Part III
Related Material

1. Mirror symmetry and string-theoretic Hodge numbers—V. Batyrev, 9/12/96

Problems arise in trying to define and compute $QH^*(X)$ when X is singular: for example, for the 'small' QH^* one does not expect the variables $q_1, \ldots, q_r$ to be defined as generators of $H^2(X)$; one expects more parameters, corresponding to a larger (conjectural) 'string-theoretic' $H_{\mathrm{st}}(X)$. This lecture deals with the problem of defining mirror symmetry for singular varieties, and more specifically with suitable Hodge numbers displaying the symmetry.

§**1. Mirror symmetry.** First a reminder of a basic example. Let A be the abelian group $(\mathbb{Z}/(n+2)\mathbb{Z})^{\oplus n+2}$, of order $(n+2)^{n+2}$. Consider the nondegenerate bilinear form $<\cdot,\cdot>: A \times A \to \mathbb{Z}/(n+2)\mathbb{Z}$ given by

$$\langle(\alpha_0, \ldots, \alpha_{n+1}), (\beta_0, \ldots, \beta_{n+1})\rangle = \sum \alpha_i \beta_i \mod (n+2)$$

B denotes the diagonal $\mathbb{Z}/(n+2)\mathbb{Z} \hookrightarrow A$; note that B is isotropic: $\langle B, B\rangle = 0$. Define $G = B^\perp/B$, another abelian group, of order $(n+2)^n$; and restrict the pairing to $G \times G \to \mathbb{Z}/(n+2)\mathbb{Z}$.

Now move to algebraic geometry. Consider the Fermat hypersurface $\phi_n \subset \mathbb{P}^{n+1}$ defined by $\sum z_i^{n+2} = 0$; G acts naturally on ϕ_n, by $(\alpha, z_i) \mapsto e^{2\pi i \alpha} z_i$.

STATEMENT. *For all $H \subset G$, and denoting by $H^\perp$ the complement of H with respect to the pairing in G, the physicists claim that*

$$(\phi_n/H) \leftrightarrow (\phi_n/H^\perp)$$

are mirror-symmetric.

EXAMPLE. $n = 3$, $H = 0$: then $G = (\mathbb{Z}/5\mathbb{Z})^3$, and the claim is that ϕ_5 and ϕ_5/G are mirror-symmetric.

Note: in this case ϕ_5 is smooth, but ϕ_5/G is singular.

Another statement: if V, V' are smooth n-dimensional Calabi-Yau, and mirror symmetric, then $h^{n-p,q}(V) = h^{p,q}(V')$ for all p, q.

How to state this if V or V' are singular? First natural conjecture: the equality of Hodge numbers should hold for some minimal desingularization. We expect that ϕ_n/H, $\phi_n/H^\perp$ have "nice" desingularizations displaying mirror symmetry of Hodge numbers. In fact, this has been proved for $n = 3$.

How to control the 'minimality' of a desingularization?

DEFINITION. X Gorenstein variety, K_X canonical divisor. A resolution $\pi : Y \to X$ is *crepant* if $\pi^* K_X = K_Y$.

(In general, $K_Y = \pi^* K_X + \sum a_i E_i$, with E_i the components of the exceptional divisors. The a_i's are called 'discrepancies'; so $a_i = 0 \leftrightarrow$ there are *no dis*crepancies $\leftrightarrow$ the resolution is crepant. This terminology is due to Miles Reid.)

Note: if $Z \subset Y$ (both nonsingular) and $Y' = B\ell_Z Y$, with exceptional divisor E, then $K_{Y'} = \pi^* K_Y + (\operatorname{codim} Z - 1)E$. Unless Y' is isomorphic to Y, $\operatorname{codim} Z - 1 > 0$; so 'crepancy' guarantees the minimality of the resolutions in the sense that blowing-up inessential loci will not give crepant desingularizations.

§**2. E-polynomials.** Assume that X is quasi-projective. Then the cohomology with compact support $H_c^*(X)$ has a mixed Hodge structure, and we may define

$$e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(H_c^*(X))$$

DEFINITION. $E(X; u, v) = \sum_{p,q} e^{p,q}(X) u^p v^q$.

Properties:

(1) If $X = \amalg X_i$, then $E(X; u, v) = \sum E(X_i; u, v)$

(2) $Y \to X$ locally trivial in the Zariski topology, with fiber $F \implies E(Y; u, v) = E(F; u, v) \cdot E(X; u, v)$.

(Note: (2) does fail unless one has Zariski-local triviality.)

Now assume that $\pi : Y \to X$ is a resolution of singularities of X, with Y smooth and compact. Then $e^{p,q}(Y) = (-1)^{p+q} h^{p,q}(Y)$ by the purity of the Hodge structure of Y; so knowing $E(Y; u, v)$ amounts to knowing the Hodge numbers of Y. On the other hand, say that we have a stratification $X = \amalg X_i$ with the analytic singularity of X constant along X_i, and $\pi^{-1}(X_i) \to X_i$ Zariski-locally trivial; then one ought to be able to compute $E(Y; u, v)$.

QUESTION. Assume that $\pi : Y \to X$ is a crepant resolution. For $x \in X$, what is $E(\pi^{-1}(x); u, v)$?

This question can be 'localized' for quotient singularities as follows: for $G \subset SL(n, \mathbb{C})$, $X = \mathbb{C}^n/G$, let $\overline{0} \in X$ be the image of $0 \in \mathbb{C}^n$. If $\pi : Y \to X$ is a minimal desingularization, what is $E(\pi^{-1}(\overline{0}); u, v)$?

An answer to this could be pasted into an answer to the global question if X has quotient singularities and $Y \to X$ is crepant. Note that crepant resolutions do not necessarily exist:

EXAMPLE. $n = 4$, $G \cong \mathbb{Z}/2\mathbb{Z}$ generated by $(-1, -1, -1, -1)$. Then $\mathbb{C}^n/G$ does not have crepant resolutions. (Crepant resolutions do exist for $n < 4$.)

However, if two crepant resolutions exist then their E-polynomial will have to be the same (see below).

§3 **Orbifold Euler number.** (Dixon, Vafa, Witten, ...) Say a group G acts on a variety X; then

$$e(X, G) = \frac{1}{|G|} \sum_{(g,h):gh=hg} e(X^g \cap X^h)$$

is the 'orbifold' or *string-theoretic* Euler number, denoted $e_{\mathrm{st}}(X/G)$.

Note: the conventional Euler number of the quotient would instead be $e(X/G) = \frac{1}{|G|} \sum_g e(X^g)$. The string-theoretic Euler number should encode more information.

EXAMPLE. $e_{\mathrm{st}}(\mathbb{C}^n/G) = \#$ of conjugacy classes, while $e(\mathbb{C}^n/G) = 1$.

Now for V, V' smooth n-dimensional mirror-symmetric Calabi-Yau, one has $e(V) = (-1)^n e(V')$. The physicists propose that for quotient varieties this should hold for d_{st}. Supporting that this is the right notion, we have

THEOREM. $e_{\mathrm{st}}(\phi_n/H) = (-1)^n e_{\mathrm{st}}(\phi_n/H^\perp)$

Now back to the local question from §2.

EXPECTATION. $E(\pi^{-1}(\overline{0}); u, v) = \sum \#(\textit{conj. classes with weight } j)(uv)^j$

Here the *weight* of a conjugacy class is defined as follows: for $g \in G$, write g in terms of roots of 1: $g = \mathrm{diag}(e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_n})$ with $0 \le \alpha_j < 1$. We are assuming $G \subset SL$, so $e^{2\pi i \sum \alpha_j} = 1$, and hence $\sum \alpha_j$ is a nonnegative integer. We let the weight of g be this integer.

It is not hard to check that the 'expectation' is correct for 2-dimensional quotient singularities. The exceptional divisor over a singularity is then a chain of say m rational curves, and one checks that the E-polynomial of such a chain is indeed $1 + (m-1)uv$ as prescribed by the above formula.

Also, the 'expectation' makes sense regardless of whether a crepant resolution should exist or not. For the singularity $\mathbb{C}^4/(-1,-1,-1,-1)$ we get $1 + (uv)^2$; and more generally, for $\mathbb{C}^{2n}/\mathbb{Z}^2$ one finds $1 + (uv)^n$.

Finally, for 'toric resolutions' one can formulate a precise results:

THEOREM. *For toric (complete intersection, Calabi-Yau) mirror pairs V, V':*

$$u^n E_{\mathrm{st}}(V; u^{-1}, v) = (-1)^n E_{\mathrm{st}}(V'; u, v)$$

where E_{st} is computed from the 'expected' E-polynomial of fibers of a resolution, as given above.

If we denote by $h^{p,q}_{\mathrm{st}}(V)$ the coefficient of $u^p v^q$ in E_{st}, the Theorem formulates then this case of mirror symmetry as the equality of 'string-theoretic' Hodge numbers

$$h^{p,q}_{\mathrm{st}}(V) = h^{p,q}_{\mathrm{st}}(V')$$

§4. **Number theory (p-adic fields).** (Only very rough ideas here)

The 'expectation' is a true formula!

To prove this, one may either use an approach via infinite dimensional geometry (suggested by M. Kontsevich), or p-adic integration theory on $\mathbb{Q}_p$. This is analogue to Lebesgue integration on $\mathbb{R}$ or $\mathbb{C}$, with Haar measure normalized by $\mu(\mathbb{Z}_p) = 1$.

A. Weil: the integral of a volume form along the maximal compact subset $X(\mathbb{Z}_p)$ of a variety X defined over $\mathbb{Q}_p$ will be $\dfrac{\#X(\mathbb{Z}_p)}{|p|^{\dim X}}$.

Now say that V, V' are birational. As integration is insensitive to proper subvarieties, and V, V' are biregularly isomorphic away from proper subvarieties, we will get the same number of points for V, V' over $\mathbb{Z}_p$. Via the Weil conjectures, this in some sense explains why one should expect the same cohomological properties for V and V'. If $Y \to X$ is a crepant resolution, a volume form Ω on X will extend with same zeros and poles on Y independently of the specific crepant resolution (by definition of crepant), and from this one can understand why one should get the same E for different resolutions.

§5. **Extending to more general singularities.** Suppose X has log terminal singularities. Can we define a string-theoretic E-polynomial for X?

Let $\pi : Y \to X$ be *any* resolution, with $D_1, \dots, D_r$ exceptional divisors. Write $K_Y = \pi^* K_X + \sum_{i \in I} a_i D_i$, with $I = 1, \dots, r$. For $J \subset I$, define

$$D_J = \cap_{j \in J} D_j \quad , \quad D_J^\circ = D_J - \cap_{i \in J} D_i$$

and $D_\emptyset = Y - \cup_{i \in J} D_j$.

Then $Y = \amalg_{J \subset I} D_J^\circ$. We define then

DEFINITION.

$$e_{\mathrm{st}}(X) = \sum_{J \subset I} e(D_J^\circ) \left(\prod_{j \in J} \frac{1}{a_j + 1} \right)$$

This agrees with the previous e_{st} for quotient singularities.

THEOREM. *This number does not depend on the resolution.*

It is immediate to check that blowing up a given resolution at a point, for example, does not change the number defined above. The present proof of this Theorem however is not so naive, and again uses p-adic integration.

This e_{st} seems the right candidate to use in a more general formulation of mirror symmetry.

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2. $\overline{M}_{0,n}$—P. Belorousski, 9/17/96

References for this material are [Knudsen], [Keel], and (for a different construction) [Kapranov].

We denote by $M_{0,n}$ the space parametrizing ordered n-tuples of distinct points on $\mathbb{P}^1$ modulo projective transformations; we work over $\mathbb{C}$.

§**1. Naive compactifications of $M_{0,n}$.** The first observation is that any three points of any ordered n-tuple of distinct points in $\mathbb{P}^1$ (say, the last three) can be placed at 0, 1, ∞ by means of a unique projective transformation. The position of the other $n-3$ points determines then the n-tuple up to projective transformation, with the only constraint that they should be distinct from each other, and distinct from 0, 1, ∞.

$n=3$: $M_{0,3} = \cong$point;

$n=4$: $M_{0,4} \cong \mathbb{P}^1 - \{0,1,\infty\}$;

more generally, $M_{0,n} \cong (\mathbb{P}^1 - \{0,1,\infty\})^{n-3}-$ diagonals, or alternatively $M_{0,n} \cong (\mathbb{C} - \{0,1\})^{n-3}-$ diagonals.

Naive compactifications of these spaces are then $(\mathbb{P}^1)^{n-3}$, or $\mathbb{P}^{n-3}$; neither of these is adequate in the sense that points at the boundary are not 'geometrically meaningful'. For example, the natural action of S_n on $M_{0,n}$ does not extend to an action on these compactifications.

Example. Consider the 5-tuple of points

$$(p_1,p_2,p_3,p_4,p_5) = (0,1,\infty,t,t^2)$$

for $t \neq 0$, then let t approach 0. In the naive compactifications, p_4 and p_5 both approach p_1. However, if we take p_1, p_4, p_5 to be the points fixed at $0,1,\infty$ by applying $z \mapsto \frac{z(1-t)}{z-t^2}$, the 5-tuple becomes

$$(p_1,p_2,p_3,p_4,p_5) = (0,\tfrac{1}{1+t},1-t,1,\infty)$$

and as t approaches 0 this time the points p_2, p_3 and p_4 come together. That is, the geometric interpretation of the limiting configuration depends on the choice of three fixed points.

§**2. The moduli problem and Knudsen's construction.**

Definition. A stable n-pointed curve of genus 0 is a connected projective nodal curve C with n distinct smooth marked points p_i, with $\dim H^1(C,\mathcal{O}_C) = 0$, and such that each component of C has at least 3 special points.

Here *special* means singular or marked. We will denote an n-pointed curve by $(C,p_1,\dots,p_n)$.

The stability condition implies that the curve has no nontrivial automorphisms fixing the marked points.

Remark. For a connected nodal curve,

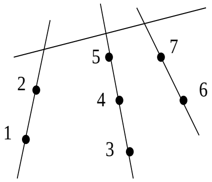
$$\text{genus} = \sum g_i + (\#\text{nodes}) - (\#\text{components}) + 1$$

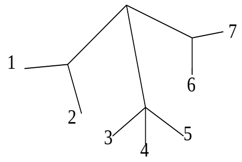
where g_i denotes the genus of the i-th irreducible component. Therefore genus$=0 \implies g_i = 0$ for all i, and $(\#\text{nodes}) - (\#\text{components}) = 1$. In particular, all components of C are necessarily copies of $\mathbb{P}^1$.

In fact, these components form a tree. It is convenient to introduce the *dual graph* of a stable curve, by setting

$$\begin{aligned} \text{vertices} &= \{\text{components of } C\} \\ \text{edges} &= \{\text{nodes}\} \\ \text{tails} &= \{\text{marked points}\} \end{aligned}$$

In this terminology, a 'vertex' is an *internal* vertex of the graph, while a 'tail' is a boundary vertex, with an edge attaching it to the rest of the graph. For example, here is a schematic representation of a 7-pointed stable curve and of the associated graph:





In terms of the graph of C, genus$=0 \implies$ the graph is a tree; and stability $\implies$ the valence (i.e. the number of edges attached to it) of each internal vertex is at least 3.

Definition. A *family* of n-pointed stable curves of genus 0 over a base scheme S is a flat projective morphism $\mathcal{C} \xrightarrow{\pi} S$ with n sections $\sigma_1,\dots,\sigma_n : S \to \mathcal{C}$ such that

$$(C_s,\sigma_1(s),\dots,\sigma_n(s))$$

is a reduced stable n-pointed curve of genus 0 for all $s \in S$. Here C_s denotes the fiber of $\mathcal{C}$ over $s \in S$,

Morphisms of families over a given base scheme are defined in the obvious way; the moduli problem is defined by the (contravariant) functor associating to each scheme the set of equivalence classes (up to isomorphism) of all families defined over it.

Theorem. *There exists a projective smooth algebraic variety $\overline{M}_{0,n}$, which is the fine moduli space for n-pointed stable genus-0 curves.*

That is, for all schemes S there is a natural bijection

$$\left\{ \begin{matrix} \text{Isom. classes of families of } n\text{-pointed} \\ \text{stable curves of genus zero over } S \end{matrix} \right\} \leftrightarrow \operatorname{Hom}(S,\overline{M}_{0,n})$$

We sketch the proof given by Knudsen.

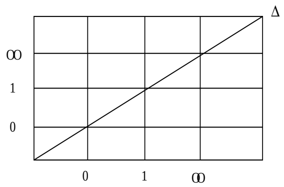
Proof sketch. Induction on n. For $n=3$, $\overline{M}_{0,3} \cong$point, since 3-pointed stable curves are necessarily irreducible. The universal family $\mathcal{C}_3 \to \overline{M}_{0,3}$ will be $\mathbb{P}^1$, with 3 marked points.

Key observation: $\mathcal{C}_n \cong \overline{M}_{0,n+1}$!

For the construction of the universal family, look first at the case $n = 4$. We have the diagram

$$\begin{array}{ccccc} \mathcal{C}_4 & \longrightarrow & \mathcal{C}_3 \times_{\overline{M}_{0,3}} \mathcal{C}_3 & \xrightarrow{\cong} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \overline{M}_{0,4} & = & \overline{M}_{0,4} = \mathcal{C}_3 & \xrightarrow{\cong} & \mathbb{P}^1 \end{array}$$

There are three obvious sections on $\mathcal{C}_3 \times_{\overline{M}_{0,3}} \mathcal{C}_3$; a fourth one is given by the diagonal:



the fibers are 4-pointed curves, but we have to separate the diagonal from the sections at the points $(0,0)$, $(1,1)$, (∞,∞); we blow-up $\mathcal{C}_3 \times_{\overline{M}_{0,3}} \mathcal{C}_3$ at the three intersection points, and this produces $\mathcal{C}_4$.

Note that the effect of the blow-up is to sprout out a new component (an exceptional divisor) on which the "glued" points will separate.

The inductive step for higher n is analogous. Assuming $\mathcal{C}_n \to \overline{M}_{0,n}$ has been constructed, take $\overline{M}_{0,n+1}$ to be $\mathcal{C}_n$, start by considering $\mathcal{C}_n \times_{\overline{M}_{0,n}} \mathcal{C}_n$, and blow up the intersections of the diagonal with the basic sections and with the singular loci of the fibers. The centers of blow-up might not be regularly embedded, so one will need to resolve the singularities introduced by blow-up; a minimal desingularization will yield $\mathcal{C}_{n+1}$. $\square$

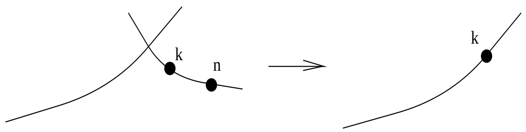
The fact that $\overline{M}_{0,n}$ is a fine moduli space yields

Corollary 1. *S_n acts on $\overline{M}_{0,n}$ by permuting the points.*

Also, we may index the points by arbitrary finite sets: if $|A| < \infty$, denote by $\overline{M}_{0,A}$ the corresponding $\overline{M}_{0,|A|}$. From Knudsen's proof one can derive

Corollary 2. *Given $B \subset A$, $|A| < \infty$, there is a contraction morphism $\overline{M}_{0,A} \to \overline{M}_{0,B}$.*

Example. Say $A = \{1,\ldots,n\}$, $B = \{1,\ldots,n-1\}$; we get a morphism $\overline{M}_{0,n} \to \overline{M}_{0,n-1}$ forgetting the n-th point. If the n-th point is on a component with only 3 special points, just forgetting it would destabilize the component; in practice, the effect of the operation is to contract such a component:



Of course, the algebra behind this is not as simple as the set-theoretic description.

Remark. Changing the order in which you forget two different points gives two possible compositions

$$\overline{M}_{0,n} \to \overline{M}_{0,n-1} \to \overline{M}_{0,n-2}$$

The resulting maps $\overline{M}_{0,n} \to \overline{M}_{0,n-2}$ must however coincide, since they clearly agree on the open locus $M_{0,n}$.

Taking this observation further, we see that for any 4-tuple of distinct indices i,j,k,ℓ in $\{1,\ldots,n\}$ there is a (unique) map

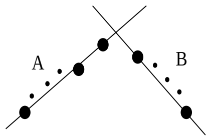
$$\overline{M}_{0,n} \to \overline{M}_{0,\{i,j,k,\ell\}} \cong \mathbb{P}^1$$

forgetting all the other points.

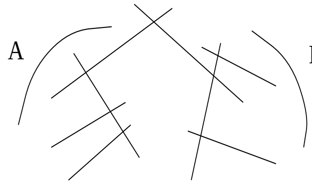
§3. Geometry of $\overline{M}_{0,n}$.

Fact. *The boundary of the compactification $M_{0,n} \hookrightarrow \overline{M}_{0,n}$ is a divisor with normal crossings.*

For all partitions $A \amalg B = \{1,\ldots,n\}$ with $|A| \geq 2$, $|B| \geq 2$, we have a divisor $D(A|B)$ at the boundary, whose general point corresponds to a curve of the type



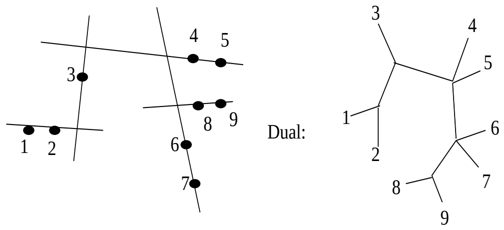
The points at the 'boundary' of $D(A|B)$ correspond to possible degenerations:



The intersection of any number of these $D(A|B)$'s is either empty or smooth. $\overline{M}_{0,n}$ has a stratification by locally closed loci indexed by the combinatorial type of the degeneration.

More precisely, the *combinatorial type* of a curve is simply its dual graph (with marked tails).

Example. In $\overline{M}_{0,9}$ (of dimension 6):



All curves with this combinatorial type form a stratum in $\overline{M}_{0,9}$. The closure of this stratum is

$$D(12|3\cdots 9)\cap D(123|4\cdots 9)\cap D(1\cdots 5|6\cdots 9)\cap D(1\cdots 7|89)$$

The boundary cycles (i.e. the closures of the strata) are themselves products of $\overline{M}_{0,k}$'s for $k<n$. For example,

$$D(A|B)\cong \overline{M}_{0,A\cup\{*\}}\times \overline{M}_{0,\{*\}\cup B}$$

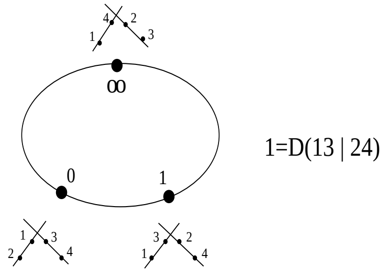
where we think of $*$ as the points at which the A-stable curve and the B-stable curve are glued.

In other words, for every k,ℓ there is a morphism

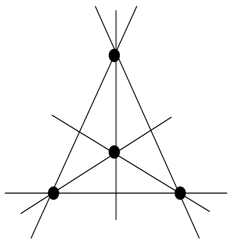
$$\overline{M}_{0,k+1}\times \overline{M}_{0,\ell+1}\hookrightarrow \overline{M}_{0,k+\ell}$$

and the image of this map is one of the boundary divisors in the target.

EXAMPLES. $\overline{M}_{0,4}\cong \mathbb{P}^1$:



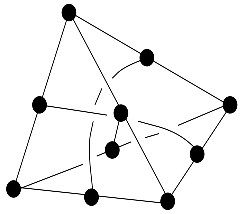
$\overline{M}_{0,5}$: blow-up $\mathbb{P}^1\times\mathbb{P}^1$ at 3 general points (as seen in Knudsen's proof in the last section). Equivalently, one can blow-up $\mathbb{P}^2$ at 4 general points, since blowing-up $\mathbb{P}^2$ at two points is the same as blowing-up $\mathbb{P}^1\times\mathbb{P}^1$ at 1 point.



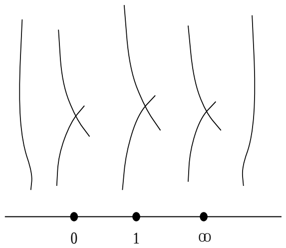
One gets the same answer by considering Procesi's compactification for this configuration. There will be 4 exceptional divisors, plus the proper transforms of the six lines through the points, which will give 6 more (-1)-curves. The blow-up

of $\mathbb{P}^2$ at four general points is a Del Pezzo surface of degree five, embedded into $\mathbb{P}^5$ by the system of cubics through the four points. The 10 special curves become lines in this embedding.

These 10 lines are the boundary divisors of $\overline{M}_{0,5}$, which are indexed as $D(A|B)$ by the $\binom{5}{2}=10$ subsets A, $|A|=2$, of $\{1,\dots,5\}$. The incidence of the $D(A|B)$'s is represented by the following graph:



Finally, $\overline{M}_{0,5}$ is the universal curve over $\overline{M}_{0,4}\cong\mathbb{P}^1$:

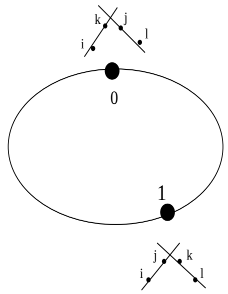


the 3 special fibers give 6 exceptional curves; plus the 4 sections, for a total of 10.

§4. Chow ring of $\overline{M}_{0,n}$. We can obtain some geometrically transparent relations in the ring as follows:

(a) $D(A|B)=D(B|A)$;

(b) the choice of distinct i,j,k,ℓ in $\{1,\dots,n\}$ gives a map $p:\overline{M}_{0,n}\to\overline{M}_{0,4}\cong\mathbb{P}^1$;



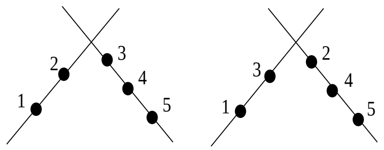
$D(ij|k\ell)=D(ik|j\ell)$ up to rational equivalence in $\mathbb{P}^1$, hence

$$p^*D(ij|k\ell)=p^*D(ik|j\ell)$$

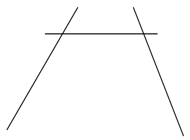
in $A^*\overline{M}_{0,n}$. That is,

$$\sum_{i,j\in A;k,\ell\in B} D(A|B) = \sum_{i,k\in A;j,\ell\in B} D(A|B)$$

(c) $D(A|B)\cdot D(C|D) = 0$ unless the curves corresponding to the general points in the divisors have common degenerations, that is unless one of the four sets A, B, C, D contains one of the others (i.e., the common refinement of the two partitions consists of 3 rather than 4 sets). For example, $A = \{1,2\}$, $B = \{3,4,5\}$, $C = \{1,3\}$, $D = \{2,4,5\}$:



there is no common degeneration to



so $D(12|345)\cdot D(13|245) = 0$.

The $D(A|B)$'s generate the Chow ring of $\overline{M}_{0,n}$ multiplicatively. Surprisingly:

THEOREM. *(a), (b), (c) are the only relations in the Chow ring of $\overline{M}_{0,n}$. Therefore,*

$$A^*\overline{M}_{0,n} = \mathbb{Z}[D(A|B), A \amalg B = \{1,\dots,n\}, |A|\geq 2, |B|\geq 2]/((a),(b),(c))$$

Also, $H^*\overline{M}_{0,n} \cong A^*\overline{M}_{0,n}$.

These results are due to Sean Keel, [Keel].

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3. Quantum cohomology, old and new—Z. Ran, 9/26/96

LEMMA. *Consider a 3-dimensional irreducible closed subvariety V of the $\mathbb{P}^N$ parametrizing plane curves of a given degree d. Suppose V parametrizes rational curves: i.e., a general point $C\in V$ corresponds to an irreducible rational curve. Then V has a degenerate member: there exists a reducible or multiple $C_0\in V$.*

Note that the bound '3' is sharp: the $\mathbb{P}^2$ of lines has no degenerate members.

PROOF. Pick general points a, b in $\mathbb{P}^2$, and consider a one-parameter subfamily B of V consisting of curves through a, b. The total space of B gives

$$\begin{array}{ccc} X & \xrightarrow{F} & \mathbb{P}^2 \\ {\scriptstyle\pi}\big\downarrow & & \\ B & & \end{array}$$

where F is a morphism and $X \to B$ is a blown-up $\mathbb{P}^1$-bundle. We may assume that F is relatively minimal, i.e., there are no vertical (-1)-curves E such that $F(E) =$pt. The general fiber of π is a $\mathbb{P}^1$; the special fibers are trees of $\mathbb{P}^1$'s. Each of these has at least two ends; these are (-1)-curves, so F is not constant along them.

Claim: there must exist at least one degenerate fiber.

Indeed, otherwise π is a $\mathbb{P}^1$-bundle; there would be sections $F^{-1}(a) = S_a$, $F^{-1}(b) = S_b$. Let then $L = F^*(\text{line})$: as F is generically finite, $L^2 > 0$; $L\cdot S_a = 0$; $L\cdot S_b = 0$. By the Hodge index theorem, $S_a^2 < 0$ and $S_b^2 < 0$. But $S_a - S_b$ is a multiple of the class of the fiber, hence

$$0 = (S_a - S_b)^2 = S_a^2 + S_b^2 < 0 \quad ,$$

contradiction. $\square$

Now we could try to get a formula from this situation for $\deg V = \deg F$, by relating it to the number of degenerate fibers containing either one or both of S_a, S_b. Using ideas from quantum cohomology, one should get

$$d - \deg V + \sum_W \deg_2 W\cdot(\deg C_2)^2 = \sum_W \deg_{1,1} W\cdot(\deg C_1)\cdot(\deg C_2)$$

where W ranges over 'boundary components' of V: W parametrizes curves of the form $C_1\cup C_2$ where

either C_1 is a 2-parameter family, and $\deg_2 W = \#\,\{C_1\cup C_2\colon\ C_1$ contains 2 general points$\}$,

or $\{C_1\}$ and $\{C_2\}$ are each a 1-parameter family, and $\deg_{1,1} W = \#\{C_1\cup C_2 :$ C_1 contains a general point, and C_2 contains a general points$\}$.

So much for the 'new' methods; move now to the 'old' method, degenerating the target.

$V_{d,\delta}$ denotes the Severi variety of nodal plane curves of degree d with δ nodes. $V_{d,\delta}$ is locally closed in $\mathbb{P}^{\binom{d+2}{2}-1}$; its dimension is

$$\binom{d+2}{2} - 1 - \delta = 3d + g - 1 \quad ,$$

where $g = \frac{(d-1)(d-2)}{2} - \delta$ is the genus. We denote by $N_{d,\delta}$ the degree of $V_{d,\delta}$.

Consider the blow-up

$$\begin{array}{ccc} \mathcal{S} = B\ell_{(0,p)}\mathbb{C}\times\mathbb{P}^2 & \longrightarrow & \mathbb{C}\times\mathbb{P}^2 \\ {\scriptstyle\pi}\big\downarrow & & \\ \mathbb{C} & & \end{array}$$

π is a flat family, and $\pi^{-1}(t) = \mathbb{P}^2$ for $t\neq 0$; $\pi^{-1}(0) = S_0 = S_1\cup S_2$, with $S_1 = B\ell_p\mathbb{P}^2$, and $S_2 = \mathbb{P}^2$.

The intersection $E = S_1\cap S_2$ sits in S_1 as the exceptional curve, and in S_2 as a line.

The blow-up sits in $(\mathbb{C}\times\mathbb{P}^2)\times\mathbb{P}^2$; so there is another map $\mathcal{S}\to\mathbb{P}^2$. We have

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "b'"] \arrow[dr, "\pi"'] & & \mathbb{C}\times\mathbb{P}^2 \arrow[dl, "pr_1"] \\ & \mathbb{C} & \end{tikzcd}$$

with $b'|_{S_2}=$ identity, and $b'|_{S_1}=$ a $\mathbb{P}^1$-bundle over image $=E\cong\mathbb{P}^1$.

Now consider $V_{d,\delta}$ as a family of curves on the general fiber of π, and take the limit as $t\to 0$. As a cycle,

$$V_{d,\delta}\to\sum m(\pi)V(d,e/\pi/d,\delta_2/\delta_1)$$

where: π is a partition, $\pi=[\ell_1,\dots,\ell_r]$, $\ell_i=\#$ of blocks of size i; the weight of π is $|\pi|=\sum i\ell_i$, and here $e=|\pi|$; $m(\pi)=\prod i^{\ell_i}$; the $\sum$ is over π, δ_1, δ_2 satisfying

$$\sum(i-1)\ell_i+\delta_1+\delta_2=\delta$$

The different π correspond to the strata of divisors of degree $|\pi|$ in $\mathbb{P}^1$: $\pi\leftrightarrow\sum_{i=1}^{r}\sum_{j=1}^{d_1} iQ_{ij}$.

$V(d,e/\pi/d,\delta_2/\delta_1)$ is the set of all $C_1\cup C_2$ such that $C_i\subset S_i$ has δ_i nodes, and is smooth near E; $C_1\cap E=C_2\cap E=$ divisor of type π in $E\cong\mathbb{P}^1$.

Note: the C_i are not necessarily irreducible—this makes the degrees easier to compute.

Next, the sought degree $N_{d,\delta}$ is the number of curves $C\in V(d,\delta)$ containing N general points $p_1,\dots,p_N$, where $N=\frac{(d+1)(d+2)}{2}-1-\delta$. Specialize $p_1,\dots,p_N$ to N_1 points on S_1 and N_2 points on S_2, $N_1+N_2=N$. To avoid trivial solutions, assume

$N_1\ge d+1$ (avoids limit components where $C_1=d$ rulings)

$N_2\ge 3$.

One choice: make N_2 as large as possible, i.e., choose $N_1=d+1$. In this case, either $e=d-1$ or else C contains a component not meeting E; call it $C_{1,0}$; this would come from a proper component of C itself, so C must be reducible. This cannot happen if $\delta<d$.

For the rest of the discussion, assume $e=d-1$. Then $C_1\in|dH-(d-1)E|$ on S_1, and

$$C_1=\sum_{i=1}^{\delta_1}R_i+C_{1,0}$$

with R_i rulings, and $C_{1,0}\in|(d-\delta_1)H-(d-\delta_1-1)E|$ a smooth rational curve. (One can in fact view this as a degeneration in the plane.)

We need to express the condition $C_1\cap E=C_2\cap E$. Let P_π be the set of divisors of type π on $E=\mathbb{P}^1$, and $P^\pi=\prod\mathbb{P}^{\ell_i}$. P^π maps birationally to P_π. The condition $C_1\cap E=C_2\cap E$ is equivalent to

$$(C_1\cap E,C_2\cap E)\in\Delta_1\times\dots\times\Delta_r\subset P^\pi\times P^\pi=(\mathbb{P}^{\ell_1}\times\mathbb{P}^{\ell_1})\times\dots\times(\mathbb{P}^{\ell_r}\times\mathbb{P}^{\ell_r})$$

Now

$$(*)\qquad \Delta_1\times\dots\times\Delta_r\sim\left(\sum_{j=0}^{\ell_1}\mathbb{P}^j\times\mathbb{P}^{\ell_1-j}\right)\times\dots\times\left(\sum_{j=0}^{\ell_r}\mathbb{P}^j\times\mathbb{P}^{\ell_r-j}\right)=\sum_{\pi'+\pi''=\pi}P^{\pi'}\times P^{\pi''}$$

where $\pi'=[\ell_1',\dots]$, $\pi''=[\ell_1'',\dots]$, and $\pi'+\pi''=[\ell_1'+\ell_1'',\dots]$.

Write $C_m\cap E=\sum\sum iQ_{ij}^m$, $m=1,2$. Fix Q_{ij}^1, $j=1,\dots,\ell_i'$, and Q_{ij}^2, $j=\ell_i'+1,\dots,\ell_i'+\ell_i''=\ell_i$. Define

$$V_{(d,e),\delta,\backslash\pi',\pi''}=\{\text{nodal curves } C_1 \text{ with } \delta \text{ nodes, and}$$
$$C_1\cap\text{ line}=D_{\pi'}^{\text{fixed}}+D_{\pi''},\ D_{\pi''}\text{ of general type }\pi'',\ |\pi'|+|\pi''|=e\}$$

There is an analogous locus $V_{e,\delta,/\pi',\pi''}$. Also, we set $N_{d,\delta,/\pi',\pi''}=\deg V_{d,\delta,/\pi',\pi''}$, etc.

Remark. From $(*)$, the condition $C_1\cap E=C_2\cap E$ is numerically equivalent to

$$C_1\in V_{(d,d-1),\delta_1,\backslash\pi',\pi''}\quad,\quad C_2\in V_{d-1,\delta_2,/\pi',\pi''}$$

for some π',π'' such that $\pi'+\pi''=\pi$.

Remark. Setting $S(\pi)=\sum\ell_i$ for $\pi=[\ell_1,\dots]$, we have $S(\pi'')=\delta_1$ (follows from #conditions on $C_1=\dim\{C_1\}$), $\delta_2=\delta-d+1+S(\pi')$.

Now we have ℓ_1' fixed, mult. 1 points on E_1 that C_1 must contain. Let δ_1-j of these lie on rulings contained in C_1. The j remaining rulings will pass through some of the $d+1$ interior points. And $j\le\ell_1''$. The number of choices is $\sum_{j=0}^{\ell_1''}\binom{\ell_1'}{\delta_1-j}\binom{d+1}{j}$.

The remaining part of C_1, that is $C_{1,0}$, is smooth. The remaining choice: degree of variety of divisors of type $\pi'''\equiv[\ell_1''-j,\ell_2'',\dots]$ on E, equal to $m(\pi''')n(\pi''')$, with $n(\pi)$ defined as $\frac{S(\pi)!}{\ell_1!\cdots\ell_r!}=\frac{(\sum\ell_i)!}{\prod(\ell_i!)}$.

Putting everything together:

$$N_{d,\delta}=\sum_{|\pi|=d-1}m(\pi)\cdot$$

$$\cdot\sum_{0\le\pi''=[\ell_i'']\le\pi,\pi'=\pi-\pi''}N_{d-1,\delta-d+1+S(\pi'),\backslash\pi'',\pi'}\cdot\sum_{j}^{\ell_1''}m(\pi''')n(\pi''')\binom{d+1}{j}\binom{\ell_1'}{S(\pi'')-j}$$

Along the same lines one can obtain true recursion formulas.

Remarks. On the right-hand-side there is precisely one term corresponding to C_2 having δ nodes: this corresponds to $N_{d-1,\delta,0,d-1}$, with the same leading term as $N_{d-1,\delta}$. One can then derive estimates for $N_{d,\delta}-N_{d-1,\delta}$.

Fix δ, consider d as a variable. Claim: as functions of d, $N_{d,\delta}$ is a polynomial of degree 2δ. One finds

$$N_{d,\delta}-N_{d-1,\delta}\sim dN_{d-1,\delta-1}\sim d^{2\delta-1}$$

There are exactly two further terms contributing to $d^{2\delta-1}$: for C_1 with one ruling, $\pi=[d-1]$; and for C_1 smooth, simply tangent to E at a unique point, $\pi=[d-3,1]$.

The leading and next-to-leading coefficients of $N_{d,\delta}$ in d have been computed by Y. Choi [Choi]:

$$a_{2\delta}^{\delta}=\frac{3^\delta}{\delta!}\quad,\quad a_{2\delta-1}^{\delta}=\frac{-2\cdot 3^\delta}{(\delta-1)!}$$

Finally, note that rational curves are the hardest to treat from this approach.

4. Results and conjectures on the tautological ring of $\mathcal{M}_g$—C. Faber, 10/3/96

First, recall Witten's conjecture=Kontsevich's theorem. Consider $\overline{M}_{g,n}$ and its universal family $\overline{\mathcal{C}}_{g,n} = \overline{M}_{g,n+1} \to \overline{M}_{g,n}$. Let ω be the relative dualizing sheaf of this map, and denote the natural sections by σ_i, $i = 1, \ldots, n$. We obtain n line bundles $\mathcal{L}_i = \sigma_i^* \omega$.

Witten's conjecture is about the intersection numbers of the $\mathcal{L}_i$:

DEFINITION. $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle := c_1(\mathcal{L}_1)^{d_1} \cdots c_1(\mathcal{L}_k)^{d_k}$ on $\overline{M}_{g,k}$ if $\sum d_j = 3g-3+k$, and 0 otherwise. (Note: so this carries the genus information.)

The conjecture gives a complete recipe to compute these numbers. The ingredients are

(1) the string equation: $\left\langle \tau_0 \prod_{i=1}^k \tau_{d_i} \right\rangle = \sum_{i:\, d_i \geq 1} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_k} \rangle$;

(2) the dilaton equation: $\left\langle \tau_1 \prod_{i=1}^k \tau_{d_i} \right\rangle = (2g-2+k) \left\langle \prod_{i=1}^k \tau_{d_i} \right\rangle$;

(3) a recursion (KdV-equation): let $T = \prod_{j=0}^m \tau_j^{e_j}$; then for $n > 0$

$$(2n+1) \left\langle \tau_n \tau_0^2 T \right\rangle = \frac{1}{4} \left\langle \tau_{n-1} \tau_0^4 T \right\rangle + \sum_{0 \leq a_j \leq e_j} \prod_{j=0}^m \binom{e_j}{a_j}$$
$$\cdot \left(\left\langle \tau_{n-1} \tau_0 T_1 \right\rangle \left\langle \tau_0^3 T_2 \right\rangle + 2 \left\langle \tau_{n-1} \tau_0^2 T_1 \right\rangle \left\langle \tau_0^2 T_2 \right\rangle \right)$$

where $T_1 = \prod_{j=0}^m \tau_j^{a_j}$, and $T = T_1 T_2$.

Witten showed that (1) and (2) hold; Kontsevich proved (3). (See [Witten], [Kontsevich2].)

For example, this allows you to compute the intersection numbers of Mumford's classes κ_i on $\overline{M}_g$. For the universal family $\overline{\mathcal{C}}_g \xrightarrow{\pi} \overline{M}_g$, with relative dualizing sheaf ω, let $K = c_1(\omega)$ and $\kappa_i = \pi_*(K^{i+1}) \in A^i \overline{M}_g$ (note: all Chow rings are taken with $\mathbb{Q}$-coefficients); here is a recipe for the intersection numbers of the κ_i's: put

$$\langle \tau_{d_1+1} \tau_{d_2+1} \cdots \tau_{d_k+1} \rangle = \sum_{\sigma \in \Sigma_k} \kappa_\sigma$$

for $\sum d_i = 3g-3$, where Σ_k denotes the symmetric group, and κ_σ is defined as follows: think of Σ_k as acting on the k-tuple $(d_1, \ldots, d_k)$; write σ as a product of disjoint cycles $\alpha_1 \alpha_2 \cdots \alpha_{\nu(\sigma)}$ (including 1-cycles). Then

$$\kappa_\sigma = \kappa_{|\alpha_1|} \kappa_{|\alpha_2|} \cdots \kappa_{|\alpha_{\nu(\sigma)}|}$$

where $|\alpha|$ = sum of the elements in the cycle α.

EXAMPLES. $k=1$: $\langle \tau_{3g-2} \rangle = \kappa_{3g-3}$;
$k=2$: $a+b = 3g-3$, $\langle \tau_{a+1} \tau_{b+1} \rangle = \kappa_a \kappa_b + \kappa_{3g-3}$;
$k=3$: $a+b+c = 3g-3$, $\langle \tau_{a+1} \tau_{b+1} \tau_{c+1} \rangle = \kappa_a \kappa_b \kappa_c + \kappa_{a+b} \kappa_c + \cdots + 2\kappa_{3g-3}$.

(Alternative formulation, due to Zagier: set $\sigma_{d_1} \cdots \sigma_{d_k} = \langle \tau_{d_1+1} \cdots \tau_{d_k+1} \rangle$; then

$$\sigma_{abc\ldots x} = \sigma_{abc\ldots} \kappa_x + \sigma_{a+x,bc\ldots} + \sigma_{a,b+x,c\ldots} + \ldots$$

allows us to translate from σ's to κ's.)

Next, let's move to M_g by restricting the classes defined above.

THEOREM. $\kappa_{g-2} \neq 0$ *in* $A^{g-2}(M_g)$.

This result should be compared with what was known before:
$\kappa_0 = 2g-2 \neq 0$ for $g=2$;
$\kappa_1 \neq 0$ for $g=3$: M_3 contains complete curves, κ_1 is ample on M_3;
$\kappa_2 \neq 0$ for $g=4$ (Faber's thesis, a long calculation). A simple proof can be obtained given that κ_1^2 and κ_2 are proportional on M_4. On $\overline{M}_g$ we have $\kappa_1 = 12\lambda_1 - \delta$ (δ = sum of boundary divisors). Showing $\lambda_1^2 \neq 0$ on M_4 is equivalent to showing $\lambda_1^2 \lambda_3 \lambda_4 \neq 0$ on $\overline{M}_4$ (as we will see in the first part of the proof below). Now map $\overline{M}_4$ to $\mathcal{A}_4^*$ (p.p.a.v.) by $C \to \mathrm{Jac} C$; the λ_i's are pull-backs from $\mathcal{A}_4^*$, and here $\lambda_1^2 \lambda_3 \lambda_4 \sim \lambda_1^9 \neq 0$ since a multiple of λ_1 embeds $\mathcal{A}_g^*$ in its Satake compactification.
Now back to the Theorem:

PROOF (SKETCH). Denote by $\mathbb{E}$ the locally free rank-g sheaf $\pi_*(\omega)$ on $\overline{M}_g$ (that is, the Hodge bundle). Then observe that, with $\lambda_i := c_i(\mathbb{E})$, $\lambda_g \lambda_{g-1}$ vanishes on $\overline{M}_g - M_g$. Indeed, $\overline{M}_g - M_g = \cup_{i=0}^{[g/2]} \Delta_i$, with Δ_0 = closure of the locus of nodal genus-$(g-1)$ curves and, for $i \geq 1$, Δ_i = closure of curves consisting of the union of a genus-i and a genus-$(g-i)$ curve. On (a finite cover of) Δ_0, we have the exact sequence

$$0 \to \mathbb{E}_{g-1} \to \mathbb{E}_g \to \mathcal{O} \to 0 \quad ,$$

so $\lambda_g = 0$ on Δ_0; on Δ_i, for $i > 0$:

$$\mathbb{E}_g = \mathbb{E}_i \oplus \mathbb{E}_{g-i}$$

and therefore $\lambda_g = (pr_i^* \lambda_i)(pr_{g-i}^* \lambda_{g-i})$, $\lambda_{g-1} = \ldots$; and $\lambda_g \lambda_{g-1} = 0$ because in every genus h, $\lambda_h^2 = 0$. And why is $\lambda_h^2 = 0$? Mumford shows that $c(\mathbb{E})^{-1} = c(\mathbb{E}^\vee)$, which implies it right away.

(Alternate argument: $\forall k \geq 1$, $\mathrm{ch}_{2k}(\mathbb{E}) = 0$; hence $\forall \ell \geq 2g$, $\mathrm{ch}_\ell(\mathbb{E}) = 0$; $\mathrm{ch}_{2g-1}(\mathbb{E}) = (\text{nonzero } \#) \lambda_g \lambda_{g-1}$ must then vanish on $\overline{M}_g - M_g$ since the components here all have genus $< g$.)

The conclusion is that $\kappa_{g-2} \neq 0$ on $M_g \iff \kappa_{g-2} \lambda_{g-1} \lambda_g \neq 0$ on $\overline{M}_g$.

Now we need Mumford's expression for $\mathrm{ch}(\mathbb{E})$ obtained in [Mumford]:

$$\mathrm{ch}_{2g-1}(\mathbb{E}) = (\neq 0) \left[\kappa_{2g-1} + \frac{1}{2} \sum_{h=0}^{g-1} (i_h)_* (K_1^{2g-2} - K_1^{2g-3} K_2 + \cdots + K_2^{2g-2}) \right]$$

Here $i_0 : \overline{M}_{g-1,2} \to \Delta_0 \subset \overline{M}_g$, and K_i =cotangent at the i^{th} point; for $h > 0$, $i_h : \overline{M}_{h,1} \times \overline{M}_{g-h,1} \to \Delta_h \subset \overline{M}_g$, and K_1, K_2 are pull-backs from the factors.

With this,

$$\frac{1}{(\neq 0)} \kappa_{g-2} \mathrm{ch}_{2g-1}(\mathbb{E}) = \langle \tau_{g-1} \tau_{2g} \rangle - \langle \tau_{3g-2} \rangle + \frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j} \tau_j \tau_{g-1} \rangle$$
$$+ \frac{1}{2} \sum_{h=1}^{g-1} \left((-1)^{g-h} \langle \tau_{3h-g} \tau_{g-1} \rangle \langle \tau_{3(g-h)-2} \rangle + (-1)^h \langle \tau_{3h-2} \rangle \langle \tau_{3(g-h)-g} \tau_{g-1} \rangle \right)$$
$$= \frac{g!}{2^{g-1}(2g)!}$$

using the recurrence relation for the τ's in Witten's conjecture.

This implies $\kappa_{g-2}\lambda_{g-1}\lambda_g \neq 0$ on $\overline{M}_g$ and concludes the proof. $\square$

REMARK. The last, computational step is still rather complicated; it requires knowing certain 'n-point functions' explicitly for $n = 3$ (the n-point functions are $\sum \langle \tau_{d_1} \cdots \tau_{d_n} \rangle x_1^{d_1} \cdots x_n^{d_n}$). There is a nice formula for the 2-point function, due to Dijkgraaf: with $\tau(w) = \sum_{n \geq 0} \tau_n w^n$,

$$\langle \tau_0 \tau(w) \tau(z) \rangle = \exp\left(\frac{w^3 + z^3}{24}\right) \sum_{n \geq 0} \frac{n!}{(2n+1)!} \left[\frac{1}{2} wz(w+z)\right]^n \quad .$$

In the proof, I make use of an explicit formula that I found for the special 3-point function $\langle \tau(-w)\tau(w)\tau(z) \rangle$. (Recently, Zagier found such a formula for the general 3-point function.) Details of the proof can be found in [Faber2].

Next, we move to a result of Looijenga [Looijenga].

Let $\mathcal{C}_g^n \xrightarrow{\pi} M_g$ be the n-fold fiber product of $\mathcal{C}_g (= M_{g,1})$ over M_g, and denote by pr_i the projections onto the factors. We let $R^*(\mathcal{C}_g^n)$ be the *tautological ring* of $\mathcal{C}_g^n$:

for $n = 0$ $\mathcal{C}_g^0 = M_g$, and $R^*(M_g) :=$ the subring of $A^*(M_g)$ generated by the κ_i (restricted to M_g);

for $n \geq 1$, $R^*(\mathcal{C}_g^n) :=$ the subring of $A^*(\mathcal{C}_g^n)$ generated by the $\pi^*\kappa_i$, and the divisor classes $K_i = pr_i^* K$ and $D_{ij} = [\{x_i = x_j\}]$ (thinking of $\mathcal{C}_g^n$ as parametrizing objects $(C; x_1, \ldots, x_n)$).

THEOREM. *(Looijenga) A degree-d element in $R^*(\mathcal{C}_g^n)$ is a linear combination of classes of the irreducible components of*

$$\Big\{(C; x_1, \ldots, x_n) \text{ such that there is an } f : C \to \mathbb{P}^1 \text{ of degree} \leq 2g - 2 + n, \text{ with}$$
$$\#f^{-1}(0) \leq g + n - 1 - d,\ \#f^{-1}(\infty) = 1,\ \{x_1, \ldots, x_n\} \subset f^{-1}(0) \cup f^{-1}(\infty)\Big\}$$

Further, for $d = g + n - 2$, all the classes of these irreducible components are proportional to the class of

$$\mathcal{H}_g^n = \{(C; x_1, \ldots, x_n) \text{ such that } C \text{ is hyperelliptic,}$$
$$\text{and } x_1 = \cdots = x_n \text{ is a Weierstrass point of } C\}$$

COROLLARY. *For $d > g + n - 2$, $R^d(\mathcal{C}_g^n) = 0$.*

COROLLARY. *For $d = g + n - 2$, $R^d(\mathcal{C}_g^n)$ is at most one-dimensional.*

The first Theorem given above, together with this last result, imply that in fact $R^{g+n-2}(\mathcal{C}_g^n) = \mathbb{Q}$.

PROOF. (Sketch of the first part.)

(1) It is enough to prove the statement for the monomials in the K_i only. We have then to prove a statement as in the theorem, but with $\deg f \leq g + n$.

(2) Simple observation: if $D_0 \sim D_\infty$ are positive disjoint divisors on a curve C, then there is a $\pi : C \to \mathbb{P}^1$ such that $\pi^*(i) = D_i$, $i = 0, \infty$; and if $p \in C$ occurs in D_i with multiplicity m_p, π induces a map $\mathbb{C} = T_i^*\mathbb{P}^1 \xrightarrow{\pi^*} T_p^* C^{\otimes m_p}$. This is not canonical because π is not canonical. But let $R =$ ramification outside 0 and ∞, and consider $\pi_* R$; fix π so that $\prod_{z \in \pi_* R} z = 1$: this determines π up to a $(\deg R)$-th root of unity.

(3) (Lemma 2.4 in loc.cit.) Relativize over a disk with generic point η and closed point 0. Consider $\mathcal{C} \to \Delta$ with section $\Delta \xrightarrow{x} \mathcal{C}$. Let $\mathcal{P}$ be a relative pencil with $d(x)$ as a member, and assume that $\mathcal{P}_\eta$ is base-point-free. $C_\eta \to \mathbb{P}^1_\eta$ is ramified at R_η outside x_η. Specialize R_η to R_0, obtain:

$\text{mult}_{x(0)} R_0 =$mult. of $x(0)$ in the fixed part of $\mathcal{P}_0$.

$\Longrightarrow$ (4) $\forall$ member $D \neq d(x)$ of $\mathcal{P}$ specializing to $d(x(0))$, the degree of the moving part of $\mathcal{P}_0$ is $\leq$ the number of η-valued points of $\{\text{supp}(D_\eta) - x(\eta)\}$

(5) Let Z be the moduli space of tuples $(C; x_1, \ldots, x_n, x; D; \mathcal{P})$ with $x_i, x \in C$; $\mathcal{P}$ a pencil on C, with $(n+g)x$ as a member; and D a degenerate member of the pencil (so $\#\text{supp}(D) < n + g$) with $\{x_1, \ldots, x_n\} \subset \text{supp}(D)$.

Stratify Z by:

$$Z^k = \{\text{supp}(D) \text{ has} \leq g + n - 1 - k \text{ points outside } x\}; \text{ so that}$$

$$Z^{n+g-1} = \{(C; \underbrace{x, \ldots, x}_{n}, x; (n+g)x, \mathcal{P}\}$$

LEMMA. *$\forall k < g + n - 1$: $Z^k - Z^{k-1}$ is quasi-affine of pure dimension $(3g - 3 + n - k)$; and $f^*K_i = 0$ on $Z^k - Z^{k+1}$ for $i = 1, \ldots, n$.*

(6) Define X^k to be the union of the irreducible components of Z^k that are not contained in Z^{n+g-1}.

(a) $f : X^0 \to \mathcal{C}_g^n$ is proper and surjective.

(b) *Claim 1: $f(X^k \cap Z^{n+g-1}) \subset f(X^{k+1})$*

Namely, for $A \in X^k \cap Z^{n+g-1}$, write $A = (C; x, \ldots, x, x; (n+g)x, P)$, so that $f(A) = (C; x, \ldots, x)$; by (4), the moving part of P has degree $\leq n + g - k - 1$ so that P has a member $E \neq (n+g)x$ with $\leq n + g - k - 2$ points outside x. Then $B = (C; x, \ldots, x, x; E, P) \in Z^{k+1}$, $\notin Z^{n+g-1}$, so $\in X^{k+1}$; $f(A) = f(B)$, proving the claim.

Now set $U^k = f^{-1}(f(X^k) - f(X^{k+1})) \subset X^k$

Claim 2: $U^k \subset Z^k - Z^{k+1}$.

This is easy to see: $U^k \cap Z^{n+g-1} = \emptyset$ (because $a \in U^k \cap Z^{n+g-1} \implies f(a) \in f(X^{k+1})$, contradiction); hence $U^k \cap Z^{k+1} = \emptyset$ (because $a \in U^k \cap Z^{k+1} \implies a \notin Z^{n+g-1} \implies a \in X^{k+1}$, contradiction).

It follows that $f^*K_i = 0$ on U^k; $f : U^k \to f(X^k) - f(X^{k+1})$ is proper, onto, finite, hence $K_i = 0$ on $f(X^k) - f(X^{k+1})$ (using $\mathbb{Q}$-coefficients).

From this: all monomials of degree d in the K_i are supported on $f(X^d)$; by (1), this is enough to prove the first part of the theorem. $\square$

The results seen so far support a standing conjecture on the tautological ring of M_g:

CONJECTURE. *(1) $R^*(M_g)$ is Gorenstein with socle in degree $(g-2)$. That is:*

(i) $R^j(M_g) = 0$ *for* $j > g-2$;
(ii) $R^{g-2}(M_g) \cong \mathbb{Q}$;
(iii) *there is a perfect pairing* $R^i(M_g) \times R^{g-2-i}(M_g) \to R^{g-2}(M_g) \overset{\text{fix}}{=} \mathbb{Q}$.

(2) $\kappa_1, \ldots, \kappa_{[g/3]}$ *generate the ring, no relations up to degree* $[g/3]$.
(3) Explicit proportionality factors in degree $(g-2)$.

As we have seen, parts (i) and (ii) of (1) are now proved. The rest of the conjecture is still open, although we have been able to check it for all $g \leq 15$ (and we have reduced it to a hard combinatorial problem for all g). A complete statement, and discussion of the evidence for this conjecture, can be found in [Faber].

5. Counting rational curves on quintic 3-folds—S. Kleiman, 10/10/96

This talk is a report on joint work with T. Johnsen, which was presented in two papers, [J-K] and `alg-geom/9601024`. The aim here is to place this work in context, to explain the main results, and to give the flavor of the proofs. The talk is organized into these three sections:

I. Context
II. Strategy
III. Proofs

I. Context. Let F be a hypersurface in $\mathbb{P}^4$ of degree 5 over the complex numbers. Assume that F is *general*, that is, represented by a point in a suitable Zariski-open set of $\mathbb{P}^{125}$. (By contrast, F is called *generic*, if it's represented by a point in the intersection of countably many Zariski-open sets. The latter condition is necessary when we consider all degrees simultaneously, but here we will consider only small degrees.)

Every irreducible rational curve C of degree d in $\mathbb{P}^4$ is given by a parameterization of the form,

$$\big(\alpha_0(t,u), \ldots, \alpha_4(t,u)\big),$$

where α_i is a homogeneous polynomial of degree d. The smooth C form an open set in $\mathrm{Hilb}(\mathbb{P}^4)$; denote it by M_d. Each parameterization is represented by a point of an affine space, and those parameterizations giving smooth curves of degree d form an open subset, which maps onto M_d; hence M_d is irreducible. This affine space has dimension $5(d+1)$, and the fiber over a C in M_d has dimension 4; hence,

$$\dim M_d = 5d + 1.$$

In fact, it's not hard to compute the dimensions of the cohomology groups of the normal bundle $\mathcal{N}_C\mathbb{P}^4$; whence, by the standard infinitesimal theory of the Hilbert scheme, M_d is smooth of dimension $5d+1$.

An arbitrary C of degree d lies on F if and only if the polynomial in t, u,

$$F(\alpha_0(t,u), \ldots, \alpha_4(t,u)),$$

is identically zero. We expect this polynomial to be homogeneous of degree $5d$, and have $5d+1$ coefficients. Their vanishing would impose $5d+1$ conditions on C. So, since $\dim M_d = 5d+1$, we expect only finitely many C on F. Denote the number of smooth C on F by n_d, and the number of all C on F by n'_d.

Schubert (1885): The number of lines on F is $n_1 = 2875$.

Clemens (1983, '84, '86): After having proved that the Griffiths group of F has infinite rational rank (that is, the vector space $(G_h(F)/G_a(F)) \otimes \mathbb{Q}$ is infinite dimensional) when F is generic, Clemens made the following series of conjectures about the irreducible rational curves C of degree d on F:

(a) $1 \leq n_d < \infty$.
(b) Each smooth C is infinitesimally rigid on F.
(c) There are no singular C on F, and so $n_d = n'_d$.
(d) Any two C, C' are disjoint.
(e) $n_d = 5^3 \cdot d \cdot *$.

S. Katz (1986): Conjectures (a) and (b) hold for $d \leq 7$. The number of conics on F is $n_2 = 609{,}250$, which is of the form $5^3 \cdot 2 \cdot *$ prescribed by (e).

THEOREM 1 (Katz for $d \leq 7$, Nijsse and Johnsen–Kleiman for $d = 8, 9$). *Conjectures* (a) *and* (b) *hold for* $d \leq 9$.

Vainsencher (1995): Conjecture (c) is false. In fact, there are $17{,}601{,}000$ six-nodal plane quintic curves on F, arising from tangent 2-planes. These curves deform to smooth irreducible curves (so they are not infinitesimally rigid); however, the corresponding maps $\mathbb{P}^1 \to F$ *are* rigid.

PROPOSITION 2 (Johnsen–Kleiman). *There are no 16-nodal 10-ics on F arising from tangent quadric surfaces.*

THEOREM 3 (Johnsen–Kleiman). *In degree d at most* 9, *the following variations of Conjectures* (c) *and* (d) *hold:*

(c′) *There are no singular C on F, other than Vainsencher's quintics.*
(d′) *Any two C, C' are disjoint if* $\deg C + \deg C' \leq 9$ (including Vainsencher's quintics).

Ellingsrud–Strømme (1991, '93): The number of twisted cubics on F is $n_3 = 371{,}206{,}375$. Note that 3 does not divide n_3, disproving part of Conjecture (e).

Candelas–de la Ossa–Green–Parkes (1990): Mirror symmetry gives an algorithm for finding a suitably defined number n'_d for all d. The values for $d \leq 10$ are given, and 5^3 *does* divide n'_d for $d \leq 10$.

Lian–Yau (1994): If 5 does not divide d, then 5^3 divides n'_d, as defined in [CDGP]. It's not ruled out that 5^3 always divides n'_d.

Kontsevich (1994): Set $N_d := c_{\mathrm{top}} E_d$, where E_d is the vector bundle on $\overline{M}_{0,0}(\mathbb{P}^4, d)$ obtained as $\phi_* \rho^* \mathcal{O}(5)$ via the diagram,

$$\begin{tikzcd} \overline{M}_{0,1}(\mathbb{P}^4, d) \arrow[r, "\rho"] \arrow[d, "\phi"] & \mathbb{P}^4 \\ \overline{M}_{0,0}(\mathbb{P}^4, d) & \end{tikzcd}$$

Kontsevich computed N_4, and it leads to the same n'_4 computed in [CDGP].

Note that there's a positive-dimensional locus in the zero set of the section of E_d defined by F, namely,

$$\{\text{maps } \mu\colon \mathbb{P}^1 \to C \subset F \text{ of degree } k\}.$$

It has dimension $(2k-2)$. To get from N_d to n'_d, use residual intersection theory, which yields the formula,

$$N_d = \sum_{k|d} \frac{n'_{d/k}}{k^3}.$$

This number n'_d is, however, not yet proved to be equal to the one computed by the physicists in [CDGP]; however, it *is* equal to the number of irreducible C on F if the latter number is finite.

II. Strategy. In two words, the strategy (due to Clemens and Katz) is this: count constants. Namely, form the incidence variety,

$$I_d := \{(C,F) | C \subset F\} \subset M_d \times \mathbb{P}^{125}.$$

The naive count of parameterizations above shows that every component of I_d has dimension at least 125.

THEOREM 4 (Clemens–Katz). *For all d, there is a pair (C,F) in I_d with F smooth along C, and with C smooth and infinitesimally rigid; in fact,*

$$\mathcal{N}_C F = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

COROLLARY 5 (Katz). *If I_d is irreducible for a given d, then Conjectures* (a) *and* (b) *hold for this d.*

PROOF. The projection $I_d \to \mathbb{P}^{125}$ is smooth and finite at a *Clemens–Katz pair* (C,F), that is, a pair given by the theorem. Therefore, the projection is smooth and finite over a generic F if I_d is irreducible. □

THEOREM 6 (Katz for $d \le 7$, Nijsse and Johnsen–Kleiman for $d = 8, 9$). *The incidence variety I_d is irreducible for $d \le 9$.*

COROLLARY 7 (Johnsen–Kleiman). *A smooth C on F of degree d at most* 9 *has the following properties:*

(1) *C spans a d-plane if $d \le 4$; otherwise, it spans $\mathbb{P}^4$.*
(2) *C is of maximal rank; that is, for each k, the restriction map,*

$$\rho_k\colon H^0(\mathcal{O}_{\mathbb{P}^4}(k)) \to H^0(\mathcal{O}_C(k)),$$

is either injective or surjective or both.
(3) *For $d = 4q + r$ with $0 \le r < 4$, the restricted twisted sheaf of* 1*-forms decomposes as follows:*

$$\Omega^1_{\mathbb{P}^4}(1)|C = \mathcal{O}_{\mathbb{P}^1}(-q-1)^r \oplus \mathcal{O}_{\mathbb{P}^1}(-q)^{4-r}.$$

PROOF. The projection $I_d \to M_d$ is surjective for $d \le 9$. Properties (1)–(3) hold for a general C by direct computation and by results of Ballico–Ellia and Verdier. □

PROPOSITION 8 (Johnsen–Kleiman). *The incidence variety I_d has the following properties:*

(1) *If $d \ge 25$, then the projection $I_d \to M_d$ is not surjective.*
(2) *If $d \ge 12$, then I_d is reducible.*
(3) *If $d \le 24$, then there exists a unique component of I_d covering M_d, and it has dimension* 125.

PROOF. (1) Since $5d+1 \ge 126$ for $d \ge 25$, the restriction map ρ_5 is not surjective. Now, the maximal rank theorem says that a general C in M_d is of maximal rank. Hence, ρ_5 is injective for a general C, and so $H^0(\mathcal{I}_C(5)) = 0$ where $\mathcal{I}_C$ is the ideal of C in $\mathbb{P}^4$. Therefore, a general C lies in no F.

(2) Consider the locus of pairs (C,F) in I_d such that C lies on a smooth quadric surface. This locus has dimension $2d+101$. On the other hand, a Clemens–Katz pair lies in a (unique) component of I_d of dimension 125.

(3) The maximal rank theorem implies that the locus,

$$I_{d,0} := \{(C,F) \in I_d | H^1(\mathcal{I}_C(5)) = 0\},$$

is nonempty if and only if $d \le 24$. Its closure $\overline{I_{d,0}}$ is the component in question. More details can be found at the beginning of the next section. □

CONJECTURE 9 (Johnsen–Kleiman). *For $d \le 24$, the complement $I'_d := I_d - I_{d,0}$ does not cover $\mathbb{P}^{125}$.*

PROPOSITION 10 (Johnsen–Kleiman). *For $d \le 24$, Conjecture* 9 *implies Conjectures* (a), (b) *and Corollary* 7.

CONJECTURE 11 (Johnsen–Kleiman). *For $d \le 24$, the component $\overline{I_{d,0}}$ of I_d contains all the Clemens–Katz pairs (C,F).*

III. Proofs. Consider the projection $\alpha\colon I_d \to M_d$, and let $C \in M_d$. The fiber $\alpha^{-1}C$ is equal to $\mathbb{P}\big(H^0(\mathcal{I}_C(5))\big)$. To compute its dimension, form the sequence,

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_C \to 0.$$

The associated long exact sequence,

$$0 \to H^0(\mathcal{I}_C(5)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \xrightarrow{\rho_5} H^0(\mathcal{O}_C(5)) \to H^1(\mathcal{I}_C(5)) \to H^1(\mathcal{O}_{\mathbb{P}^4}(5)),$$

in which the last term vanishes, yields the formulas,

$$\dim \alpha^{-1}C = h^0(\mathcal{I}_C(5)) - 1 = 124 - 5d + h^1(\mathcal{I}_C(5)).$$

Set $M_{d,i} := \{C \in M_d | h^1(\mathcal{I}_C(5)) = i\}$ and $I_{d,i} := \alpha^{-1}M_{d,i}$. Then $M_{d,0}$ is open, and the maximal rank theorem implies that, if $d \le 24$, then $M_{d,0}$ is nonempty. Recall that M_d is irreducible. Therefore, if $d \le 24$, then $I_{d,0}$ is irreducible of dimension 125, and its closure is the component in question in Part (3) of Proposition 8.

By a theorem of Gruson, Lazarsfeld, and Peskine, if $d \le 7$, then $M_{d,0} = M_d$, and so I_d is irreducible. For $d = 8, 9$, we have to work a little harder, and prove the following lemma. It implies that $I'_d := I_d - I_{d,0}$ has dimension at most 124. Since every component of I_d must have dimension at least 125, again I_d is irreducible. Thus Theorem 6 holds, and Theorem 1 follows because of Corollary 5.

LEMMA 12. *For $d = 8, 9$, if $i \geq 1$, then* $\operatorname{codim}(M_{d,i}, M_d) \geq i + 1$.

PROOF. Assume $d = 8$. Then $\dim M_8 = 5(8+1) - 4 = 41$. Now, the work of Gruson, Lazarsfeld, and Peskine tells us that

$$h^1(\mathcal{I}_C(5)) = \begin{cases} 0, & \text{if } C \text{ spans } \mathbb{P}^4; \\ 1, & \text{if } C \subset \mathbb{P}^3 \text{ and } C \not\subset \text{ a smooth quadric } Q; \\ 5, & \text{if } C \subset \text{ a smooth quadric } Q \subset \mathbb{P}^3. \end{cases}$$

Hence $\dim M_{8,1} = 4(8+1) - 4 + 4 = 36$: the first term in the middle is the number of parameterized C in $\mathbb{P}^3$; the second is the number of reparameterizations, and the third is the number of $\mathbb{P}^3$s in $\mathbb{P}^4$. Similarly, we have

$$\begin{aligned} \dim M_{8,5} &= \dim\{C \subset Q\} + \dim\{Q \subset \mathbb{P}^3\} + \dim\{\mathbb{P}^3 \subset \mathbb{P}^4\} \\ &= 15 + 9 + 4 = 28. \end{aligned}$$

Assume $d = 9$. By the work of Gruson, Lazarsfeld, and Peskine again and by an extension of it due to d'Almeida, there are five cases to consider:

$$h^1(\mathcal{I}_C(5)) \leq \begin{cases} 0, & C \not\subset \text{hyperplane, has no 7-secants;} \\ 0, & C \subset \text{hyperplane, has no 7-secants;} \\ 1, & C \not\subset \text{hyperplane, has 7-secants;} \\ 10, & C \subset \text{smooth quadric surface (so has 7-secants);} \\ 8, & C \subset \text{hyperplane, } \not\subset \text{smooth quadric, has 7-secants.} \end{cases}$$

In fact, equality holds in the first four cases, and a certain amount of direct analysis is required to handle the last two cases. Given these bounds, the assertion is established by counting the number of C that appear in each of the five cases. $\square$

The proof of Theorem 3 is, in spirit, like that of Theorem 6. Namely, we decompose a suitable incidence variety of pairs (C, F) into manageable locally closed pieces, whose dimensions we can bound from above using the work of Gruson, Lazarsfeld, and Peskine and of d'Almeida. We conclude that these pieces do not cover $\mathbb{P}^{125}$, and so a general F contains no C in question.

For (c$'$), the C in question are the *singular* irreducible rational curves of degree d at most 9. Since we only need to prove crude bounds on dimensions, we may work with pieces of the space of parameterized C rather than the Hilbert scheme, and we do so to simplify the job. On the other hand, each C has a nonzero arithmetic genus g, which enters the scene via the Riemann–Roch theorem,

$$h^0(\mathcal{O}_C(5)) = 5d + 1 - g + h^1(\mathcal{O}_C(5)).$$

So we must use the Castelnuovo–Halphen bounds on g in terms of d. There are many cases to analyze, and the analysis is at times a bit tedious.

For (d$'$), the C in question are the reducible curves of degree d at most 9 with two intersecting components A and B, which are either both smooth curves with the three properties listed in the corollary to Theorem 3 or else one is such a smooth curve and the other is a six-nodal plane quintic. The main ingredients in the proof are the following two lemmas. The first is proved via a direct case-by-case analysis, and the second is proved using Hirschowitz's "méthode d'Horace" in the case where one of the components is a six-nodal plane quintic.

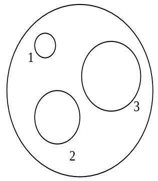
LEMMA 13. *We have* $\operatorname{codim}\{C \mid \#(A \cap B) \geq n+1\} \geq n$.

LEMMA 14. *We have* $h^1(\mathcal{I}_C(5)) = 0$.

6. Operads and associativity of QH^*—P. Aluffi, 10/15/96

This lecture is prompted by a remark in §10 of [F-P], stating that the associativity of the quantum product is in a suitable sense equivalent to a certain map being a morphism of *operads*. I will set up the definitions necessary to understand this statement, and give a sketchy indication of why it holds. The plan is as follows:
—*§1. Example*
—*§2. Formal definition*
—*§3. The endomorphism operad of a vector space V, End_V*
—*§4. The moduli space operad, $\mathcal{M}$, and its homology, $H_*\mathcal{M}$*
—*§5. Gromov-Witten numbers and Gromov-Witten classes*
—*§6. GW on X induce $H_*\mathcal{M}(n) \to End_{H^*X}(n)$*
—*§7. Properties of GW and morphisms of operads*

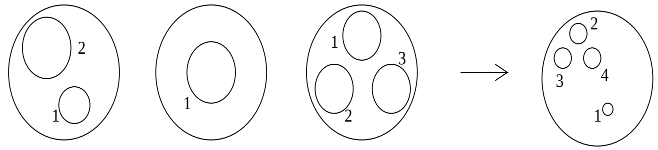
§**1. Example.** A "disk arrangement" is a disjoint union of labeled disks within the unit circle:



Let $O(n)$ denote the set of all n-disk arrangements ($n \geq 0$). S_n acts on $O(n)$ by relabeling the subdisks. We can define an operation

$$\rho : O(k) \times O(\ell_1) \times \cdots \times O(\ell_k) \to O(\ell_1 + \cdots + \ell_k)$$

by the following recipe: given $(o(k); o(\ell_1), \ldots, o(\ell_k))$ in the source,
—scale the unit circle in $o(\ell_i)$ down to the size of the i^{th} subdisk in $o(k)$;
—replace the i^{th} subdisk of $o(k)$ with the resized $o(\ell_i)$;
—remove the boundary of the resized $o(\ell_i)$;
—label the total $\sum \ell_i$ subdisks in the natural way (starting from the first ℓ_1, etc.).
For example, $\rho : O(2) \times O(1) \times O(3) \to O(4)$ acts:



We should think of each $o(k) \in O(k)$ as giving a different 'multiplication' from $O(\ell_1) \times \cdots \times O(\ell_k)$ to $O(\ell_1 + \cdots + \ell_k)$.

This system of operations, for all k, satisfies an obvious "associativity" rule. Suppose given

—a disk arrangement $o(k)$;

—k disk arrangements $o(\ell_1), \dots, o(\ell_k)$; and

—for each i, ℓ_i disk arrangements $o(m_{i1}), \dots, o(m_{i\ell_i})$ (for a total of $\sum_{i,j} m_{ij}$ subdisks).

Then we can do two different things:

—first apply ρ to each $(o(\ell_i); o(m_{i1}), \dots, o(m_{i\ell_i}))$, producing an $o(\sum_j m_{ij})$ for each i; then apply to $(o(k); o(\sum_j m_{1j}), \dots, o(\sum_j m_{kj}))$; or

—first apply ρ to $(o(k); o(\ell_1), \dots, o(\ell_k))$, obtaining an $o(\sum \ell_i)$; then apply to $(o(\sum \ell_i); o(m_{11}), \dots, o(m_{k\ell_k}))$.

It is clear that these two operations produce the same $o(\sum_{ij} m_{ij})$. In other words, the following **operadic diagram** commutes:

$$\begin{array}{ccc}
O(k) \times (O(\ell_1) \times \prod_j O(m_{1j})) \times \cdots \times (O(\ell_k) \times \prod_j O(m_{kj})) & \xrightarrow{id \times \rho \times \cdots \times \rho} & \cdots \\
\downarrow{\scriptstyle \text{shuffle}} & & \\
(O(k) \times O(\ell_1) \times \cdots \times O(\ell_k)) \times O(m_{11}) \times \cdots \times O(m_{k\ell_l}) & \xrightarrow{\rho \times id \times \cdots \times id} & \cdots
\end{array}$$

$$\begin{array}{ccc}
\cdots \longrightarrow \quad O(k) \times O(\sum_j m_{1j}) \times \cdots \times O(\sum_j m_{kj}) & \xrightarrow{\rho} & O(\sum_{ij} m_{ij}) \\
& & \| \\
\cdots \longrightarrow O(\ell_1 + \cdots + \ell_k) \times O(m_{11}) \times \cdots \times O(m_{k\ell_k}) & \xrightarrow{\rho} & O(\sum_{ij} m_{ij})
\end{array}$$

There is more structure in this simple example. In $O(1)$ there is a special Q, that is the 'unit disk inside the unit circle'. It's clear that this acts as a unit, for example in the sense that via $O(1) \times O(n) \to O(n)$, for all $o(n) \in O(n)$ we have $(Q, o(n)) \to o(n)$. To state this in the proper generality, we can say that *there exists a map q from* the singleton, that is *the unit element Q for product* in the category of sets, *to* $O(1)$, *such that the following two* **unit diagrams** *commute:*

$$\begin{tikzcd}
Q \times O(n) \arrow[r, "\sim"] \arrow[d, "q \times id"'] & O(n) \\
O(1) \times O(n) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
O(n) \times Q^n \arrow[r, "\sim"] \arrow[d, "id \times q^n"'] & O(n) \\
O(n) \times O(1)^n \arrow[ur] &
\end{tikzcd}$$

There is even more structure: there are two obvious equivariance properties satisfied by ρ under the action of the permutation groups on the $O(n)$. These can be expressed by two **equivariance diagrams,** which are conceptually elementary but notationally demanding. These are left to the reader to write out; or see [May], for example.

§**2. Formal definition.** Let $\mathcal{S}$ be a symmetric monoidal category (that is, $\mathcal{S}$ has an associative product with unit, and shuffling of factors gives canonical isomorphisms) with product $\times$ and unit object Q. Examples we will consider will be

—Sets with product $\times$ and Q = singleton;

—Topological Spaces, with product $\times$ and Q = singleton;

—Vector Spaces over a field k, with product $\otimes$ and $Q = k$.

DEFINITION. An operad O in $\mathcal{S}$ consists of objects $O(n)$ for all $n \geq 0$, a unit map $Q \xrightarrow{q} O(1)$, a right action of the symmetric group S_n on $O(n)$ for each n, and operations

$$\rho : O(k) \times O(\ell_1) \times \cdots \times O(\ell_k) \to O(\ell_1 + \cdots + \ell_k)$$

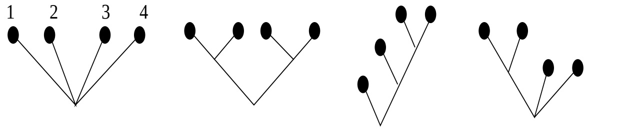
for $k \geq 1$, satisfying the operadic, unit, and equivariance diagrams of §1.

REMARK. $O(0)$ plays little rôle in this definition, and no rôle in this lecture, so we will ignore it here.

Morphisms of operads $\mathcal{C} \to O$ are defined in the natural way.

MORE EXAMPLES. • $\mathcal{C}_n(k)$ = affine embeddings of k disjoint copies of the standard cube I^n in I^n ($\mathcal{C}_n(k)$ can be suitably topologized, making this a topological operad). This is a straightforward generalization of the first example, and in some sense it was the 'first' operad. It is called the *Boardman-Vogt little n-cubes operad* (see [B-V]. There are no 'operads' there, as the name had not yet entered into use; there are however 'PROPS', 'cherry trees', etc.). The name and formal definition of operad were introduced by J. P. May, [May2]. A brief (pre)historical sketch on operads is in [Stasheff].

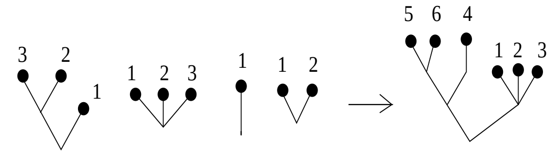
• *Oriented trees.* $\mathcal{T}(n)$ = the set of trees with one root and n labeled tails. For example, $\mathcal{T}(4)$ consists of



etc.

The operation is by 'grafting'. For example,

$$\mathcal{T}(3) \times \mathcal{T}(3) \times \mathcal{T}(1) \times \mathcal{T}(2) \to \mathcal{T}(6) \qquad \text{acts}$$



These are the 'cherry trees' of [B-V]. As graphs can be thought of as a generalization of trees, so there is a corresponding generalization of the notion of operads, that is *modular operads,* introduced by Getzler and Kapranov.

§**3. The Endomorphism operad.** Let V be a vector space (or more generally an element of a category as above and with internal Hom).

DEFINITION. The *Endomorphism operad* End_V of V is defined by setting

$$\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V) \quad .$$

The unit is the map $k \to \mathrm{Hom}(V,V)$ sending 1 to the identity; the action of S_n is by permutation of the factors in $V^{\otimes n}$; and the operation

$$\mathrm{End}_V(n) \otimes \mathrm{End}_V(\ell_1) \otimes \cdots \otimes \mathrm{End}_V(\ell_n) \to \mathrm{End}_V(\ell_1 + \cdots + \ell_n)$$

acts on basic tensors as follows: for $(\phi \otimes \alpha_1 \otimes \cdots \otimes \alpha_n)$ in the source, that is $\phi \in \mathrm{Hom}(V^{\otimes n}, V)$, $\alpha_i \in \mathrm{Hom}(V^{\otimes \ell_i}, V)$, the corresponding homomorphism $V^{\otimes(\sum \ell_i)} \to V$ is induced by

$$(v_{11}, \dots, v_{1\ell_1}, \dots, v_{k1}, \dots, v_{k\ell_k}) \mapsto \phi(\alpha_1(v_{11}, \dots, v_{1\ell_1}), \dots, \alpha_n(v_{n1}, \dots, v_{n\ell_n}))$$

The operad axioms should be clear for End_V, from the associativity of composition.

Aside on terminology. If we have a morphism of operads $\mathcal{C} \to \mathrm{End}_V$, we may say that '$V$ is a $\mathcal{C}$-algebra', or that we have defined an 'action of $\mathcal{C}$ on V', or that V is realized as a representation of $\mathcal{C}$ (again, these notions may be defined not just for vector spaces, but for objects in any reasonable category). May's original result was that "a connected space admits an action of the little cube operad $\mathcal{C}_n$ if and only if it has the homotopy type of an n-fold iterated loop space." There are a number of fancy-sounding terms that translate into "representation of (a certain) operad": see several papers by T. Kimura et al. For example, in [KSV] we read that "… a conformal field theory at the tree level is equivalent to an algebra over the operad of Riemann spheres with punctures." Operads are increasingly relevant to physics, as are other fields that I usually would not associate with physics. In the same paper, I was surprised to read: "We recall the Deligne-Knudsen-Mumford compactification of $M_{0,n}$ … see [11,23,24,25] or *any review of two-dimensional quantum field theory.*"

§4. The moduli space operad.

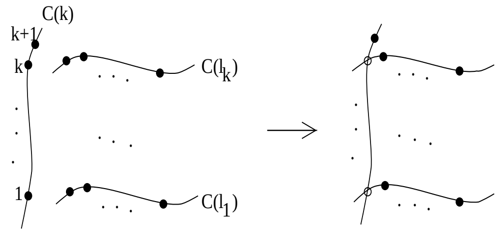
Definition. The moduli space operad $\mathcal{M}$ is defined by setting $\mathcal{M}(n) = \overline{M}_{0,n+1}$ for $n \geq 2$, and $\mathcal{M}(1) =$pt., to be pictured as a $\mathbb{P}^1$ with two marked points (that is, a component which will automatically contract to a point, by the stability requirement).

The unit of $\mathcal{M}$ consists of the single point in $\mathcal{M}(1)$; the action of S_n on $\mathcal{M}(n) = \overline{M}_{0,n+1}$ is by permutation of the first n points. For the operation

$$\rho : \mathcal{M}(k) \times \mathcal{M}(\ell_1) \times \cdots \times \mathcal{M}(\ell_k) \to \mathcal{M}(\ell_1 + \cdots + \ell_k) \quad ,$$

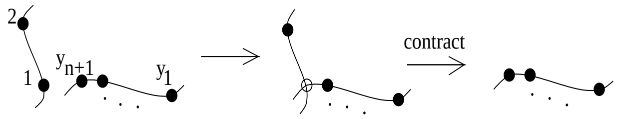
say that $(C(k); C(\ell_1), \dots, C(\ell_k))$ is in the source (that is, $C(k)$ is a stable $(k+1)$-pointed rational curve, etc.) The operation joins $C(\ell_i)$ to $C(k)$ by identifying the last, 'free' marked point of $C(\ell_i)$ with the i^{th} point of $C(k)$. After collapsing unstable components, this produces a stable curve with $(\ell_1 + \cdots + \ell_k) + 1$ marked points, as needed. A schematic representation of this operation on general points

of the factors is:



It is a good exercise to see what this operation does to the 'combinatorial type' of stable curves, in the sense of Belorousski's lecture on $\overline{M}_{0,n}$.

The sense in which $\mathcal{M}(1)$ acts as a unit is by collapsing the corresponding tail: for example, via $\mathcal{M}(1) \times \mathcal{M}(n) \to \mathcal{M}(n)$



The operad $\mathcal{M}$ lives (for example) in the topological category. We can get a related operad in Vector Spaces over $\mathbb{Q}$ by taking homology: set

$$H_*\mathcal{M}(n) = H_*(\mathcal{M}(n), \mathbb{Q})$$

The operation is obtained by composing

$$H_*\mathcal{M}(k) \otimes H_*\mathcal{M}(\ell_1) \otimes \cdots \otimes H_*\mathcal{M}(\ell_k) \xrightarrow{\text{Künneth}} H_*(\mathcal{M}(k) \times \mathcal{M}(\ell_1) \times \cdots \times \mathcal{M}(\ell_k))$$
$$\xrightarrow{H_*(\rho)} H_*\mathcal{M}(\sum \ell_i)$$

Next, the Gromov-Witten invariants on a variety X allow us to define linear maps $H_*\mathcal{M}(n) \to \mathrm{End}_{H^*X}(n)$ for all $n \geq 1$. Then the properties of Gromov-Witten invariants will imply that this is a map of operads.

§5. Gromov-Witten invariants and classes. Reminder: for nice X and $\beta \in A^1X$ there is a space $\overline{M}_{0,n}(X,\beta)$ and n evaluation maps $\eta_1, \dots, \eta_n : \overline{M}_{0,n}(X,\beta) \to X$, acting $\eta_i : (C; p_1, \dots, p_n; f) \mapsto f(p_i)$. The number associated with classes $\gamma_1, \dots, \gamma_n \in A^*X$ is

$$\int_{\overline{M}_{0,n}(X,\beta)} \eta_1^*\gamma_1 \cup \cdots \cup \eta_n^*\gamma_n$$

Now we need to shift the focus a little: first, we want to consider the whole class $\eta_1^*\gamma_1 \cup \cdots \cup \eta_n^*\gamma_n$ (note: not gaining information, according to the 'first reconstruction theorem' in [K-M]); second, we transfer the class to $\overline{M}_{0,n}$. For this, consider the diagram

$$\begin{tikzcd} & & \overline{M}_{0,n} \\ \overline{M}_{0,n}(X,\beta) \arrow[r, "m"] & X^n \times \overline{M}_{0,n} \arrow[ur, "\pi_2"] \arrow[dr, "\pi_1"'] & \\ & & X^n \end{tikzcd}$$

Here $(\pi_1 \circ m)(C; p_1, \dots, p_n; f) = (f(p_1), \dots, f(p_n))$, that is, $\pi_1 \circ m = (\eta_1, \dots, \eta_n)$; we can then map $\gamma_1 \otimes \dots \otimes \gamma_n$ in $H^*(X^n)$ to a class in $\overline{M}_{0,n}$ by pushing-forward via $(\pi_2 \circ m)$ the pull-back via $(\pi_1 \circ m)$.

Definition. Call this class $I^X_{0,n,\beta}(\gamma_1 \otimes \dots \otimes \gamma_n)$.

This definition reflects the *motivic axiom* of [K-M]. *Motivic* here means that the map $H^* X^n \to H^* \overline{M}_{0,n}$ is obtained via a correspondence, that is a class in $H_*(X^n \times \overline{M}_{0,n})$. For genus=0, and nice X, the image from $\overline{M}_{0,n}(X,\beta)$ of the fundamental class can be used for this purpose, and this leads to the $I^X_{0,n,\beta}$ above. In other cases the situation is considerably more involved; the construction of Behrend and Fantechi produces a good candidate in great generality.

Remark. Our previous Gromov-Witten number is still $\int_{\overline{M}_{0,n}} I^X_{0,n,\beta}(\gamma_1 \otimes \dots \otimes \gamma_n)$.

Summarizing: the Gromov-Witten information is encoded in maps

$$I^X_{0,n,\beta} : (H^* X)^{\otimes n} \to H^*(\overline{M}_{0,n})$$

We can also put all of them together if we want, by taking as many copies of the target as there are effective β's:

$$I^X_{0,n} : (H^* X)^{\otimes n} \to \oplus_{\beta \in \mathcal{B}} H^*(\overline{M}_{0,n})_{(\beta)}$$

where $\mathcal{B} = \{$effective β's$\}$, and the product on the $\oplus$ respects the $\mathcal{B}$-grading. Put it otherwise, we should consider

$$I^X_{0,n} : (H^* X)^{\otimes n} \to H^*(\overline{M}_{0,n})$$

as a map between $\mathcal{B}$-graded objects in order to carry along the information about β. *We will do this implicitly in what follows.*

§6. GW on X induce $H_*\mathcal{M}(n) \to \mathrm{End}_{H^*X}(n)$. At this point we have maps

$$(H^* X)^{\otimes n} \otimes H^* X \to H^*(\overline{M}_{0,n+1})$$
$$(\gamma_1 \otimes \dots \otimes \gamma_n) \otimes \gamma_{n+1} \mapsto I(\gamma_1 \otimes \dots \otimes \gamma_{n+1})$$

where $I(\gamma_1 \otimes \dots \otimes \gamma_{n+1}) = \oplus_\beta I^X_{0,n+1,\beta}(\gamma_1 \otimes \dots \otimes \gamma_{n+1})$ (again, source and target of I are $\mathcal{B}$-graded, see §5; this will be hidden in the notations). Dualizing this map, we obtain a map

$$\begin{aligned} H_*(\overline{M}_{0,n+1}) \to & \mathrm{Hom}((H^* X)^{\otimes n} \otimes H^* X, \mathbb{Q}) \\ &= \mathrm{Hom}((H^* X)^{\otimes n}, (H^* X)^\vee) \\ &= \mathrm{Hom}((H^* X)^{\otimes n}, H^* X) \end{aligned}$$

by Poincaré duality. That is, we now have a map

$$H_*\mathcal{M}(n) \to \mathrm{End}_{H^*X}(n)$$

for $n \geq 2$, which we proceed to write out explicitly.

First, Poincaré duality works $H^* X \to (H^* X)^\vee$ by sending c to $\int_X c \cup \cdot = \alpha_c(\cdot)$. Say T_i form a basis for $H^* X$, and $(g_{ij}) = \int T_i \cup T_j$, $(g^{ij}) = (g_{ij})^{-1}$ as usual. Then, writing c in terms of this basis:

$$c = \sum c^i T_i \mapsto \left(T_j \mapsto \alpha_c(T_j) = \int_X \sum_i c^i T_i \cup T_j = \sum_i g_{ij} c^i \right) \quad \text{, and hence}$$
$$c^k = \sum_i \delta^k_i c^i = \sum_{ij} g^{jk} g_{ij} c^i = \sum_j g^{jk} \alpha_c(T_j) \quad \text{, or}$$
$$c = \sum_k c^k T_k = \sum_{jk} g^{jk} \alpha_c(T_j)\, T_k$$

Now start with $Z \in H_*\mathcal{M}(n) = H_*\overline{M}_{0,n+1} = (H^*\overline{M}_{0,n+1})^\vee$, and get

$$\begin{aligned} Z \mapsto \int_Z \cdot \mapsto & \left(\gamma_1 \otimes \dots \otimes \gamma_{n+1} \mapsto \int_Z I(\gamma_1 \otimes \dots \otimes \gamma_{n+1}) \right) \in \mathrm{Hom}((H^* X)^{\otimes n+1}, \mathbb{Q}) \\ &= \left(\gamma_1 \otimes \dots \otimes \gamma_n \mapsto \int_Z I(\gamma_1 \otimes \dots \otimes \gamma_n \otimes \cdot) \right) \in \mathrm{Hom}((H^* X)^{\otimes n}, (H^* X)^\vee) \\ &= \left(\gamma_1 \otimes \dots \otimes \gamma_n \mapsto \sum_{jk} g^{jk} \int_Z I(\gamma_1 \otimes \dots \otimes \gamma_n \otimes T_j)\, T_k \right) \end{aligned}$$

in $\mathrm{Hom}((H^* X)^{\otimes n}, H^* X) = \mathrm{End}_{H^*X}(n)$. This defines

$$H_*\mathcal{M}(n) \to \mathrm{End}_{H^*X}(n)$$

for all $n \geq 2$. For $n = 1$, there is little choice:

$$\begin{aligned} H_*\mathcal{M}(1) = H_*(\mathrm{pt}) = \mathbb{Q} &\to \mathrm{Hom}(H^* X, H^* X) = \mathrm{End}_{H^*X}(1) \\ 1 &\mapsto \text{identity} \end{aligned}$$

§7. Properties of Gromov-Witten invariants and morphisms of operads. The (vague) claim is now that *the general properties of Gromov-Witten invariants amount to the fact that the maps*

$$(*) \qquad H_*\mathcal{M}(n) \to End_{H^*X}(n)$$

defined in §6 preserve the operad structures on source and target.

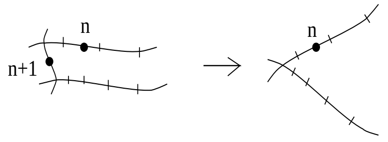
Remark/Example. Both operads involved have in fact a little more structure, and this is also preserved. On the End side, we have

$$\begin{aligned} \psi : \mathrm{End}_{H^*X}(n) &\to \mathrm{End}_{H^*X}(n-1) \\ \alpha &\mapsto (\gamma_1 \otimes \dots \otimes \gamma_{n-1} \mapsto \alpha(\gamma_1 \otimes \dots \otimes \gamma_{n-1} \otimes 1)) \end{aligned}$$

On the $H_*\mathcal{M}$ side, we have the maps induced in homology by the maps

$$\phi : \mathcal{M}(n) = \overline{M}_{0,n+1} \to \overline{M}_{0,n} = \mathcal{M}(n-1)$$

obtained by forgetting the last point (and collapsing unstable components):



Claim. *The map $H_*\mathcal{M} \to End_{H^*X}$ defined above preserves this structure; that is, the diagrams*

$$\begin{array}{ccc} H_*\mathcal{M}(n) & \longrightarrow & End_{H^*X}(n) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ H_*\mathcal{M}(n-1) & \longrightarrow & End_{H^*X}(n-1) \end{array}$$

commute.

Proof. These (and all other analogous facts) should be straightforward from the explicit formula given for (*) in §6. Going first right and then down gives

$$Z \mapsto \left(\gamma_1 \otimes \cdots \otimes \gamma_n \mapsto \sum_{jk} g^{jk} \int_Z I(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes T_j) T_k\right)$$
$$\mapsto \left(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \mapsto \sum_{jk} g^{jk} \int_Z I(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes 1 \otimes T_j) T_k\right) \quad ;$$

going first down and then right gives

$$Z \mapsto \phi_* Z \mapsto \left(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \mapsto \sum_{jk} g^{jk} \int_{\phi_* Z} I(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes T_j) T_k\right)$$

Using that (g^{jk}) is nonsingular, reading components, and replacing T_j by γ_n we see then that the commutativity of the diagrams is equivalent to

$$\int_Z I(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes 1) = \int_{\phi_* Z} I(\gamma_1 \otimes \cdots \otimes \gamma_n)$$

for all $Z \in H_*\overline{M}_{0,n+1}$ and $\gamma_1, \ldots, \gamma_n \in H^*X$. This holds by the projection formula, and is essentially equivalent to property (1) of Gromov-Witten invariants in Ranestad's lecture. $\square$

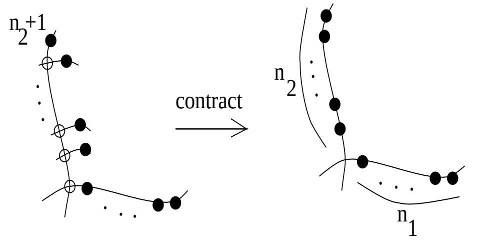
Concerning associativity, writing that $(*)$ induces a morphism of operads also amounts to the commutativity of certain diagrams. Writing this out leads essentially to the WDVV equations, which also appeared in Ranestad's lecture. A key formula from that lecture, rewritten slightly to match notations, reads

$$\sum \int_{\overline{M}_{0,A\cup\{\bullet\}}} I((\otimes_{a\in A}\gamma_a) \otimes T_\ell)\, g^{\ell m} \int_{\overline{M}_{0,B\cup\{\bullet\}}} I(T_m \otimes (\otimes_{b\in B}\gamma_b)) = \int_{D(A|B)} I(\otimes_{c\in A\cup B}\gamma_c)$$

(for the notations, see also Belorousski's lecture on $\overline{M}_{0,n}$. And note that the dependence on β is implicit in I once the relevant objects are $\mathcal{B}$-graded, see §5; the β-component of the left-hand-side is a $\sum_{\beta_1+\beta_2=\beta}$ of terms $\int I_{\beta_1} \cdots \int I_{\beta_2} \cdots$). We want to show that this follows from the commutativity of one of the diagrams expressing that $(*)$ is a morphism of operads. Conversely, these equalities and the structure of the boundary of the spaces $\overline{M}_{0,n}$ ought to imply that *all* such diagrams commute; but this seems substantially more involved, and we will not attempt to discuss it here.

For $|A| = n_1$, $|B| = n_2$, with $n_1, n_2 \geq 2$, consider

$$\mathcal{M}(n_2) \times \mathcal{M}(n_1) \times \underbrace{\mathcal{M}(1) \times \cdots \times \mathcal{M}(1)}_{n_2-1} \to \mathcal{M}(n_1+n_2-1)$$



This map realizes the map

$$\overline{M}_{0,A\cup\{\bullet\}} \times \overline{M}_{0,B\cup\{\bullet\}} \to \overline{M}_{0,A\cup B}$$

whose image is the divisor $D(A|B)$ of $\overline{M}_{0,A\cup B}$. Now take H_* and apply $(*)$:

$$\begin{array}{ccc} H_*\mathcal{M}(n_2) \otimes H_*\mathcal{M}(n_1) \otimes H_*\mathcal{M}(1)^{\otimes(n_2-1)} & \longrightarrow & H_*\mathcal{M}(n_1+n_2-1) \\ \big\downarrow & & \big\downarrow \\ \mathrm{End}_{H^*X}(n_2) \otimes \mathrm{End}_{H^*X}(n_1) \otimes \mathrm{End}_{H^*X}(1)^{\otimes(n_2-1)} & \longrightarrow & \mathrm{End}_{H^*X}(n_1+n_2-1) \end{array}$$

Assume $(*)$ induces a morphism of operads; then this diagram commutes. Chase

$$[\overline{M}_{0,B\cup\{\bullet\}}] \otimes [\overline{M}_{0,A\cup\{\bullet\}}] \otimes [\mathrm{pt}]^{\otimes n_2-1} \quad :$$

going first right and then down, this maps to $[D(A|B)]$ and then to

$$\gamma_1 \otimes \cdots \otimes \gamma_{n_1+n_2-1} \mapsto \sum_{jk} g^{jk} \int_{D(A|B)} I(\gamma_1 \otimes \cdots \otimes \gamma_{n_1+n_2-1} \otimes T_j)\, T_k \quad ;$$

going first down and then right, it maps to

$$\left(\sum_{jk} g^{jk}\int_{\overline{M}_{B\cup\{\bullet\}}} I(\underbrace{\cdot\otimes\cdots\otimes\cdot}_{n_2}\otimes T_j)\,T_k\right)$$
$$\otimes\left(\gamma_1\otimes\cdots\otimes\gamma_{n_1}\mapsto \sum_{\ell m} g^{\ell m}\int_{\overline{M}_{A\cup\{\bullet\}}} I(\gamma_1\otimes\cdots\otimes\gamma_{n_1}\otimes T_\ell)\,T_m\right)$$
$$\otimes(\gamma_{n_1+1}\mapsto\gamma_{n_1+1})\otimes\cdots\otimes(\gamma_{n_1+n_2-1}\mapsto\gamma_{n_1+n_2-1})$$

and then (by the obvious linearity of I) to

$$\gamma_1\otimes\cdots\otimes\gamma_{n_1+n_2-1}\mapsto\sum_{jk} g^{jk}\left(\sum_{\ell m} g^{\ell m}\int_{\overline{M}_{A\cup\{\bullet\}}} I(\gamma_1\otimes\cdots\otimes\gamma_{n_1}\otimes T_\ell)\right.$$
$$\left.\cdot\int_{\overline{M}_{B\cup\{\bullet\}}} I(T_m\otimes\gamma_{n_1+1}\otimes\cdots\otimes\gamma_{n_1+n_2-1}\otimes T_j)\,T_k\right)$$

Comparing the two results, we see that we must have

$$\sum\int_{\overline{M}_{A\cup\{\bullet\}}} I(\gamma_1\otimes\cdots\otimes\gamma_{n_1}\otimes T_\ell)g^{\ell m}\int_{\overline{M}_{B\cup\{\bullet\}}} I(T_m\otimes\gamma_{n_1+1}\otimes\cdots\otimes\gamma_{n_1+n_2-1}\otimes T_j)$$
$$=\int_{D(A|B)} I(\gamma_1\otimes\cdots\otimes\gamma_{n_1+n_2-1}\otimes T_j)$$

for all $\gamma_1,\ldots,\gamma_{n_1+n_2-1}$, and therefore (again by linearity of I)

$$\sum\int_{\overline{M}_{A\cup\{\bullet\}}} I(\gamma_1\otimes\cdots\otimes\gamma_{n_1}\otimes T_\ell)\,g^{\ell m}\int_{\overline{M}_{B\cup\{\bullet\}}} I(T_m\otimes\gamma_{n_1+1}\otimes\cdots\otimes\gamma_{n_1+n_2})$$
$$=\int_{D(A|B)} I(\gamma_1\otimes\cdots\otimes\gamma_{n_1+n_2})$$

which is our basic identity. $\square$

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7. Axioms for Gromov-Witten invariants, I—B. Fantechi, 10/24/96

We introduce and motivate the axioms from [K-M] for Gromov-Witten invariants, then discuss approaches for a construction of classes satisfying these axioms, with emphasis on Behrend's work [B-M]. This talk and the next one are a rough and oversimplified outline of the contents of [K-M] and [B-M]. They contain *intentional mistakes* (and probably unintentional ones as well).

Let X be a smooth complex projective variety.

DEFINITION. A *system of Gromov-Witten classes* on X is the datum, for every $\beta\in H_2(X)$ and for every $g,n\geq 0$ such that $2g+n\geq 3$, of a linear map

$$I^X_{g,n,\beta}: H^*(X)^{\otimes n}\to H^*(\overline{M}_{g,n})$$

satisfying the properties

(GW0) Effectivity

(GW1) S_n-equivariance
(GW2) Grading
(GW3) Fundamental class
(GW4) Divisors
(GW5) Mapping to a point ($\beta=0$)
(GW6) Splitting
(GW7) Genus reduction
(GW8) Motivic axiom

A *tree level system* of Gromov-Witten classes is the same, with $g=0$ (so the genus reduction axiom becomes irrelevant).

Here $H_2(X)=H_2(X,\mathbb{Z})$, and cohomology is taken with $\mathbb{Q}$-coefficients.

Idea behind the axioms: suppose we are in the best possible world; in particular, nontrivial finite groups of automorphisms do not exist, and for a generic map $f: C\to X$, $h^1(f^*T_X)=0$. (Note: this is *never* true if $g(C)\geq 1$.) Let $\overline{M}_{g,n}(X,\beta)$ be the moduli space of maps (as defined in previous lectures). What is its dimension supposed to be? 'Coarse' reasoning: look at the map $\overline{M}_{g,n}(X,\beta)\to\overline{M}_{g,n}$, and pretend it simply forgets the extra data. The dimension of $\overline{M}_{g,n}$ is $(3g-3+n)$; we are pretending that there is no (or no generic) obstruction from H^1, so (by Riemann-Roch) the fibers would have dimension

$$h^0(g^*T_X)=\chi(f^*T_X)=\dim X(1-g)+\beta\cdot c_1(X)\quad,\quad\text{hence}$$

$$\dim\overline{M}_{g,n}(X,\beta)=(\dim X-3)(1-g)+n+\beta\,c_1(X)$$

We call this number the *expected dimension* of $\overline{M}_{g,n}(X,\beta)$. The natural map

$$\overline{M}_{g,n}(X,\beta)\to X^n\times\overline{M}_{g,n}\quad:$$

gives (via Poincaré duality) a class

$$c^X_{g,n,\beta}=[\overline{M}_{g,n}(X,\beta)]\in H^*(X^n\times\overline{M}_{g,n})$$

Consider the two projection

$$\begin{tikzcd} X^n\times\overline{M}_{g,n} \arrow[r, "q"] \arrow[d, "p"'] & \overline{M}_{g,n} \\ X^n & \end{tikzcd}$$

and define (note: $H^*(X^n)=H^*(X)^{\otimes n}$)

$$I^X_{g,n,\beta}=q_*(p^*(\cdot)\cap c^X_{g,n,\beta})$$

(motivated by the intuitive enumerative meaning).

We will now state each of the axioms and 'verify' some of them in the very optimistic assumptions above, in the hope to give an intuitive motivation for them.

(GW0) Let $H_2(X)_+=\{\beta\in H_2(X)/\beta\cdot c_1(\mathcal{L})\geq 0$ for all ample $\mathcal{L}\}$. Then $I^X_{g,n,\beta}=0$ unless $\beta\in H_2(X)_+$. Verification: $\overline{M}_{g,n}(X,\beta)$ is clearly empty unless either $\beta=0$, or β is the class of an effective case. In either case, $\beta\in H_2(X)_+$.

(GW1) $I^X_{g,n,\beta}$ is S_n-equivariant.

(GW2) $\deg I^X_{g,n,\beta} = 2[(g-1)\dim X + \beta \cdot K_X]$. Verification: This is the degree induced by the dimension estimate above (the '2' comes in passing from the complex to the real dimension).

(GW3) Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the map forgetting the last point (and stabilizing); then $I^X_{g,n+1,\beta}(\cdot \otimes 1_X) = \pi^* \circ I^X_{g,n,\beta}$. Moreover,

$$I^X_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes 1_X) = \begin{cases} \int_X \gamma_1 \cup \gamma_2 & \text{if } \beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

(GW4) If $\gamma \in H^2(X)$, then

$$\pi_* I^X_{g,n+1,\beta}(\cdot \otimes \gamma) = (\beta \cdot \gamma)\, I^X_{g,n,\beta}$$

(GW5) (Case $\beta = 0$) $\overline{M}_{g,n}(X,0) = \overline{M}_{g,n} \times X$. Here

$$\begin{aligned} \dim \overline{M}_{g,n}(X,0) &= 3g-3+n+\dim X \\ \text{exp. } \dim \overline{M}_{g,n}(X,0) &= 3g-3+n+\dim X(1-g) \end{aligned}$$

do not match for $g > 0$. Now the obstruction space is:

$$\mathcal{T}^1_{(C,x,f)} = H^1(C, f^*T_X) = H^1(C, \mathcal{O}_C) \otimes T_{x,f(C)} \quad ;$$

that is, with

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "g"'] & X \\ \overline{M}_{g,n}(X,0) & \end{tikzcd} \quad ,$$

$\mathcal{T}^1 = R^1 g_*(f^*T_X)$. This has rank $g \dim X$, accounting for the difference between actual and expected dimension. Use $\mathcal{T}^1$ then to 'correct' the fundamental class:

$$c^X_{g,n,0} = [\overline{M}_{g,n}(X,0)] \cup c_{\mathrm{top}}\mathcal{T}^1 \quad .$$

Note: in the real world, $\mathcal{C}$ does not exist; this is one reason to consider $\overline{M}_{g,n}(X,\beta)$ as a Deligne-Mumford stack: so it is a smooth, fine moduli space, can work with cohomology (with $\mathbb{Q}$-coefficients), etc. Define

$$I^X_{g,n,0} = p_{2*}(p_1^*(\cdot) \cdot c^X_{g,n,0})$$

where p_i are the projections from $\overline{M}_{g,n} \times X$.

(GW6) Fix g_1, g_2, g, n_1, n_2, n such that $2g_i + n_i + 1 \geq 3$, $g = g_1 + g_2$, $n = n_1 + n_2$. Let $\varphi : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$ be the usual glueing map. Then

$$\varphi^* \circ I^X_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \sum_{\beta_1+\beta_2=\beta} I^X_{g_1,n_1+1,\beta_1} \otimes I^X_{g_2,n_2+1,\beta_2}(\gamma_1 \otimes \cdots \otimes \gamma_{n_1} \otimes [\Delta] \otimes \gamma_{n_1+1} \otimes \cdots \otimes \gamma_n)$$

where $[\Delta]$ is the class of the diagonal in X^2 (this can also be written in terms of a basis and of the usual g^{ab}). Note that the sum is finite, by effectivity.

(GW7) Let $\psi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ be the map joining two of the points and increasing the genus. Then

$$\psi^* \circ I^X_{g,n,\beta} = I^X_{g-1,n+2,\beta}(\cdot \otimes [\Delta])$$

(GW8) There exists $c^X_{g,n,\beta} \in A_*(X^n \times \overline{M}_{g,n})$ such that

$$I^X_{g,n,\beta} = q_*(p^*(\cdot) \cap c^X_{g,n,\beta})$$

with q, p the projections, as before.

Remark. Once we take (GW8) for granted, all other axioms can be formulated in terms of $c^X_{g,n,\beta}$. For example, both (GW3) and (GW4) boil down to

$$c^X_{g,n+1,\beta} = (id_{X^n} \times \pi)^* c^X_{g,n,\beta}$$

We now try to describe how to construct a system of classes $c^X_{g,n,\beta}$ such that the axioms hold.

Constructions. If $g = 0$ and X is convex, then everything works fine, with $c^X_{0,n,\beta} = [\overline{M}_{0,n}(X,\beta)]$. The expectation is that this should be the case whenever the dimension equals the expected dimension. *Main problem: define* $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}} \in A_*(\overline{M}_{g,n}(X,\beta))$ *with the correct dimension and properties.*

Symplectic approach. Deform enough data so that $\overline{M}_{g,n}(X,\beta)$ is of the correct dimension. This requires a sophisticated analysis. Solutions have been given by Fukaya-Ono [F-O], Li-Tian [Li-Tian2], Siebert [Siebert].

Algebraic approach, Li & Tian's method [Li-Tian]. We first describe the situation locally in the euclidean topology. Let M be a moduli space. Usually from deformation theory one has for all $m \in M$ an obstruction space $O_{M,m}$. Choose m_0, and let $T = T_{M,m_0}$, $O = O_{M,m_0}$. Then there is a map of germs of analytic spaces $f : (T,0) \to (O,0)$ such that $(M, m_0) = f^{-1}(0)$; for $m \in (M, m_0)$, $T_{M,m} = \ker df(m)$ and $O_{M,m} = \operatorname{Coker} df(m)$. Then Li and Tian define a normal cone

$$C_{M/T} \subset O \times M \quad ,$$

of pure dimension $= \dim T$, and T-invariant fiberwise.

Global story: Given $E_T \xrightarrow{g} E_O$ morphism of vector bundles over M, such that $T_{M,m} = \ker g(m)$, $O_{M,m} = \operatorname{Coker} g(m)$ (the obstruction complex), there is a cone $\mathcal{C} \subset E_0$ of pure dimension $= \operatorname{rk} T$, and we can define

$$[M]^{\mathrm{virt}} = [\mathcal{C}] \cup (\text{zero section of } E_0)$$

Note $\dim[M]^{\mathrm{virt}} = \operatorname{rk} E - \operatorname{rk} F = \dim T(m) - \dim O(m)$ (=expected dimension) for all m.

The hard work goes now in defining everything rigorously and proving the independence on the choices, and the relevant properties.

Behrend-Fantechi: For all m, $\mathcal{C}_m$ comes from a cone in $O(m)$.

(1) The cone only depends only on the deformation functor;

(2) invariance means it comes from $[E_O/E_T]$ (stack quotient);

(3) The stack is an easy version of the deformation functor.

Result: given any M, Deligne-Mumford stack, there exists a pure dimensional $\underline{C}_M$ (the *intrinsic normal cone*, see [B-F]); and given any $E_T \to E_O$, the obstruction complex $\underline{C}_M \hookrightarrow [E_O/E_T] = \underline{E}$ yields a $[M]^{\mathrm{virt}} = [\underline{C}_M] \cap$ zero-section of $\underline{E}$. This only depends on $[E_O/E_T]$, i.e., on $E_T \to E_O$ as an object in the derived category.

There is also a relative version over any smooth Artin stack.

Definition. For any g, any n, any X and any β, let $\mathcal{M} = \mathcal{M}_{g,n}$, the moduli stack of prestable curves of genus g with n marked points: $\mathcal{M}$ is a smooth Artin stack, containing $\overline{M}_{g,n}$ as an open and proper subset ($\mathcal{M}$ is not separated). There is a natural morphism $\overline{M} := \overline{M}_{g,n}(X,\beta) \to \mathcal{M}$, sending (C, x_i, f) to (C, x_i) (without stabilizing). Let $\mathcal{C} \to \overline{M}$ be the universal curve, and consider the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \\ \overline{M} & & \end{array}$$

The complex $Rp_*(f^*T_X)$, a well-defined object in the derived category, is a relative obstruction theory for $\overline{M}$ over $\mathcal{M}$, and yields a $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}}$ of the correct dimension.

This implies immediately (GW0), (GW1), (GW2), (GW8); and (GW5) with a little care. To prove (GW3) and (GW4), one must consider

$$\overline{M}_{g,n+1}(X,\beta) \xrightarrow{\pi} \overline{M}_{g,n}(X,\beta)$$

The map π is flat of relative dimension 1. Then one needs

$$[\overline{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}} = \pi^*[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}}$$

This follows from properties of the relative intrinsic normal cone.

To prove (GW6) and (GW7), and for any further work we will have to face *graphs* and axioms from [B-M].

8. Axioms for Gromov-Witten invariants, II—B. Fantechi, 10/31/96

This lecture is based on [B-M] and [Behrend] (the same disclaimer for the previous lecture applies here). Among the problems left open from the first lecture, we had the *splitting axiom*, dealing with the behavior of the Gromov-Witten classes on sets of reducible curves. This leads to considering more complicated curves.

Definition. A *modular graph* τ is the datum of

(1) a finite set V of *vertices;*

(2) a finite set F of *flags;*

(3) a map $\partial : F \to V$;

(4) an involution $j : F \to F$ (denote $\overline{f} = j(f)$)

(5) (modular) a map, the *genus,* $g : V \to \mathbb{Z}_{\geq 0}$

Notation: T = set of *tails* $= \{f \in F/\overline{f} = f\}$; E = set of *edges* $= \{\{f, \overline{f}\}/f \neq \overline{f}\}$; for $v \in V$, $F_v = \{f/\partial f = v\}$.

Definition. The *topological realization* of τ, denoted $|\tau|$: start with a point for every vertex; for every edge $\{e, \overline{e}\}$ glue $[0,1]$ at ∂e, $\partial\overline{e}$; for every tail t, glue $[0,1]$ at ∂t.

Intuitive relation with pointed curves: every vertex v corresponds to a curve of genus $g(v)$; for every edge we make the corresponding curves intersect transversally; for every tail we put a marked point. Hence a modular graph describes the structure of a prestable curve.

Definition. A vertex v of τ is *stable* if $2g(v) + \#F_v \geq 3$.

We will assume all vertices of all graphs under consideration to be stable.

Example. An irreducible n-pointed genus-g curve corresponds to a single vertex v, with $g(v) = g$, and $F = F_v = \{1, \ldots, n\}$.

Fix a smooth projective variety X over $\mathbb{C}$.

Definition. Let τ be a modular graph. An X-marking of τ is a map $\alpha : V(\tau) \to H_2(X)^+$.

Definition. Fix (τ, α). A *prestable (τ,α)-map to* X is the datum (C_v, x_f, μ), where

(1) $\forall v \in V(\tau)$, C_v is a prestable curve of genus $g(v)$;

(2) $\forall f \in F(\tau)$, $x_f \in C_{\partial f}$;

(3) μ maps $C = \amalg C_v \to X$ so that $\mu(x_f) = \mu(x_{\overline{f}})$ for all $f \in F$; and $\mu_*[C_v] = \alpha(v) \in H_2(X)^+$.

Moreover, (C_v, x_f, μ) is *stable* if $\forall v \in V$, $(C_v, \{x_f\}_{f \in F_v}, \mu_{C_v})$ is a stable map.

Remark. If X =pt, α and μ bear no information. We speak then simply of a τ-(pre)stable curve.

Remark. Note that the domain of a prestable (τ, α) map has at least as many irreducible components as the vertices of τ, but it might have more (i.e., some curve C_v may be reducible, or singular).

Definition. A family of (τ,α)-prestable maps over S is the datum (C_v, x_f, μ) where

(1) $C_v \to S$ is a flat proper map, and μ is a map $C = \amalg C_v \to X$;

(2) $x_f : S \to C_v$ is a section (for $f \in F_v$), and $\mu \circ x_f = \mu \circ x_{\overline{f}}$;

(3) $\forall s \in S$, the fiber of (C_v, x_f, μ) over s is a (τ, α)-prestable curve.

Remark. Families of (τ, α) (pre)stable maps pull-back; so we may consider the corresponding functor, as usual, and as usual we wish to represent it.

THEOREM. *Let X be a projective variety, (τ,α) an X-marked modular graph.*

(1) *There exists a fine moduli space $\overline{M}_\tau(X,\alpha)$ for (τ,α)-stable maps to X; $\overline{M}_\tau(X,\alpha)$ is a proper, separated Deligne-Mumford stack of finite type.*
(2) *There exists a fine moduli space $\mathcal{M}(\tau)$ for τ-prestable curves; $\mathcal{M}(\tau)$ is a smooth Artin stack.*

For example, for τ a graph with a single vertex of genus g and n flags around it, $\overline{M}_\tau(X,\alpha)$ is our usual $\overline{M}_{g,n}(X,\alpha)$.

DEFINITION. Let (τ,α) be a marked graph.

(1) $\chi(\tau)=\chi(|\tau|)-\sum_{v\in V(\tau)} g(v)$.
(2) $g(\tau)=1-\chi(\tau)$.
(3) $\dim\tau=-3\chi(\tau)+\#T-\#E$.
(4) The class of $\tau=\alpha(\tau)=\sum_{v\in V}\alpha(v)$.
(5) $\dim(\tau,\alpha)=\dim(\tau)+\chi(\tau)\dim X+\alpha(\tau)c_1(T_X)$.

Exercise: prove that $\overline{M}_\tau$ has dimension $=\dim(\tau)$.

There are natural maps

$$\overline{M}_\tau(X,\alpha)\to\mathcal{M}(\tau)\to\overline{M}_\tau$$

given by $(C_v,x_f,\mu)\mapsto(C_v,x_f)\mapsto(C_v,x_f)^{\mathrm{stab}}$.

The *expected dimension* of $\overline{M}_\tau(X,\alpha)$ is $\dim(\tau,\alpha)$.

DEFINITION. An *orientation* for $\overline{M}$ over X is the datum, for every X-marked graph (τ,α), of a class

$$\mathcal{I}_\tau(X,\alpha)\in A_{\dim(\tau,\alpha)}\overline{M}_\tau(X,\alpha)$$

satisfying 5 compatibility axioms, to be discussed shortly.

An orientation determines a good system of Gromov-Witten classes, in the sense of the previous lecture, as follows. Omitting X from the notations for convenience, we have the diagram:

$$\begin{tikzcd} & & \overline{M}_\tau \\ \overline{M}_\tau(\alpha) \arrow[r, "ev"] & \overline{M}_\tau\times X^{T(\tau)} \arrow[ur, "p_\tau"] \arrow[dr, "q_\tau"'] & \\ & & X^{T(\tau)} \end{tikzcd}$$

and we set $\widetilde{\mathcal{I}}_\tau(\alpha)=ev_*\mathcal{I}_\tau(\alpha)$, and define

$$I_\tau^\alpha : H^*\otimes T(\tau)\to H^*(\overline{M}_\tau)\quad\text{by}$$

$$I_\tau^\alpha(\gamma)=p_{\tau *}\left(q_\tau^*(\gamma)\cap\widetilde{\mathcal{I}}_\tau(\alpha)\right)\quad .$$

We list the compatibility axioms and give for each of them a short, informal description.

(BM1) Mapping to a point
(BM2) Products
(BM3) Cutting edges
(BM4) Forgetting tails
(BM5) Isogeny

(BM1) This is the case $\alpha=0$ ($\iff\ \alpha(\tau)=0$). The axiom is the obvious generalization of the corresponding GW axiom (GW5).

(BM2) If (τ,α) is the 'disjoint union' of (τ_1,α_1) and (τ_2,α_2), then $\mathcal{I}_\tau(\alpha)=\mathcal{I}_{\tau_1}(\alpha_1)\times\mathcal{I}_{\tau_2}(\alpha_2)$.

(BM3) Cutting an edge: the modular graph σ is obtained from τ by 'cutting the edge' $\{e,\overline{e}\}$ if $V(\sigma)=V(\tau)$, $F(\sigma)=F(\tau)$, $\partial\sigma=\partial\tau$, $g_\sigma=g_\tau$, and

$$j_\sigma(f)=\begin{cases} j_\tau(f) & f\notin\{e,\overline{e}\}\\ f & f\in\{e,\overline{e}\}\end{cases}$$

(so the edge $\{e,\overline{e}\}$ between internal vertices is replaced by two tails). We have the cartesian diagram

$$\begin{tikzcd} \overline{M}_\tau(\alpha) \arrow[r] \arrow[d] & \overline{M}_\sigma(\alpha) \arrow[d] \\ X \arrow[r, "\Delta"] & X^2 \end{tikzcd}$$

where the first vertical map sends (C_v,x_f,μ) to $\mu(x,e)=\mu(x,\overline{e})$, while the second sends (C_v,x_f,μ) to the pair $(\mu(x,e),\mu(x,\overline{e}))$. The axiom states then that the orientation for τ comes from σ:

$$\mathcal{I}_\tau(\alpha)=\Delta^!(\mathcal{I}_\sigma(\alpha))\quad .$$

This corresponds to gluing two curves at one marked point on each.

(BM4) Forgetting tails: σ is obtained from τ by forgetting the tail $t\in T(\tau)$ if $V(\sigma)=V(\tau)$, $F(\sigma)=F(\tau)-\{t\}$, $j_\sigma=j_\tau$, $g_\sigma=g_\tau$, $\partial_\sigma=\partial_\tau$ (warning: we are cheating a little because σ here could become unstable; pretend it stays stable, for simplicity). This situation generalizes the map from $\overline{M}_{g,n}$ forgetting one of the marked points: we get a flat, proper map

$$\overline{M}_\tau(X,\alpha)\xrightarrow{\pi}\overline{M}_\sigma(X,\alpha)$$

and the axiom states that

$$\mathcal{I}_\tau(\alpha)=\pi^*\mathcal{I}_\sigma(\alpha)\quad .$$

This corresponds to forgetting one of the marked points.

(BM5) Isogeny: for simplicity, we will consider here only the case of 'contracting an edge'. The modular graph σ is obtained from τ by contracting the edge $\{e,\overline{e}\}$ if $F(\sigma)=F(\tau)-\{e,\overline{e}\}$, $\partial_\sigma=\partial_\tau$, $j_\sigma=j_\tau$, and

—if $\{e,\overline{e}\}$ forms a loop in τ, that is $\partial e=\partial\overline{e}$, then $V(\sigma)=V(\tau)$ and $g_\sigma(\partial e)=g_\tau(\partial e)+1$;

—if $\{e, \overline{e}\}$ does not form a loop in τ, $V(\sigma) = (V(\tau) - \{\partial e, \partial\overline{e}\}) \cup \{*\}$, with $g_\sigma(*) = g_\tau(\partial e) + g_\tau(\partial\overline{e})$
(and $g_\sigma = g_\tau$ on unaffected vertices).

This corresponds to the smoothing of a node.

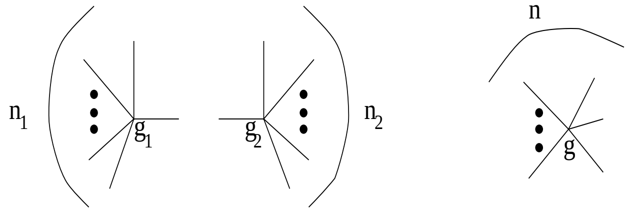
Further, for α a marking on σ consider all possible compatible markings α_i on τ. We get a diagram

$$\begin{array}{ccc} \amalg_i \overline{M}_\tau(X, \alpha_i) & \longrightarrow & \overline{M}_\sigma(X, \alpha) \\ \downarrow & & \downarrow \\ \overline{M}_\tau & \xrightarrow{\phi} & \overline{M}_\sigma \end{array}$$

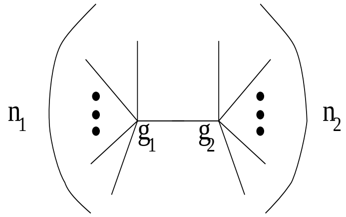
and the axiom (in a slightly simplified form) prescribes that

$$\sum \mathcal{I}_\tau(\alpha_i) = \phi^! \mathcal{I}_\sigma(\alpha) \quad .$$

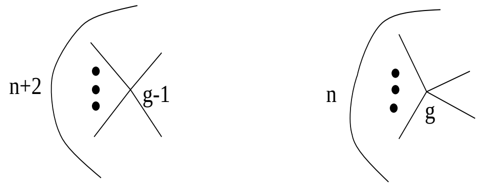
As mentioned above, it can be shown that an orientation defines a system of Gromov-Witten classes satisfying the axioms (GW0)—(GW8) of the previous lecture. For example, the splitting axiom compares classes for $\overline{M}_{g_1, n_1+1}$, $\overline{M}_{g_2, n_2+1}$ with classes for $\overline{M}_{g,n}$ with $g = g_1 + g_2$ and $n = n_1 + n_2$:



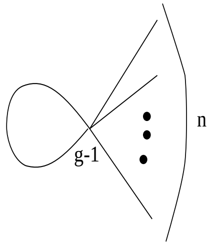
These are related to each other by respectively cutting an edge or contracting it from



Thus the 'splitting axiom' (GW6) follows by judiciously applying (BM3) and (BM5). Similarly,



are obtained by respectively cutting the loop, or contracting it, in



(BM3) and (BM5) can then be used to prove the 'genus reduction axiom' (GW7) for Gromov-Witten classes.

Main result. *(Behrend) The class defined by*

$$\mathcal{I}_\tau(\alpha) = [\overline{M}_\tau(\alpha), E^\bullet_{\tau,\alpha}]^{\text{virt}}$$

gives an orientation for $\overline{M}$ over X.

In this statement, $[\cdot]^{\text{virt}}$ is the (relative) virtual fundamental class in the sense of the previous lecture, and $E^\bullet_{\tau,\alpha}$ is the relative obstruction theory over $\mathcal{M}_\tau$: $E^\bullet_{\tau,\alpha} = R\pi_*\mu^*T_X$ in

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mu} & X \\ {\scriptstyle\pi}\downarrow & & \\ \overline{M}_\tau(X, \alpha) & & \end{array} \quad ,$$

where $(\mathcal{C}, \mu, X)$ is the universal family over $\overline{M}_\tau(X, \alpha)$.

This result produces then a system of Gromov-Witten classes satisfying the axioms from [K-M], for an arbitrary smooth, projective variety X.

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9. Kapranov's work on $\overline{M}_{0,n}$—C. Faber, 10/29/96

The reference for this is [Kapranov]; we work over $\mathbb{C}$.

§1. The main result.

Theorem. *Let $p_1, \ldots, p_n$ be n points in $\mathbb{P}^{n-2}$ in general position. Let $V_0(p) = V_0(p_1, \ldots, p_n)$ be the space of Veronese curves in $\mathbb{P}^{n-2}$ through $p_1, \ldots, p_n$. Then:*

(a) $V_0(p) \cong M_{0,n}$.

Consider $V_0(p)$ as a subvariety of the Hilbert scheme $\mathcal{H}$ parametrizing all subschemes of $\mathbb{P}^{n-2}$; let $V(p) = V(p_1, \ldots, p_n)$ be the closure of $V_0(p)$ in $\mathcal{H}$. Then:

(b) $V(p) \cong \overline{M}_{0,n}$. The subschemes of $\mathbb{P}^{n-2}$ representing points of $V(p)$ are, considered together with the n points, stable n-pointed curves of genus 0.

(c) Analogous statement for Chow variety instead of Hilbert scheme.

Remarks. In suitable coordinates, $p_i = e_i = (0 : \cdots : 0 : \underset{i}{1} : 0 : \cdots : 0)$ for $0 \le i \le n-1$, and $p_n = e_n = (1 : \cdots : 1)$. Also, by a classical result of Castelnuovo, through any $(n+1)$ points in $\mathbb{P}^{n-2}$ in general position passes a unique Veronese curve.

§2. Discussion. Fix $p_1, \ldots, p_n$ in general position in $\mathbb{P}^{n-2}$, $\overline{M}_{0,n} \cong V(p)$.

CLAIM. *Every component of the curve $C \in V(p)$ is a Veronese curve in its span.*

A more precise statement requires a few preliminaries:

(a) Let $\mathcal{A}_{n-1}$ be the configuration of $\binom{n}{2}$ hyperplanes Λ_{ij}, Λ_{ij} =span of$\{p_k\}$, $k \neq i, j$ $(i \neq j)$. A *face* of $\mathcal{A}_{n-1}$ is a projective subspace which is the intersection of some of the Λ_{ij}.

The hyperplanes in $\mathcal{A}_{n-1}$ are isomorphic to the projectivization of the hyperplanes $\{t_i = t_j\}$ in the subspace $\mathbb{C}_0^{n-1} \subset \mathbb{C}^n$ with equation $t_1 + \cdots + t_n = 0$. (The 'mirror' of the root system A_{n-1}.)

(b) The faces in $\mathcal{A}_{n-1}$ correspond to equivalence relations on $\underline{n} = \{1, 2, \ldots, n\}$: for a relation R, $\Lambda(R) = \cap_{iRj} \Lambda_{ij}$.

(c) So there are exactly $2^{n-1} - 1$ 0-dimensional faces, corresponding to equivalence relations with exactly 2 equivalence classes (e.g., $p_i \leftrightarrow \{i\} \amalg \{\underline{n} - \{i\}\}$).

(d) Intersecting Λ_{ij} with a face Λ, we obtain an $\mathcal{A}_m$, with $m = \dim \Lambda + 1$.

(e) Let $\mathcal{T}$ be a tree with tails (=endpoints) $A_1, \ldots, A_n$, and let v, w be vertices (possibly endpoints) of $\mathcal{T}$. Let $[v, w]$ be the unique geodesic from v to w. Define for v an internal vertex of $\mathcal{T}$ an equivalence relation $\cong_v$ on $\underline{n}$:

$$i \cong_v j \iff v \notin [A_i, A_j]$$

The equivalence classes of $\cong_v$ correspond to the edges incident at v.

(f) For e an edge in $\mathcal{T}$, define

$$i \cong_e j \iff e \notin [A_i, A_j]$$

Note: for each e, this equivalence relation has exactly two equivalence classes.

(g) With the above understood, recall that every n-pointed stable curve corresponds to a tree (see Belorousski's lecture on $\overline{M}_{0,n}$). Then the claim is:

THEOREM. *For $C \in V(p)$, consider its tree $\mathcal{T}$. Let v be an internal vertex of $\mathcal{T}$, and let C_v be the corresponding component of C. Then*

(1) *C_v is a Veronese curve in its span $< C_v >$; $< C_v >$ is the face of $\mathcal{A}_{n-1}$ corresponding to $\cong_v$. Its dimension is*

$$\text{(\# equivalence classes)}-2 = \text{(\# edges incident to } v)-2$$
$$= \text{(\# special points on } C_v)-2$$

(2) *Let e be an edge of $\mathcal{T}$ connecting internal vertices; that is, e corresponds to a node z of C. Then z corresponds to the 0-dimensional face of $\mathcal{A}_{n-1}$ corresponding to $\cong_e$. Therefore, the possible singular points of any $C \in V(p)$ belong to a fixed finite subset of $2^{n-1} - 1 - n$ points in $\mathbb{P}^{n-2}$.*

Next, recall that for $i = 1, \ldots, n$ there is a map $\pi_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ forgetting the i-th point. Thus there must be corresponding maps $V(p_1, \ldots, p_n) \to V(q_1, \ldots, q_{n-1})$.

CLAIM. *These are induced by the projections*

$$\mathbb{P}^{n-2} - \{p_i\} \dashrightarrow \mathbb{P}_i^{n-3}$$

This seems rather reasonable.

Next, there are basic line bundles $\mathcal{L}_i$ whose fiber at C is $T^*_{x_i} C$. These can be realized as follows: consider the map:

$$\sigma_i : V(p) \to \mathbb{P}_i^{n-3}$$

sending C to the embedded tangent line to C at p_i (which determines a point in $\mathbb{P}^{n-3}$ under the i-th projection as above). Then $\mathcal{L}_i = \sigma_i^*(\mathcal{O}_{\mathbb{P}_i^{n-3}}(1))$. Also, we get maps $\gamma_{\mathcal{L}_i} : \overline{M}_{0,n} \dashrightarrow \mathbb{P}H^0(\overline{M}_{0,n}, \mathcal{L}_i)^*$.

Results and comments:

(1) $\dim H^0(\overline{M}_{0,n}, \mathcal{L}_i) = n - 2$;

(2) $\gamma_{\mathcal{L}_i}$ is a birational morphism; $\gamma_{\mathcal{L}_i} = \sigma_i$ after the identification $\mathbb{P}_i^{n-3} = \mathbb{P}(H^0(\overline{M}_{0,n}, \mathcal{L}_i)^*$;

(3) σ_i can be decomposed explicitly into blow-ups; there are other constructions for this, due to Fulton-MacPherson and Keel;

(4) the identifications induce rational maps $\mathbb{P}_i^{n-3} \dashrightarrow \mathbb{P}_j^{n-3}$; these turn out to be Cremona transformations.

(5) Finally, this gives an interpretation for the Witten τ-numbers in genus 0:

$$< \tau_{d_1} \cdots \tau_{d_n} > = \int_{\overline{M}_{0,n}} \prod_i c_1(\mathcal{L}_i)^{d_i}$$

with $\sum d_i = n - 3$. This is the number of Veronese curves through a certain assortment of points, and tangent to certain codimension-d_i planes at these points. This might lead to a computation of these numbers (which are however already known and rather easy to obtain).

Next, we have the relative dualizing sheaf ω_C on a stable n-pointed curve C of genus 0: obtained by gluing ω_i on components C_i, where the ω_i are regular on the smooth part, and at a point of intersection of two components they may have simple poles with opposite residues. By Knudsen's work, $\omega_C(x_1 + \cdots + x_n)$ is very ample, and has $(n-1)$ independent global sections. Via the corresponding map $C \hookrightarrow \mathbb{P}^{n-2}$, the images p_i of the x_i are in general position (as shown by a computation).

Now for part (a) of the main theorem, consider $x_1, \ldots, x_n$ distinct points on $\mathbb{P}^1$, and embed $\mathbb{P}^1$ in $\mathbb{P}^{n-2}$ with $\omega_{\mathbb{P}^1}(x_1 + \cdots + x_n)$. By a projective transformation, the image is moved to a Veronese curve through $p_1, \ldots, p_n$, given points in general position in $\mathbb{P}^{n-2}$. So each curve $\in M_{0,n}$ can be realized as stated. If f is an isomorphism of two n-pointed curves, f induces an isomorphism of the $(n-2)$-nd symmetric products $\check{\mathbb{P}}^{n-2}$, hence of their duals $\mathbb{P}^{n-2}$, fixing n generic points; it follows that f is the identity. This essentially establishes $M_{0,n} \cong V_0(p_1, \ldots, p_n)$.

For part (b), consider a stable n-pointed genus-0 curve over any base S: $\mathcal{C} \xrightarrow{\pi} S$, with sections s_i. We have an embedding of $\mathcal{C}$ in $\mathbb{P}(\pi_*(\omega_{\mathcal{C}/S}(s_1 + \cdots + s_n))^*)$, a projective bundle with n sections $s_1, \ldots, s_n$, in general position in every fiber.

Trivialize the situation by moving every fiber to pass through a fixed frame in $\mathbb{P}^{n-2}$. The map π is flat, so we get a map $\gamma_{S,\mathcal{C}}$ from S to the Hilbert scheme $\mathcal{H}$, such that $\mathcal{C}$ is the pull-back via $\gamma_{S,\mathcal{C}}$ of the universal flat family of subschemes Ξ over $\mathcal{H}$:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \Xi \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S & \xrightarrow{\gamma_{S,\mathcal{C}}} & \mathcal{H} \end{array}$$

Now for all S this determines a map

$$\gamma_S : \mathrm{Mor}(S, \overline{M}_{0,n}) \to \mathrm{Mor}(S, \mathcal{H})$$

by sending a family $\mathcal{C}$ (which determines and is determined by a unique morphism $S \to \overline{M}_{0,n}$) to $\gamma_{S,\mathcal{C}}$. In fact, taking $S = \overline{M}_{0,n}$ itself gives a map from this to $\mathcal{H}$, and the dense open $M_{0,n}$ of $\overline{M}_{0,n}$ maps to $V_0(p) \subset \mathcal{H}$; so $\overline{M}_{0,n}$ maps to the closure $V(p)$ of $V_0(p)$ in $\mathcal{H}$, and γ_S must factor through

$$\gamma_S : \mathrm{Hom}(S, \overline{M}_{0,n}) \to \mathrm{Hom}(S, V(p))$$

We want to show that this is a bijection for all S. The injectivity follows from the fact that every $\mathcal{C}$ is a pull-back from Ξ; for the surjectivity, we have the diagram

$$\begin{array}{ccc} \overline{C}_{0,n} & \longrightarrow & \Xi \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \overline{M}_{0,n} & \longrightarrow & V(p) \end{array}$$

and since the fibers of Ξ over V are stable n-pointed curves of genus 0, there must be a map $V(p) \to \overline{M}_{0,n}$ as $\overline{M}_{0,n}$ is a fine moduli space. □

Note that we are assuming already that $\overline{M}_{0,n}$ is a fine moduli space; it would be interesting to use the construction to show directly that $V(p)$ is a fine moduli space.

For (c) in the main theorem, let $\mathcal{C}$ be the Chow variety; we have $\mathcal{H} \to \mathcal{C}$ and $\overline{M}_{0,n} \cong V(p)$ surjecting on the closure $W(p)$ of $V_0(p)$ in $\mathcal{C}$; we would show that this latter map is an isomorphism. One proves that it is a set bijection, then argues that tangent vectors to $V(p)$ (which is smooth) are not contracted in $W(p)$.

§**3. Problem.** Let $x_1(t), \ldots, x_n(t) \in \mathbb{C}((t))$ be distinct formal Laurent series in t. Assume that for $t \neq 0$, $(\mathbb{P}^1; x_1(t), \ldots, x_n(t))$ is a stable n-pointed curve in $M_{0,n} \subset \overline{M}_{0,n}$. The limit $\lim_{t\to 0}(\mathbb{P}^1; x_1(t), \ldots, x_n(t))$ is then a certain $C \in \overline{M}_{0,n}$; the question is: can this C be determined?

Choosing a suitable coordinate on $\mathbb{P}^1$, and multiplying by a common power of t if necessary, we may assume that $x_n(t)$ is identically ∞, and that $x_1(t), \ldots, x_{n-1}(t) \in \mathbb{C}[[t]]$. Baby example: something like

$$(t^2, t^3, 2t^3, 3t^3, \infty)$$

Set-up. We define a tree, the 'tree of infinitely near points in $\mathbb{C}$': the vertices will be pairs (m, f) with $m \in \mathbb{Z}_{\geq -1}$, and f a polynomial in $\mathbb{C}[t]$ of degree $\leq m$ (the

only polynomial of degree ≤ -1 is 0); the edges will be between pairs (m, f) and $(m+1, g)$ if and only if g agrees with f modulo t^{m+1}.

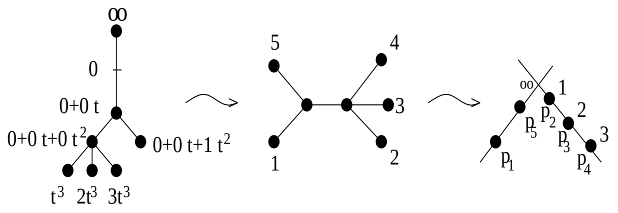
The ends of this tree correspond to formal Taylor series $\in \mathbb{C}[[t]]$.

For $(x_1(t), \ldots, x_{n-1}(t), \infty)$ as prescribed above, look at the subtree $\mathcal{T}$ of the tree $\mathcal{T}_0$ of infinitely near points in $\mathbb{C}$ obtained as the union of paths in $\mathcal{T}_0$ connecting the $x_i(t)$ and the root $v_0 = (-1, 0)$; and let $\mathcal{T}'$ be the simplest topologically equivalent version of $\mathcal{T}$ (where only internal vertices of degree ≥ 3 survive). Then:

Claim. *(1) $\mathcal{T}'$ is the tree of the limit curve;*

(2) to determine the isomorphism class of the limit, one needs the projective equivalence class of the special points on every component C_v of C. This can be read off from $\mathcal{T}'$: the edges departing from v correspond to the special points on C_v, and each has a number naturally associated with it (∞ for the edge connecting back towards the root, and the coefficients of the next term in the Taylor series represented by the other edges).

In the example above: $(t^2, t^3, 2t^3, 3t^3, \infty)$:



Exercise. For a slightly different flavor, take $K = \mathbb{Q}_2$, with ring of integers $\mathbb{Z}_2$, and determine the stable reduction (mod 2) of $(\mathbb{P}^1; 0, 1, 2, 3, 4, 5, 6, 7, 8)$.

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10. Enumeration of rational curves, after Kontsevich—C. Faber, 11/28/96

Three examples:

(1) Rational curves on $\mathbb{P}^2$. Consider the map $\varphi_k : \overline{M}_{0,k}(\mathbb{P}^2, d) \to (\mathbb{P}^2)^k$ defined by $\varphi_k(C; x_i; f) = (f(x_i))$, and let

$P_d :=\#\{$rational curves on $\mathbb{P}^2$ of degree $d \geq 1$, through $(3d-1)$ generic points$\}$
$=$degree of φ_{3d-1}
$= \int_{\overline{M}_{0,3d-1}(\mathbb{P}^2,d)} \prod_{i=1}^{3d-1} \varphi^*\left(c_1(\mathcal{O}(1)_i)^2\right)$
with evident notations.

(2) Rational curves on quintic threefolds. Let V be a threefold given by a section Q of $\mathcal{O}(5)$ on $\mathbb{P}^4$. Consider

$$\begin{array}{ccc} & & \overline{M}_{0,1}(\mathbb{P}^4, d) \xrightarrow{\varphi} \mathbb{P}^4 \\ & & \downarrow{\scriptstyle\pi} \\ Z(\widetilde{Q}_d) = \overline{M}_{0,0}(V, d[\mathbb{P}^1]) & \xrightarrow{\subset} & \overline{M}_{0,0}(\mathbb{P}^4, d) \end{array}$$

Here $\widetilde{Q}_d$ is the section of $\mathcal{E}_d := \pi_*(\varphi^*\mathcal{O}(5))$ determined by Q. Note that $\mathcal{E}_d$ is a vector bundle: for $f : C \to \mathbb{P}^4$ a genus-0 stable map, $H^1(C, f^*\mathcal{O}(5)) = 0$. Then look at the number

$$N_d := \int_{\overline{M}(\mathbb{P}^4,d)} c_{5d+1}(\mathcal{E}_d) \quad :$$

this ought to count the cardinality of $Z(\widetilde{Q}_d)$, that is the weighted number of rational curves on a quintic threefold. Problem: $N_d \notin \mathbb{Z}$ in general; however, one can see that

$$N_d = \sum_{k|d} k^{-3} N^o_{d/k}$$

with N^o_d counting the actual number of rational curves (without contributions of multiple coverings from lower d's, see (3)). This should be an integer.

(3) Multiple coverings of rational curves on Calabi-Yau threefolds. Consider a $C_0 \cong \mathbb{P}^1$ on a Calabi-Yau threefold V with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The space $\overline{M}_{0,0}(V, d[C_0])$ has a connected component $\overline{M}_{0,0}(C_0, d[C_0]) \cong \overline{M}_{0,0}(\mathbb{P}^1, d)$. Its dimension is $(2d-2)$, while the virtual dimension is 0; the obstruction sheaf $\mathcal{F}_d$ is the rank-$(2d-2)$ vector bundle with fiber $H^1(C, f^*(\mathcal{T}_V/\mathcal{T}_{C_0})) \cong \mathbb{C}^2 \otimes H^1(C, f^*(\mathcal{O}(-1)))$ at each point $f : C \to C_0$ (a degree-d stable map). By definition, the contribution of this component will be

$$M_d := \int_{\overline{M}(\mathbb{P}^1,d)} c_{2d-2}(\mathcal{F}_d) \quad .$$

Expectation: $M_d = d^{-3}$ (checked by Manin). This explains then the relation between N_d's and N^o_d's.

This relation can be inverted by using the Möbius function:

$$N^o_d = \sum_{k|d} \mu(k) k^{-3} N_{d/k}$$

Goal of the lecture: to explain how Kontsevich ([Kontsevich]) computes in principle the numbers P_d, N_d, M_d. The main tool is *Bott's formula.*

Situation: X is a (smooth) complex projective manifold, $\mathcal{E}$ is a holomorphic vector bundle on X; a complex torus $T \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ acts algebraically on $(X, \mathcal{E})$.

Fact: then X^T is smooth. On each connected component X^γ, $\mathcal{E} = \oplus_{\lambda \in T^\vee} E^{\gamma,\lambda}$, a decomposition into 'eigenbundles' for characters λ. The same for the normal bundles to the fixed loci: for $\mathcal{N}^\gamma := \mathcal{T}_X/\mathcal{T}_{X^\gamma}$, $\mathcal{N}^\gamma = \oplus_{\lambda \in T^\vee, \lambda \neq 0} \mathcal{N}^{\gamma,\lambda}$.

To write the formula, we introduce the Chern roots of these bundles: e_i for $\mathcal{E}$; $e_i^{\gamma,\lambda}$ for $E^{\gamma,\lambda}$; $n_i^{\gamma,\lambda}$ for $\mathcal{N}^{\gamma,\lambda}$. So $\sum_{k \geq 0} c_k(\mathcal{E}) = \prod_i (1 + e_i)$, and so on.

Now let P be a homogeneous polynomial of degree $\dim X$ in the Chern classes $c_i(\mathcal{E})$ (considered as indeterminates of degree i); that is, a symmetric homogeneous polynomial in the e_i, of degree $\dim X$. Bott's formula then states that

$$\int_X P(e_i) = \sum_\gamma \int_{X^\gamma} \frac{P(e_i^{\gamma,\lambda} + \lambda)}{\prod (n_i^{\gamma,\lambda} + \lambda)} \quad .$$

Every character λ defines a linear form on $\mathrm{Lie}(T)$, so the right-hand-side is a rational function on $\mathrm{Lie}(T)$ (which is constant by the statement, and equals $\int_X P(e_i)$).

We will use Bott's formula on *orbifolds.*

Description of fixed points of the natural action of $T \cong (\mathbb{C}^*)^{n+1}$ on $\overline{M}_{g,k}(\mathbb{P}^n, d)$. This is induced from the action of T on $\mathbb{P}^n$.

Notation: p_i, $i = 1, \ldots, n+1$, are the fixed points of T on $\mathbb{P}^n$. We think of T acting diagonally on $(n+1)$-tuples of homogeneous coordinates, so p_i is the projectivization of the i-th coordinate line in $\mathbb{C}^{n+1}$. For $i \neq j$, let $\ell_{ij} = \ell_{ji}$ be the line in $\mathbb{P}^n$ through p_i and p_j.

Suppose that the stable map $f : C \to \mathbb{P}^n$ represents a point of $\overline{M}_{g,k}(\mathbb{P}^n, d)^T$. Then

(1) $f(C)$ is T-invariant, so a union of lines ℓ_{ij} (points with > 2 nonzero coordinates have ≥ 2-dimensional orbits);
(2) the images of all marked and singular points, and of all contracted components, are points p_i;
(3) a component C^α of V not contracted by f maps onto a line ℓ_{ij}, say with degree d_α. The map $C^\alpha \to \ell_{ij}$ can only be ramified over p_i and p_j; by the Hurwitz formula, necessarily C^α has genus 0 and is totally ramified over p_i and p_j. Note that in particular C^α is smooth; the map is $f(z_1 : z_2) = (0 : \cdots : z_1^{d_\alpha} : \cdots : z_2^{d_\alpha} : \cdots : 0)$.

To each T-fixed $(C; x_1, \ldots, x_k; f)$ we associate a graph Γ (that is, a 1-dimensional finite CW-complex):

(a) Vertices $v \in \mathrm{Vert}(\Gamma)$ correspond to connected components C_v of $f^{-1}(p_i)$ (note: C_v may be a point);
(b) Edges $\alpha \in \mathrm{Edge}(\Gamma)$ correspond to irreducible components C_α (of genus 0) mapping onto lines ℓ_{ij}.

So, the edges α at a vertex v correspond to the non-contracted irreducible components C_α having non-empty intersection with the connected component C_v. The two vertices connected by the edge α corresponding to a C_α, say mapping to ℓ_{ij}, correspond to the unique connected components $C_{v_i,\alpha} \subset f^{-1}(p_i)$, $C_{v_j,\alpha} \subset f^{-1}(p_j)$ resp. which contain $C_\alpha \cap f^{-1}(p_i)$, $C_\alpha \cap f^{-1}(p_j)$ resp. Note that these two vertices are distinct, so Γ has no simple loops.

In short, Γ is made from C by contracting each C_v to a vertex v. In particular, Γ is connected.

(c) Labels on Γ: the vertices v get a number f_v via $f(C_v) = p_{f_v}$; the edges α get labeled by the degree d_α of $f : C_\alpha \to \ell_{ij}$. Further, we define $g_v :=$ the arithmetic genus of the 1-dimensional part of C_v; and $S_v \subset \{1, \ldots, k\}$ to be the set of indices of marked points lying on C_v.

Claim. *The connected components of $\overline{M}_{g,k}(\mathbb{P}^n, d)^T$ are naturally labeled by the equivalence classes of connected graphs Γ (with specifications) such that*

(1) *if an edge α connects the vertices u, v, then $f_u \neq f_v$ (in particular, Γ cannot have simple loops);*
(2) $1 - \chi(\Gamma) + \sum_{v \in \mathrm{Vert}(\Gamma)} g_v = g$;

(3) $\sum_{\alpha\in \mathrm{Edge}(\Gamma)} d_\alpha = d$;
(4) $\amalg_v S_v = \{1,\dots,k\}$.

Kontsevich says (in words): $\overline{M}_{g,k}(\mathbb{P}^n,d)^\Gamma \cong \left(\prod_v \overline{M}_{g_v,\mathrm{val}(v)+\#S_v}\right)/\mathrm{Aut}(\Gamma)$. This isn't quite true, as we will see.

Now for the computation. Assumptions: (1) $g=0$, so all Γ's are trees, all interior genera $g_v = 0$;
(2) for simplicity, we forget the marked points (for the time being);
(3) notation:
(a) $\overline{M} = \overline{M}(\mathbb{P}^n,d)$ (n, d are fixed);
(b) $[\mathcal{E}]$ for the class in the equivariant K-group with $\mathbb{Q}$-coefficients of a T-equivariant vector bundle $\mathcal{E}$ on $\overline{M}^\Gamma$:

$$K^0_T(\overline{M}^\Gamma)\otimes\mathbb{Q} \cong K^0(\overline{M}^\Gamma)\otimes\mathbb{Q}[T^\vee]$$

(c) $[\chi]$ for the class of a trivial line bundle with T-action given by $\chi\in T^\vee\otimes\mathbb{Q}$.
(4) We systematically decompose fibers of vector bundles as formal linear combinations of other vector spaces, in order to compute the characters. First, the normal bundle:

$$[N_{\overline{M}^\Gamma}] = [T_{\overline{M}}] - [T_{\overline{M}^\Gamma}]$$

$$[T_{\overline{M}}] = [H^0(C,f^*T_{\mathbb{P}^n})] + \sum_{y\in C^\alpha\cap C^\beta,\alpha\neq\beta}[T_y(C^\alpha)\otimes T_y(C^\beta)]$$
$$+\left(\sum_{y\in C^\alpha\cap C^\beta,\alpha\neq\beta}([T_y(C^\alpha)]+[T_y(C^\beta)]) - \sum_\alpha [H^0(C^\alpha, T_{C^\alpha})]\right)$$

(Recall: (i) no marked points; (ii) for one component, this is OK; (iii) if we add a component, then we subtract an extra 3-dimensional thing, but we also add 3 dimensions: 1 for smoothing the node, and 2 for moving it. Compare with [F-P].)

$$[T_{\overline{M}^\Gamma}] = 0 + \sum_{y\in C^\alpha\cap C^\beta,\alpha\neq\beta,\alpha,\beta\notin\mathrm{Edge}}[T_y(C^\alpha)\otimes T_y(C^\beta)]$$
$$+\sum_{y\in C^\alpha\cap C^\beta,\alpha\neq\beta,\alpha\notin\mathrm{Edge}}[T_y(C^\alpha)] - \sum_{\alpha\notin\mathrm{Edge}}[H^0(C^\alpha,T_{C^\alpha})]$$

(Namely, we can only smooth nodes for which both branches are contracted; we can move a node only on a contracted component; and we have infinitesimal automorphisms only on contracted components.) So,

$$[N_{\overline{M}^\Gamma}] = [H^0(C,f^*T_{\mathbb{P}^n})] + [N^{\mathrm{abs}}_{\overline{M}^\Gamma}] \quad \text{, where}$$

$$[N^{\mathrm{abs}}_{\overline{M}^\Gamma}] = \sum_{y;\alpha,\beta\in\mathrm{Edge}}[T_y(C^\alpha)\otimes T_y(C^\beta)] + \sum_{y;\alpha\in\mathrm{Edge};\beta\notin\mathrm{Edge}}[T_y(C^\alpha)\otimes T_y(C^\beta)]$$
$$+\left(\sum_{y;\alpha\in\mathrm{Edge}}[T_y(C^\alpha)] - \sum_{\alpha\in\mathrm{Edge}}[H^0(C^\alpha,T_{C^\alpha})]\right)$$

The only nontrivial vector bundle term: $T_y(C^\beta)$ for β not an edge, in the second summand. However, this term has trivial character (which will simplify the computation).

More notions and notations. A *flag* is an edge α with arrow: (v,α), $v\in\mathrm{Vert}(\Gamma)$ thought of as the source of the arrow. This is unambiguous, as Γ has no simple loops. For $F=(v,\alpha)$ a flag, the *weight* of F is

$$w_F := \frac{(\lambda_{f_v}-\lambda_{f_u})}{d_\alpha}$$

where u is the other vertex of α. Here λ_i, $i=1,\dots,n+1$, are the natural coordinates on $\mathrm{Lie}(T)$; w_F is the character of T for the action on $T_{C^\alpha,C_v\cap C^\alpha}$. If $\overline{F}=(u,\alpha)$ is the dual flag then of course $w_{\overline{F}} = -w_F$.

We need the following integral:

$$I(w_1,\dots w_k) := \int_{\overline{M}_{0,k}}\frac{1}{w_i + c_1(T_{x_i}(C))}$$
$$= \sum_{d_i\geq 0,\sum d_i = k-3}\prod_{i=1}^k w_i^{-d_i-1}\underbrace{\langle\tau_{d_1}\cdots\tau_{d_k}\rangle}_{=\frac{(k-3)!}{d_1!\cdots d_k!}} \qquad \text{(Witten's notation)}$$
$$=\left(\sum w_i^{-1}\right)^{k-3}\left(\prod w_i^{-1}\right)$$

Recall that we are to compute integrals $\int_{\overline{M}^\gamma}\frac{P(e_i^{\Gamma,\lambda}+\lambda)}{\prod_i(n_i^{\Gamma,\lambda}+\lambda)}$. We will see that in all three examples the equivariant vector bundles are trivial as vector bundles: $e_i^{\Gamma,\lambda}=0$ always. So $P(\cdots)$ is just a constant for the integral. Similarly for many of the $n_i^{\Gamma,\lambda}$, but not for all: exactly the terms

$$\sum_{y;\alpha\in\mathrm{Edge},\beta\notin\mathrm{Edge}}[T_y(C^\alpha)\otimes T_y(C^\beta)]$$

in $[N]$ (written additively) are nontrivial bundles. Writing multiplicatively, and putting it in the denominator, we get

$$\int_{\overline{M}^\Gamma}\frac{1}{\prod_{y;\alpha\in\mathrm{Edge},\beta\notin\mathrm{Edge}}(w_F + c_1(T_y(C^\beta)))}$$

(with $F=(y,\alpha)$) as the only part that needs to be integrated.

Now $\overline{M}^\Gamma$ "=" $\left(\prod_v \overline{M}_{0,\mathrm{val}(v)+\#S_v}\right)/\mathrm{Aut}(\Gamma) \mapsto \prod_{v,\mathrm{val}(v)\geq 3}\overline{M}_{0,\mathrm{val}(v)}$, forgetting marked points and the action of $\mathrm{Aut}(\Gamma)$. So this becomes

$$\prod_{v,\mathrm{val}(v)\geq 3}\left\{\left(\sum_{F=(v,\alpha)} w_F^{-1}\right)^{\mathrm{val}(v)-3}\prod_{F=(v,\alpha)} w_F^{-1}\right\} =: \prod_{v,\mathrm{val}(v)\geq 3} (*)$$

by the integral computed above.

Now two remarkable steps recover the vertices of lower valence:

(1) $\sum_{y;\alpha,\beta\in\mathrm{Edge}}[T_y(C^\alpha)\otimes T_y(C^\beta)]$ corresponds to the vertices with valency 2. The contribution (to the denominator) is $\prod_{v,\mathrm{val}(v)=2}(w_{F_1(v)}+w_{F_2(v)})^{-1}$. Since $(\frac{1}{a}+\frac{1}{b})^{-1}\frac{1}{a}\frac{1}{b}=\frac{1}{a+b}$, this gives exactly

$$\prod_{v,\mathrm{val}(v)=2}(*)$$

(2) $-\sum_{\alpha\in\mathrm{Edge}}[H^0(C^\alpha,T_{C^\alpha})]=-\sum_{\alpha\in\mathrm{Edge}}([-w_{F(\alpha)}]+[0]+[w_{F(\alpha)}])$
$=-\sum_{\text{flags }F}[w_F]-(\#\text{edges})[0]$. On the other hand, $\sum_{y,\alpha\in\mathrm{Edge}}[T_y(C^\alpha)]$
$=\sum_{F=(v,\alpha),\mathrm{val}(v)\geq 2}[w_F]$. All in all, we get

$$-\sum_{F=(v,\alpha),\mathrm{val}(v)=1}[w_F]-(\#\text{edges})[0]$$

The term $-(\#\text{edges})[0]$ will cancel out. The rest gives $\prod w_F$, and after all $w_F=(w_F^{-1})((w_F)^{-1})^{-2}$; putting all together, the contribution of $[N^{abs}_{\frac{}{M}\Gamma}]+(\#\text{edges})[0]$ is

$$\text{(A)}\qquad \prod_v\left\{\left(\sum_{F=(v,\alpha)}w_F^{-1}\right)^{\mathrm{val}(v)-3}\prod_{F=(v,\alpha)}w_F^{-1}\right\}$$

Next, we examine $[H^0(C,f^*T_{\mathbb{P}^n})]$. We have a short exact sequence

$$0\to H^0(C,f^*T_{\mathbb{P}^n})\to\oplus_{\alpha\in\mathrm{Edge}}H^0(C^\alpha,f^*T_{\mathbb{P}^n})\to\oplus_v T_{p_{f_v}}\mathbb{P}^n\otimes\mathbb{C}^{\mathrm{val}(v)-1}\to 0$$

(global sections are tuples of sections over edges, that agree at each v for all edges at v). To see what the middle term is, recall that f on C_α is given via

$$X_i(f(z))=z_1^{d_\alpha},X_j(f(z))=z_2^{d_\alpha},X_k(f(z))=0\quad\forall k\neq i,j$$

for $z=(z_1:z_2)$ a coordinate on $\mathbb{P}^1\cong C^\alpha$ mapping $d_\alpha:1$ onto ℓ_{ij}, and the X_i homogeneous coordinates on $\mathbb{P}^n$.

A calculation shows that the following elements form a basis for $H^0(C^\alpha,f^*T_{\mathbb{P}^n})$:

(1) $z^aX_i\frac{\partial}{\partial X_i}\quad -d_\alpha\leq a\leq d_\alpha$;
(2) $z_1^az_2^b\frac{\partial}{\partial X_k}\quad a+b=d_\alpha,0\leq a,b;k\neq i,j(k\in\{1,\ldots,n+1\})$

There is a unique element with trivial T-action: $X_i\frac{\partial}{\partial X_i}$, yielding $(\#\text{edges})[0]$ and giving the promised cancellation.

Now z corresponds to $\frac{\lambda_i-\lambda_j}{d_\alpha}=w_F$ ($F=(v,\alpha)$ with $f(C_V)=p_i$); X_k corresponds to λ_k, z_1 to λ_i/d_α, and z_2 to λ_j/d_α.

The third term in the short exact sequence gives

$$(1-\mathrm{val}(v))\sum_{j\neq f_v}[\lambda_{f_v}-\lambda_j]$$

since the elements $X_{f_v}\frac{\partial}{\partial X_j}$ with $j\neq f_v$ form a basis of $T_{p_{f_v}}\mathbb{P}^n$. In total, we find

(B)

$$\prod_{\alpha\text{ joining }v_1,v_2}\left\{\frac{\left(\frac{d_\alpha}{\lambda_{f_{v_1}}-\lambda_{f_{v_2}}}\right)^{2d_\alpha}}{(-1)^{d_\alpha}((d_\alpha)!)^2}\prod_{k\neq f_{v_1},k\neq f_{v_2}}\prod_{a+b=d_\alpha;a,b\geq 0}\frac{1}{\frac{a}{d_\alpha}\lambda_{f_{v_1}}+\frac{b}{d_\alpha}\lambda_{f_{v_2}}-\lambda_k}\right\}$$
$$\cdot\prod_v\left(\prod_{j\neq f_v}(\lambda_{f_v}-\lambda_j)\right)^{\mathrm{val}(v)-1}$$

Marked points. Claim: the only effect of allowing marked points is that in term (A), $\mathrm{val}(v)$ must be replaced by $\mathrm{val}(v)+\#S_v$. Also, the graphs considered must allow extra tails, corresponding to the markings.

The last contribution (C) we need to consider is the contribution of the vector bundles $\mathcal{E}$ in the three examples: trivial bundles, twisted with suitable characters.

(1) $\prod_v(\lambda_{f_v})^{2\#S_v}$;
(2) The exact sequence

$$0\to H^0(C,f^*\mathcal{O}(5))\to\oplus_{\alpha\in\mathrm{Edge}}H^0(C^\alpha,f^*\mathcal{O}(5))\to\oplus_v\mathcal{O}(5)_{p_{f_v}}\otimes\mathbb{C}^{\mathrm{val}(v)-1}\to 0$$

is used to obtain

$$\prod_{\alpha\text{ joining }v_1,v_2}\left(\prod_{a,b\geq 0,a+b=5d_\alpha}\frac{a\lambda_{f_{v_1}}+b\lambda_{f_{v_2}}}{d_\alpha}\right)\prod_v(5\lambda_{f_v})^{1-\mathrm{val}(v)}$$

(3) Similarly,

$$0\to\oplus_v(\mathcal{O}(-1)_{p_{f_v}}\otimes\mathbb{C}^{\mathrm{val}(v)-1})\to H^1(C,f^*\mathcal{O}(-1))$$
$$\to\oplus_{\alpha\in\mathrm{Edge}}H^1(C^\alpha,f^*\mathcal{O}(-1))\to 0$$

yields

$$\left\{\prod_{\alpha\text{ joining }v_1,v_2}\left(\prod_{a,b<0,a+b=-d_\alpha}\frac{a\lambda_{f_{v_1}}+b\lambda_{f_{v_2}}}{d_\alpha}\right)\cdot\prod_v(-\lambda_{f_v})^{\mathrm{val}(v)-1}\right\}^2$$

The final sum is

$$\sum_{\hat\Gamma}\frac{1}{\mathrm{Aut}(\hat\Gamma)}(A)(B)(C)$$

where $\hat\Gamma$ are the graphs with tails (for marked points), and as mentioned above (A) has $\mathrm{val}(v)+\#S_v$ instead of $\mathrm{val}(v)$ if there are marked points.

This is what Kontsevich writes.

One hitch: we are doing integrals over orbifolds, so we should not forget automorphisms of general elements (maps). The order of this group is $(\#\mathrm{Aut}(\hat\Gamma))\cdot(\prod_{\alpha\in\mathrm{Edge}}d_\alpha)$; we then need to divide by this.

Moral. If you can formulate your favorite counting problem as the computation of the degree of a Chern class on a Kontsevich space $\overline{M}_{g,k}(\mathbb{P}^n, d)$, you have a good chance of reducing it to a sum over graphs. This should at least enable you to calculate the first few cases by computer. Also, physicists and combinatorists have tricks to do sums over trees and graphs.

Examples. First example, $d = 1$. Since $d_\alpha \geq 1$ and $\sum_\alpha d_\alpha = d = 1$, we can only have one edge. There are 4 ways to distribute the 2 marked points, and 3 different labelings: (1)(2), (1)(3), (2)(3). For (1)(2): let w_1 be the flag at 1, etc. So $w_1 = \lambda_1 - \lambda_2$, $w_2 = \lambda_2 - \lambda_1$, and

(A)= $(w_1^{-1})^{\#S_1-1}(w_2^{-1})^{\#S_2-1} = (-1)^{\#S_2-1}$;

(B)= $\frac{-1}{(\lambda_1-\lambda_2)^2}\frac{1}{\lambda_1-\lambda_3}\frac{1}{\lambda_2-\lambda_3}$ (all valencies are 1);

(C)= $(\lambda_1)^{2\#S_1}(\lambda_2)^{2\#S_2}$.

Total for (1)(2):

$$\frac{1}{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}(\lambda_1^4+\lambda_2^4-2\lambda_1^2\lambda_2^2) = \frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$

Now take the corresponding terms for (1)(3) and (2)(3), and discover the nice identity

$$\frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} + \frac{(\lambda_1+\lambda_3)^2}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{(\lambda_2+\lambda_3)^2}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} = 1 \quad ,$$

the number of lines through 2 points.

The second example, for $d = 1$, also involves trees with a single edge. Carefully evaluating each term gives the total

$$\begin{aligned}
&\frac{25\,\lambda_1\,\lambda_2\,(4\,\lambda_1+\lambda_2)\,(3\,\lambda_1+2\,\lambda_2)\,(2\,\lambda_1+3\,\lambda_2)\,(\lambda_1+4\,\lambda_2)}{(\lambda_1-\lambda_3)\,(\lambda_2-\lambda_3)\,(\lambda_1-\lambda_4)\,(\lambda_2-\lambda_4)\,(\lambda_1-\lambda_5)\,(\lambda_2-\lambda_5)}\\
&+\frac{25\,\lambda_1\,\lambda_3\,(4\,\lambda_1+\lambda_3)\,(3\,\lambda_1+2\,\lambda_3)\,(2\,\lambda_1+3\,\lambda_3)\,(\lambda_1+4\,\lambda_3)}{(\lambda_1-\lambda_2)\,(-\lambda_2+\lambda_3)\,(\lambda_1-\lambda_4)\,(\lambda_3-\lambda_4)\,(\lambda_1-\lambda_5)\,(\lambda_3-\lambda_5)}\\
&+\frac{25\,\lambda_2\,\lambda_3\,(4\,\lambda_2+\lambda_3)\,(3\,\lambda_2+2\,\lambda_3)\,(2\,\lambda_2+3\,\lambda_3)\,(\lambda_2+4\,\lambda_3)}{(-\lambda_1+\lambda_2)\,(-\lambda_1+\lambda_3)\,(\lambda_2-\lambda_4)\,(\lambda_3-\lambda_4)\,(\lambda_2-\lambda_5)\,(\lambda_3-\lambda_5)}\\
&+\frac{25\,\lambda_1\,\lambda_4\,(4\,\lambda_1+\lambda_4)\,(3\,\lambda_1+2\,\lambda_4)\,(2\,\lambda_1+3\,\lambda_4)\,(\lambda_1+4\,\lambda_4)}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_3)\,(-\lambda_2+\lambda_4)\,(-\lambda_3+\lambda_4)\,(\lambda_1-\lambda_5)\,(\lambda_4-\lambda_5)}\\
&+\frac{25\,\lambda_2\,\lambda_4\,(4\,\lambda_2+\lambda_4)\,(3\,\lambda_2+2\,\lambda_4)\,(2\,\lambda_2+3\,\lambda_4)\,(\lambda_2+4\,\lambda_4)}{(-\lambda_1+\lambda_2)\,(\lambda_2-\lambda_3)\,(-\lambda_1+\lambda_4)\,(-\lambda_3+\lambda_4)\,(\lambda_2-\lambda_5)\,(\lambda_4-\lambda_5)}\\
&+\frac{25\,\lambda_3\,\lambda_4\,(4\,\lambda_3+\lambda_4)\,(3\,\lambda_3+2\,\lambda_4)\,(2\,\lambda_3+3\,\lambda_4)\,(\lambda_3+4\,\lambda_4)}{(-\lambda_1+\lambda_3)\,(-\lambda_2+\lambda_3)\,(-\lambda_1+\lambda_4)\,(-\lambda_2+\lambda_4)\,(\lambda_3-\lambda_5)\,(\lambda_4-\lambda_5)}\\
&+\frac{25\,\lambda_1\,\lambda_5\,(4\,\lambda_1+\lambda_5)\,(3\,\lambda_1+2\,\lambda_5)\,(2\,\lambda_1+3\,\lambda_5)\,(\lambda_1+4\,\lambda_5)}{(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_3)\,(\lambda_1-\lambda_4)\,(-\lambda_2+\lambda_5)\,(-\lambda_3+\lambda_5)\,(-\lambda_4+\lambda_5)}\\
&+\frac{25\,\lambda_2\,\lambda_5\,(4\,\lambda_2+\lambda_5)\,(3\,\lambda_2+2\,\lambda_5)\,(2\,\lambda_2+3\,\lambda_5)\,(\lambda_2+4\,\lambda_5)}{(-\lambda_1+\lambda_2)\,(\lambda_2-\lambda_3)\,(\lambda_2-\lambda_4)\,(-\lambda_1+\lambda_5)\,(-\lambda_3+\lambda_5)\,(-\lambda_4+\lambda_5)}\\
&+\frac{25\,\lambda_3\,\lambda_5\,(4\,\lambda_3+\lambda_5)\,(3\,\lambda_3+2\,\lambda_5)\,(2\,\lambda_3+3\,\lambda_5)\,(\lambda_3+4\,\lambda_5)}{(-\lambda_1+\lambda_3)\,(-\lambda_2+\lambda_3)\,(\lambda_3-\lambda_4)\,(-\lambda_1+\lambda_5)\,(-\lambda_2+\lambda_5)\,(-\lambda_4+\lambda_5)}\\
&+\frac{25\,\lambda_4\,\lambda_5\,(4\,\lambda_4+\lambda_5)\,(3\,\lambda_4+2\,\lambda_5)\,(2\,\lambda_4+3\,\lambda_5)\,(\lambda_4+4\,\lambda_5)}{(-\lambda_1+\lambda_4)\,(-\lambda_2+\lambda_4)\,(-\lambda_3+\lambda_4)\,(-\lambda_1+\lambda_5)\,(-\lambda_2+\lambda_5)\,(-\lambda_3+\lambda_5)}
\end{aligned}$$

which simplifies to 2875, the number of lines on a quintic threefold.

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11. Equivariant cohomology—P. Belorousski, 12/12/96

The basic reference is [A-B]; also, see [Ginzburg].

General theory. Let G be a compact connected Lie group, and consider its universal principal bundle $EG \to BG$. Here EG is a contractible space, with a *free* (right) G-action (this defines it uniquely up to homotopy), and $BG = EG/G$. One often works with finite–dimensional approximations BG_N, EG_N (which can be chosen smooth and compact).

Let X be a smooth manifold with a smooth left G-action. The following is what is known as the *Borel's mixing construction*. Consider $EG \times X$ (on which G acts freely by $(\alpha, x) \xrightarrow{g} (\alpha g^{-1}, gx)$) and the quotients:

$$\begin{tikzcd}
EG \arrow[d] & EG \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
BG & (EG \times X)/G \arrow[l, "\pi"'] \arrow[r, "\sigma"] & X/G
\end{tikzcd}$$

$X_G := (EG \times X)/G$ is the *homotopy quotient* of X by G.

Definition. The *equivariant cohomology* of X is $H^*_G(X) := H^*(X_G)$.

The map $\pi : X_G \to BG$ is a bundle with fiber X; in the associated Leray spectral sequence, $E_2^{p,q} = H^p(BG, H^q(X)) \implies H^{p+q}(X_G)$. In our case, BG will be simply connected and we will work over $\mathbb{C}$, so by Künneth $H^p(BG, H^q(X)) \cong H^p(BG) \otimes H^q(X)$.

Also, we have a pull-back map $\pi^* : H^*(BG) \to H^*_G(X)$, which gives $H^*_G(X)$ an $H^*(BG)$-module structure. We will write H^*_G for the coefficient ring $H^*(BG) = H^*_G(\mathrm{pt})$.

The map σ is not a fibration. For x a point in X, O_x its orbit, $\sigma^{-1}(O_x) = EG/G_x = BG_x$, where G_x is the stabilizer of x. If G acts freely on X, then σ is a homotopy equivalence, and $H^*_G(X) \cong H^*(X/G)$. In this sense, equivariant cohomology can say something new only if the action of G on X is not free to start with.

We also have the fiber inclusion over the base point of BG, $i : X \hookrightarrow X_G$, and the corresponding pull-back map $i^* : H^*_G(X) \to H^*(X)$. Finally, we have the push-forward $\pi_* : H^*_G(X) \to H^{*-n}_G$; and a pairing

$$\langle \cdot, \cdot \rangle : H^i_G(X) \otimes H^j_G(X) \to H^{i+j-n}_G$$

with $n = \dim X$.

We can say a lot more in the case of the *Hamiltonian actions* on *symplectic manifolds*.

Reminder. A smooth manifold X^{2n} is *symplectic* if it is endowed with a form $\omega \in \Omega^2(X)$ such that

(1) ω is closed;
(2) the symplectic volume form $\omega^n/n!$ is nowhere 0 on X.

This gives $C^\infty(X)$ a Lie algebra structure, with the Poisson bracket $\{f,g\} = df(g)$ as the Lie bracket, after identifying df with a vector field via ω. $C^\infty(X)$ is an extension of the Lie subalgebra $\mathrm{Vect}_\omega(X) \subset \mathrm{Vect}(X)$ of vector fields preserving ω.

An action of a group G on X is *symplectic* if it preserves the form ω. We can ask more: the action of G on X gives a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(X)$, and we can require that there be a lifting

$$\begin{tikzcd} & C^\infty(X) \arrow[d] \\ \mathfrak{g} \arrow[r] \arrow[ur, dashed] & \mathrm{Vect}(X) \end{tikzcd}$$

If such a lifting exists, the action is called *Hamiltonian* (or *Poisson*).

THEOREM. *If X has a Hamiltonian action of G, then the Leray spectral sequence degenerates at the second term. As a consequence, $H^*_G(X)$ is a free H^*_G-module, isomorphic to $H^*(X) \otimes H^*_G$.*

Also, the restriction map $i^* : H^*_G(X) \to H^*(X)$ is surjective. Furthermore, the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate, so we have an isomorphism

$$H^*_G(X) \xrightarrow{\sim} \mathrm{Hom}_{H^*_G}(H^*_G(X), H^*_G)$$

Thus if the action is Hamiltonian we may pick a basis $\{h_1, \ldots, h_m\}$ of the free H^*_G-module $H^*_G(X)$, and get that $\det(\langle h_i, h_j \rangle) \in H^0_G \cong \mathbb{C}$ (since the pairing is nondegenerate, the determinant must be invertible).

We will look at algebraic actions of reductive groups. Fact: they are all Hamiltonian.

Classifying spaces. First, take $G = \mathbb{C}^*$. Then $EG_N = \mathbb{C}^{N+1} \setminus \{0\}$, and $EG_N \to BG_N$ is simply the factor map $\mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$. In the limit, $EG \to BG$ is the map $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^\infty$.

We can think of EG as the complement to the zero section in the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^\infty$. We get $H^*_{\mathbb{C}^*} = H^*(\mathbb{P}^\infty) = \mathbb{C}[\hbar]$, where $\deg \hbar = 2$, $\hbar = c_1(\mathcal{O}(1))$.

For $G = T^n = (\mathbb{C}^*)^n$, $EG = (\mathbb{C}^\infty \setminus \{0\})^n$ and $BG = (\mathbb{P}^\infty)^n$. The coefficient ring is $H^*_{T^n} = \mathbb{C}[\hbar_1, \ldots, \hbar_n]$, with $\hbar_i = c_1(\mathcal{O}_i(1))$.

For $G = \mathrm{GL}(n)$ the finite dimensional approximation is $EG_N = \mathrm{Fr}(n,N)$, the bundle of n-frames in $\mathbb{C}^n$ over the Grassmannian $\mathrm{Gr}(n,N) = BG_N$. In the limit, $EG \to BG$ is $\mathrm{Fr}(n,\infty) \to \mathrm{Gr}(n,\infty)$. $H^*_G = \mathbb{C}[s_1, \ldots, s_n]$, where $s_i = c_i(\mathcal{S}^n)$, with $\mathcal{S}^n$ the universal subbundle over $\mathrm{Gr}(n,\infty)$. Of course $\deg s_i = 2i$. We can think of the s_i's as the elementary symmetric functions (up to sign) in the $\hbar_i$'s:

$$\mathbb{C}[s_1, \ldots, s_n] = \mathbb{C}[\hbar_1, \ldots, \hbar_n]^{S_n}, \qquad s_i = (-1)^i s_i(\hbar).$$

Important example. $T = T^{n+1} = (\mathbb{C}^*)^{n+1} \hookrightarrow \mathrm{GL}(n+1)$ acting on $\mathbb{P}^n$:

$$(t_0, \ldots, t_n) \mapsto \begin{pmatrix} t_0 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & t_n \end{pmatrix}$$

Here EG sits as the complement to the zero section in $E = \mathcal{O}_0(-1) \oplus \cdots \oplus \mathcal{O}_n(-1)$ over $(\mathbb{P}^\infty)^{n+1}$. The homotopy quotient $\mathbb{P}^n_T$ is $\mathbb{P}E$. So

$$\begin{aligned} H^*_T(\mathbb{P}^n) &= \mathbb{C}[\hbar_0, \ldots, \hbar_n; y]/(y^{n+1} + y^n c_1(E) + \cdots + c_{n+1}(E)) \\ &= \mathbb{C}[\hbar_0, \ldots, \hbar_n; y]/(y - \hbar_0) \cdots (y - \hbar_n) \end{aligned}$$

The base ring here is $\mathbb{C}[\hbar_0, \ldots, \hbar_n]$ as seen above, and the module structure is the one obvious from the presentation.

Next note that this action has $(n+1)$ fixed points: $p_k = (0 : \cdots : \overset{k}{1} : \cdots : 0)$. The inclusion $i_k : p_k \hookrightarrow \mathbb{P}^n$ is equivariant, so we have a push-forward on $H^*_T(\ )$:

$$(i_k)_* : \mathbb{C}[\hbar_0, \ldots, \hbar_n] \to \mathbb{C}[\hbar_0, \ldots, \hbar_n; y]/(y - \hbar_0) \cdots (y - \hbar_n)$$

What is this map? It corresponds to the embedding $\mathbb{P}(\mathcal{O}_k(-1)) \hookrightarrow \mathbb{P}(E)$ of the projectivisation of the k-th copy of $\mathcal{O}(-1)$ over $(\mathbb{P}^\infty)^{n+1}$. Now if

$$0 \to S \to E \to Q \to 0$$

is an exact sequence of bundles over X, then $\mathbb{P}(S) \hookrightarrow \mathbb{P}(E)$ is the zero locus of a section of $p^*Q \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$, where p denotes the projection $\mathbb{P}(E) \to X$. So, $[\mathbb{P}(S)] = c_{top}(p^*Q \otimes \mathcal{O}_{\mathbb{P}(E)}(1))$.

Using this, it is straightforward to compute the class of $\mathbb{P}(\mathcal{O}_k(-1))$ and find that under $(i_k)_*$

$$1 \mapsto (y - \hbar_0) \cdots \widehat{(y - \hbar_k)} \cdots (y - \hbar_n)$$

What about the pull-back

$$(i_k)^* : \mathbb{C}[\hbar_0, \ldots, \hbar_n; y]/(y - \hbar_0) \cdots (y - \hbar_n) \to \mathbb{C}[\hbar_0, \ldots, \hbar_n] \qquad ?$$

Here $\hbar_i \mapsto \hbar_i$ for all i; as for the image of y, the $\mathcal{O}_{\mathbb{P}(E)}(1)$ restricts to $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{O}_k(-1))$. This says that $y \mapsto \hbar_k$.

The composition

$$(i_k)^*(i_k)_* : \mathbb{C}[\hbar_0, \ldots, \hbar_n] \to \mathbb{C}[\hbar_0, \ldots, \hbar_n]$$

is given by

$$1 \mapsto (\hbar_k - \hbar_0) \cdots \widehat{(\hbar_k - \hbar_k)} \cdots (\hbar_k - \hbar_n)$$

Remark: this is precisely the equivariant top Chern class of the (equivariant) normal bundle of p_k in $\mathbb{P}^n$.

Localization. Let G act on X (assume that we are in the algebraic situation for simplicity), and let X^G be the fixed point locus. The inclusion $X^G \subset X$ is equivariant, so we have a H^*_G-module restriction homomorphism $H^*_G(X) \to H^*_G(X^G)$.

CLAIM. *This map is injective, and its cokernel is H^*_G-torsion.*

Hence, it is an isomorphism after localization at a suitable $f \in H^*_G$. Since G acts trivially on X^G, we get $H^*_G(X^G) = H^*(X^G) \otimes H^*_G$, and hence

$$H^*_G(X) \cong H^*(X^G) \otimes H^*_G$$

after a suitable localization.

Appendix: The computer program `farsta`—A. Kresch

I. Purpose

[DF-I]: Look at examples.

For simple varieties,

Given associativity relations + several N's,

derive more N's.

`farsta`: Automate this process.

To obtain `farsta`

Go to `http://www.math.uchicago.edu/~kresch`
follow the link to `farsta`
uncompress, untar, and compile by typing `make`

Documentation (view with `more`, print with `lpr`)

`farsta-documentation`
`farsta-examples`

[F-P]: X, $\operatorname{rk} H^*X = m+1$

Get $(m^4 - 2m^3 + 3m^2 - 2m)/8$ associativity relations;

each is an equation of formal power series.

Isolating coefficients, each associativity relation yields a family of relations in N's.

What `farsta` can do:

1. From description of H^*X and K_X, derive associativity relations.
2. Given associativity relation and particular degree,

   derive equation among N's.
3. Substitute known numbers; solve if equation reduces to just one unknown.
4. Store linear relations, and do linear algebra to solve for N's.

II. Example

$$X = \mathbb{P}^3$$
$$H^*X = \mathbb{Z}\langle T_0, T_1, T_2, T_3\rangle$$
$$-K_X = 4T_1$$

Relations:

6 relations $\qquad R1$–$R6$.

E.g.

$$(R2)\ {}^{1}_{2}\!\succ\!\!-\!\!\prec{}^{3}_{2} \qquad \Gamma_{233} = \Gamma_{113}\Gamma_{222} - \Gamma_{112}\Gamma_{223}$$

where

$$\Gamma = \sum_{a+2b=4c} N(c;a,b)\, e^{cy_1}\, \frac{y_2^a}{a!}\, \frac{y_3^b}{b!}.$$

Equating coefficients of $e^{2y_1}\, y_2\, y_3$ yields

$$N(2;2,3) = N(1;0,2)N(1;4,0) - N(1;2,1)^2.$$

Same relation, coefficient of $e^{2y_1}\, y_2^3$:

$$\frac{1}{6}N(2;4,2) = \frac{1}{3}N(1;2,1)N(1;4,0).$$

From these equations, the numbers

$$N(1;0,2) = 1$$
$$N(1;2,1) = 1$$
$$N(1;4,0) = 2$$

determine

$$N(2;2,3) = 1$$
$$N(2;4,2) = 4.$$

Computer:

```
>A P3
dim(P3) = ?         >3
basis H^*(P3) = ?   >y0 y1 y2 y3
rank H^2(P3) = ?    >1
     (complex) codim(y2) = ?         >2
     (complex) codim(y3) = ?         >3
enter classical potential function:
(1/2) y0 2 y3 + y0 y1 y2 + (1/6) y1 3
rank Pic(P3) = ?    >1
      PD(B_1)  in H^2(P3) = ?          >y1
TP3 = tangent bundle on P3
      int_(B_1)  c_1(TP3) = ?          >4

>R 2
(2) (2,3,3,4) 1 G_y2,y3,y3 + 1 G_y1,y1,y2
   G_y2,y2,y3 - 1 G_y1,y1,y3 G_y2,y2,y2 = 0

>N 1 0 2 1
>N 1 2 1 1
>N 1 4 0 2
>E 2 2 1 1
Relation (2)   2 ; 1 1  gives  N(2;2,3) = 1
>E 2 2 3 0
Relation (2)   2 ; 3 0  gives  N(2;4,2) = 4
```


III. Exhaustive search

try lots of E's

```
>A P3
   .
   .     (enter description of P^3).
   .
>N 1 0 2 1
>X 1 6 1 4
```

… tells the computer to try exhaustively:

$*$ relations 1 through 6, (i.e., all)

$*$ curve class $1T_1$ through $4T_1$.

```
Relation (1)  1 ; 0 0  gives  N(1;2,1) = 1
Relation (4)  1 ; 1 0  gives  N(1;4,0) = 2
Relation (2)  2 ; 3 0  gives  N(2;4,2) = 4
Relation (3)  2 ; 2 0  gives  N(2;2,3) = 1
Relation (3)  2 ; 0 1  gives  N(2;0,4) = 0
Relation (4)  2 ; 3 1  gives  N(2;6,1) = 18
Relation (6)  3 ; 5 0  gives  N(2;8,0) = 92
Relation (2)  3 ; 5 1  gives  N(3;6,3) = 190
    ...
Relation (3)  4 ; 0 5  gives  N(4;0,8) = 4
```

IV. Linear algebra:

```
>L1
```

tells the computer to keep linear relationships among N's in memory.

Recursively substitutes and backsubstitutes to determine more N's

Necessary, e.g., for $G(2,4)$.

V. Observations:

1. Specifying cone of effective classes is unimportant —

   `farsta` figures it out and gives zeros for noneffective β (even w/o any input!).

2. Can change canonical class to get new solutions.

3. Can work with H^* of an orbifold — any ring with $\mathbb{Q}$-Poincaré duality works.

4. Choice of basis can have big impact on performance.

VI. Performance (on a Sun SparcSTATION 20):

* $G(2,4)$ $m = 5$ 55 relations
  $deg = 1, 2, 3,$ 4 in 90 seconds.
  —all— most

* $\mathrm{Hilb}_2\,\mathbb{P}^2$ $m = 8$ 406 relations
  $(0,0) \le (d_1, d_2) \le (2,2)$ in 3 minutes;
  $(0,0) \le (d_1, d_2) \le (2,4)$ in 10 hours;
  $(0,0) \le (d_1, d_2) \le (4,4)$ in 45 hours.

* $G(2,5)$ $m = 9$ 666 relations
  — basis of Schubert cycles
  89 out of 139 $d = 1$ N's in 3 seconds;
  775 out of 865 $d = 2$ N's in 5 minutes;
  3478 out of 3608 $d = 3$ N's in 4 hours;
  11512 out of 11682 $d = 4$ N's in 104 hours.

* $G(2,5)$ $m = 9$ — Tom Graber's basis
  About 10 percent faster.

* $G(2,6)$ $m = 14$ 4186 relations
  573 out of 877 $d = 1$ N's in 6 minutes;
  7675 out of 8820 $d = 2$ N's in 36 hours.

* $G(3,6)$ $m = 19$ 14706 relations
  Can't even get started.

VII. Limitations

* No odd cohomology.

* No free parameters —

  numbers only (rational, multi-precision numerator and denominator).

* Memory limitations —

  Linear expression buffer tends to fill up.

* Practical limit on $m = \mathrm{rk}\, H^*X - 1$:

$m < 10$	usually fine;
$10 \le m \le 15$	questionable;
$m > 15$	probably hopeless.

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