, with $\sum_{\ell(w)=k} a_w \mathfrak{S}^q_w \in QH_+$.*

Given (a), (b), (c) and the Lemma, we can prove Theorem 2:

Fix $w \in S_n$, let $k = \ell(w)$, and for $\ell \le \binom{n}{2}$ let $\ell' = \binom{n}{2} - \ell$.

(1) $[\Omega_w] = \mathfrak{S}^q_w + \sum_{\ell<k} P_{\ell w}$, with $P_{\ell w} = \sum_{\ell(v)=\ell} c_{vw}\mathfrak{S}^q_v$.

We have to show that all $P_{\ell w} = 0$, or equivalently that $< P_{\ell w} \cdot \mathfrak{S}^q_v >= 0$ for all v with $\ell(v) = \ell'$; or, equivalently, that

$$< P_{\ell w} \cdot E_J >= 0 \quad \text{for all } J \text{ with } |J| = \ell'.$$

(2) Now

$$< P_{\ell w} \cdot E_J >=< \sum \Omega_w \cdot E_j > \in K_+ \quad .$$

But $QH_+ \ni \sum_w a_w \mathfrak{S}^q_w = \sum a_w[\Omega_w] - \sum_{w,\ell} a_w P_{\ell w}$, so $-\sum_{w,\ell} a_w P_{\ell w} \in QH_+$, and

$$-\sum a_w < P_{\ell w} \cdot E_J > \in K_+ \quad .$$

It follows $< P_{\ell w} \cdot E_J >= 0$, as needed. □

Remark. (c) can be proved in the form (c'): $< E_I \cdot E_J >= 0$ if $|I| + |J| \ge \binom{n}{2}$.

Comments on the algebra. Define operators $X_1, \ldots, X_n$ on $K[x_1, \ldots, x_n]$ by

$$X_k = x_k - \sum_{i<k} q_{ik}\partial_{t_{ik}} + \sum_{j>k} q_{kj}\partial_{t_{kj}}$$

(1) These operators commute, and commute with the $\cdot e_p^n$'s.
(2) $\forall f \in K[x]$, there is a unique $F \in K[X_1, \ldots, X_n]$ with $F(1) = f$.
(3) If $f = e_J$, then $F = E_J$; if $f = \mathfrak{S}_w(x)$, then $F = \mathfrak{S}^q_w(X)$.

The algebra of these operators streamlines the proofs considerably. For example, Quantum Monk amounts to proving

$$(X_1 + \cdots + X_i)\mathfrak{S}^q_w(X) = \sum_{\text{classical}} \mathfrak{S}^q_{wt_{ab}}(X) + \sum_{\text{quantum}} q_{cd}\mathfrak{S}^q_{wt_{cd}}(X) \quad :$$

and this is shown by evaluating at 1 and applying the above and the classical Monk.

Example. Finally, here are the quantum Schubert polynomials for $n = 3$:

$$\begin{tikzcd}
& \overset{321}{x_1^2x_2 + q_1x_1} \arrow[dl] \arrow[dr] & \\
\overset{231}{x_1x_2 + q_1} \arrow[d] & & \overset{312}{x_1^2 - q_1} \arrow[d] \\
\overset{213}{x_1} \arrow[dr] & & \overset{132}{x_1 + x_2} \arrow[dl] \\
& \overset{123}{1} &
\end{tikzcd}$$

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7. The small QH^*-ring of flag manifolds, I —I. Ciocan-Fontanine, 11/14/96

Fix an n-dimensional complex vector space V. We will use the following notations (differing slightly from Fulton's notations):

$F = F\ell(V)$ will be the manifold of flags $\{U_1 \subset \cdots \subset U_{n-1} \subset V\}$; V_F will denote $V \otimes \mathcal{O}_F$; there are tautological bundles

$$E_1 \subset \cdots \subset E_{n-1} = E_n = V_F \twoheadrightarrow L_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow L_1$$

on F, with $L_i = V_F/E_{n-i}$. We let x_i be $c_1(\ker(L_i \to L_{i-1}))$. Fact (see Fulton's lecture):

$$H^*(F) = \mathbb{Z}[x_1, \ldots, x_n]/(e_1^n, \ldots, e_n^n)$$

with e_i^k = the i-th symmetric polynomial in $x_1, \ldots, x_k$.

Next, fix a reference flag $V_1 \subset \cdots \subset V_{n-1} \subset V$. For $w \in S_n$ (=the symmetric group), let

$$r_w(q,p) = \#\{i \le q : w(i) \le p\} \quad \text{and set}$$

$$\Omega_w = \{y \in F / \operatorname{rk}_y(V_p \otimes \mathcal{O}_F \to L_q) \le r_w(q,p), \forall q, p\}$$

The set Ω_w is irreducible, of ($\mathbb{C}$-)codimension $\ell(w)$ = length of w = the number of 'inversions' in w. We will denote by Ω_w also the corresponding elements in H_*F

and H^*F. The set $\{\Omega_w : w \in S_n\}$ forms an additive basis for $H^*(F)$. Further, we have the duality

$$\int_F \Omega_w \Omega_{w_0 v} = \delta_{wv}$$

where w_0 is the permutation of maximal length, that is, $w_0(i) = n+1-i$.

The *Giambelli formula* tells us that $\Omega_w = \mathfrak{S}_w(x)$ in $H^*(F)$, with $\mathfrak{S}_w(x)$ =the corresponding Schubert polynomial, as defined in Fulton's lecture.

For $s_i = (i, i+1)$, the set $\{\Omega_{s_1}, \dots, \Omega_{s_{n-1}}\}$ is a basis for $H^2(F)$. Aiming to quantize the situation, we let Y_i be $\Omega_{w_0 s_i}$, so that $\{Y_1, \dots, Y_{n-1}\}$ gives a basis for $H_2(F)$, and let indeterminates q_i correspond to the Y_i. We consider the ring $K = \mathbb{Z}[q_1, \dots, q_{n-1}]$ and define the (small) *quantum cohomology ring* of F to be

$$QH^*(F) = H^*(F) \otimes_{\mathbb{Z}} K$$

as a K-module, with product

$$\Omega_u * \Omega_v = \sum \overline{q}^{\overline{d}} I_{\overline{d}}(\Omega_u \Omega_v \Omega_w) \Omega_{w_0 w}$$

where $\overline{d} = (d_1, \dots, d_{n-1})$, $\overline{q}^{\overline{d}} = \prod q_i^{d_i}$, etc. Note: F is a linear section of the Segre embedding of the product of the relevant Grassmannians in their Plücker embedding; Ω_{s_i} is the pull-back of $\mathcal{O}(1)$ from the i^{th} Grassmannians. In particular, the Ω_{s_i} are nef, and it follows that if $f : C \to F$ is a map from a curve, then $f_*[C] = \sum d_i Y_i$, with all $d_i \geq 0$.

Next, we operate on polynomials $P(x, q)$ by replacing the intersection $\cdot$ with $*$: that is, we leave the q_i's alone, and we replace monomials in the x's with the corresponding quantum products. Note that as it turns out that $\Omega_{s_i} = \mathfrak{S}_{s_i}(x) = x_1 + \cdots + x_i = e_1^i$, we have $x_i = \Omega_{s_i} - \Omega_{s_{i-1}}$. So, for example, $P = q_4 x_1 x_2$ yields $\hat{P} = q_4(\Omega_{s_1} * (\Omega_{s_2} - \Omega_{s_1}))$. We want to write the Ω's in terms of this operation: that is,

the *Quantum Giambelli* problem is then to find $\mathfrak{S}_w^q(x, q)$ such that $\widehat{\mathfrak{S}_w^q} = \Omega_w$.

Necessarily, $\mathfrak{S}_w^q = \mathfrak{S}_w +$ a quantum correction; that is, $\mathfrak{S}_w^q(x, 0) = \mathfrak{S}_w$.

A presentation of QH^* is given by $\mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}]/I_q$, where I_q is the ideal generated by 'quantum perturbations of the e_i^n'. It will follow from Theorem 1 below that these can be taken to be the polynomials $E_1^n, \dots, E_n^n$:

$$E_i^k = \; i^{\text{th}} \text{ elementary quantum symmetric polynomial in } x_1, \dots, x_k, q_1, \dots, q_{k-1}$$

(see also Fulton's lecture). These can be defined in terms of the characteristic polynomial of

$$\begin{pmatrix} x_1 & q_1 & 0 & \dots & 0 \\ -1 & x_2 & q_2 & \dots & 0 \\ & & \ddots & & \\ & & \dots & x_{k-1} & q_{k-1} \\ 0 & 0 & 0 & -1 & x_k \end{pmatrix} ,$$

and satisfy the recursion

$$(*) \qquad E_i^k = E_i^{k-1} + x_k E_{i-1}^{k-1} + q_{k-1} E_{i-2}^{k-2}$$

Since $e_i^k = e_i^{k-1} + x_k e_{i-1}^{k-1}$, writing $E_i^k = e_i^k + {}^q E_i^k$, we have that $(*)$ is equivalent to

$$(**) \qquad {}^q E_i^k = {}^q E_i^{k-1} + x_k {}^q E_{i-1}^{k-1} + q_{k-1} {}^q E_{i-2}^{k-2} \quad .$$

Now denote by A_i^k the class represented by e_i^k in $H^*(F)$, $1 \leq i \leq k \leq n$. We have

$$A_i^k = \begin{cases} \Omega_{\alpha_{i,k}} & k < n \\ 0 & k = n \end{cases}$$

with $\alpha_{i,k} =$ the cycle $(k-i+1, \dots, k+1)$.

THEOREM 1. *$\widehat{E_i^k} = A_i^k$ in $QH^*(F)$.*

In particular, $\widehat{E_i^n} = 0$ in $QH^*(F)$; that is, $E_i^n \in I_q$ as promised.

Theorem 1 will follow from

THEOREM 2. *(Geometric formulation of a special case of the Quantum Monk formula.)*

$$\Omega_{s_j} * \Omega_{\alpha_{i,k}} = \textit{(classical term)} + \delta_{jk} q_j \Omega_{\alpha_{i,k} \cdot s_j} = \begin{cases} \textit{classical} & j \neq k \\ \textit{classical} + q_k \Omega_{\alpha_{i-1,k-1}} & j = k \end{cases}$$

PROOF. (Of Theorem 1, assuming Theorem 2.) This is done by induction on k. The statement is trivial for $k = 1 \implies i = 1$: $e_1^1 = E_1^1 = x_1$. Assume proven for $\leq (k-1)$; then

$$\begin{aligned} \widehat{e_i^k} &= \widehat{e_i^{k-1}} + \widehat{x_k e_{i-1}^{k-1}} = \widehat{e_i^{k-1}} + x_k * \widehat{e_{i-1}^{k-1}} \\ &= A_i^{k-1} - \widehat{{}^q E_i^{k-1}} + x_k * A_{i-1}^{k-1} - x_k * \widehat{{}^q E_{i-1}^{k-1}} \\ &= A_i^{k-1} - \widehat{{}^q E_i^{k-1}} + x_k * A_{i-1}^{k-1} - \widehat{x_k {}^q E_{i-1}^{k-1}} \end{aligned}$$

Now by Theorem 2

$$\begin{aligned} x_k * A_{i-1}^{k-1} &= \Omega_{s_k} * \Omega_{\alpha_{i-1,k-1}} - \Omega_{s_{k-1}} * \Omega_{\alpha_{i-1,k-1}} \\ &= A_i^k - A_i^{k-1} - q_{k-1} \Omega_{\alpha_{i-2,k-2}} \end{aligned}$$

and therefore

$$\begin{aligned} \widehat{e_i^k} &= A_i^k - \widehat{{}^q E_i^{k-1}} - q_{k-1} \widehat{E}_{i-2}^{k-2} - \widehat{x_k {}^q E_{i-1}^{k-1}} \\ &= A_i^k - \widehat{B_i^k} \end{aligned}$$

with $B_i^k = {}^q E_i^{k-1} + x_k {}^q E_{i-1}^{k-1} + q_{k-1} E_{i-2}^{k-2} = {}^q E_i^k$ by the recursion. That is,

$$\widehat{E_i^k} = \widehat{e_i^k} + \widehat{{}^q E_i^k} = A_i^k \quad ,$$

completing the induction. □

8. The small QH^*-ring of flag manifolds II —I. Ciocan-Fontanine, 11/21/96

Plan: as much as we can of the following

(0) What's left to prove;
(1) $\mathrm{Hom}_{\overline{d}}$ and its compactification $\mathcal{H}Q_{\overline{d}}$;
(2) Three-point (GW) numbers $I_{\overline{d}}(\Omega_{w_1}\Omega_{w_2}\Omega_{w_3})$ via $\mathcal{H}Q_{\overline{d}}$;
(3) Structure of the boundary;
(4) Proof of the moving lemma;
(5) Idea for computing $I_{\overline{d}}(\dots)$;
(6) Proof of 'special' Quantum-Monk formula;
(7) General Quantum-Monk formula.

(0) From the first lecture: $V \cong \mathbb{C}^n$, $F = F(V) =$ the space of complete flags in V.
Universal quotients: $V_F = V \otimes \mathcal{O}_F \to Q_{n-1} \to \cdots \to Q_1$;
Fixed reference flag: $V_1 \subset V_2 \subset \cdots \subset V_n = V$.
Every $w \in S_n$ determines a rank function,

$$r_w(q,p) = \#\{i : i \le q; w(i) \le p\}$$

The corresponding *Schubert variety* is $\Omega_w = \{\mathrm{rk}(V_p \otimes \mathcal{O} \to Q_q) \le r_w(q,p)\forall q,p\}$; that is, the degeneracy locus of the universal quotients, using r_w for degeneracy conditions.

The codimension of Ω_w is the length of w, denoted $\ell(w)$.

If s_i denotes the transposition $(i,i+1)$, then $\{\Omega_{s_i}\}$ gives a (group) basis for $H^2(F)$.

The *quantum product* is defined by

$$\Omega_{w_1} * \Omega_{w_2} = \sum_{\overline{d}} \overline{q}^{\overline{d}} I_{\overline{d}}(\Omega_{w_1}\Omega_{w_2}\Omega_{w_3})\Omega_{w_0w_3}$$

where $\overline{d} = (d_1,\dots,d_{n-1})$, $\overline{q} = (q_1,\dots,q_{n-1})$, and w_0 is the longest permutation.

Ω_{w_0w} is 'Poincaré dual' to Ω_w.

Then we have to show the following (*'Special Quantum-Monk'*):

$$\Omega_{s_j} * \Omega_{\alpha_{i,k}} = \begin{cases} \text{classical} & , \quad j \ne k \\ \text{classical} + q_k\Omega_{\alpha_{i-1,k-1}} & , \quad j = k \end{cases}$$

with $\alpha_{i,k} = s_{k-i+1}\cdots s_k$.

(1) Equivalent formulation:

$$I_{\overline{d}}(\Omega_{s_j}\Omega_{\alpha_{i,k}}\Omega_w) = \begin{cases} 1 & \overline{d} = \overline{e}_k, \text{ and } w = w_0\alpha_{i-1,k-1} \\ 0 & \text{otherwise} \end{cases}$$

Here $\overline{e}_k = (0,\dots,\underset{k}{1},0,\dots,0)$.

The idea is to compute the three-point functions geometrically, that is as intersection numbers on some compactification of $\mathrm{Hom}_{\overline{d}}$.

We will denote $H_{\overline{d}} := \mathrm{Hom}_{\overline{d}}(\mathbb{P}^1, F) = M_{0,3}(F,\overline{d})$, that is

$$H_{\overline{d}} = \{[f] : f : \mathbb{P}^1 \to F, f_*[\mathbb{P}^1] = \sum d_i\Omega_{w_0s_i}\}$$

We will use a compactification other than $\overline{M}_{0,3}(F,\overline{d})$.

Since F is homogeneous, $H_{\overline{d}}$ will be smooth of dimension $\binom{n}{2} + 2\sum d_i$. We have the evaluation map $\mathbb{P}^1 \times H_{\overline{d}} \to F$, $(t,[f]) \mapsto f(t)$. For fixed distinct $t_1,t_2,t_3 \in \mathbb{P}^1$,

$$I_{\overline{d}}(\Omega_{w_1}\Omega_{w_2}\Omega_{w_3}) = \#\{[f] : f(t_i) \in \Omega_{w_i} \quad , \quad i = 1,2,3\} \quad .$$

We set $\Omega_w(t) := ev^{-1}(\Omega_w) \cap \{t\} \times H_{\overline{d}}$, so heuristically

$$I_{\overline{d}}(\Omega_{w_1}\Omega_{w_2}\Omega_{w_3}) = \#\{\Omega_{w_1}(t_1) \cap \Omega_{w_2}(t_1) \cap \Omega_{w_3}(t_1)\}$$

Explicitly,

$$\Omega_w(t) = \{\mathrm{rk}(V_p \otimes \mathcal{O} \to ev^*Q_q) \le r_w(q,p)\forall p,q\} \cap \{t\} \times H_{\overline{d}}$$

Idea (Bertram): Compactify $H_{\overline{d}}$ so that there exist natural vector bundle extensions of ev^*Q_q across the boundary, and use the same conditions to define $\Omega_w(t)$'s there.

The data of $\{f : \mathbb{P}^1 \to F, \overline{d}\}$ determines a sequence of surjections $\{V_{\mathbb{P}^1} \to L_{n-1} \to \cdots \to L_1\}$ with $\mathrm{rk}\, L_i = i$, $\deg L_i = d_i$. Dualize this sequence, and get

$$S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset V^*_{\mathbb{P}^1}$$

with $\mathrm{rk}\, S_i = i$, and $\deg S_i = -d_i$. And next consider the surjections

$$V^*_{\mathbb{P}^1} \to V^*_{\mathbb{P}^1}/S_1 \to \cdots \to V^*_{\mathbb{P}^1}/S_{n-1}$$

Degenerate this to get $\mathcal{H}Q_{\overline{d}}$, then set

$$\overline{\Omega}_w(t) := \{\mathrm{rk}(V_p \otimes \mathcal{O} \to \mathcal{S}^*_q) \le r_w(q,p)\} \cap \{t\} \times \mathcal{H}Q_{\overline{d}} \quad .$$

The $\mathcal{S}_i$ are defined below:

$$\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-1} \to V^* \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}}}$$

THEOREM 1. *(Laumon, C–F, Kim) (i) $\mathcal{H}Q_{\overline{d}}$ is a smooth, irreducible projective variety of dimension $\binom{n}{2} + 2\sum d_i$, containing $H_{\overline{d}}$ as an open dense subscheme. There exists a universal sequence of quotients*

$$V^* \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}}} \to \mathcal{T}_{k-1} \to \cdots \to \mathcal{T}_1$$

with $\mathcal{T}_i$ flat over $\mathcal{H}Q_{\overline{d}}$ and fixed relative Hilbert polynomial

$$\chi(\mathcal{T}_i(m)) = (m+1)i + d_{n-i}$$

(This is an 'extension' of Grothendieck's quot-schemes).

(ii) $\mathcal{S}_i := \ker(V^* \to J_{n-i})$ are vector bundles on $\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}}$; there are injections of sheaves

$$\mathcal{S}_1 \subset \cdots \subset S_{n-1} \subset V^*$$

(which may degenerate as maps of vector bundles).

(2) The following results show that we can use $\mathcal{H}Q_{\overline{d}}$ to define $I_{\overline{d}}(\Omega_{w_1}\Omega_{w_2}\Omega_{w_3})$:

THEOREM 2. *('Moving Lemma') (i)* $\forall w_1, \dots, w_N \in S_n$, $t_1, \dots, t_N \in \mathbb{P}^1$, *and general* $g_1, \dots, g_N \in SL_n$, *the intersection* $\cap_{i=1}^N g_i \Omega_{w_i}(t_i)$ *is either empty, or of pure codimension* $\sum \ell(w_i)$ *in* $H_{\overline{d}}$.

(ii) If in addition the t_i*'s are distinct, then* $\cap g_i \overline{\Omega}_{w_i}(t_i)$ *is either empty, or of codimension* $\sum \ell(w_i)$ *in* $\mathcal{H}Q_{\overline{d}}$, *and equals the closure of the* $\cap\Omega$ *in (i).*

In particular, when the codimension is maximal (that is, when the intersection consists of points) it all happens in the open part, so it counts what it is supposed to count.

COROLLARY 1. *The class of* $\overline{\Omega}_w(t)$ *in* $A^{\ell(w)}(\mathcal{H}Q_{\overline{d}})$ *is independent of* t *and the fixed flag* $V_\bullet \subset V$.

COROLLARY 2. *If* $\sum \ell(w_i) = \binom{n}{2} + 2\sum d_i$ *and* $t_1, \dots, t_N$ *are distinct, then the number of intersection points of general translates is given by*

$$\# \cap_{i=1}^N \Omega_{w_i}(t_i) = \int_{\mathcal{H}Q_{\overline{d}}} [\overline{\Omega}_{w_1}(t_1)] \cup \cdots \cup [\overline{\Omega}_{w_N}(t_N)]$$

So we define

$$\langle \Omega_{w_1} \cdots \Omega_{w_N} \rangle_{\overline{d}} = \begin{cases} \int_{\mathcal{H}Q_{\overline{d}}} \cdots & \text{if } \sum \ell(w_i) = \binom{n}{2} + 2\sum d_i \\ 0 & \text{otherwise} \end{cases}$$

(Difference with the usual GW: here we take a fixed, albeit general, configuration of points in $\mathbb{P}^1$.)

COROLLARY 3. $\langle \Omega_{w_1}\Omega_{w_2} \rangle_{\overline{d}} = 0$ *for* $\overline{d} \neq 0$.

(Indeed, if we get one we must get a whole $\mathbb{C}^*$ of intersections.)

However, since ev does not extend to $\mathcal{H}Q_{\overline{d}}$, a different strategy is needed to prove (ii). We need to analyze the restriction of $\cap\overline{\Omega}_{w_i}(t_i)$ to the boundary, and show that it has 'large enough' codimension.

(3): Structure of the boundary. Recall the universal sequence on $\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}}$:

$$0 \to \mathcal{S}_1^{\overline{d}} \to \cdots \to \mathcal{S}_{n-1}^{\overline{d}} \to V^*_{\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}}} \to \mathcal{T}_{n-1}^{\overline{d}} \to \cdots \to \mathcal{T}_1^{\overline{d}}$$

$H_{\overline{d}}$ is the largest open subscheme in $\mathcal{H}Q_{\overline{d}}$ such that on $\mathbb{P}^1 \times H_{\overline{d}}$ all these maps are nondegenerate as vector bundle maps.

Next, we 'stratify' according to the degeneracies of the map, more precisely according to the ranks of $\mathcal{T}_i^{\overline{d}}$.

Idea: how to construct $V^*_{\mathbb{P}^1} \to T_{n-1} \to \cdots \to T_1$ on $\mathbb{P}^1$ with assigned Hilbert polynomials and with prescribed ranks at $t \in \mathbb{P}^1$? Let $\mathrm{rk}_t T_{n-i} = n - i + e_i$, with $e_i \geq 0$; start with $S_1 \subset \cdots \subset S_{n-1} \subset V^*_{\mathbb{P}^1}$, a point in $\mathcal{H}Q_{\overline{d}-\overline{e}}$, together with quotients $S_i(t) \to \mathbb{C}^{e_i}(t)$. Let $\widetilde{S}_i = \ker(S_i \to \mathbb{C}^{e_i}(t))$, a vector bundle of rank i and degree $-d_i$ on $\mathbb{P}^1$, and a subsheaf of $V^*_{\mathbb{P}^1}$. If we want $\widetilde{S}_i \to V^*$ to factor through $\widetilde{S}_{i+1}$, we should start with quotients $S_i(t) \to C^{e_i}(t)$ together with compatible maps $\mathbb{C}^{e_i}(t) \to \mathbb{C}^{e_{i+1}}(t)$. Set then $T_{n-i} = V^*/\widetilde{S}_i$.

The following construction and theorem are a globalization of this idea, showing also that every degeneration is obtained by the above construction.

Let $\overline{e} = (e_1, \dots, e_{n-1})$ such that $0 \leq e_i \leq \min(i, d_i)$, and $e_i - e_{i-1} \leq 1$. Consider the universal sequence

$$0 \to \mathcal{S}_1^{\overline{d}-\overline{e}} \hookrightarrow \cdots \hookrightarrow \mathcal{S}_{n-1}^{\overline{d}-\overline{e}} \to V^*_{\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}}}$$

Let $G_i \xrightarrow{\pi_i} \mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}}$ be the Grassmann bundle of e_i-dimensional quotients of $\mathcal{S}_i^{\overline{d}-\overline{e}}$, with universal sequence

$$0 \to K_i \to \pi_i^* \mathcal{S}_i^{\overline{d}-\overline{e}} \to Q_i \to 0$$

and set

$$X_{\overline{e}} = G_1 \times_{\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}}} \cdots \times_{\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}}} G_{n-1} \quad .$$

Define $\mathcal{U}_{\overline{e}} \subset X_{\overline{e}}$ as the subset where $K_i \to Q_{i+1}$ vanishes, and $K_i \to V^*$ is injective as a vector bundle map.

THEOREM 3. *(i)* $\mathcal{U}_{\overline{e}}$ *is irreducible, Cohen-Macaulay, of dimension* $1 + \binom{n}{2} + 2\sum d_i - \sum e_i - \sum e_i(e_i - e_{i-1})$. *The projection* $\pi : \mathcal{U}_{\overline{e}} \to \mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}}$ *is flat, and its image contains* $\mathbb{P}^1 \times H_{\overline{d}-\overline{e}}$.

(ii) There exist maps $h_{\overline{e}} : \mathcal{U}_{\overline{e}} \to \mathcal{H}Q_{\overline{d}}$ *satisfying*

(a) *if* $\mathrm{rk}_{(t,x)} \mathcal{T}_{n-i}^{\overline{d}} = n - i + e_i$, *then* $x \in h_{\overline{e}}(\mathcal{U}_{\overline{e}})$;
(b) *the restriction of* $h_{\overline{e}}$ *to* $\pi^{-1}(\mathbb{P}^1 \times H_{\overline{d}-\overline{e}})$ *is an isomorphism onto its image.*

EXAMPLES. 1) $\overline{e}_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$. Then X_{e_i} is a $\mathbb{P}^{i-1}$-bundle over $\mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}_i}$ and $\mathcal{U}_{\overline{e}_i}$ is a section over an open set, that is, $\pi : \mathcal{U}_{\overline{e}_i} \to \mathbb{P}^1 \times \mathcal{H}Q_{\overline{d}-\overline{e}_i}$ is an open immersion.

2) $n = 3$, $\overline{d} = (1,1)$. We have the strata D_1, D_2 from above and a codimension 2 stratum $E = h_{(1,1)}(\mathcal{U}_{(1,1)})$; $\mathcal{U}_{(1,1)} = X_{(1,1)} = \check{\mathbb{P}}(\mathcal{S}_2^{(0,0)})$; $h_{(1,1)}$ maps $\mathcal{U}_{(1,1)}$ isomorphically onto E.

LEMMA 1. $h_{\overline{e}}^{-1}(\overline{\Omega}_w(t)) = \pi^{-1}(\mathbb{P}^1 \times \overline{\Omega}_w(t)) \cup \widetilde{\Omega}_w^{\overline{e}}(t)$

where $\widetilde{\Omega}_w^{\overline{e}}(t) \subset \mathcal{U}_{\overline{e}}(t) := \pi^{-1}(\{t\} \times \mathcal{H}Q_{\overline{d}-\overline{e}})$ is defined by $\{\mathrm{rk}(V_p \otimes \mathcal{O} \to K_q^*) \leq r_w(q,p), \forall p, q\}$.

On $\mathcal{U}_{\overline{e}}(t)$ we have the flag of quotients

$$V \otimes = \mathcal{O} \twoheadrightarrow K_{n-1}^* \twoheadrightarrow K_{n-2}^* \twoheadrightarrow \cdots \twoheadrightarrow K_1^*$$

with $\mathrm{rk}\, K_i^* = i - e_i$, implying that some of these maps are isomorphisms. Let $k = \#\{i - e_i\}$. Define a partition of $[0, n]$ as follows: $i_0 = 0$, $i_j = \min\{i | i - e_i > i_{j-1} - e_{i_{j-1}}\}$, $i_{k+1} = n$. Let $n_j = i_j - e_{i_j}$, i.e., $i \in [i_j, i_{j+1} - 1] \implies \mathrm{rk}\, K_i^*$ is n_j (constant). Since the matrix $(r_w(q,p))$ has nondecreasing columns, $\widetilde{\Omega}_w^{\overline{e}}$ is defined by $\{\mathrm{rk}(V_p \otimes \mathcal{O}) \to K_j^* \leq r_w(i_j, p), j = 1, \dots, k, \forall p\}$.

Let $F(n_1,\dots,n_k,V)$ be the *partial* flag variety parametrizing flags of quotients of V, with ranks given by the n_i and with universal sequence

$$(*) \qquad V\otimes\mathcal{O}\twoheadrightarrow Q_{n_k}\twoheadrightarrow\cdots\twoheadrightarrow Q_{n_1}$$

There exists a $\phi_{\overline{e}}(t):\mathcal{U}_{\overline{e}}(t)\to F(n_1,\dots,n_k,V)$ such that

$$V\otimes\mathcal{O}\twoheadrightarrow K^*_{i_k}Y\cdots\twoheadrightarrow K^*_{i_1}$$

is $\phi_{\overline{e}}(t)^*$ of $(*)$, and $\widetilde{\Omega}^{\overline{e}}_w=\phi_{\overline{e}}^{-1}(t)(D_{w,\overline{e}})$, with $D_{w,\overline{e}}$ defined by the 'same' degeneration condition in $F(n_1,\dots,n_k,V)$.

LEMMA 2. *$D_{w,\overline{e}}$ is irreducible, of codimension a, satisfying $a\geq\ell(w)-\sum e_i$.*

LEMMA 3. *Let $\overline{e}$ be as above, and assume $\overline{e}\neq 0$. Then*
(i) *$\sum e_i(e_i-e_{i-1})\geq 1$;*
(ii) *We have equality in (i) if and only if $\overline{e}=\overline{e}_{k\ell}=(0,\dots,0,\underset{k}{1},\dots,\underset{\ell}{1},0,\dots,0)$ for some $1\leq k\leq\ell\leq n-1$;*
(iii) *Let $\overline{e}=\overline{e}_{k\ell}$ as above, and $w\in S_n$ any permutation, $D_{w,\overline{e}}$ as in Lemma 2. Then $a=\ell(w)-\sum e_i=\ell(w)-(\ell-k+1)$ if and only if $w(k)>max\{w(k+1),\dots,w(\ell+1)\}$.*

Remark: (ii) and (iii) will be used for the proof of Quantum Monk.

(4): Proof of Theorem 1 (ii). This is done by induction on $\overline{d}$: for $\overline{d}=(0,\dots,0)$, $\mathcal{H}Q_{\overline{d}}=H_{\overline{d}}=F$, OK. Assume then $\overline{d}\neq 0$.

Let $c=\sum\ell(w_i)$. It is enough to show that $h_{\overline{e}}(\mathcal{U}_{\overline{e}})\cap(\cap\Omega_{w_i}(t_i))$ has codimension $>c$ in $\mathcal{H}Q_{\overline{d}}$ for every $\overline{e}$. Now $h_{\overline{e}}$ is birational onto its image, hence it suffices to prove that the codimension of $\cap h_{\overline{e}}^{-1}(\Omega_{w_i}(t_i))$ in $\mathcal{U}_{\overline{e}}$ is greater than

$$c-(\dim\mathcal{H}Q_{\overline{d}}-\dim\mathcal{U}_e)=c+1-\sum e_i-\sum e_i(e_i-e_{i-1})$$

By Lemma 1,

$$\cap h_{\overline{e}}^{-1}(\overline{\Omega}_{w_i}(t_i))=\cap(\pi^{-1}(\mathbb{P}^1\times\Omega_{w_i}(t_i)\cup\widetilde{\Omega}^{\overline{e}}_{w_i}(t_i)))$$

$\widetilde{\Omega}^{\overline{e}}_{w_i}(t_i)$ is supported on $\mathcal{U}_{\overline{e}}(t_i)=\pi^{-1}(\{t_i\}\times\mathcal{H}Q_{\overline{d}-\overline{e}})$ and $t_1,\dots,t_N$ are *distinct*, hence there are only two types of nonempty intersections:

$$(*) \qquad \cap_{i=1}^N\pi^{-1}(\mathbb{P}^1\times\overline{\Omega}_{w_i}(t_i))$$

and

$$(**) \qquad \cap_{i=1}^{N-1}\pi^{-1}(\mathbb{P}^1\times\overline{\Omega}_{w_i}(t_i))\cap\widetilde{\Omega}^{\overline{e}}_{w_N}(t_N)$$

The estimate for $(*)$ is immediate by induction (as π is flat). As for $(**)$, write $W=\cap_{i=1}^{N-1}\pi^{-1}(\{t_N\}\times\overline{\Omega}_{w_i}(t_i))$, so $(**)=W\cap\widetilde{\Omega}^{\overline{e}}_{w_N}(t_N)$. The codimension of W

in $\mathcal{U}_{\overline{e}}(t_N)$ is $\sum_{i=1}^{N-1}\ell(w_i)=c-\ell(w_N)$. Now use Kleiman's theorem to deduce that the codimension of $(**)$ in $\mathcal{U}_{\overline{e}}(t_N)$ is $c-\ell(w_N)+a$, and then

$$\operatorname{codim}_{\mathcal{U}_{\overline{e}}}(**)=1+c-\ell(w_N)+a\geq 1+c-\sum e_i>1+c-\sum e_i-\sum e_i(e_i-e_{i-1})$$

as needed (the first inequality by Lemma 2, the second by Lemma 3(i)). $\square$

(5): $\langle\Omega_{s_i},\Omega_w,\Omega_{w'}\rangle_{\overline{d}}=?$ (with $1+\ell(w)+\ell(w')=\binom{n}{2}+2\sum d_i$).

This is $\#(\overline{\Omega}_{s_i}(u)\cap\overline{\Omega}_w(v)\cap\overline{\Omega}_{w'}(t))$ for $u,v,t\in\mathbb{P}^1$ distinct; that is,

$$\int_{HQ_{\overline{d}}}[\overline{\Omega}_{s_i}(u)]\cup[\overline{\Omega}_w(v)]\cup[\overline{\Omega}_{w'}(t)]$$

Suppose that when $u=v$, $Z:=\overline{\Omega}_{s_i}(u)\cap\overline{\Omega}_w(u)\cap\overline{\Omega}_w(t)$ is still top-codimensional. Then $\langle\Omega_{s_i},\Omega_w,\Omega_{w'}\rangle_{\overline{d}}=$the length of Z. However, Z is supported in the boundary! (this follows from the moving lemma). In fact, one can be much more precise:

PROPOSITION.
(i) *Z is either empty, or has pure codimension $\binom{n}{2}+2\sum d_i$ in $\mathcal{H}Q_{\overline{d}}$;*
(ii) *Z is contained in $\cup_{\overline{e}_{k\ell}}h_{\overline{e}_{k\ell}}(\mathcal{U}_{\overline{e}_{k\ell}}(u))$, with $\overline{e}_{k\ell}$ as in Lemma 3(ii), and $k\leq i\leq\ell$;*
(iii) *If $Z\cap h_{\overline{e}_{k\ell}}(\mathcal{U}_{\overline{e}_{k\ell}}(u))\neq\emptyset$, then w satisfies the condition of Lemma 3(iii): $w(k)>max\{w(k+1),\dots,w(\ell+1)\}$.*

PROOF. The same argument as in the Moving lemma gives that $(*)$ and $(**)$ are empty, as now we have top codimension. One additional case:

$$\widetilde{\Omega}^{\overline{e}}_{s_i}(u)\cap\widetilde{\Omega}^{\overline{e}}_w(u)\cap\pi^{-1}(\{u\}\times\overline{\Omega}_{w'}(t))\subset\mathcal{U}_{\overline{e}}(u)$$

Since $\ell(s_i)=1$, we can gain at most one dimension; the strict inequality may become an equality.

(ii), (iii) follow from Lemma 3(ii), (iii). $\square$

(6): Proof of Quantum Monk. Denote by $\alpha_{i,k}$ the product $s_{k-i+1}\cdots s_k$.

$$\langle\Omega_{s_j},\Omega_{\alpha_{i,k}},\Omega_{w'}\rangle_{\overline{d}}=\text{length}(Z)$$

The only index $\overline{e}_{k\ell}$ for which $\alpha_{i,k}$ satisfies condition (iii) above is $\overline{e}_k=\overline{e}_{kk}$. Hence, by (ii) above, Z is empty when $j\neq k$.

Assume $j=k$. Then $Z=h_{\overline{e}_k}(W)\subset h_{\overline{e}_k}(\mathcal{U}_{\overline{e}_k}(u))$, with $W=\widetilde{\Omega}^{\overline{e}_k}_{s_k}(u)\cap\Omega^{\overline{e}}_{\alpha_{i,k}}(u)\cap\pi^{-1}(\{u\}\times\overline{\Omega}_{w'}(t))$. Recall that $\pi:\mathcal{U}_{\overline{e}_k}(u)\to\{u\}\times\mathcal{H}Q_{\overline{d}-\overline{e}_k}$ is an open immersion. One checks easily that $\widetilde{\Omega}^{\overline{e}_k}_{s_k}(u)=\mathcal{U}_{\overline{e}_k}(u)$ and $\widetilde{\Omega}^{\overline{e}}_{\alpha_i,k}(u)=\overline{\Omega}_{\alpha_{i-1,k-1}}(u)\cap\mathcal{U}_{\overline{e}_k}(u)$; so $W=\emptyset$ unless $\overline{d}-\overline{e}_k=0$ and w' is the 'dual' permutation $w_0\cdot\alpha_{i-1,k-1}$. In this case $\mathcal{U}_{\overline{e}_k}(u)=\{u\}\times F$ and $W=\{pt\}$; moreover $h_{\overline{e}_k}$ is an isomophism onto its image. Hence $Z=\{pt\}$, length $Z=1$, as needed. $\square$

9. Enumerative geometry for hyperelliptic curves—T. Graber, 12/5/96

Question. How many hyperelliptic curves of genus g, degree d in $\mathbb{P}^2$ pass through $(3d+1)$ general points?

Strategy. The data of a map from a hyperelliptic curve C to $\mathbb{P}^2$ is almost the same as the data of a map $\mathbb{P}^1 \to H(2,\mathbb{P}^2)$, the Hilbert scheme of pairs of points in $\mathbb{P}^2$. Lifting such a map:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "2:1"] & \mathcal{X} \arrow[r] \arrow[d, "2:1"] & \mathbb{P}^2 \\ \mathbb{P}^1 \arrow[r] & H(2,\mathbb{P}^2) & \end{tikzcd}$$

one gets a honest hyperelliptic curve in $\mathbb{P}^2$ if the bottom map does not land in the diagonal.

Here is the plan:

I. Calculate the genus 0 Gromov-Witten invariants of $H = H(2,\mathbb{P}^2)$;
  —associativity;
  —geometry of H, 'virtual' considerations (H is not convex).

II. Relate the GW-invariants to enumerative geometry;
  —use natural PGL(3) action to control moduli spaces;
  —understand virtual contributions.

Geometry of H. Points of H correspond to either pairs of points in $\mathbb{P}^2$, or points with tangent directions: so there will always be exactly one line containing a given one. This gives a map

$$\pi : H \to \mathbb{P}^{2^\vee}$$

and $\pi^{-1}([L]) = \mathrm{Sym}^2 L \cong \mathbb{P}^2$. In fact, $H \cong \mathbb{P}(\mathrm{Sym}^2 T\mathbb{P}^2)$. This description allows us to calculate most standard invariants of H. In particular, the Chow ring $A^*(H)$ is generated by divisors:

$T_1 = \pi^*(\mathcal{O}(1))$

$T_2 =$\{set of subschemes incident to a fixed line\}.

These span the nef cone. A dual basis is given by

$B_1 =$\{subschemes supported at a fixed point\};

$B_2 =$line in a fiber of π;

so the effective curves are of the form $(a,b) = aB_1 + bB_2$ with $a, b \geq 0$.

Given a hyperelliptic plane curve C as above, we can recover the degree and genus of C from the homology class (a,b) of the associated rational curve in H. If C has degree d and genus g, then intersecting with T_2 gives $d = b$. To recover the genus of C, note that the branch points of the hyperelliptic involution correspond exactly to the intersections of the rational curve with Δ, the divisor in H parametrizing non-reduced subschemes. It is easy to show that $\Delta = 2(T_2 - T_1)$. We conclude that $2g + 2 = (a,b)\cdot\Delta = 2b - 2a$ so $g = b - a - 1$.

Looking at $H \to \mathbb{P}^2$, the diagonal Δ is realized as a conic bundle inside H: $\Delta = \mathbb{P}(T\mathbb{P}^2) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 T\mathbb{P}^2)$ by Veronese. The Chow ring $A^*(\Delta)$ is generated by T_1 and $\frac{1}{2}T_2$. Curves in Δ are of the form (a,b) with b even.

Next, $c_1(TH) = 3T_2$. Hence, the expected dimension of $\overline{M}_{0,0}(H,(a,b))$ is $3b + 4 - 3 = 3b + 1$; in particular, it does not depend on the genus of (a,b).

Review of GW invariants:

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n)$$

with usual notations, where the $\int$ is taken on a fundamental class which equals $[\overline{M}_{0,n}(X,\beta)]$ if X is convex. If X is not convex (which is the case at hand), the $\int$ must be taken over a $V \in A_*(\overline{M}_{0,n}(X,\beta)$ of the expected dimension. Heuristically,

$$I_\beta(\cdots) = \#\{\rho_1^{-1}(\Gamma_1) \cap \cdots \rho_n^{-1}(\Gamma_n) \cap V\}$$

with evident notations.

First reconstruction theorem (from [K-M]): if $A^*(X)$ is generated by divisors, then all genus 0 GW-invariants of X can be determined from the 2-point numbers $I_\beta(\gamma_1\gamma_2)$.

So we look at $I_{(a,b)}(\gamma_1\gamma_2)$. The class γ_1 imposes at most 3 conditions on curves; two classes impose at most 6 conditions. However, if $b \geq 2$ then the expected dimension is ≥ 7, so the only 2-point numbers come from curve classes $(a,0)$ or $(a,1)$.

Start with $(a,0)$-curves. These lie in Δ, since $(a,0)\cdot\Delta = -2a < 0$. Curves $(1,0)$ are fibers of the map $\Delta \to \mathbb{P}^2$ given by support. In other words, all representatives of this class are of the form originally described. So the moduli space $\overline{M}_{0,0}(H,(1,0))$ is isomorphic to $\mathbb{P}^2$ with universal curve Δ. Curves of type $(a,0)$ are a-sheeted covers of such fibers. The expected dimension for $(a,0)$ curves is $3\cdot 0 + 1 = 1$; so we consider $I_{(a,0)}(\gamma)$ for $\gamma \in A^2(H)$. $A^2(H)$ is 3-dimensional, we'll just look at 2 elements here.

Candidates for γ:

$T_4 =$\{all subschemes incident to a fixed point\};

$T_3 =$\{subschemes contained in a fixed line\}.

$I_{(a,0)}(T_4) = I_{(1,0)}(T_4) = 0$ becauses T_4 really imposes two conditions. That is, if we look in $\overline{M}_{0,0}(H,(1,0)) = \mathbb{P}^2$, the virtual class is an element V of A_1. The locus corresponding to curves meeting a representative of T_4 is just a single point, though, so they don't intersect. Similarly, an $(a,0)$ curve will meet a representative cycle for T_4 if and only if the $(1,0)$ curve that it covers meets it, so again the codimension is too high. (There is a second class in $A^2(H)$ which gives a zero GW-invariant for the same reason.)

For T_3 we actually have to worry about the virtual class. It is easy to see that $I_{(1,0)}(T_3) = \deg(V)$, the degree of the virtual class.

We can actually compute this virtual number (although it will also follow from associativity): consider again $\Delta \subset H$; Δ is made of flags, and hence it is homogeneous. $\overline{M}_{0,n}(\Delta,(a,0))$ maps isomorphically to $\overline{M}_{0,n}(H,(a,0))$ since curves of class $(a,0)$ in H are automatically contained in Δ. We want to find $V \in A_1(\overline{M}_{0,n}(H,(a,0))$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "f"] \arrow[d] & \Delta \arrow[r] & H \\ \overline{M}_{0,n}(H,(a,0)) & & \end{tikzcd}$$

By the fancy definition, $V = c_{top}(R^1\pi_* f^*(N))$. Here N denotes the normal bundle of Δ in H. In the $(1,0)$ case we have a fine moduli space, so we can actually compute this Chern class using Grothendieck-Riemann-Roch. (The answer is 3.) For $(a,0)$ with $a \geq 2$ we would have to bring in stacks, and we can do it otherwise anyway.

Next, consider $I_{(a,1)}(\gamma_1\gamma_2)$. For $a > 1$, $(a,1)\cdot\Delta < 0$, so have at least a component in Δ. However, the second index is not even, so these curves are all reducible. Curves of type $(a,1)$ map to lines in $\mathbb{P}^2$; they will consist of a $(0,1)$ curve, meeting Δ in two points, with a total $(a,0)$ attached: we glue an $(a_1,0)$ at the first point and an $(a_2,0)$ at the second, with $a_1 + a_2 = a$. Because we have such an explicit description of the moduli space, we can identify the space

$$\{\rho_1^{-1}(\Gamma_1) \cap \rho_2^{-1}(\Gamma_2)\}$$

which occurs in the interpretation of the GW-invariants. (Actually we will just identify the image of this locus in the 0-pointed space.)

EXAMPLE. $I_{(a,1)}(\text{pt. class}, T_3)$. The point class should be thought of as a pair of points, $\{p,q\}$ in $\mathbb{P}^2$, and T_3 corresponds to a choice of line l in $\mathbb{P}^2$. Since $\{p,q\}$ is not contained in Δ, the $(0,1)$ component must hit this point. This determines both the fiber of π in which this line lives, and fixes one point through which it must pass. Another point is determined by the fact that one of the $(a_i,0)$ components must meet the T_3. As these can be attached only at points of intersection of the $(0,1)$ curve with Δ, it follows that the curve must contain a double point supported at the intersection of l and $\overline{pq}$. The choice of a_1 and a_2 as well as the choice of particular a_i-sheeted covers has no effect on the incidence relation we are concerned with, except for the condition that a_1 must be non-zero. (a_1 is the degree of the curve glued at the special intersection point.)

The moduli space of solutions splits up into connected components determined by the partition of a between the two intersection points. Once this is decided, all that remains is to choose at each point an a_i sheeted cover of $\mathbb{P}^1$ and a particular point of that cover to glue to the $(0,1)$ curve. We denote the space of such data by $M(a_i)$. ($M(a)$ is naturally isomorphic to a fiber of the evaluation map from $\overline{M}_{0,1}(\mathbb{P}^1, a)$ to $\mathbb{P}^1$.)

The moduli space of solutions is

$$\amalg_{a_1+a_2=a; a_1>0; a_2\geq 0} M(a_1) \times M(a_2) \quad ;$$

if either a_1 or a_2 is > 1, the corresponding component has positive dimension, and again we need to understand the virtual class in $A_0(M(a_1)\times M(a_2))$. Because the virtual class is constructed from the deformation theory of the stable map, and because the deformations of our $(a,1)$ curve naturally split into deformations of the $(a_1,0)$ and $(a_2,0)$ curves, it follows that this virtual class splits as $V_{a_1}\cdot V_{a_2}$. In conclusion,

$$I_{a,1}(\text{pt}, T_3) = \sum_{a_1+a_2=a, a_1>0, a_2\geq 0} V_{a_1}\cdot V_{a_2}$$

Similar phenomena happen in many cases. In the end, the only unknowns are V_{a_i} and $I_{(a,0)}(T_3)$.

Now, remarkably as usual, the associativity equations are enough to compute

$$\begin{cases} I_{(a,0)}(T_3) = 3/a^2 \\ V_a = \begin{cases} 1 & \text{if } a = 0,1 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

From this, all numbers can be computed.

Enumerative significance. The number $I_{(a,b)}(T_4^{3b+1})$ should count hyperelliptic curves through $(3d+1)$ points. Unfortunately, this is not literally true. Since H is not convex, there will be unwanted contributions to this GW-invariant. We have unusually good control over these contributions for H however, because of the action of PGL(3). Observe that H is almost homogeneous: the PGL(3) action has only two orbits. So TH is generated by global sections outside Δ. For $f: \mathbb{P}^1 \to H$ with image not contained in the diagonal, $f^*(TH)$ is generically generated by global sections, hence it *is* generated by global sections. Therefore, $H^1(f^*TH) = 0$: H is almost convex. In a neighborhood of such $[f]$, $\overline{M}_{0,n}(\cdots)$ is of the expected dimension. The same holds for reducible curves, as long as no component lies in Δ.

For $f: \mathbb{P}^1 \to \Delta \subset H$ representing a class (a,b), observe that Δ *is* homogeneous; the dimension at such $[f]$ is computed to be $2a+b$. This is $> 3b+1$ if and only if $a > b$, that is if and only if $(a,b)\cdot\Delta < 0$.

One could hope that if $b > a$, then $\overline{M}_{0,0}(H,(a,b))$ has the expected dimension. However this is not the case, as one can have a curve consisting of two components C_1, C_2, and an f mapping C_2 to Δ and with $f_*[C_1] = (a_1,b_1)$, $f_*[C_2] = (a_2,b_2)$: $b = b_1 + b_2 > a = a_1 + a_2$ doesn't prevent $a_2 > b_2$, in which case $[f]$ moves too much.

However, most of the extraneous components do not contribute. This is because curves in Δ do not hit enough points.

EXAMPLE. $C = C_1 \cup C_2$. Need to hit $3(b_1+b_2)+1$ T_4's. Say C_1 moves correctly; then C_1 can only hit $3b_1+1$ T_4's. This leaves $3b_2$ for C_2. Now a rational curve in Δ gives rise to a nonreduced subvariety of $\mathbb{P}^2$ supported on a rational curve. The degree of the subscheme is b_2, so this rational curve has degree $b_2/2$. This curve can then hit only $\frac{3}{2}b_2 - 1$ points, $< 3b_2$.

One exception: $b_2 = 0$, $(a,b) = (a_1,b) + (a_2,0)$. We get solutions here. This is fairly clear. Because the expected dimension doesn't depend on a, and since the dimension of the locus of irreducible curves is equal to the expected dimension, we should see finitely many irreducible curves in class (a_1,b) satisfying the desired incidence conditions for any a_1. Such a curve will meet Δ in $2(b-a_1)$ points. At each of these points you can glue on a $(c_i,0)$ curve in such a way that $a_1 + \sum c_i = a$. Any such partition of a gives a component of the moduli space which looks like

$$\amalg M(c_1)\times M(c_2)\times\cdots\times M(c_n).$$

The deformation theory for these curves is identical to the deformation theory on the similar moduli spaces we saw earlier. So again the virtual class splits up across the factors, giving a contribution $\sum V_{c_1}V_{c_2}\cdots V_{c_n}$. In fact, all $c_i = 0, 1$ (else $V_{c_i} = 0$). Because of this, we can identify all solutions to the enumerative problem which the GW-invariant is actually solving. Namely, we get the irreducible curves of type (a, b) that we want, but we also get the irreducible solutions of type (a_1, b) for all $a_1 < b$ decorated with $(1, 0)$ curves at exactly $(a - a_1)$ of the points of intersection of the irreducible curve with Δ.

So: define

$$E_{(a,b)}(T_4^{3d+1}) = \#\{\text{irreducible curves of type } (a, b) \text{ meeting the cycles and transverse to } \Delta\}$$

that is the number of honest hyperelliptic curves of degree $d = b$ and genus $b-a-1$ through $(3d + 1)$ points in $\mathbb{P}^2$. The result is that

$$I_{(a,b)}(T_4^{3d+1}) = \sum_{i=0}^{a} \binom{2b-2a+2i}{i} E_{(a-i,b)}$$

and this relation can be inverted to find the $E_{(a,b)}$ in terms of the $I_{(a,b)}$. For example, for genus-2 curves of degree d through $(3d + 1)$ points, we find 27 curves for $d = 4$, 36855 for $d = 5$, 58444767 for $d = 6$, and so on.

For genus 0 and 1, extra care has to be taken to account for 'extra' g_2^1's, as maps $\mathbb{P}^1 \to H$ parametrize a choice of hyperelliptic curve in $\mathbb{P}^2$ *and* a choice of hyperelliptic involution.

10. Quantum differential equations and equivariant quantum cohomology, I—R. Pandharipande, 12/3/96

Example. $\mathbb{C}^*$-equivariant GW-invariants of $\mathbb{P}^2$. Let $\mathbb{C}^*$ act on $\mathbb{P}^2$ by

$$t \mapsto \begin{pmatrix} t^a & & \\ & t^b & \\ & & t^c \end{pmatrix}$$

and let $e_1 = a+b+c$, $e_2 = ab+ac+bc$, $e_3 = abc$. Finite-dimensional approximation of $E\mathbb{C}^* \to B\mathbb{C}^*$: $E\mathbb{C}^*_n = \mathbb{P}(\mathcal{O}(-a) \oplus \mathcal{O}(-b) \oplus \mathcal{O}(-c)) \to B\mathbb{C}^*_n = \mathbb{P}^n$. A module basis of the (ordinary) equivariant cohomology of $\mathbb{P}^2$ over $H^*_{\mathbb{C}^*} = \mathbb{C}[t]$ is

$$\begin{cases} 1 \\ T_1 = \xi = c_1(\mathcal{O}_{\mathbb{P}}(1)) \\ T_2 = \xi^2 \end{cases}$$

The equivariant pairing matrix is

$$(g_{ef}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & e_1 t \\ 1 & e_1 t & (e_1^2 - e_2)t^2 \end{pmatrix}$$

So we can compute (g^{ef}). Proceed as usual, setting up a 'quantum equivariant potential'

$$\Gamma = \sum_{d>0}\sum_{k\geq 0} I_d(T_2^{3d-1+k}) e^{dy_1} \frac{y_2^{3d-1+k}}{(3d-1+k)!}$$

By dimension reasoning, $I_d(T_2^{3d-1+k}) = N_{d,k}t^k$ with $N_{d,k}$ numbers. Associativity as in the usual story: read the coefficient of T_0 in $(T_1 * T_1) * T_2 = T_1 * (T_1 * T_2)$ and obtain

$$\Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111}\Gamma_{122} + 2e_1 t\Gamma_{122} - (e_1^2 + e_2)t^2\Gamma_{112} + (e_1e_2 - e_3)t^3\Gamma_{111}$$

Then find equations for the $N_{d,k}$'s: for $d > 0$, $k \geq 0$, $(d, k) \neq (1, 0)$,

$$\begin{aligned} N_{d,k} = &\sum_{d_1,d_2\geq 1; k_1,k_2\geq 0; d_1+d_2=d; k_1+k_2=k} N_{d_1,k_1}N_{d_2,k_2}d_1d_2 \cdot \\ &\cdot \left[d_1d_2\binom{3d-4+k}{3d_1-2+k_1} - d_1^2\binom{3d-4+k}{3d_1-1+k_1} \right] \\ &+ 2e_1dN_{d,k-1} - (e_1^2+e_2)d^2N_{d,k-2} + (e_1e_2-e_3)d^3N_{d,k-3} \end{aligned}$$

The old equation is just a subrecursion here ($k = 0$; set $N_{d,k} = 0$ if $k < 0$). This equation determines all the numbers recursively, from $N_{1,0} = 1$. For example, $N_{1,1} = 2e_1$.

Note: the $N_{d,k}$ are symmetric functions of degree k in a, b, c.

Dubrovin formalism. Let $\mathcal{S}$ be a trivial bundle over $M = \mathbb{R}^n$, with sections $\hat{s}_1, \ldots, \hat{s}_m$ which trivialize it. We consider a connection $\nabla : H^0(\mathcal{S}) \to H^0(\mathcal{S} \otimes \Omega^1_M)$, that is a map on C^∞ sections, satisfying

$$\nabla(f \cdot s) = f\nabla s + s \otimes df$$

for f a function. Vector fields $\frac{\partial}{\partial x_i}$ determine covariant derivatives $\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}$ by contracting ∇s by $\frac{\partial}{\partial x_i}$.

In local coordinates, we may write

$$\nabla(\hat{s}_j) = \Gamma_{ij}^k \hat{s}_k \otimes dx_i$$

(omitting obvious $\sum$), and for $g^j\hat{s}_j \in H^0(\mathcal{S})$

$$\nabla_i(g^j\hat{s}_j) = \frac{\partial g^j}{\partial x_i}\hat{s}_j + \Gamma_{ij}^k g^j \hat{s}_k$$

We can think of a connection as a way of lifting vector fields from M to $\mathcal{S}$ (that is, a *distribution* in the sense of Frobenius): for each point $s \in \mathcal{S}$, we have n independent vectors in $T_s\mathcal{S}$:

$$\frac{\partial}{\partial x_p} \mapsto \frac{\partial}{\partial x_p} - \Gamma_{pj}^k s_j \frac{\partial}{\partial s_k}$$

(in coordinates $x_1, \ldots, x_n, s_1, \ldots, s_m$ on $\mathcal{S}$).

Now we want sections s for which $\nabla s = 0$. We get Frobenius integrability conditions: if $\mathcal{L}_1$, $\mathcal{L}_2$ are vector fields in the distribution, then $[\mathcal{L}_1, \mathcal{L}_2]$ must also be in it. This is where the curvature comes up: denoting by L_p the lift of $\frac{\partial}{\partial x_p}$ is $[L_p, L_q]$ the lift of a vector field? Computation:

$$[L_p, L_q] = \left\{ -\frac{\partial}{\partial x_p}\Gamma^k_{qj} + \frac{\partial}{\partial x_q}\Gamma^k_{pj} + \Gamma^\ell_{pj}\Gamma^k_{q\ell} - \Gamma^\ell_{qj}\Gamma^k_{p\ell} \right\} s_j \frac{\partial}{\partial s_k}$$

No x-term, so the part in $\{\}$ must be 0 for integrability. This term is denoted by R^k_{jpq}, the coefficients of the curvature form in local coordinates.

Dubrovin. Apply this to $M = V = H^*(X)$ (say $\mathbb{C}$ coefficients), with basis $T_0, \ldots, T_m$, and $S = T_V$: the trivial bundle with fiber V. The coordinates on the manifold V will be denoted y_i, and ∂_i will be coordinates on the fiber. There is a metric on T_V: $(g_{ij}) = \langle \partial_i, \partial_j \rangle$. Define a connection by setting

$$\nabla_i \partial_j = \Phi_{ije} g^{(ek)} \partial_k$$

that is by prescribing Christoffel symbols $\Gamma^k_{ij} = \Phi_{ije} g^{ek}$. Here Φ is the GW potential, that is

$$\sum_{n \geq 3} \frac{1}{n!} \sum_\beta I_\beta(\gamma^{\otimes n})$$

Now compute the curvature (R^k_{jpq}) and impose its vanishing (that is, the integrability condition): $\frac{\partial}{\partial y_p}\Gamma^k_{qj} = \frac{\partial}{\partial y_q}\Gamma^k_{pj}$ comes for free from the definition, and the rest says

$$\Phi_{pje} g^{e\ell} \Phi_{q\ell f} = \Phi_{qje} g^{e\ell} \Phi_{p\ell f}$$

(times the invertible matrix (g^{fh})). This are just the WDVV equations!

So at least formally there are parallel sections. Givental (in one part of [Givental]) writes down such sections. His goal in doing this: operators that kill these sections will give relations in QH^*.

Givental. $\nabla_\hbar = \hbar d - \sum (p_\alpha *) dt_\alpha \wedge$

Here $\hbar$ is just a parameter; $p_1, \ldots, p_n$ form a basis of $H = H^*X$, and for its (trivial) tangent bundle T_H (same notation). $\nabla_\hbar$ acts $H^0(T_H) \to H^0(T_H \otimes \Omega^1)$.

We will obtain sections $\gamma = \sum t_i p_i$, with coordinates $(t) = (t_1, \ldots, t_n)$. Consider vector fields $F \in H^0(T_H)$, in coordinates:

$$F(t) = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \sum F^j p_j$$

Covariant derivatives:

$$\nabla_{\hbar,i}(F(t)) = \sum_k \left(\hbar \frac{\partial F^k}{\partial t_i} - \Phi_{ije} g^{ek} F^j \right) p_k$$

(so $\nabla_{\hbar,i}(p_j) = -p_i * p_j$, the quantum product). If $\hbar = 1$, $\nabla_\hbar$ is a honest connection; else, strictly speaking, $\nabla_\hbar$ is only a connection "up to scalar"; but the same Frobenius equations hold.

Consider now $\mathbb{P}^1 = \mathbb{P}\mathbb{C}^2$; $\mathbb{C}^*$ acts by $\begin{pmatrix} t^0 & \\ & t^1 \end{pmatrix}$. Equivariant cohomology: $H^*_{\mathbb{C}^*}\mathbb{P}^1 = \mathbb{C}[p,\hbar]/(p^2 - p\hbar)$. In terms of the basis $\{1, p\}$, $(g_{ef}) = \begin{pmatrix} 0 & 1 \\ 1 & \hbar \end{pmatrix}$. Given polynomials $f(p,\hbar)$, $g(p,\hbar)$, we have an equivariant pairing $\langle f, g \rangle \in H^*_{\mathbb{C}^*} = \mathbb{C}[\hbar]$. In fact, one checks that

$$\langle f, g \rangle = \frac{1}{2\pi i} \int \frac{fg\, dp}{p(p-\hbar)}$$

Over $\mathbb{C}(\hbar)$ we can take the basis $\frac{p}{\hbar}$, $\frac{\hbar - p}{\hbar}$, for which the intersection form diagonalizes to $\begin{pmatrix} \frac{1}{\hbar} & 0 \\ 0 & -\frac{1}{\hbar} \end{pmatrix}$.

Back to our manifold X, $V = H^*X$, ∇ on T_V. We want sections s with $\nabla s = 0$. Look then at $X \times \mathbb{P}^1$, $\mathbb{C}^*$ acting trivially on X and as above on $\mathbb{P}^1$; and use equivariant GW invariants on $X \times \mathbb{P}^1$. The equivariant cohomology is

$$H^*_{\mathbb{C}^*}(X \times \mathbb{P}^1) = H^*X \otimes_{\mathbb{C}} \mathbb{C}[p,\hbar]/(p(p-\hbar))$$

Look for a $\mathbb{C}[\hbar]$-module basis: we have $\{p_i\}$ for H^*X; and over $\mathbb{C}(\hbar)$, we have $p_i \otimes \frac{p}{\hbar}$, $p_i \otimes \frac{\hbar - p}{\hbar}$. Write

$$\varphi = \text{“}(\tau, t)\text{”} = \sum t_i(p_i \otimes \frac{p}{\hbar}) + \tau_i(p_i \otimes \frac{\hbar - p}{\hbar})$$

for elements of $H^*_{\mathbb{C}^*}(X \times \mathbb{P}^1) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}(\hbar)$. Then the equivariant intersection pairing is nice:

$$\langle \varphi, \varphi' \rangle = \langle (\tau, t), (\tau', t') \rangle = \frac{\langle t, t' \rangle - \langle \tau, \tau' \rangle}{\hbar}$$

where $\langle t, t' \rangle = \langle \sum t_i p_i, \sum t'_i p_i \rangle$ and $\langle \tau, \tau' \rangle = \langle \sum \tau_i p_i, \sum \tau'_i p_i \rangle$ are both calculated on X alone.

In fact, for more equivariant classes, the equivariant push-forward is given by

$$\int \varphi \cup \varphi' \cup \varphi'' = \frac{(\int_X t \cup t' \cup t'') - (\int_X \tau \cup \tau' \cup \tau'')}{\hbar}$$

Here τ's and t's do not mix because $p(\hbar - p)$ in the numerator kills the poles, so the residue is 0.

Next, consider the following function G of $(t, \tau, \hbar, q, q_0)$:

$$G = \sum_{n \geq 0} \sum_\beta \sum_d \frac{1}{n!} q^\beta q_0^d I^{\text{equiv.}}_{n,(\beta,d)}(\varphi^{\otimes n})$$

Here (β, d) is a curve class of $X \times \mathbb{P}^1$ in the evident fashion, and φ is written in terms of $\hbar$ etc., as above. Note that the $q^\beta q_0^d$ terms 'pull apart' the curve classes; that's what Givental always does. Write

$$G = G^{(0)} + q_0 G^{(1)} + q_0^2 G^{(2)} + \ldots$$

CLAIM. $G^{(0)} = \dfrac{\Phi(t,q) - \Phi(\tau,q)}{\hbar}$, *with* $\Phi =$ *ordinary GW-potential on* X.

This comes down to the preceding equations for the equivariant push–forward. Another basis, e.g. without $1/\hbar$, would not simplify so much.

Next, set $\theta_{\alpha\beta} = \dfrac{\partial^2 G^{(1)}}{\partial\tau_\alpha \partial\tau_\beta}$.

CLAIM. *Fix* β. *Set* $F = \sum_\alpha \theta_{\alpha\beta} g^{\alpha j} p_j$. *Then*

$$-\hbar \frac{\partial}{\partial\tau_\gamma} F = p_\gamma(\tau) * F$$
$$\hbar \frac{\partial}{\partial t_\gamma} F = p_\gamma(t) * F$$

(Warning: typo in [Givental]: t and τ are swapped.)

Note: the entries of F involve p_α's, and $p_\gamma * p_\alpha = \sum$ power series(var.)p_f. "$p_\gamma(\tau)$" means: set the variable to τ in the power series. That is, say $F = \sum F^i p_i$; now $p_\gamma * \sum F^i p_i = \sum F^i p_\gamma * p_i$; $p_\gamma * p_i = \sum \Phi_{\gamma i e} g^{ef} p_f$. So we can define $p_\gamma(\tau) * F$ to be $\sum_{i,e,f} F^i \Phi_{\gamma i e}(\tau) g^{ef} p_f$, and similarly for $p_\gamma(t) * F$.

The F given by the last Claim are the 'parallel sections', but they involve τ's and t's.

To prove the claim, one essentially uses the fact that G satisfies the equivariant WDVV equation for $X \times \mathbb{P}^1$.

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11. Quantum differential equations and equivariant quantum cohomology, II—R. Pandharipande, 12/10/96

Review of the previous lecture: X is a variety; $V = H^*(X,\mathbb{C})$; we have a natural identification $V \cong T_V$; $\{p_1,\dots,p_N\}$ is a basis of V, and $\partial_1,\dots,\partial_N$ the corresponding basis of T_V; we let t_i be coordinates on V. We have defined a connection 'up to scalar':

$$\nabla_\hbar = \hbar d - \sum_\alpha (p_\alpha *) dt_\alpha \wedge$$

acting in the following way: for a vector field $F(t) = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix} = \sum F^j p_j = \sum F^j \partial_j$,

we have the covariant derivative

$$\nabla_{\hbar,i} F = \sum_k \left(\hbar \frac{\partial F^k}{\partial t_i} - \Phi_{ije} g^{ek} F^j \right) \partial_k$$

where Φ denotes the GW-potential.

We have seen that Φ satisfies the WDVV equations $\iff$ the corresponding formal connection is flat. We are looking for solutions of $\nabla_{\hbar,i} F = 0\ \forall i$; that is, $\forall i,k$

$$\hbar \frac{\partial}{\partial t_i} F^k = \Phi_{ije} g^{ek} F^j$$

Givental's approach: to get solutions, consider $X \times \mathbb{P}^1$ with $\mathbb{C}^*$ acting on the $\mathbb{P}^1$ factor by $(t^0 : t^1)$. Fact:

$$H^*_{\mathbb{C}^*}\mathbb{P}^1 = \mathbb{C}[p,\hbar]/(p^2 - p\hbar)$$

We found a better basis here, over $\mathbb{C}(\hbar)$: $p/\hbar$, $(\hbar - p)/\hbar$.

REMARK. By *localization*, the equivariant cohomology of $\mathbb{P}^1$ is essentially concentrated at $0,\infty$; this basis is the natural one from this point of view.

We then have a basis for $H^*_{\mathbb{C}^*}(X \times \mathbb{P}^1) = H^*X \otimes \mathbb{C}[p,\hbar]/(p^2 - p\hbar)$: $p_i \otimes p/\hbar$, $p_i \otimes (\hbar - p)/\hbar$. So we write $\varphi \in H^*_{\mathbb{C}^*}(X \times \mathbb{P}^1)$ as

$$\varphi = \sum t_i \left(p_i \otimes \frac{p}{\hbar} \right) + \sum \tau_i \left(p_i \otimes \frac{\hbar - p}{\hbar} \right)$$

Next, consider the *equivariant* GW-potential $G(t,\tau,\hbar,q,q_0)$; we expand it in terms of powers of the $\mathbb{P}^1$-curve class:

$$G = G^{(0)} + q_0 G^{(1)} + q_0^2 G^{(2)} + \cdots$$

We have seen that

(I) $G^{(0)} = \dfrac{\Phi(t,q) - \Phi(\tau,q)}{\hbar}$, the usual GW-potential;

(II) letting $\theta_{ab} := \dfrac{\partial^2 G^{(1)}}{\partial\tau_a \partial t_b}$ and $F := \sum_a \theta_{ab} g^{aj} \partial_j$, then we have the two equations

$$\begin{cases} -\hbar \dfrac{\partial}{\partial\tau_\gamma} F = \partial_\gamma(\tau) * F \\[2ex] \hbar \dfrac{\partial}{\partial t_\gamma} F = \partial_\gamma(t) * F \end{cases}$$

(end of the review)

Localization. Let Y be a manifold with a $\mathbb{C}^*$ action, and denote by $Y^{\mathbb{C}^*}$ the set of fixed points. We have $\pi_* : H^*_{\mathbb{C}^*}(Y) \to H^*_{\mathbb{C}^*}$ as usual, and also $H^*_{\mathbb{C}^*}(Y^{\mathbb{C}^*}) \cong H^*(Y^{\mathbb{C}^*}) \otimes H^*_{\mathbb{C}^*}$ since the action is trivial on $Y^{\mathbb{C}^*}$. The equivariant inclusion $Y^{\mathbb{C}^*} \hookrightarrow Y$ gives a diagram

$$\begin{tikzcd} \omega \in H^*\mathbb{C}^*(Y) \arrow[r] \arrow[d] & \omega \in H^*_{\mathbb{C}^*}(Y^{\mathbb{C}^*}) = H^*(Y^{\mathbb{C}^*}) \otimes H^*_{\mathbb{C}^*} \arrow[dl, dashed] \\ \pi_*\omega \in H^*_{\mathbb{C}^*} & \end{tikzcd}$$

The diagonal arrow can be filled in: it works $\omega \mapsto \displaystyle\int_{Y^{\mathbb{C}^*}} \frac{\omega}{c^{\mathrm{eq}}_{top} N}$, where N =normal bundle of $Y^{\mathbb{C}^*}$ in Y.

That is, $\pi_*\omega$ can be evaluated on fixed points. In this sense, equivariant cohomology is 'concentrated at the fixed points'.

Equivariant GW-invariants.

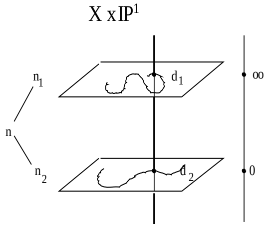
$$I_{n,d+1}\left\langle p_a\left(\frac{\hbar-p}{\hbar}\right)\cdot p_b\left(\frac{p}{\hbar}\right)\cdot\varphi^{\otimes n}\right\rangle$$

is the equivariant push-forward of ω from $Y=\overline{M}_{0,n+2}(X\times\mathbb{P}^1,d+1)$, where "$(d+1)$" stands for the sum of a class d from the X factor and of the class of a point from $\mathbb{P}^1$, and (with evident notations)

$$\omega=e_1^*\left(p_a\left(\frac{\hbar-p}{\hbar}\right)\right)\cup e_2^*\left(p_b\left(\frac{p}{\hbar}\right)\right)\cup e^*(\varphi)^{n\text{ times}}$$

The idea is to compute the fixed point locus $Y^{\mathbb{C}^*}$, then use localization to compute the push-forward.

What is $Y^{\mathbb{C}^*}$? We want the class to be 1 on the $\mathbb{P}^1$ factor, so a map $\mathbb{P}^1\to X\times\mathbb{P}^1$ must travel up exactly once; and must go straight up for a fixed point (else a surface is swept). Next,



$p_a((\hbar-p)/\hbar)$ must sit on the ∞-plane, and $p_b(p/\hbar)$ must sit on 0; the other $n=n_1+n_2$ points can be split arbitrarily among the two planes. This describes $Y^{\mathbb{C}^*}$.

Count the conditions: two nodes (along the vertical component), fixed at 0 and ∞; total of four conditions, so expect $\operatorname{codim}_Y Y^{\mathbb{C}^*}$ to be 4. As for the normal bundle, the tangent representation is t at ∞, corresponding to $-\hbar$; and t^{-1} at 0, corresponding to $\hbar$; smoothing the nodes, we get $-\hbar+c(\infty)$ at 0, and $\hbar+c(0)$ at ∞. The localization formula gives then the push-forward as

$$\int_{Y^{\mathbb{C}^*}}\frac{\omega}{-\hbar^2(-\hbar+c(\infty))(\hbar+c(0))}$$

This is all modulo a slight lie: $Y^{\mathbb{C}^*}$ is not codimension 4 everywhere. All n points may be concentrated on one of the planes, leaving no nodes at the other plane: the two possibilities give two codimension 3 components, with normal bundles whose classes are resp.

$$-\hbar^2(\hbar+c(0))\quad,\quad-\hbar^2(-\hbar+c(\infty))$$

Putting all together, $I_{n,d+1}\langle\cdots\rangle$ is the sum of three terms (one from cod. 4, two from cod. 3), involving only 'ordinary' GW-invariants of X. The first term is

$$\sum\frac{1}{-\hbar^2}\frac{n!}{n_1!n_2!}I_{n_1+2,d_1}\left\langle p_a\cdot\tau^{\otimes n_1}\frac{p_\epsilon}{-\hbar+c(\infty)}\right\rangle g^{\epsilon\epsilon'}I_{n_2+2,d_2}\left\langle p_b\cdot t^{\otimes n_2}\frac{p_{\epsilon'}}{-\hbar+c(0)}\right\rangle$$

where the $\sum$ is over $d_1+d_2=d; n_1+n_2=n; (d_1,n_1)\neq(0,0),(d_2,n_2)\neq(0,0)$; the $g^{\epsilon\epsilon'}$ factor makes sure the vertical component goes straight up: the point at 0 equals the point at ∞. The second and third term are similar:

$$\frac{1}{-\hbar^2}g_{a\epsilon}g^{\epsilon\epsilon'}I_{n+2,d}\left\langle p_b\cdot t^{\otimes n}\frac{p_{\epsilon'}}{\hbar+c(0)}\right\rangle$$
$$+\frac{1}{-\hbar^2}I_{n+2,d}\left\langle p_a\cdot\tau^{\otimes n}\frac{p_\epsilon}{-\hbar+c(\infty)}\right\rangle g^{\epsilon\epsilon'}g_{\epsilon' b}$$

To simplify notations, define a new function:

$$\psi_{ef}(t,\hbar)=g_{ef}+\sum_{n\geq 0,(n,d)\neq(0,0)}\sum_d\frac{1}{n!}I_{n+2,d}\left\langle\frac{p_t}{\hbar+c}\cdot t^{\otimes n}\cdot p_a\right\rangle$$

Then in short localization gives

$$-\hbar^2\theta_{ab}=\sum_{\epsilon,\epsilon'}\psi_{a\epsilon}(\tau,-\hbar)g^{\epsilon\epsilon'}\psi_{b\epsilon'}(t,\hbar)$$

(note: typos in [Givental]).

Now recall that θ must satisfy equations (II) (from the review):

$$-\hbar\frac{\partial}{\partial\tau_\gamma}\theta_{ab}g^{ak}\partial_k=p_\gamma(\tau)*\theta_{ab}g^{ak}\partial_k$$

Substituting the expression obtained above into this:

$$-\hbar\frac{\partial}{\partial\tau_\gamma}\psi_{a\epsilon}(\tau,-\hbar)g^{\epsilon\epsilon'}\psi_{\beta\epsilon'}(t,\hbar)g^{ak}\partial_k=p_\gamma(\tau)*\psi_{a\epsilon}(\tau,-\hbar)g^{\epsilon\epsilon'}\psi_{b\epsilon'}(t,\hbar)g^{ak}\partial_k$$

Taking out the invertible part:

$$-\hbar\frac{\partial}{\partial\tau_\gamma}\psi_{a\epsilon}(\tau,-\hbar)g^{\alpha k}\partial_k=p_\gamma(\tau)*\psi_{a\epsilon}(\tau,-\hbar)g^{\alpha k}\partial_k$$

or, after $\hbar\leftrightarrow-\hbar$:

$$\boxed{\hbar\frac{\partial}{\partial\tau_\gamma}\psi_{a\epsilon}(\tau,\hbar)g^{\alpha k}\partial_k=p_\gamma(\tau)*\psi_{a\epsilon}(\tau,\hbar)g^{\alpha k}\partial_k}$$

From this, one can obtain n independent solutions.

A different approach is obtained via *Gravitational descendents,* going back to Witten. We have defined ordinary GW-invariants by

$$\int_{[\overline{M}_{g,n}(X,\beta)]}e_1^*(\gamma_1)\cup\cdots\cup e_n^*(\gamma_n)$$

with notations as usual (in particular, with due care if X is non convex, $g > 0$, etc.). We can more generally define invariants

$$\int_{[\overline{M}_{g,n}(X,\beta)]} e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cup c_1^{\alpha_1} \cup \cdots \cup c_n^{\alpha_n}$$

where c_i =first Chern class of i-th *co*tangent line. Notation:

$$I_{n,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_n,\alpha_n)\rangle = \int_{[\overline{M}_{g,n}(X,\beta)]} \prod e_i^*(\gamma_i) \cup c_i^{\alpha_i}$$

The ordinary GW-invariants satisfy properties:

(I) *Fundamental class:* $I_{n+1,\beta}\langle\gamma_1\cdots\gamma_n\cdot 1\rangle = 0$ if $n \geq 3$ or $\beta \neq 0$, $n \geq 0$;
(II) *Divisor:* for $\gamma \in H^2(X)$, $I_{n+1,\beta}\langle\gamma_1\cdots\gamma_n\cdot\gamma\rangle = \int_\beta \gamma \cdot I_{n,\beta}\langle\gamma_1\cdots\gamma_n\rangle$.

Now the fancier invariants satisfy upgraded properties:

(I) *Fundamental class:*

$$I_{n+1,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_n,\alpha_n)(1,0)\rangle = \sum_{i=1}^n I_{n,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_i,\alpha_i-1)\cdots(\gamma_n,\alpha_n)\rangle$$

(of course, setting to 0 terms involving negative α-components);

(II) *Divisor:* for $\gamma \in H^2(X)$,

$$\begin{aligned} I_{n+1,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_n,\alpha_n)(\gamma,0)\rangle &= \int_\beta \gamma \cdot I_{n,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_n,\alpha_n)\rangle \\ &\quad + \sum_{i=1}^n I_{n,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_i\cdot\gamma,\alpha_i-1)\cdots(\gamma_n,\alpha_n)\rangle \end{aligned}$$

The underlying reason why these hold: consider the map forgetting the $(n+1)$-st point:

$$\nu : \overline{M}_{0,n+1}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$$

At the first point, we have c_1, $\nu^* c_1$: one can show that

$$c_1 = \nu^* c_1 + [D_{1,n+1}]$$

where $D_{1,n+1}$ is the divisor obtained by splitting (on ≥ 2 components) the points as $(1,n+1)|(2,\ldots,n)$, and the class as $0|\beta$.

For $g = 0$, the fancier invariants also satisfy suitable WDVV equations. As in the ordinary GW-case, the splitting axiom and the linear equivalence in $\overline{M}_{0,4}$ yield recursive relations for the invariants.

The claim is that the fundamental class and divisor properties, plus the upgraded WDVV, imply that the function $\psi_{ef}(t,\hbar)$ defined above is a solution of the main differential equation. This gives a second, and non-equivariant proof, of the main theorem.

In [Dubrovin], all this is seen from an axiomatic point of view. There a reconstruction theorem for $g = 0$ is stated:

Proposition. *The tree-level gravitational descendents can be uniquely reconstructed from the tree-level system of Gromov-Witten invariants.*

Sketch of proof. Seek an inductive scheme

$$\begin{aligned} &I_{n,\beta}\langle(\gamma_1,\alpha_1)\cdots(\gamma_n,\alpha_n)\rangle \\ &\qquad = \sum(\text{either lower curve class, or fewer cotangent line classes}) \end{aligned}$$

For this: the gravitational descendents are obtained by integrating

$$e_1^*(\gamma_1)c_1^{\alpha_1}\cdots e_n^*(\gamma_n)c_n^{\alpha_n}$$

over $\overline{M}_{0,n}(X,\beta)$. Instead, integrate

$$e_1^*(\gamma_1)c_1^{\alpha_1}\cdots e_n^*(\gamma_n)c_n^{\alpha_n} \cup e_{n+1}^*(H) \cup e_{n+2}^*(H) \cup e_{n+3}^*(1)$$

(with H =hyperplane class) over $\overline{M}_{0,n+3}(X,\beta)$. If we are not yet done, then some of the α_i is $\neq 0$, say α_n. Apply then the basic linear equivalence $D(12|34) \sim D(14|23)$ with "1"$= e_n^*(\gamma_n)c_n^{\alpha_n}$, "2"$= e_{n+1}^*(H)$, "3"$= e_{n+2}^*(H)$, and "4"$= e_{n+3}^*(1)$. This makes the induction click. □

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12. Quantum double Schubert polynomials—W. Fulton, 12/12/96

This is joint work with Ciocan-Fontanine.

Reminder on quantum Schubert polynomials (after Fomin–Gel'fand–Postnikov): for $w \in S_n$, we can define a polynomial $\mathfrak{S}_w^q(x)$, as follows. Quantum elementary symmetric polynomials are defined by $E_i(x_1,\ldots,x_p) := \sum$products of vertices and edges which cover exactly i vertices once in

$$\begin{array}{ccccccccc} x_1 & & x_2 & & x_3 & & \ldots & & x_p \\ \bullet & q_1 & \bullet & q_2 & \bullet & q_3 & \ldots & q_{p-1} & \bullet \end{array}$$

$(\deg x_i = 1, \deg q_i = 2)$. Clearly $E_i(x_1,\ldots,x_p)$ specialize to the usual elementary symmetric polynomials $e_i(x_1,\ldots,x_p)$ as the $q_j \mapsto 0$.

Definition. Write the usual Schubert polynomial corresponding to $w \in S_n$ as

$$\mathfrak{S}_w(x) = \sum a_J e_J(x)$$

with $a_J \in Z$, and denoting $e_J(x) = e_{j_1}(x_1)e_{j_2}(x_1,x_2)\cdots e_{j_{n-1}}(x_1,\ldots,x_{n-1})$, with $0 \leq j_i \leq i$. Set analogously $E_J(x) = E_{j_1}(x_1)\cdots E_{j_{n-1}}(x_1,\ldots,x_{n-1})$; then define

$$\mathfrak{S}_w^q(x) = \sum a_J E_J(x)$$

The theorem ('Quantum Giambelli', see the previous lecture on quantum Schubert polynomials) is that these $\mathfrak{S}_w^q(x)$ represent the Schubert varieties in the quantum cohomology of flag manifolds.

Now for the classical Schubert polynomials there is a concrete description: write $w = w_0 s_{i_1} \cdots s_{i_\ell}$ in the shortest possible way, where $w_0 = n(n-1)\cdots 1$ is the 'longest' permutation, s_i is the transposition $(i, i+1)$, and $\ell = \ell(w_0) - \ell(w)$. Then

$$\mathfrak{S}_w(x) = \partial^x_{i_\ell} \circ \cdots \circ \partial^x_{i_1}(\mathfrak{S}_{w_0}(x))$$

where $\mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_1$ and $\partial^x_i(P) = \dfrac{P - s^x_i(P)}{x_i - x_{i-1}}$ (see also previous lectures). The goal here is to find a similar formula for $\mathfrak{S}^q_w(x)$.

The idea comes from the following notion. There exist 'double Schubert polynomials' $\mathfrak{S}_w(x,y)$, obtained as

$$\mathfrak{S}_w(x,y) = \partial^x_{i_\ell} \circ \cdots \circ \partial^x_{i_1}(\mathfrak{S}_{w_0}(x,y))$$

with $\mathfrak{S}_{w_0}(x,y) = \prod_{i+j \le n}(x_i + y_j)$ (note: in the usual definition, one finds $-$ instead of $+$). It would seem as if the y_j are useless; on the contrary:

FACT. $\mathfrak{S}_w(x,y) = \mathfrak{S}_{w^{-1}}(y,x)$.

Equivalently, write $w = s_{j_\ell} \cdots s_{j_1} w_0$ again in the shortest possible way; then

$$\mathfrak{S}_w(x,y) = \partial^y_{j_\ell} \circ \cdots \circ \partial^y_{j_1}(\mathfrak{S}_{w_0}(x,y))$$

DEFINITION. With w as above, define *quantum double Schubert polynomials* by

$$\mathfrak{S}_w(x,y,q) = \partial^y_{j_\ell} \circ \cdots \circ \partial^y_{j_1}(\mathfrak{S}_{w_0}(x,y,q))$$

where

$$\mathfrak{S}_{w_0}(x,y,q) = \prod_{p=1}^{n-1} E_p(x_1 + y_{n-p}, \ldots, x_p + y_{n-p})$$

Note that $\mathfrak{S}_w(x,y,0) = \mathfrak{S}_w(x,y)$ by definition. The interesting fact is that

THEOREM. $\mathfrak{S}_w(x,0,q) = \mathfrak{S}^q_w(x)$.

PROOF. Claim: $\mathfrak{S}_w(x,y,q) = \sum a_J(y) E_J(x)$ with $a_J(y) \in \mathbb{Z}[y]$.

Granted this, $\mathfrak{S}_w(x,0,q) = \sum a_J(0) E_J(x)$, and the $a_J(0)$ are the right coefficients to give $\mathfrak{S}^q_w(x)$ since $\mathfrak{S}_w(x) = \mathfrak{S}_w(x,0,0) = \sum a_J(0) e_J(x)$.

So we only need to prove the Claim.

PROOF OF THE CLAIM. (1) True for $w = w_0$: use that

$$E_p(x_1 + y_{n-p}, \ldots, x_p + y_{n-p}) = \sum_{i=0}^{p} E_i(x_1 \ldots x_p)(y_{n-p})^{p-i}$$

which is clear from the definition.

(2) The ∂^y_i's only work on the coefficients $a_J(y)$. $\square$

Geometric origin. On the hyperquot scheme$\times\mathbb{P}^1$ we have

$$E_1 \subset \cdots \subset E_{n-1} \subset V \to Q_{n-1} \to \cdots \to Q_1$$

where the E_i's are subbundles of V, the Q_i's are bundles, but the maps on the right are not necessarily surjective. We want the locus Ω'_w where heuristically

$$\text{“rk}(E_p \to Q_q) \le \#\{i \le q, w(i) \le p\}\text{”}$$

If each $V \to Q_i$ is surjective, then $[\Omega'_w] = \mathfrak{S}_w(x,y)$, where x_i, y_i denote (up to sign) the class of successive quotients of the Q_j's, E_j's respectively. When the maps are not surjective, higher Chern classes of the Q_i's must be used.

For Ω'_{w_0}, we need the locus where $E_1 \to Q_{n-1}$ vanishes; this is

$$c_{top}(E_1^\vee \otimes Q_{n-1}) = \sum_{i=0}^{n-1} c_i(Q_{n-1}) y_1^{n-1-i} = E_{n-1}(x_1 + y_1, \ldots, x_{n-1} + y_1)$$

On this locus, look further where $E_2/E_1 \to Q_{n-2}$ vanishes. A computation gives that this is $E_{n-2}(x_1 + y_2, \ldots, x_{n-2} + y_2)$, and so on.

Remark: here $x_1 + \cdots + x_i = c_1(Q_i)$, and $q_i = c_2(Q_{i+1} - Q_i)$; all other Chern classes are determined by c_1's and c_2's. This must be a special property of the universal bundles on the hyperquot scheme.