Belyi functions for hyperbolic hypergeometric-to-Heun transformations

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Abstract

One place where Belyi functions occur is pullback transformations of hypergeometric differential equations to Fuchsian equations with few singularities. This paper presents a complete classification of Belyi functions for transforming certain hypergeometric equations to Heun equations (i.e., canonical Fuchsian equations with 4 singularities). The considered hypergeometric equations have the local exponent differences $1/k, 1/\ell, 1/m$ that satisfy $k, \ell, m \in \mathbb{N}$ and the hyperbolic condition $1/k + 1/\ell + 1/m < 1$. In total, we find 872 such Belyi functions up to Möbius transformations, in 366 Galois orbits. Their maximal degree is 60, which is well beyond reach of standard computational methods. To obtain these Belyi functions, we developed two efficient algorithms that exploit the implied hypergeometric-to-Heun transformations.

1 Introduction

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Belyi functions and *dessins d'enfants* [24] is a captivating field of research in algebraic geometry, complex analysis, Galois theory and related fields. However, computation of Belyi functions of degree over 20 is still considered hard [12, Example 2.4.10] even for genus 0 Belyi coverings $\mathbb{P}^1 \to \mathbb{P}^1$. This computational difficulty promises a long lasting appeal, both for constructivists and theoreticians. Grothendieck [8, pg. 248] doubted that *"there is a uniform method for solving the problem by computer"*. The subject of this paper is effective computation of certain Belyi functions $\mathbb{P}^1 \to \mathbb{P}^1$, of degree up to 60, utilizing the fact that those functions transform Fuchsian differential equations with a small number of singularities.

In this paper, a *Belyi function*[1] (or a *Belyi covering*) is a rational function $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ that branches only in the 3 fibers $z = \varphi(x) \in \{0, 1, \infty\}$. We

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[1]Generally, a Belyi function is a map $\varphi : S \to \mathbb{P}^1_z$ from a Riemann surface S to $\mathbb{P}^1(\mathbb{C})$ that only branches in the fibers $z \in \{0, 1, \infty\}$. Definitions 1.1–1.3 could be applied to general Belyi maps $S \to \mathbb{P}^1_z$. Often a Belyi function is defined by requiring branching in at most 3 fibers rather than in the fixed fiber set $\{0, 1, \infty\}$. Although any 3 fibers can be moved to $\{0, 1, \infty\}$ by a Möbius transformation of $\mathbb{P}^1$, the definition difference is significant in §4.5.

distinguish the two projective lines by the indices x, z just as in [22]. To describe the Belyi functions we classify, we introduce the following definitions.

Definition 1.1. *Given positive integers k, ℓ, m, a Belyi function $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is called* (k, ℓ, m)-regular *if all points above $z = 1$ have branching order k, all points above $z = 0$ have branching order ℓ, and all points above $z = \infty$ have branching order m.*

Examples of $(2, 3, m)$-regular Belyi functions with $m \in \{3, 4, 5\}$ are the well-known Galois coverings $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 12, 24, 60 with the tetrahedral A_4, octahedral S_4 or icosahedral A_5 monodromy groups, respectively. Platonic solids give their dessins d'enfant [15].

Definition 1.2. *Given yet another positive integer n, a Belyi function $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is called* (k, ℓ, m)-minus-n-regular *if, with exactly n exceptions, all points above $z = 1$ have branching order k, all points above $z = 0$ have branching order ℓ, and all points above $z = \infty$ have branching order m. We will also use the shorter term* (k, ℓ, m)-minus-n.

Examples of (k, ℓ, m)-minus-2 functions are quotients of the just mentioned Galois coverings by a cyclic monodromy group.If $1/k+1/\ell+1/m > 1$ and $n \geq 3$, there are (k, ℓ, m)-minus-n Belyi functions of arbitrary high degree. They give Kleinian pull-back transformations [10, 18] to second order Fuchsian equations with finite monodromy (i.e., a basis of algebraic solutions) from a few standard hypergeometric equations. An example of a $(2, 3, 5)$-minus-3 Belyi function of degree 1001 is given online at [17, NamingConvention]. As Remark 5.1 here shows, (k, ℓ, m)-minus-1-regular Belyi functions exist only if $1 \in \{k, \ell, m\}$.

Definition 1.3. *A Belyi function φ is called* minus-n-hyperbolic *if:*

(i) there are positive integers k, ℓ, m satisfying $1/k + 1/\ell + 1/m < 1$ (the hyperbolic condition*) such that φ is (k, ℓ, m)-minus-n-regular;*

(ii) there is at least one point of branching order k above $z = 1$, a point of order ℓ above $z = 0$, and a point of order m above $z = \infty$.

Minus-3-hyperbolic Belyi functions are listed in [20, §9]. Table 3 in [20] lists nine[2] Galois orbits of such Belyi functions, of degree 6, 8–10, 12, 18 or 24.

This paper gives all *minus-4-hyperbolic Belyi functions* $\mathbb{P}^1 \to \mathbb{P}^1$. The motivation is that they give transformations of Gauss hypergeometric differential equations without Liouvillian [24] solutions to Heun equations (i.e., Fuchsian equations with 4 singularities). This allows to express non-Liouvillian Heun functions in terms of better understood Gauss hypergeometric functions. The application to these transformations of Fuchsian equations is discussed in §5. This paper, combined with the list of *parametric* hypergeometric-to-Heun transformations in [23], covers all non-Liouvillian cases of hypergeometric-to-Heun transformations.

We used two algorithms to compute the minus-4-hyperbolic Belyi functions. They both utilize the fact that these Belyi functions give hypergeometric-to-Heun transformations. One algorithm is probabilistic and uses modular lifting.

[2] Minus-3-hyperbolic Belyi functions give rise to the hypergeometric transformations described in [20, §9]. There are 10 different such Belyi functions up to Möbius transformations, in 9 Galois orbits. The degree 18 Belyi function there is defined over $\mathbb{Q}(\sqrt{-7})$.

It exploits the fact that Heun's equation is represented by few parameters. The other algorithm is deterministic, and uses existence of a hypergeometric-to-Heun transformation to get more algebraic equations for the (a priori) undetermined coefficients of a Belyi function.

The branching patterns are enumerated in §3, following the approach from [22]. Section 4 discusses special *obstructed* cases of encountered Belyi functions. The application to hypergeometric-to-Heun transformations is explained in §5. Our algorithms are presented in §6. The Appendix sections give ordered lists A–J of computed Belyi functions, discusses composite Belyi functions, and compares our results with Felixon's list [7] of *Coxeter decompositions* in the hyperbolic plane. All dessins d'enfant of computed Belyi coverings are depicted in this paper, most of them next to the A–J tables of §B. Our list of dessins is larger than [1, 4, 13] combined.

2 Organizing definitions, examples

We start with a few definitions that will help us to organize the list of Belyi functions. Then we take a relaxed look at a few examples, including those of the largest degree 60. At the same time, dessins d'enfant are introduced more fully, setting a geometric tone of our presentation.

Definition 2.1. *Let φ be a (k,ℓ,m)-minus-n-regular Belyi function for some n. The* regular branchings *of φ are the points above $z=1$ of order k, the points above $z=0$ of order ℓ, and the points above $z=\infty$ of order m. The other n points in the three fibers are called* exceptional points *of φ. A* branching fraction *of φ is a rational number A/B, where A is a branching order at an exceptional point Q, and $B \in \{k,\ell,m\}$ is the prescribed branching order for the fiber of Q.*

Definition 2.2. *Let $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ be a (k,ℓ,m)-minus-4 Belyi function. Let q_1,q_2,q_3,q_4 denote its exceptional points. The j-*invariant *of φ is the j-invariant of the elliptic curve $Y^2 = \prod_{q_i\neq\infty}(X-q_i)$, given by formula (2.2) below. It is invariant under Möbius transformations of $\mathbb{P}^1_x$.*

A canonical form *of φ is a composition of φ with a Möbius transformation that has three exceptional points at $x=0,1,\infty$. The fourth exceptional point then becomes $x=t$, where t is a cross-ratio of q_1,q_2,q_3,q_4. The cross-ratio depends on the order of q_1,q_2,q_3,q_4, and there is an S_3-orbit ($S_3 \cong S_4/V_4$)*

$$\left\{t,1-t,\frac{t}{t-1},\frac{1}{t},\frac{1}{1-t},1-\frac{1}{t}\right\} \tag{2.1}$$

*of related cross-ratios. Any of these values is a t-*value *of φ. The j-invariant is*

$$j(t) = \frac{256\,(t^2-t+1)^3}{t^2(t-1)^2}. \tag{2.2}$$

As an example, $t \in \{-1,2,\frac{1}{2}\}$ gives $j=1728$. If $j \notin \{0,1728\}$ then the six t-values in above S_3-orbit are distinct. The S_3-action gives a homomorphism $S_3 \to \mathrm{Gal}(\mathbb{Q}(t)/\mathbb{Q}(j))$, hence $[\mathbb{Q}(t):\mathbb{Q}(j)] \in \{1,2,6\}$.

Definition 2.3. *The t-*field *resp. j-*field *of φ is the number field generated by a t-value resp. the j-invariant. The r-field (*canonical realization field*) of φ is* *the smallest field over which a canonical form of φ is defined. These fields do not depend on the ordering of the* 4 *exceptional points.*

Example 2.4. *The degree 12 rational function*

$$\varphi(x) = \frac{64x^2(x-3)^9(x-9)}{27(x-1)(8x^3-72x^2-27x+27)^3}$$

is a $(2,3,9)$*-minus-*4 *Belyi map. Indeed, the numerator of* $\varphi(x)-1$ *is a full square. It is already in a canonical form, as* $x=0$*,* $x=1$ *and* $x=\infty$ *(of branching order 2) are exceptional points. The fourth exceptional point* $x=9$ *is a t-value. The j-invariant is equal to* $2^2 73^3/3^4$ *by formula* (2.2)*.*

The branching pattern *of* φ *is given by three partitions of the degree* $d=12$ *into branching orders above* $0,1,\infty$*. Using the notation in [22], we express the* branching pattern *of* φ *shortly as follows:*

$$6\,[2] = 3\,[3]+2+1 = [9]+2+1.$$

The prescribed branching orders are indicated with square brackets, with their multiplicity in front. The 4 branching orders that are not enclosed in square brackets represent the 4 exceptional points. Dividing them by their prescribed branching order(s) produces the 4 branching fractions: $1/3, 2/3, 1/9, 2/9$.

In the application setting of hypergeometric-to-Heun transformations in §5, the regular branchings will become *regular points* (after a proper projective normalization) of the pulled-back Heun equation H; the exceptional points will be the *singularities* of H; and the branching fractions will be the *exponent differences* of H. The exponent differences of the hypergeometric equation under transformation will be $1/k, 1/\ell, 1/m$. Example 2.4 will be continued in §5.

Definitions 2.2, 2.3 will be used to group the obtained Belyi functions into manageable classes. The Belyi functions will be listed twice in this paper. The first list is Tables 2.3.7–3.4.4 of §3. Its ordering by the (k,ℓ,m)-triples and branching patterns reflects the classification scheme. In Appendix §B, the list of Galois orbits is grouped and ordered by the j-fields, t-fields, branching fractions. This order allows quick recognition whether a given Heun function is reducible to a hypergeometric function with a rational argument φ.

Belyi functions nicely correspond to certain graphs called *dessins d'enfant*[3]. Mimicking [4, Section 2], we spell out standard correspondences for genus 0 Belyi functions as follows. There are 1-1 correspondences between these objects:

(I) Belyi functions $\mathbb{P}^1_z \to \mathbb{P}^1_x$, up to Möbius transformations $x \mapsto (ax+b)/(cx+d)$.

[3]Generally, a *dessin d'enfant* [24] is a bi-colored graph (possibly with multiple edges), with a cyclic order of edges around each vertex given. This defines a unique (up to homotopy) embedding of the bi-colored graph into a Riemann surface. Customarily, the vertex colors are black and white. The dessins d'enfant of genus 0 Belyi coverings can be drawn on a plane, as the Riemann sphere minus a point is homeomorphic to a plane. Given a Belyi covering φ, its dessin d'enfant is realized as the pre-image of the interval segment $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ onto its Rieman surface, with the vertices above $z=0$ colored black and the vertices $z=1$ colored white. The branching pattern of φ determines the degrees (i.e., valencies) of vertices of both colors of its dessin d'enfant, and the degrees of cells on the Riemann surface. The cell degree is determined by counting vertices of one color while tracing its boundary. The degree of a dessin d'enfant is the degree of the corresponding Belyi function.

(II) Plane dessins d'enfant, up to homotopy on the Riemann sphere.

(III) The triples (g_0, g_1, g_∞) of elements in a symmetric group S_d, such that:

- $g_0 g_1 g_\infty = 1$;
- the total number of cycles in g_0, g_1, g_∞ is equal to $d + 2$;
- g_0, g_1, g_∞ generate a transitive action on a set of d elements;

up to simultaneous conjugacy of g_0, g_1, g_∞ in S_d.

(IV) Field extensions of $\overline{\mathbb{Q}}(z)$ of genus 0, unramified outside $z = 0, 1, \infty$.

Part (III) gives the monodromy presentation of a Belyi covering, and d is the degree. The dessins d'enfant is basically a graphical representation of the combinatorial data in (III). This paper presents all obtained dessins pictorially, while the accompanying website [17] gives the Belyi maps (I), the permutations in (III) and other data (such as j, t, r-fields). The number $d + 2$ is illuminated in the proof of Lemma 3.1. For each fiber $z \in \{0, 1, \infty\}$, the conjugacy class of g_z is determined by the partition of d that reflects the branching pattern in the fiber. Part (IV) is convenient for considering the composition structure of Belyi maps; see Appendix C.

The considered Belyi functions have rather regular dessins d'enfant. Definitions 1.1–1.3 are easy to reformulate for dessins d'enfant;

Definition 2.5. *A dessin d'enfant is called* (k, ℓ, m)-minus-n-regular *if, with exactly n exceptions, all white vertices have degree k, all black vertices have degree ℓ, and all cells have degree m.*

Definition 2.6. *A dessin d'enfant Γ is called* minus-n-hyperbolic *if:*

(i) there are positive integers k, ℓ, m satisfying $1/k + 1/\ell + 1/m < 1$ such that Γ is (k, ℓ, m)-minus-n-regular;

(ii) there is at least one white vertex of degree k, a black vertex of degree ℓ, and a cell of degree m.

All minus-4-hyperbolic dessins d'enfants could be found by a combinatorial search on a computer. But with our Maple implementations it was faster to compute first the minus-4-hyperbolic Belyi functions, and then compute their monodromy permutations in (III). This paper presents all minus-4-hyperbolic dessins (up to complex conjugation), most of them next to the tables of Appendix §B.

In total, there are 872 Belyi functions of the minus-4-hyperbolic type, up to Möbius transformations in both x and z. They come in 366 Galois orbits[4]. In leap years we could decorate a calendar with the minus-4-hyperbolic dessins d'enfant, one Galois orbit per day. We categorize and label the Galois orbits of the objects in (I)–(IV) as A1–J28; see §3.1 and Appendix §B. The largest Galois orbit J28 has 15 dessins, for a $(2, 3, 7)$-minus-4 branching pattern of degree 37. Completeness is checked with two independent algorithms and other checks, see §6 and Appendix §D.

[4]Belyi functions are explicitly defined over algebraic number fields, and the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes Belyi coverings with the same branching pattern. The size of a Galois orbit of dessins d'enfant is the degree of the moduli field; see §4. Given a branching pattern, the set of Belyi coverings with that branching pattern is finite (up to Möbius transformations), possibly empty. The Galois action does not need to be transitive on this set, and several Galois orbits with the same branching pattern may appear.

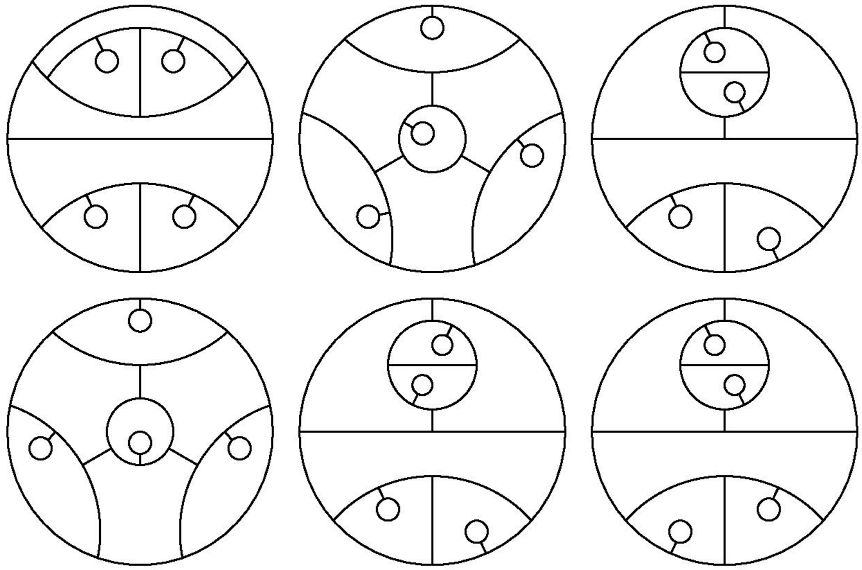


Figure 1: The degree 60 dessins d'enfant

The highest degree of a minus-4-hyperbolic Belyi function is 60. Its branching pattern is $30\,[2] = 20\,[3] = 8\,[7] + 1 + 1 + 1 + 1$. There are two Galois orbits for this branching pattern, with three dessins each. We identify the two Galois orbits as H14 and H46. The dessins d'enfant for these Belyi functions are depicted[5] in Figure 1. The 4 exceptional points in each dessin are represented by circular loops; they could be assumed to lie in the center of each cell of degree 1. The other cells (including the outer ones) have degree 7. The left-most dessins of H14 and H46 clearly have a reflection symmetry, hence they are defined over $\mathbb{R}$. The other two dessins of H46 are mirror images of each other, and are related by the complex conjugation.

The Belyi functions of degree 60 are composite. H46 is a composition of a degree 15 Belyi function H45 with two consecutive quadratic substitutions. The intermediate Belyi function J19 of degree 30 has a Galois orbit of size 9, while H45 is defined over the same cubic field as H46. The branching fractions of H45 are $1/2, 1/2, 1/2, 1/7$. The dessins d'enfant for the component functions H45 and J19 are depicted in Figure 2.

The covering H14 is a composition of degree 30 Belyi function H10 with a quadratic substitution. The functions H10 and J19 have the same branching fractions $1/2, 1/2, 1/7, 1/7$. The Belyi functions H10, H14 are examples that

[5]The dessins in Figure 1 have all white vertices of order 2, hence they are examples of *clean* dessins d'enfant. It is customary to depict clean dessins without white vertices, so that edges connect black vertices directly, and loops are possible. A white vertex is then implied in the middle of each edge. Our policy of drawing dessins is the following. White vertices of order 2 are not shown, but the edges going through them are drawn thick. Other white vertices are shown, but the incident edges are drawn thin. A black vertex of degree ≥ 2 is not drawn (as it is a clear branching point), unless it is incident to a thin edge. Take a look at Figure 2 to see effects of this policy. Figures 1, 2 depict all dessins d'enfant for H14, H45, H46, J19, including complex-conjugated pairs. In other pictures, we will not draw dessins whose complex conjugates are already present.

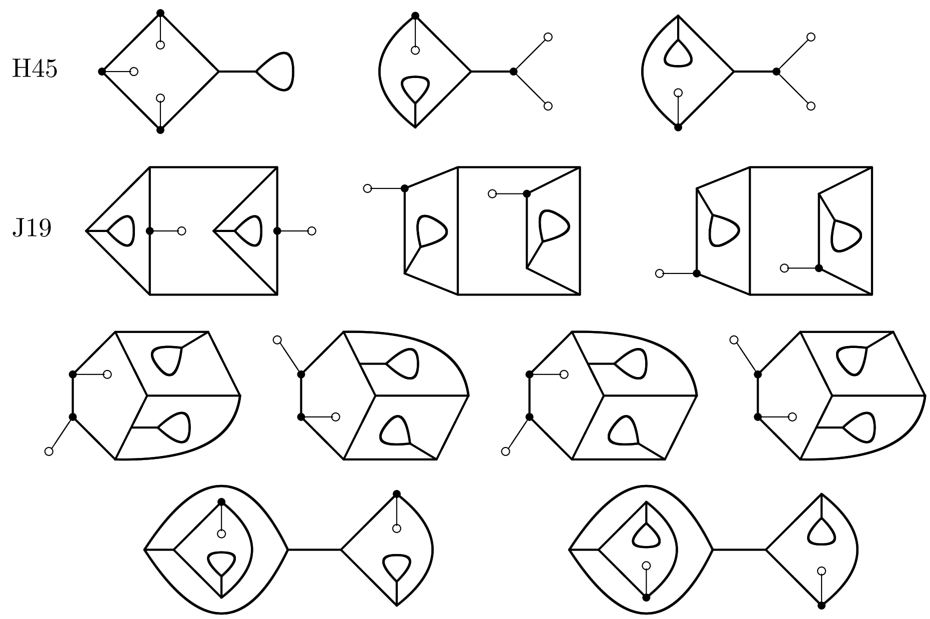


Figure 2: The degree 15 and 30 components of H46.

have an *obstruction*, as described in §4. This has interesting geometric consequences for the dessins d'enfant. Although both have a totally real moduli field $\mathbb{Q}(\cos\frac{2\pi}{7})$, not all dessins of H10 and H14 have a reflection symmetry. Rather, the complex conjugation may give a homeomorphic dessin, identifiable with the original only after an automorphism of the Riemann sphere. For example, consider the middle and the right-most dessins of H14 in Figure 1. The dessins d'enfant for H10 are depicted in Figure 3, together with most of other examples with an obstruction.

3 The branching patterns

We enumerate the possible branching patterns in the same way as was done for *parametric* hypergeometric-to-Heun transformations in [22]. To end up with a finite number of cases, we use Hurwitz formula and the hyperbolic condition $1/k + 1/\ell + 1/m < 1$. Without loss of generality, we assume the non-decreasing order $k \le \ell \le m$ for the regular branching orders from now on.

Lemma 3.1. *Let φ be a minus-4-hyperbolic Belyi covering of degree d, with the regular branching orders $k \le \ell \le m \in \mathbb{Z}_{>0}$. Then*

(i) There are exactly $d-2$ regular branchings and 4 exceptional points.

(ii) $d - \left\lfloor \dfrac{d}{k} \right\rfloor - \left\lfloor \dfrac{d}{\ell} \right\rfloor - \left\lfloor \dfrac{d}{m} \right\rfloor \le 2.$

(iii) Let S denote the sum of 4 branching fractions. Then $d = \dfrac{2-S}{1-\frac{1}{k}-\frac{1}{\ell}-\frac{1}{m}}$.

(iv) $\left(1 - \frac{1}{k} - \frac{1}{\ell}\right) m^2 - 3m + 4 \le 0.$

(v) $\frac{1}{2} \le \frac{1}{k} + \frac{1}{\ell} < 1.$

Proof. By Hurwitz formula (or [20, Lemma 2.5]), there are $3d-(2d-2) = d+2$ distinct points above $\{0, 1, \infty\}$ when φ is a Belyi map $\mathbb{P}^1 \to \mathbb{P}^1$. The first claim follows. The number of regular branchings is at most $\lfloor d/k \rfloor + \lfloor d/\ell \rfloor + \lfloor d/m \rfloor$. This gives the inequality in *(ii)*. The number $d-2$ of regular branchings is also equal to $d/k + d/\ell + d/m - S$, giving the degree formula in *(iii)*.

We have $d \ge m$, otherwise condition *(ii)* of Definition 1.2 is not satisfied. Combining this with the degree formula gives the inequality in *(iv)*. Together with $m \ge 4$, the inequality in *(iv)* gives $1-1/k-1/\ell \le 1/2$. Part *(v)* follows. $\square$

The inequalities in *(iv)*, *(v)* gives a finite list of triples (k, ℓ, m). Setting $S = 4/m$ in part *(iii)* gives an upper bound for d, leaving the following candidates for (k, ℓ, m, d):

$$\begin{array}{cccc}
(2,3,7,\le 60), & (2,3,8,\le 36), & (2,3,9,\le 28), & (2,3,10,\le 24),\\
(2,3,11,\le 21), & (2,3,12,\le 20), & (2,3,13,\le 18), & (2,3,14,\le 18),\\
(2,3,15,\le 17), & (2,3,16,\le 16), & (2,4,5,\le 24), & (2,4,6,\le 16),\\
(2,4,7,\le 13), & (2,4,8,\le 12), & (2,4,9,\le 11), & (2,4,10,\le 10),\\
(2,5,5,\le 12), & (2,5,6,\le 10), & (2,5,7,\le 9), & (2,5,8,\le 8),\\
(2,6,6,\le 8), & (2,6,7,\le 7), & (3,3,4,\le 12), & (3,3,5,\le 9),\\
(3,3,6,\le 8), & (3,3,7,\le 7), & (3,4,4,\le 6), & (3,4,5,\le 5),\\
(4,4,4,\le 4). & & &
\end{array}$$

The last two candidates give less than 4 exceptional points.

Given a candidate tuple (k, ℓ, m, d), it is straightforward to enumerate the correspondent branching patterns. Let h_1, h_0, h_∞ denote the eventual number of regular branchings in the fibers $z = 0, 1, \infty$, respectively. Then we have $h_1 + h_0 + h_\infty = d - 2$ by Lemma 3.1*(i)*, and $0 < h_1 \le \lfloor d/k \rfloor$, etc. With a possible integer solution (h_1, h_0, h_∞) at hand, we have to partition the numbers $d - kh_1$, $d - \ell h_0$, $d - mh_\infty$ into total 4 positive parts, not equal to the respective regular orders k, ℓ, m. For example, if $(k, \ell, m, d) = (2, 3, 7, 28)$ then we are lead to partition either 7 into 3 parts (in the $m = 7$ fiber) or 4 into 4 parts (in the $\ell = 3$ fiber). There are four such partitions of 7 and one of 4, hence five branching patterns.

In total, there are 378 branching patterns[6] for minus-4-hyperbolic Belyi functions. We list them in the first two columns of Tables 2.3.7–3.4.4, by giving their branching fractions and the degree. The table titles refer to the tuple (k, ℓ, m). The branching fractions are left unsimplified (e.g. 4/8 instead of 1/2) to keep the fibers and branching orders of exceptional points plainly visible. The branching patterns are uniquely determined by the unsimplified branching fractions.

[6] One branching pattern $6\,[3] = 9\,[2] = 8 + 7 + 1 + 1 + 1$ is counted twice. It appears in Tables 2.3.7 and 2.3.8 because it is $(2, 3, m)$-minus-4-regular for two values of m (=7 or 8). It turns out, however, there are no Belyi functions with this branching pattern.

3.1 Summary of computed results

The last two columns of Tables 2.3.7–3.4.4 give information about found coverings with the considered branching patterns. Computation of Belyi coverings for each possible branching pattern is the most demanding step. The algorithms used to generate and verify the list of Belyi functions are presented in Section §6.

The third column of Tables 2.3.7–3.4.4 gives a label for every Galois orbit with the branching pattern defined by a sequence of 4 branching fractions in the first column. With the application to Heun equations in mind, we group the Belyi functions by the $\mathbb{Q}$-extension of the j-invariant. The cases with $j \in \mathbb{Q}$ are further grouped by the t-field. We group the computed minus-4-hyperbolic Belyi functions into the following 10 classes, labeled alphabetically from A to J:

A1–A24:	the Belyi functions with $j = 1728$, that is $t \in \{-1, 2, 1/2\}$;
B1–B34:	the other Belyi functions with $t \in \mathbb{Q}$;
C1–C42:	the Belyi functions with $j \in \mathbb{Q}$ and a real quadratic t-field;
D1–D50:	the Belyi functions with $j \in \mathbb{Q}$ and an imaginary quadratic t-field;
E1–E25:	the Belyi functions with $j \in \mathbb{Q}$, when the t-field is the splitting field (of degree 6 always) of a cubic polynomial in $\mathbb{Q}[z]$;
F1–F25:	the Belyi functions with a real quadratic j-field;
G1–G52:	the Belyi functions with an imaginary quadratic j-field;
H1–H53:	the Belyi functions with a cubic j-field;
I1–I33:	the Belyi functions with a j-field of degree 4 or 5;
J1–J28:	the Belyi functions with a j-field of degree at least 6 (and ≤ 15).

In each class, the Galois orbits of Belyi functions are ordered by the criteria described in Appendix §A. A numbered label refers to a whole Galois orbit of Belyi functions (or dessins d'enfant), as already mentioned in §2. If there is more than one Galois orbit with the same branching pattern, a line is devoted to each Galois orbit in Tables 2.3.7–3.4.4.

The last column of the same tables gives basic information about the size of Galois orbits, j-fields, t-fields of the computed Belyi functions. The j-field is indicated as follows:

- by the field degree d, in the power notation j^d;
- if the field is quartic, $j^4(a, b, c)$ conveys a minimal field polynomial $X^4 + aX^2 + bX + c$;
- if the field is cubic, $j^3(a, b)$ conveys a minimal field polynomial X^3+aX+b;
- if the field is quadratic, $j^2(\sqrt{a})$ means the field $\mathbb{Q}(\sqrt{a})$;
- if $j = 0$, it is stated so;
- for $j \in \mathbb{Q} \setminus \{0\}$, no j-notation is given, but the t-field and (possibly) the moduli field are indicated.

The t-field is specified as follows:

- if the j-field is indicated, the t-field degree d is given (in the power notation t^d) only if $j \neq 0$ and the t-field is an extension of the j-field;

Table 2.3.7.

1/7,1/7,1/7,1/7	60	H14	$j^3(-7,7)$, t^6
		H46	$j^3(-7,14), t^{18}$
1/7,1/7,1/7,2/7	54	D28	$t(\sqrt{-3})$
1/3,1/7,1/7,1/7	52	D35	$t(\sqrt{-5})$
1/7,1/7,1/7,3/7	48	—	no covering
1/7,1/7,2/7,2/7		A21	$t=-1$
		I5	$j^4(-7,0,14), t^8$
1/3,1/7,1/7,2/7	46	J7	j^6, t^{12}
1/2,1/7,1/7,1/7	45	G44	$j^2(\sqrt{-7})$, t^{12}
1/3,1/3,1/7,1/7	44	H11	$j^3(-7,7)$, t^6
		J26	j^{13}, t^{26}
1/7,1/7,1/7,4/7	42	G33	$j^2(\sqrt{-7})$, t^4
1/7,1/7,2/7,3/7		—	no covering
1/7,2/7,2/7,2/7		—	no covering
1/3,1/7,1/7,3/7	40	J16	j^7, t^{14}
1/3,1/7,2/7,2/7		G34	$j^2(\sqrt{-7})$, t^4
1/2,1/7,1/7,2/7	39	G43	$j^2(\sqrt{-7})$, t^4
1/3,1/3,1/7,2/7	38	J13	j^7, t^{14}
2/3,1/7,1/7,1/7		I6	$j^4(-7,0,14), t^{24}$
1/2,1/3,1/7,1/7	37	J28	j^{15}, t^{30}
1/3,1/3,1/3,1/7	36	I14	$j^4(0,14,21), t^{24}$
1/7,1/7,1/7,5/7		—	no covering
1/7,1/7,2/7,4/7		A22	$m^2(\sqrt{-7}), t{=}-1$
		D40	$t(\sqrt{-7})$
1/7,1/7,3/7,3/7		B12	$t=3^2$
		I28	j^5, t^{10}
1/7,2/7,2/7,3/7		—	no covering
2/7,2/7,2/7,2/7		A23	$t=-1$
		E22	$t^{\mathrm{spl}}(9,2)$
1/3,1/7,1/7,4/7	34	D17	$t(\sqrt{-2})$
1/3,1/7,2/7,3/7		J10	j^6
1/3,2/7,2/7,2/7		—	no covering
1/2,1/7,1/7,3/7	33	I32	j^5, t^{10}
1/2,1/7,2/7,2/7		G41	$j^2(\sqrt{-7})$, t^4
1/3,1/3,1/7,3/7	32	A10	$t=-1$
		F21	$j^2(\sqrt{7})$, t^4
1/3,1/3,2/7,2/7		C6	$t(\sqrt{3})$
		I23	j^5, t^{10}
2/3,1/7,1/7,2/7		G32	$j^2(\sqrt{-7})$
1/2,1/3,1/7,2/7	31	J24	j^{13}
1/2,1/2,1/7,1/7	30	H10	$j^3(-7,7)$, t^6
		J19	j^9, t^{18}
1/3,1/3,1/3,2/7		—	no covering
1/3,2/3,1/7,1/7		J6	j^6, t^{12}
1/7,1/7,1/7,6/7		D22	$j=0$
1/7,1/7,2/7,5/7		D5	$t(\sqrt{-1})$
1/7,1/7,3/7,4/7		—	no covering
1/7,2/7,2/7,4/7		B13	$t=3^2$
1/7,2/7,3/7,3/7		C18	$t(\sqrt{5})$
2/7,2/7,2/7,3/7		D23	$j=0$
1/2,1/3,1/3,1/7	29	J27	j^{14}, t^{28}
1/3,1/3,1/3,1/3	28	G36	$j^2(\sqrt{-7})$, t^4
		H13	$j^3(-7,7)$, t^6
1/3,1/7,1/7,5/7		I8	$j^4(0,3,3)$, t^8
1/3,1/7,2/7,4/7		B16	$t=3^3/2$
1/3,1/7,3/7,3/7		—	no covering
1/3,2/7,2/7,3/7		—	no covering
1/2,1/7,1/7,4/7	27	D46	$t(\sqrt{-15})$
1/2,1/7,2/7,3/7		B2	$m^2(\sqrt{-7}), t{=}2^2$
		I25	j^5
1/2,2/7,2/7,2/7		—	no covering
1/3,1/3,1/7,4/7	26	J12	j^6, t^{12}
1/3,1/3,2/7,3/7		H51	$j^3(4,2)$, t^6
2/3,1/7,1/7,3/7		C12	$t(\sqrt{3})$
2/3,1/7,2/7,2/7		D47	$t(\sqrt{-15})$
1/2,1/3,1/7,3/7	25	J23	j^{11}
1/2,1/3,2/7,2/7		H43	$j^3(2,2)$, t^6
1/2,1/2,1/7,2/7	24	I4	$j^4(-7,0,14)$, t^8
1/3,1/3,1/3,3/7		D20	$m^2(\sqrt{-3}), j{=}0$
1/3,2/3,1/7,2/7		B6	$m^2(\sqrt{-3}), t{=}3^2$
		I2	$j^4(0,0,7)$
1/7,1/7,2/7,6/7		—	no covering
1/7,1/7,3/7,5/7		F24	$j^2(\sqrt{21})$, t^4
1/7,1/7,4/7,4/7		G40	$j^2(\sqrt{-7})$, t^4
1/7,2/7,2/7,5/7		—	no covering
1/7,2/7,3/7,4/7		—	no covering
1/7,3/7,3/7,3/7		—	no covering
2/7,2/7,2/7,4/7		—	no covering
2/7,2/7,3/7,3/7		F20	$j^2(\sqrt{7})$, t^4
1/2,1/3,1/3,2/7	23	J14	j^7, t^{14}
1/2,2/3,1/7,1/7		J17	j^7, t^{14}
1/2,1/2,1/3,1/7	22	J25	j^{13}, t^{26}
1/3,1/3,2/3,1/7		J15	j^7, t^{14}
1/3,1/7,1/7,6/7		H36	$j^3(9,2)$, t^6
1/3,1/7,2/7,5/7		B34	$t=2^4 19^2/7^4$
1/3,1/7,3/7,4/7		H52	$j^3(-3,9)$
1/3,2/7,2/7,4/7		G42	$j^2(\sqrt{-7})$, t^4
1/3,2/7,3/7,3/7		—	no covering
1/2,1/3,1/3,1/3	21	I9	$j^4(0,3,3)$, t^{24}
1/2,1/7,1/7,5/7		H44	$j^3(2,2)$, t^6
1/2,1/7,2/7,4/7		G30	$j^2(\sqrt{-7})$
1/2,1/7,3/7,3/7		C36	$t(\sqrt{21})$
1/2,2/7,2/7,3/7		C39	$t(\sqrt{105})$
1/3,1/3,1/7,5/7	20	H42	$j^3(2,2)$, t^6
1/3,1/3,2/7,4/7		A11	$t=-1$
1/3,1/3,3/7,3/7		H33	$j^3(-5,5)$, t^6
2/3,1/7,1/7,4/7		A14	$t=-1$
		B32	$t=2^8/11^2$
2/3,1/7,2/7,3/7		F23	$j^2(\sqrt{21})$
2/3,2/7,2/7,2/7		A15	$t=-1$

1/2,1/3,1/7,4/7	19	J11	j^6
1/2,1/3,2/7,3/7		I20	$j^4(3,7,4)$
1/2,1/2,1/7,3/7	18	I27	j^5, t^{10}
1/2,1/2,2/7,2/7		A3	$t=-1$
		H37	$j^3(9,2)$, t^6
1/3,1/3,1/3,4/7		E6	$t^{\mathrm{spl}}(-3,10)$
1/3,2/3,1/7,3/7		I26	j^5
1/3,2/3,2/7,2/7		—	no covering
1/7,1/7,1/7,8/7		—	no covering
1/7,1/7,3/7,6/7		C4	$t(\sqrt{2})$
1/7,1/7,4/7,5/7		—	no covering
1/7,2/7,2/7,6/7		—	no covering
1/7,2/7,3/7,5/7		B31	$t=7^4$
1/7,2/7,4/7,4/7		—	no covering
1/7,3/7,3/7,4/7		D29	$t(\sqrt{-3})$
2/7,2/7,2/7,5/7		—	no covering
2/7,2/7,3/7,4/7		—	no covering
2/7,3/7,3/7,3/7		—	no covering
1/2,1/3,1/3,3/7	17	H49	$j^3(-17,51), t^6$
1/2,2/3,1/7,2/7		I33	j^5
1/2,1/2,1/3,2/7	16	I22	j^5, t^{10}
1/3,1/3,2/3,2/7		A7	$m^2(\sqrt{-3}), t{=}-1$
		H38	$j^3(21,14)$, t^6
1/3,1/7,2/7,6/7		G19	$j^2(\sqrt{-3})$
1/3,1/7,3/7,5/7		B17	$t=3^3/2$
1/3,1/7,4/7,4/7		—	no covering
1/3,2/7,2/7,5/7		D33	$t(\sqrt{-5})$
1/3,2/7,3/7,4/7		F3	$j^2(\sqrt{2})$
1/3,3/7,3/7,3/7		—	no covering
2/3,2/3,1/7,1/7		A12	$t=-1$
		H48	$j^3(7,42)$, t^6
1/2,1/2,1/2,1/7	15	H45	$j^3(-7,14), t^{18}$
1/2,1/3,2/3,1/7		J22	j^{10}
1/2,1/7,1/7,6/7		G25	$j^2(\sqrt{-3})$, t^4
1/2,1/7,2/7,5/7		H40	$j^3(2,2)$
1/2,1/7,3/7,4/7		B33	$t=3^7 5/11^3$
1/2,2/7,2/7,4/7		—	no covering
1/2,2/7,3/7,3/7		C35	$t(\sqrt{21})$
1/2,1/2,1/3,1/3	14	G39	$j^2(\sqrt{-7})$, t^4
		H12	$j^3(-7,7)$, t^6
		I3	$j^4(0,0,7)$, t^8
1/3,1/3,1/3,2/3		E19	$t^{\mathrm{spl}}(0,28)$
1/3,1/3,1/7,6/7		D26	$t(\sqrt{-3})$
1/3,1/3,2/7,5/7		—	no covering
1/3,1/3,3/7,4/7		—	no covering
2/3,1/7,1/7,5/7		C27	$t(\sqrt{7})$
2/3,1/7,2/7,4/7		G31	$j^2(\sqrt{-7})$
2/3,1/7,3/7,3/7		C34	$t(\sqrt{21})$
2/3,2/7,2/7,3/7		C26	$t(\sqrt{7})$
1/2,1/3,1/7,5/7	13	I21	$j^4(0,4,48)$
1/2,1/3,2/7,4/7		H50	$j^3(4,2)$
1/2,1/3,3/7,3/7		C33	$t(\sqrt{13})$
1/2,1/2,1/7,4/7	12	G38	$j^2(\sqrt{-7})$, t^4
1/2,1/2,2/7,3/7		F22	$j^2(\sqrt{7})$, t^4
1/3,1/3,1/3,5/7		E14	$t^{\mathrm{spl}}(3,1)$
1/3,2/3,1/7,4/7		B25	$t=3^5$
1/3,2/3,2/7,3/7		B20	$t=2^7/3$
1/2,1/3,1/3,4/7	11	H27	$j^3(-4,4)$, t^6
1/2,2/3,1/7,3/7		H53	$j^3(-42,140)$
1/2,2/3,2/7,2/7		C42	$t(\sqrt{385})$
1/2,1/2,1/3,3/7	10	H34	$j^3(-5,5)$, t^6
1/3,1/3,2/3,3/7		D30	$t(\sqrt{-5})$
2/3,2/3,1/7,2/7		C40	$t(\sqrt{105})$
4/3,1/7,1/7,1/7		E20	$t^{\mathrm{spl}}(21,14)$
1/2,1/2,1/2,2/7	9	E21	$t^{\mathrm{spl}}(9,2)$
1/2,1/3,2/3,2/7		I15	$j^4(-24,34,141)$
1/2,1/2,2/3,1/7	8	H47	$j^3(7,42)$, t^6
1/3,2/3,2/3,1/7		C29	$t(\sqrt{7})$
1/2,1/2,1/2,1/3	7	G35	$j^2(\sqrt{-7})$, t^4
1/2,1/3,1/3,2/3		H35	$j^3(0,28)$, t^6

Table 2.3.8.

1/8,1/8,1/8,1/8	36	F6	$j^2(\sqrt{2})$, t^4
		G18	$j^2(\sqrt{-2})$, t^{12}
2/8,2/8,1/8,1/8	30	B10	$t=3^2$
		G5	$j^2(\sqrt{-1})$, t^4
1/8,1/8,1/8,3/8		D24	$m^2(\sqrt{-2}), j{=}0$
1/3,2/8,1/8,1/8	28	G8	$j^2(\sqrt{-1})$, t^4
1/2,1/8,1/8,1/8	27	G6	$j^2(\sqrt{-1})$, t^4
1/3,1/3,1/8,1/8	26	C22	$t(\sqrt{6})$
		I19	$j^4(-3,2,6)$, t^8
4/8,2/8,1/8,1/8	24	A5	$t=-1$
2/8,2/8,2/8,2/8		A18	$t=-1$
2/8,2/8,1/8,3/8		—	no covering
1/8,1/8,1/8,5/8		—	no covering
1/8,1/8,3/8,3/8		B14	$t=3^2$
		G15	$j^2(\sqrt{-2})$, t^4
4/8,1/3,1/8,1/8	22	—	no covering
1/3,2/8,2/8,2/8		—	no covering
1/3,2/8,1/8,3/8		I16	$j^4(2,8,8)$
1/2,2/8,2/8,1/8	21	—	no covering
1/2,1/8,1/8,3/8		G16	$j^2(\sqrt{-2})$, t^4
1/3,1/3,2/8,2/8	20	A9	$t=-1$
		H22	$j^3(5,10)$, t^6
1/3,1/3,1/8,3/8		D16	$t(\sqrt{-2})$
2/3,2/8,1/8,1/8		A13	$m^2(\sqrt{-2}), t{=}-1$
		G3	$j^2(\sqrt{-1})$, t^4
1/2,1/3,2/8,1/8	19	J21	j^{10}

1/2,1/2,1/8,1/8	18	F4	$j^2(\sqrt{2})$, t^4
		J1	j^6, t^{12}
4/8,4/8,1/8,1/8		—	no covering
4/8,2/8,2/8,2/8		—	no covering
4/8,2/8,1/8,3/8		B1	$t = 2^2$
1/3,1/3,1/3,2/8		E5	$t^{\mathrm{spl}}(-3,10)$
1/3,2/3,1/8,1/8		G12	$j^2(\sqrt{-2})$, t^4
2/8,2/8,1/8,5/8		D1	$t(\sqrt{-1})$
2/8,2/8,3/8,3/8		C5	$t(\sqrt{3})$
2/8,6/8,1/8,1/8		—	no covering
1/8,1/8,1/8,7/8		—	no covering
1/8,1/8,3/8,5/8		—	no covering
1/8,3/8,3/8,3/8		—	no covering
1/2,1/3,1/3,1/8	17	J20	j^9, t^{18}
4/8,1/3,2/8,2/8	16	—	no covering
4/8,1/3,1/8,3/8		—	no covering
1/3,1/3,1/3,1/3		D8	$t(\sqrt{-2})$
		F2	$j^2(\sqrt{2})$
1/3,2/8,1/8,5/8		B30	$t = 2^2 13^2$
1/3,2/8,3/8,3/8		—	no covering
1/3,6/8,1/8,1/8		G22	$j^2(\sqrt{-3})$, t^4
1/2,4/8,2/8,1/8	15	G1	$j^2(\sqrt{-1})$
1/2,2/8,2/8,3/8		B15	$t = 3^4$
1/2,1/8,1/8,5/8		G9	$j^2(\sqrt{-1})$, t^4
1/2,1/8,3/8,3/8		C8	$t(\sqrt{3})$
4/8,1/3,1/3,2/8	14	F8	$j^2(\sqrt{2})$, t^4
4/8,2/3,1/8,1/8		—	no covering
1/3,1/3,1/8,5/8		H23	$j^3(5,10)$, t^6
1/3,1/3,3/8,3/8		C28	$t(\sqrt{7})$
2/3,2/8,2/8,2/8		—	no covering
2/3,2/8,1/8,3/8		B29	$t = 5^3/2^2$
1/2,4/8,1/3,1/8	13	I18	$j^4(-3,2,6)$
1/2,1/3,2/8,3/8		H29	$j^3(-1,2)$
1/2,1/2,2/8,2/8	12	C1	$t(\sqrt{2})$
1/2,1/2,1/8,3/8		D9	$t(\sqrt{-2})$
		G14	$j^2(\sqrt{-2})$, t^4
4/8,1/3,1/3,1/3	12	E2	$t^{\mathrm{spl}}(3,2)$
1/3,2/3,2/8,2/8		C21	$t(\sqrt{6})$
1/3,2/3,1/8,3/8		G10	$j^2(\sqrt{-2})$
1/2,1/3,1/3,3/8	11	F25	$j^2(\sqrt{22})$, t^4
1/2,2/3,2/8,1/8		I10	$j^4(-2,4,-1)$
1/2,1/2,1/3,2/8	10	H20	$j^3(5,10)$, t^6
1/3,1/3,2/3,2/8		F19	$j^2(\sqrt{6})$, t^4
2/3,2/3,1/8,1/8		C32	$t(\sqrt{10})$
1/2,1/2,1/2,1/8	9	G17	$j^2(\sqrt{-2})$, t^{12}
1/2,1/3,2/3,1/8		J2	j^6
1/2,1/2,1/3,1/3	8	F5	$j^2(\sqrt{2})$, t^4
		G13	$j^2(\sqrt{-2})$, t^4
1/3,1/3,1/3,2/3		D13	$t(\sqrt{-2})$

Table 2.3.9.

3/9,1/9,1/9,1/9	24	—	no covering
1/9,1/9,2/9,2/9		A24	$t = -1$
		H8	$j^3(-3,4)$, t^6
1/3,1/9,1/9,2/9	22	H9	$j^3(-3,4)$, t^6
1/2,1/9,1/9,1/9	21	G52	$j^2(\sqrt{-15})$, t^{12}
1/3,1/3,1/9,1/9	20	H1	$j^3(-3,1)$
		J3	j^6, t^{12}
3/9,3/9,1/9,2/9	18	—	no covering
3/9,1/9,1/9,4/9		—	no covering
3/9,2/9,2/9,2/9		—	no covering
6/9,1/9,1/9,1/9		E3	$t^{\mathrm{spl}}(3,2)$
1/9,1/9,2/9,5/9		C23	$t(\sqrt{6})$
1/9,2/9,2/9,4/9		—	no covering
1/3,3/9,3/9,1/9	16	A6	$t = -1$
1/3,3/9,2/9,2/9		C7	$t(\sqrt{3})$
1/3,1/9,1/9,5/9		H25	$j^3(3,1)$, t^6
1/3,1/9,2/9,4/9		H6	$j^3(3,2)$
1/2,3/9,1/9,2/9	15	H4	$j^3(0,2)$
1/2,1/9,1/9,4/9		D49	$t(\sqrt{-39})$
1/2,2/9,2/9,2/9		—	no covering
1/3,1/3,3/9,2/9	14	D39	$t(\sqrt{-7})$
1/3,1/3,1/9,4/9		G21	$j^2(\sqrt{-3})$, t^4
3/9,2/3,1/9,1/9		D2	$t(\sqrt{-1})$
2/3,1/9,2/9,2/9		C9	$t(\sqrt{3})$
1/2,1/3,3/9,1/9	13	I17	$j^4(0,6,9)$
1/2,1/3,2/9,2/9		C41	$t(\sqrt{273})$
1/2,1/2,1/9,2/9	12	H7	$j^3(-3,4)$, t^6
1/3,1/3,1/3,3/9		D19	$j = 0$
1/3,2/3,1/9,2/9		B7	$t = 3^2$
		H2	$j^3(0,3)$
1/2,1/3,1/3,2/9	11	H32	$j^3(6,1)$, t^6
1/2,2/3,1/9,1/9		H31	$j^3(-3,8)$, t^6
1/2,1/2,1/3,1/9	10	J4	j^6, t^{12}
1/3,1/3,2/3,1/9		H3	$j^3(0,3)$, t^6
1/2,1/3,1/3,1/3	9	G26	$j^2(\sqrt{-3})$, t^{12}

Table 2.3.10.

1/10,1/10,1/10,1/10	24	D6	$t(\sqrt{-1})$
		F17	$j^2(\sqrt{5})$, t^4
5/10,1/10,1/10,1/10	18	E7	$t^{\mathrm{spl}}(5,10)$
2/10,2/10,2/10,2/10		—	no covering
2/10,2/10,1/10,3/10		B24	$t = 2^7/3$
2/10,4/10,1/10,1/10		D4	$t(\sqrt{-1})$
1/10,1/10,3/10,3/10		C25	$t(\sqrt{6})$
1/3,2/10,2/10,2/10	16	—	no covering
1/3,2/10,1/10,3/10		G2	$j^2(\sqrt{-1})$
1/3,4/10,1/10,1/10		D42	$t(\sqrt{-14})$
1/2,2/10,2/10,1/10	15	C15	$t(\sqrt{5})$
1/2,1/10,1/10,3/10		D44	$t(\sqrt{-15})$
		D50	$t(\sqrt{-39})$

1/3,1/3,2/10,2/10	14	C38	$t(\sqrt{21})$
1/3,1/3,1/10,3/10		D48	$t(\sqrt{-35})$
2/3,2/10,1/10,1/10		F14	$j^2(\sqrt{5})$, t^4
1/2,1/3,2/10,1/10	13	J5	j^6
1/2,1/2,1/10,1/10	12	F10	$j^2(\sqrt{5})$
		G7	$j^2(\sqrt{-1})$, t^4
1/3,1/3,1/3,2/10		E13	$t^{\rm spl}(3,1)$
1/3,2/3,1/10,1/10		H30	$j^3(0,10)$, t^6
1/2,1/3,1/3,1/10	11	J8	j^6, t^{12}
1/3,1/3,1/3,1/3	10	B19	$t=2^5/5$

Table 2.3.11.

1/11,1/11,1/11,4/11	18	—	no covering
1/11,1/11,2/11,3/11		D15	$t(\sqrt{-2})$
1/11,2/11,2/11,2/11		—	no covering
1/3,1/11,1/11,3/11	16	D41	$t(\sqrt{-7})$
1/3,1/11,2/11,2/11		C13	$t(\sqrt{3})$
1/2,1/11,1/11,2/11	15	H26	$j^3(-4,4)$, t^6
1/3,1/3,1/11,2/11	14	H28	$j^3(-4,4)$, t^6
2/3,1/11,1/11,1/11		E15	$t^{\rm spl}(-11,22)$
1/2,1/3,1/11,1/11	13	J9	j^6, t^{12}
1/3,1/3,1/3,1/11	12	G45	$j^2(\sqrt{-11}), t^{12}$

Table 2.3.12.

3/12,1/12,1/12,1/12	18	D21	$j=0$
2/12,2/12,1/12,1/12		B11	$t=3^2$
1/3,2/12,1/12,1/12	16	D14	$t(\sqrt{-2})$
		G24	$j^2(\sqrt{-3}), t^4$
1/2,1/12,1/12,1/12	15	G27	$j^2(\sqrt{-3}), t^{12}$
1/3,1/3,1/12,1/12	14	F9	$j^2(\sqrt{3})$
		G23	$j^2(\sqrt{-3})$, t^4

Table 2.3.13.

1/13,1/13,1/13,2/13	18	E16	$t^{\rm spl}(-1,2)$
1/3,1/13,1/13,1/13	16	G28	$j^2(\sqrt{-3}), t^{12}$

Table 2.3.14.

1/14,1/14,1/14,1/14	18	C2	$t(\sqrt{2})$

Table 2.4.5.

1/5,1/5,1/5,1/5	24	A20	$t=-1$
		F12	$j^2(\sqrt{5})$
		G48	$j^2(\sqrt{-15})$, t^4
1/4,1/4,1/5,1/5	22	I30	j^5, t^{10}
1/4,1/4,1/4,1/4	20	C17	$t(\sqrt{5})$
		D45	$t(\sqrt{-15})$
		E11	$t^{\rm spl}(5,10)$
1/5,1/5,1/5,2/5		E9	$t^{\rm spl}(5,10)$
2/4,1/5,1/5,1/5	18	—	no covering
1/4,1/4,1/5,2/5		G51	$j^2(\sqrt{-15})$, t^4
1/2,1/4,1/5,1/5	17	J18	j^8, t^{16}
2/4,1/4,1/4,1/5	16	H18	$j^3(5,10)$, t^6
1/5,1/5,1/5,3/5		E18	$t^{\rm spl}(0,10)$
1/5,1/5,2/5,2/5		H16	$j^3(5,10)$
1/2,1/4,1/4,1/4	15	E25	$t^{\rm spl}(25,50)$
2/4,1/5,1/5,2/5	14	D38	$t(\sqrt{-7})$
1/4,1/4,1/5,3/5		C10	$t(\sqrt{3})$
1/4,1/4,2/5,2/5		H41	$j^3(2,2)$, t^6
1/2,1/4,1/5,2/5	13	I31	j^5
1/2,1/2,1/5,1/5	12	F16	$j^2(\sqrt{5})$, t^4
		I11	$j^4(-3,0,6), t^8$
2/4,2/4,1/5,1/5		A2	$t=-1$
		G50	$j^2(\sqrt{-15})$, t^4
2/4,1/4,1/4,2/5		A4	$m^2(\sqrt{-1}), t{=}{-1}$
1/4,3/4,1/5,1/5		F15	$j^2(\sqrt{5})$, t^4
1/5,1/5,1/5,4/5		D3	$t(\sqrt{-1})$
1/5,1/5,2/5,3/5		B23	$t=2^7/3$
1/5,2/5,2/5,2/5		—	no covering
1/2,2/4,1/4,1/5	11	I29	j^5
1/2,1/2,1/4,1/4	10	C14	$t(\sqrt{5})$
		C30	$t(\sqrt{10})$
		H17	$j^3(5,10)$, t^6
2/4,2/4,1/4,1/4		F13	$j^2(\sqrt{5})$, t^4
2/4,1/5,1/5,3/5		D43	$t(\sqrt{-15})$
2/4,1/5,2/5,2/5		C16	$t(\sqrt{5})$
1/4,1/4,1/4,3/4		—	no covering
1/4,1/4,1/5,4/5		B8	$t=3^2$
1/4,1/4,2/5,3/5		—	no covering
1/2,1/4,1/5,3/5	9	I24	j^5
1/2,1/4,2/5,2/5		D27	$t(\sqrt{-3})$
1/2,1/2,1/5,2/5	8	H19	$j^3(5,10)$, t^6
2/4,2/4,1/5,2/5		—	no covering
2/4,1/4,1/4,3/5		C19	$t(\sqrt{6})$
1/4,3/4,1/5,2/5		B26	$t=3^4/2^5$
1/2,2/4,1/4,2/5	7	H39	$j^3(2,2)$
1/2,3/4,1/5,1/5		G29	$j^2(\sqrt{-5})$, t^4
1/2,1/2,2/4,1/5	6	G46	$j^2(\sqrt{-15})$, t^4
1/2,1/2,1/2,1/4	5	E10	$t^{\rm spl}(5,10)$

Table 2.4.6.

1/6,1/6,1/6,1/6	16	B28	$t=3^4/2^5$
		D11	$t(\sqrt{-2})$
1/4,1/4,1/6,1/6	14	C11	$t(\sqrt{3})$
		I13	$j^4(0,4,12), t^8$
3/6,1/6,1/6,1/6	12	D18	$j=0$
2/6,2/6,1/6,1/6		B5	$t=3^2$
1/4,1/4,1/4,1/4		B4	$t=2^2$
		B22	$t=2^7/3$
2/4,2/6,1/6,1/6	10	—	no covering
3/6,1/4,1/4,1/6		H5	$j^3(0,2)$, t^6
2/6,2/6,1/4,1/4		D32	$t(\sqrt{-5})$

k, ℓ, m	Max.	Br. patterns		Coverings		Moduli field degree						
	degree	Total	No cov.	Orb.	Total	1	2	3	4	5	6	≥ 7
2, 3, 7	60	152	30	140	427	51	27	25	11	8	5	13
2, 3, 8	36	65	16	58	130	23	24	4	4		2	2
2, 3, 9	24	32	6	29	67	12	3	11	1		2	
2, 3, 10	24	20	2	21	38	13	5	1			2	
2, 3, 11..14	18	18	2	18	33	9	6	2			1	
2, 4, 5	24	40	5	42	91	21	10	6	1	4		1
2, 4, 6/7/8	16	24	5	25	43	16	4	1	4			
2, 5/6/7	12	12	1	15	22	10	3	2				
3, 3/4/5	12	15	2	16	21	12	3	1				
Total	—	378	69	366	872	167	85	53	21	12	12	16

Table 1: Statistics of Belyi maps

Table 2.4.6 (continued)

1/2,2/6,1/4,1/6	9	I1	$j^4(0,8,12)$
1/2,1/2,1/6,1/6	8	C3	$t(\sqrt{2})$
		G11	$j^2(\sqrt{-2})$, t^4
2/4,2/4,1/6,1/6		—	no covering
2/4,2/6,1/4,1/4		D12	$t(\sqrt{-2})$
1/4,3/4,1/6,1/6		G20	$j^2(\sqrt{-3})$, t^4
1/2,2/4,1/4,1/6	7	I12	$j^4(0,4,12)$
1/2,1/2,1/4,1/4	6	C24	$t(\sqrt{6})$
		D25	$t(\sqrt{-3})$

Table 2.4.7.

1/7,1/7,1/7,2/7	12	—	no covering
2/4,1/7,1/7,1/7	10	—	no covering
1/4,1/4,1/7,2/7		D34	$t(\sqrt{-5})$
1/2,1/4,1/7,1/7	9	I7	$j^4(2,8,9)$, t^8
2/4,1/4,1/4,1/7	8	G37	$j^2(\sqrt{-7})$, t^4
1/2,1/4,1/4,1/4	7	E24	$t^{\rm spl}(-7,14)$

Table 2.4.8.

1/8,1/8,1/8,1/8	12	—	no covering
1/4,1/4,1/8,1/8	10	B9	$t = 3^2$
		G4	$j^2(\sqrt{-1})$, t^4
1/4,1/4,1/4,1/4	8	A16	$t = -1$

Table 2.5.5.

1/5,1/5,1/5,1/5	12	A19	$t = -1$
		F11	$j^2(\sqrt{5})$
		G47	$j^2(\sqrt{-15})$, t^4
1/5,1/5,1/5,2/5	10	E8	$t^{\rm spl}(5,10)$
1/5,1/5,1/5,3/5	8	E17	$t^{\rm spl}(0,10)$
1/5,1/5,2/5,2/5		H15	$j^3(5,10)$
1/2,1/5,1/5,2/5	7	D37	$t(\sqrt{-7})$
1/2,1/2,1/5,1/5	6	A1	$t = -1$
		G49	$j^2(\sqrt{-15})$, t^4

Table 2.5.6.

1/6,1/6,1/6,1/6	10	C31	$t(\sqrt{10})$
2/6,1/5,1/5,1/5	8	—	no covering
1/5,2/5,1/6,1/6		D36	$t(\sqrt{-6})$
1/2,1/5,1/5,1/6	7	H24	$j^3(3,1)$, t^6

Table 2.5.7.

1/5,1/5,1/5,1/7	8	E23	$t^{\rm spl}(2,2)$

Table 2.6.6.

1/6,1/6,1/6,1/6	8	B27	$t = 3^4/2^5$
		D10	$t(\sqrt{-2})$

Table 3.3.4.

1/4,1/4,1/4,1/4	12	A17	$t = -1$
1/3,1/3,1/4,1/4	10	A8	$t = -1$
		H21	$j^3(5,10)$, t^6
2/4,1/4,1/4,1/4	9	—	no covering
1/3,1/3,1/3,1/4		E4	$t^{\rm spl}(-3,10)$
1/3,1/3,1/3,1/3	8	D7	$t(\sqrt{-2})$
		F1	$j^2(\sqrt{2})$
2/4,1/3,1/3,1/4	7	F7	m^4, $j(\sqrt{2})$, t^4
2/4,1/3,1/3,1/3	6	E1	$t^{\rm spl}(3,2)$
1/3,2/3,1/4,1/4		C20	$t(\sqrt{6})$
1/3,1/3,2/3,1/4	5	F18	$j^2(\sqrt{6})$, t^4

Table 3.3.5.

1/5,1/5,1/5,1/5	9	—	no covering
1/3,1/3,1/5,1/5	7	C37	$t(\sqrt{21})$
1/3,1/3,1/3,1/5	6	E12	$t^{\rm spl}(3,1)$
1/3,1/3,1/3,1/3	5	B18	$t = 2^5/5$

Table 3.4.4.

1/4,1/4,1/4,1/4	6	B3	$t = 2^2$
		B21	$t = 2^7/3$
1/3,1/3,1/4,1/4	5	D31	$t(\sqrt{-5})$

- if $j \in \mathbb{Q} \setminus \{0\}$ and $t \in \mathbb{Q}$, a value of t is given in the factorized form (as motivated by §E);
- if $j \in \mathbb{Q} \setminus \{0\}$ and the t-field is quadratic, $t(\sqrt{a})$ means the field $\mathbb{Q}(\sqrt{a})$;
- if $j \in \mathbb{Q} \setminus \{0\}$ and the t-field degree is greater than 2, $t^{\mathrm{spl}}(a,b)$ means the splitting field (of degree 6 in our examples) of the polynomial $X^3 + aX + b$.

The size of the Galois orbit[7] is equal to the degree of the moduli field.In most cases, the j-field and the moduli field coincide[8]. The moduli field is indicated only if it differs from the j-field, either with $m^2(\sqrt{a})$ if its degree is 2, or with m^d if its degree $d > 2$.

More information about each computed Galois orbit can be found in the tables of Appendix §B and our website [17]. Statistics of computed Belyi coverings is presented in Table 1. There are no coverings for 69 out of 378 derived branching patterns.

4 The moduli field and obstruction conics

In this section, let O denote the group of Möbius transformations:

$$O = \left\{ \frac{ax+b}{cx+d} \mid a,b,c,d \in \overline{\mathbb{Q}} \text{ with } ad - bc \neq 0 \right\} \cong \mathrm{Aut}(\overline{\mathbb{Q}}(x)/\overline{\mathbb{Q}}).$$

Two rational functions $\varphi_1, \varphi_2 \in \overline{\mathbb{Q}}(x)$ are called *Möbius-equivalent*[9], denoted $\varphi_1 \sim \varphi_2$, if there exists $\mu \in O$ with $\varphi_1 \circ \mu = \varphi_2$.

A *realization field* of a Belyi covering φ is a number field over which a Möbius equivalent function $\varphi \circ \mu$ is defined. The r-field from Definition 2.3 is such a field, but often not of minimal degree. This section describes how, for each Belyi function φ in our table, we determined its moduli field and realization fields.

Definition 4.1. *Let $\varphi \in \overline{\mathbb{Q}}(x)$ be a Belyi function. Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The* moduli field *$M_\varphi$ is the fixed field of $\{\sigma \in G | \varphi \sim \sigma(\varphi)\}$.*

Clearly $M_\varphi \subseteq K_\varphi$ for any explicit $\varphi \in K_\varphi(x)$ over some realization field K_φ. The moduli field is known to be equal the intersection of the realization fields of φ. It can be computed directly from the given definition, by checking which Galois conjugates of φ are Möbius-equivalent to φ. The computed Belyi functions φ always had $[K_\varphi : \mathbb{Q}(j)] \leq 2$, where $\mathbb{Q}(j)$ is the j-field. But $\mathbb{Q}(j) \subseteq M_\varphi \subseteq K_\varphi$. Therefore $M_\varphi = K_\varphi$ if $K_\varphi = \mathbb{Q}(j)$. For those $K_\varphi \neq \mathbb{Q}(j)$, let σ be the non-trivial element of $\mathrm{Gal}(K_\varphi/\mathbb{Q}(j))$. Our moduli fields are then determined by just checking whether $\varphi \sim \sigma(\varphi)$.

[7] Our notation allows to count the total number of dessins d'enfant in selected Galois orbits rather quickly in Tables 2.3.7–3.4.4. Each fourth column starts either with the m^d or j^d notation (where d is the size of the Galois orbit), or starts with a symbol without a numeric power (in which case $d = 1$).

[8] The moduli field always contains the j-field, as the j-value is an invariant of Möbius transformations. In our encountered cases, the moduli field is at most a quadratic extension of the j-field.

[9] To determine whether $\varphi_1 \sim \varphi_2$, we can factor the numerator of $\varphi_1(x) - \varphi_2(y)$. If it has a factor $p(x,y) \in \overline{\mathbb{Q}}[x,y]$ with $\deg_x(p) = \deg_y(p) = 1$, then $\varphi_1 \sim \varphi_2$ and one finds μ by solving $p(x,y) = 0$ with respect to y.

Id	branching fractions	d	$[k\ell m]$	Moduli field	Obstruction conic	Bad primes
B12	$1/7, 1/7, 3/7, 3/7$	36	[237]	$\mathbb{Q}$	$u^2 + v^2 + 7$	$7, \infty$
C6	$1/3, 1/3, 2/7, 2/7$	32	[237]	$\mathbb{Q}$	$u^2 + v^2 + 1$	$2, \infty$
C30	$1/2, 1/2, 1/4, 1/4$	10	[245]	$\mathbb{Q}$	$u^2 + 2v^2 + 5$	$5, \infty$
D45	$1/4, 1/4, 1/4, 1/4$	20	[245]	$\mathbb{Q}$	$u^2 + 2v^2 + 5$	$5, \infty$
F1	$1/3, 1/3, 1/3, 1/3$	8	[334]	$\mathbb{Q}(\sqrt{2})$	$u^2 + 3v^2 + \sqrt{2} - 1$	$3, \infty$
F4	$1/2, 1/2, 1/8, 1/8$	18	[238]	$\mathbb{Q}(\sqrt{2})$	$u^2 + v^2 + 1$	∞, ∞
F6	$1/8, 1/8, 1/8, 1/8$	36	[238]	$\mathbb{Q}(\sqrt{2})$	$u^2 + v^2 + 1$	∞, ∞
F11	$1/5, 1/5, 1/5, 1/5$	12	[255]	$\mathbb{Q}(\sqrt{5})$	$u^2 + 2v^2 + \sqrt{5}$	$5, \infty$
H1	$1/3, 1/3, 1/9, 1/9$	20	[239]	$\mathbb{Q}(\operatorname{Re}\zeta_9)$	$u^2 + v^2 + \operatorname{Re}\zeta_9$	∞, ∞
H10	$1/2, 1/2, 1/7, 1/7$	30	[237]	$\mathbb{Q}(\operatorname{Re}\zeta_7)$	$u^2 + v^2 - \operatorname{Re}\zeta_7$	∞, ∞
H11	$1/3, 1/3, 1/7, 1/7$	44	[237]	$\mathbb{Q}(\operatorname{Re}\zeta_7)$	$u^2 + v^2 - \operatorname{Re}\zeta_7$	∞, ∞
H12	$1/2, 1/2, 1/3, 1/3$	14	[237]	$\mathbb{Q}(\operatorname{Re}\zeta_7)$	$u^2 + v^2 - \operatorname{Re}\zeta_7$	∞, ∞
H13	$1/3, 1/3, 1/3, 1/3$	28	[237]	$\mathbb{Q}(\operatorname{Re}\zeta_7)$	$u^2 + v^2 - \operatorname{Re}\zeta_7$	∞, ∞
H14	$1/7, 1/7, 1/7, 1/7$	60	[237]	$\mathbb{Q}(\operatorname{Re}\zeta_7)$	$u^2 + v^2 - \operatorname{Re}\zeta_7$	∞, ∞

Table 2: Belyi functions with an obstruction.

Among the minus-4-hyperbolic Belyi functions, there are 14 Galois orbits for which the moduli field not a realization field. They are given in Table 2. The column "Obstruction conic" gives an answer to the question: What are the realization fields of a Belyi map φ? This question is illuminated in §4.1 and §4.2. The last column gives an alternative characterization of obstruction conics, as explained in §4.3.

Two Belyi functions are Möbius-equivalent if and only if they have the same dessin d'enfant up to homotopy. Thus, the moduli field of a dessin d'enfant is well defined. The number of different dessins (up to homotopy) in a Galois orbit is equal to the degree of the number field. The dessins d'enfant of most of the Belyi maps of Table 2 are depicted in Figure 3. The other Galois orbits with obstructed dessins can be found in Figures 1 and 4.

The interesting questions whether a dessin has a moduli field $\subset \mathbb{R}$, and if so, does it have a realization over $\mathbb{R}$, are considered in [5]. Although all moduli fields in the obstructed cases are real, not all their dessins have a reflection symmetry (i.e., have a realization over $\mathbb{R}$). Rather, their complex conjugates are equivalent to the original up to homotopy that permutes the cells, reflecting a non-trivial Möbius equivalence. The number of these skew-symmetric dessins depends on the number of bad ∞-primes shown in the last column of Table 2. The moduli field for H1, H10 − H14 has three infinite primes, but only two of them are bad. Therefore one dessin in those orbits has a reflection symmetry, and the other two are skew symmetric. Likewise, F1 and F11 each have one dessin with $\mathbb{R}$-realization and one dessin without.

4.1 Obstructions on realization fields

If the moduli field M_φ is a realization field, the other realization fields are extensions of M_φ. Otherwise, the realization fields are determined by a *conic obstruction.* For each of the cases of Table 2, the realization fields are those

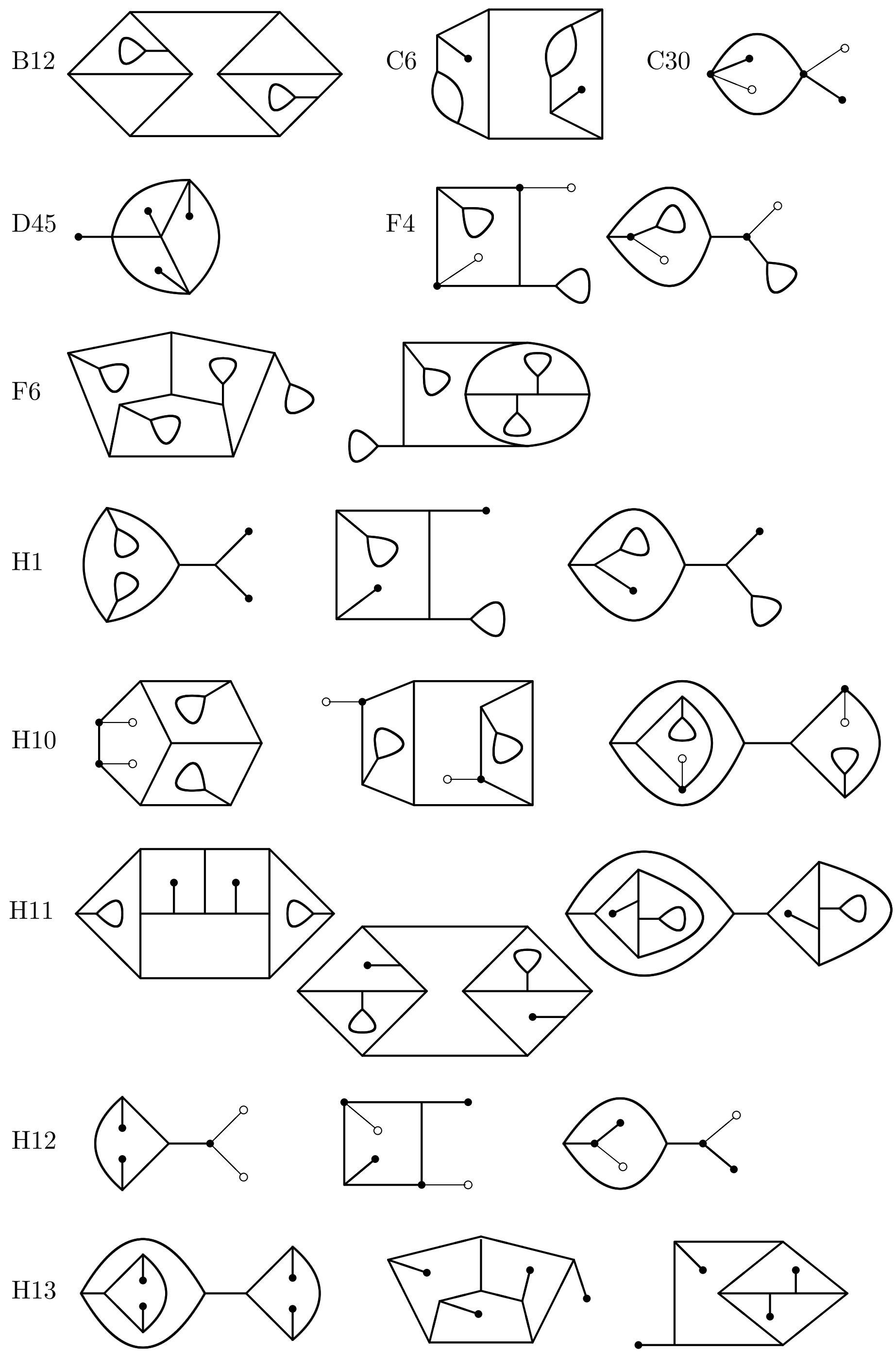


Figure 3: The coverings (except F1, F11, H14) with an obstruction

extensions of M_φ that have a rational point on the conic curves given in the sixth column.

If $\varphi \in \overline{\mathbb{Q}}(x)$, we denote

$$O_\varphi = \{\mu \in O | \varphi \circ \mu = \varphi\} \cong \mathrm{Aut}(\overline{\mathbb{Q}}(x)/\overline{\mathbb{Q}}(\varphi)).$$

We have $|O_\varphi| > 1$ in the following four cases: D45, F6, H13, H14. Then $|O_\varphi| = 2$, and the dessins have a non-trivial automorphism.

Lemma 4.2. *If $\varphi \in K(x)$ is a Belyi function, and $|O_\varphi| = 2$, then we can write $\varphi = f \circ f_2$ for some $f, f_2 \in K(x)$ with f_2 of degree 2. If φ has a realization over L, then so does f.*

Proof: Let μ be the non-identity element of O_φ. Since φ is invariant under $\mathrm{Gal}(\overline{K}/K)$, the same must be true for μ, and so $\mu \in K(x)$. Since μ has order two in $\mathrm{Aut}(\overline{\mathbb{Q}}(x)/\overline{\mathbb{Q}})$, its fixed field $F \subseteq K(x)$ has index 2. Define $f_2 \in F$ as $x\mu$ if $x\mu \notin K$, and $x + \mu$ otherwise. Then f_2 has degree 2, and hence it generates F. Since $\varphi \in F$, it follows that $\varphi = f \circ f_2$ for some $f \in K(x)$.

If φ has a realization $\tilde{\varphi} \in L(x)$, then from the non-identity element of $O_{\tilde{\varphi}}$ we can compute explicit $\tilde{f}_2$ and $\tilde{f}$ in $L(x)$ in exactly the same way. So if φ has a realization over L, then so does f. $\square$

Lemma 4.3. *Let L be a field of characteristic 0. Then*

$$\text{C30 has a realization over } L \Longleftrightarrow \text{D45 has a realization over } L.$$

Proof: "$\Longrightarrow$": Let $\varphi \in L(x)$ be a realization for C30. The branching structure of C30 above 1 is $1, 1, 2, 2, 2, 2$ and so $\varphi^{-1}(\{1\}) \subseteq \overline{\mathbb{Q}} \bigcup \{\infty\}$ contains 6 points, two of which (call them P_1, P_2) are unramified. Let $S := \{P_1, P_2\}$. Since φ is invariant under $\mathrm{Gal}(\overline{L}/L)$, the same must be true for S. So either $P_1, P_2 \in L \bigcup \{\infty\}$, or P_1, P_2 are the roots of an irreducible quadratic polynomial $Q \in L[x]$. In both cases one can construct an explicit $f_2 \in L(x)$ of degree 2 that branches above P_1 and P_2. Then $\varphi \circ f_2 \in L(x)$ is a Belyi function with the same branching type as D45. The Heun table shows that among the Belyi functions with this branching type, the only one that allows C30 as a decomposition factor is D45. Hence $\varphi \circ f_2$ is a realization of D45 over L. The converse statement follows from Lemma 4.2. $\square$

The same proof also works for F6, F4, for H13, H12, and for H14, H10. So it suffices to answer Question (3) for the cases with $|O_\varphi| = 1$. We shall do this for an example, namely C30. The same works for the remaining cases, and the results are listed in Table 2.

4.2 The obstruction conic for C30

The moduli field for C30 is $M = \mathbb{Q}$. Our website [17] lists this Belyi function

$$\varphi = \frac{2((x^2+5)\sqrt{-3} - 3x^2 - 60x + 15)(x^2 + 5x - 5)^4}{(12x)^5}$$

in $K(x)$ where $K = \mathbb{Q}(\sqrt{-3})$. Suppose $\varphi \sim g$ for some $g \in L(x)$ with $\sqrt{-3} \notin L$. Write $\varphi = g \circ \mu$ for some $\mu \in O$, which must be unique because O_φ is trivial.

That implies $\mu \in L(\sqrt{-3})(x)$ since $\varphi, g \in L(\sqrt{-3})(x)$. So we can write $\mu = (ax+b)/(cx+d)$ with $a,b,c,d \in L(\sqrt{-3})$ and $ad - bc \neq 0$.

Without loss of generality we may assume $c = 0$ or $c = 1$. The case $c = 0$ is a quick computation, and does not lead to solutions (with $ad - bc \neq 0$ and $\sqrt{-3} \notin L$) so we set $c := 1$. Write $a = a_0 + a_1\sqrt{-3}$, $b = b_0 + b_1\sqrt{-3}$ and $d = d_0 + d_1\sqrt{-3}$, for some $a_0, a_1, b_0, b_1, d_0, d_1 \in L$. We can replace μ by $\mu - a_0$, this corresponds to replacing $g(x)$ by $g(x + a_0)$, which is still in $L(x)$. After this, we may assume $a_0 = 0$. The case $a_1 = 0$ does not lead to solutions so we may suppose $a_1 \neq 0$. Replacing μ by μ/a_1 corresponds to replacing $g(x)$ by $g(a_1 x)$, after this, we may assume $a_1 = 1$. Then we can write $\varphi = g \circ \mu$ where

$$\mu = \frac{x\sqrt{-3} + b_0 + b_1\sqrt{-3}}{x + d_0 + d_1\sqrt{-3}} \tag{4.1}$$

for some $b_0, b_1, d_0, d_1 \in L$.

Let $\sigma : \sqrt{-3} \mapsto -\sqrt{-3}$ be the non-trivial element of $\mathrm{Gal}(L(\sqrt{-3})/L)$. Now $\sigma(\varphi) \sim \varphi$ (otherwise the moduli field would have been K). We compute the unique $\nu \in O$ with $\sigma(\varphi) = \varphi \circ \nu$ by factoring the numerator of $\varphi(y) - \sigma(\varphi(x))$, and find $\nu = -5/x$ (uniqueness implies $\nu \in K(x)$. Now

$$g \circ \sigma(\mu) = \sigma(g) \circ \sigma(\mu) = \sigma(g \circ \mu) = \sigma(\varphi) = \varphi \circ \nu = g \circ \mu \circ \nu$$

and hence $\sigma(\mu) = \mu \circ \nu$ (using again $|O_g| = |O_\varphi| = 1$). Write the numerator of $\sigma(\mu) - \mu \circ \nu$ as $\sum_{i=0}^{2} \sum_{j=0}^{1} C_{i,j}\, x^i (\sqrt{-3})^j$ with $C_{i,j} \in \mathbb{Q}[b_0, b_1, d_0, d_1]$. The equation $\sigma(\mu) = \mu \circ \nu$ corresponds to setting all $C_{i,j}$ to 0, resulting in:

$$b_1 + d_0 = b_0 - 3d_1 = 5 + b_0 d_1 - b_1 d_0 = 0$$

which reduces to

$$b_1 = -d_0, \quad b_0 = 3d_1, \quad \text{and} \quad d_0^2 + 3d_1^2 + 5 = 0. \tag{4.2}$$

So if φ has a realization over L then $d_0^2 + 3d_1^2 + 5 = 0$ has a solution $d_0, d_1 \in L$. Conversely, for any solution $d_0, d_1 \in L$, one obtains an explicit realization of C30 over L, namely $g = \varphi \circ \mu^{-1}$, with μ as in (4.1),(4.2).

4.3 Cohomological obstructions

The conic (4.2) describes all fields L over which C30 has a realization. The conic $s^2 + 2t^2 + 5 = 0$ is birational[10] to (4.2) and hence it describes the same fields. C30 has an *obstruction*; it has no realization over its moduli field $\mathbb{Q}$ because (4.2) has no $\mathbb{Q}$-rational points. Table 2 lists all entries of the Heun table that have an obstruction.

Obstructions can be described in several ways: skew fields, 2-cocycles, or bad primes. If $\mathfrak{p}$ is a prime of the number field M_φ, then $\mathfrak{p}$ is called a *bad prime* if φ has no realization over the completion of M_φ at $\mathfrak{p}$ ($\mathbb{R}$ is denoted by $\mathfrak{p} = \infty$). The set of bad primes completely describes the obstruction (a conic is determined up to birational equivalence by its bad primes).

[10] We have an implementation [17] to find birational maps for conics over $\mathbb{Q}$. For number fields, we followed the same method, computing F1,F4,…,H14 case by case.

If $\varphi \in \overline{\mathbb{Q}}(x)$ has $|O_\varphi| = 1$ then φ defines a 2-cocycle in a natural way, representing an element of $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\ O)$. Section 4.2 spells this out in an explicit example; we computed the image of (id, σ) under the 2-cocycle (the $\nu \in O$ for which $\mathrm{id}(\varphi) \circ \nu = \sigma(\varphi)$), and then used that to determine the fields L for which this 2-cocycle maps to the identity in $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/L),\ O)$.

A conic over M_φ with a point defined over an odd-degree extension of M_φ will necessarily have a point over M_φ. So an obstruction can only occur for branching patterns where in each branching index appears an even number of times in each partition. In particular, the branching fractions in Table 2 must be of the form a, a, b, b for some a, b.

While the conics for C6 and F4 look the same, their sets of bad primes differs. The explanation is that the moduli field of F4 has two infinite primes (two real embeddings) while the moduli field of C6 has only one real embedding.

4.4 Conic-realizations

We say that a Belyi function φ has a conic-realization if there exist a conic C defined over M_φ such that the function field of C over M_φ has a Belyi function $f : C \to P^1$ with the same dessin as φ.

In the example C30, let $L = \mathbb{Q}(d_0)[d_1]/(d_0^2 + 3d_1^2 + 5)$ be the function field of the conic. Then $\varphi \sim g$ where $g := \varphi \circ \mu^{-1} \in L(x)$ with μ as in (4.1),(4.2). Substituting an arbitrary rational number for x in g produces an element of L. This element turns out to be a conic-realization for C30. In the same way we can find a conic-realization for all cases with $|O_\varphi| = 1$.

Given a realization for C30 over L, we can use lemma 4.3 to obtain a realization $h \in L(x)$ for D45. However, if we substitute a rational number for x, the resulting element of L will not have the D45 dessin, instead, it has the C30 dessin. It follows from [14] that the $|O_\varphi| = 2$ cases can not have a conic-realization (in fact, [5, Section 4] explains why no element of $\mathbb{R}(u)[v]/(u^2+v^2+1)$ can have D45 or F6 as a dessin).

Sometimes a conic-realization is a compact way to represent a Belyi function, for example:

$$\text{C6}: \ \frac{(u-5)(u-1)^3(v^4+18v^2+8v(v-58)(u-4)-3403)^3}{3456(v(u-4)-13)^7} \in \mathbb{Q}(u)[v]/(E_1)$$

where $E_1 = u^2 + v^2 + 2$,

$$\text{C30}: \ (v-u-10)\left(\frac{u}{6}+\frac{5}{12}\right)^4 \in \mathbb{Q}(u)[v]/(u^2+3v^2+5),$$

and

$$\text{F11}: \quad v\left(\frac{u-1}{v-u}\right)^5 \in \mathbb{Q}(\sqrt{5})(u)[v]/(E_2)$$

where $E_2 = 29u^2 - 4uv + v^2 - 4u + 1102v + 1 + 2(6u^2 - uv - u + 246v)\sqrt{5}$ which is much shorter than any $\varphi \in \overline{\mathbb{Q}}(x)$ we found for F11.

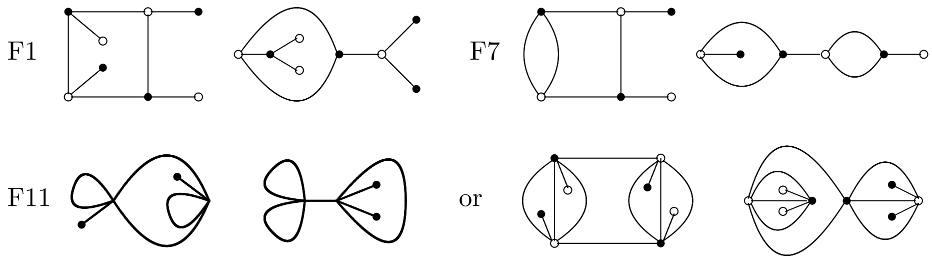


Figure 4: Indefinite cases of moduli fields

4.5 Ambiguous moduli fields

The moduli field for the Galois orbit F7 is $M = \mathbb{Q}(\sqrt{3+6\sqrt{2}})$ by the standard definitions. However, the branching pattern is $[4]+2+1 = 2\,[3]+1 = 2\,[3]+1$ has two symmetric fibers $2\,[3]+1$. By using a Möbius transformation on $\mathbb{P}^1_z$, a Belyi function for F7 can be expressed over $\mathbb{Q}(\sqrt{2})$:

$$\frac{\big(x-\sqrt{2}\big)\big(16x^3-32x^2+\big(26-7\sqrt{2}\big)x-14+10\sqrt{2}\big)\big(16x^2\big(x+1-\sqrt{2}\big)+\big(58-41\sqrt{2}\big)(x-1)\big)}{256\,x^2(x-1)}. \tag{4.3}$$

The branching fibers for this rational function are $z = \infty$ and

$$z = \pm\sqrt{3+6\sqrt{2}}\,\frac{753-531\sqrt{2}}{128}, \tag{4.4}$$

with two fibers conjugated in $M \supset \mathbb{Q}(\sqrt{2})$. Defining a Belyi function by requiring branching in (at most) any 3 fibers, not specifically $\{0,1,\infty\}$, would make no geometrical difference because of Möbius transformations. But evidently, there are arithmetic consequences for moduli and realization fields.

The number of dessins for F7 is 2 or 4 depending of whether the dessins are counted up to Möbius equivalence on $\mathbb{P}^1_z$ or not. Figure 4 depicts two of the dessins for F7. The other two are obtained by swapping the color labeling of black and white vertices, which corresponds to swapping the two symmetric fibers. If the symmetric fibers are put at $z = 0$, $z = 1$, the transformation $z \mapsto 1-z$ swaps the two symmetric fibers and changes the sign of $\sqrt{3+6\sqrt{2}}$. The second depicted dessin for F7 is realized over $\mathbb{R}$.

Even more interestingly, the Galois orbits F1 and F11 demonstrate an interesting mix of a conic obstruction and ambiguous moduli field. Their realization fields are obstructed by the conics in Table 2 if we insist in having the branching fibers at $\{0,1,\infty\}$. But the Belyi functions can be expressed (after a Möbius transformation on $\mathbb{P}^1_z$) over their moduli fields if the location of fibers is not fixed. In particular, the following rational function in $\mathbb{Q}(\sqrt{2})(x)$ gives F1:

$$\frac{x^6+(10-2\sqrt{2})x^5+(2-35\sqrt{2})x^4+(176+84\sqrt{2})x^3+(66+68\sqrt{2})x^2+(44+28\sqrt{2})x-20-14\sqrt{2}}{\big(x^2+2+\sqrt{2}\big)\,\big(x^2-2\sqrt{2}x-2-\sqrt{2}\big)^4}.$$

The ramified fibers are $z = \infty$ and $z = \pm\frac{3}{8}\sqrt{-6\sqrt{2}}$. Reflecting this, the first dessin of F1 in Figure 4 is symmetric if vertex coloring is ignored, but the black and white vertices are interchanged by the complex conjugation. The dessins for F11 are drown in two variations: first compactly, by hiding white vertices of order 2; then assigning the black and white vertices to represent points of order 5 to show the fiber interchanging symmetry.

Examples of coverings with this dual interpretation of the moduli field are given in [6]. The examples of Pharamond have the branching patterns $4+2 = 4+1+1 = 4+1+1$ or $4+2+1 = 4+2+1 = 4+2+1$. The moduli field for the first pattern is $\mathbb{Q}(\sqrt{5})$, but the rational function can be written over $\mathbb{Q}$ if the fiber location is not fixed. There are two orbits for the pattern with three symmetric fibers. A rational function for the first orbit can be similarly written in $\mathbb{Q}(\sqrt{2})$, though the moduli field is $\mathbb{Q}(\sqrt{-1-2\sqrt{2}})$ by definition. A function for the other orbit can be written in $\mathbb{Q}(\sqrt{-6})$, while the moduli field is of degree 12, obtained by adjoining the roots of the polynomial $z^3 - z^2 + (3+\sqrt{-6})z - 3$.

5 Application to Heun functions

The minus-4-hyperbolic Belyi functions have application to transformations between hypergeometric and Heun functions (or their differential equations). This allows to express some Heun functions in terms of better understood hypergeometric functions. In fact, we utilize this application in our algorithms to compute the Belyi functions.

The Gauss hypergeometric equation

$$\frac{d^2y(z)}{dz^2} + \left(\frac{C}{z} + \frac{A+B-C+1}{z-1}\right)\frac{dy(z)}{dz} + \frac{A\,B}{z\,(z-1)}\,y(z) = 0 \tag{5.1}$$

and the Heun differential equation

$$\frac{d^2y(x)}{dx^2} + \left(\frac{c}{x} + \frac{d}{x-1} + \frac{a+b-c-d+1}{x-t}\right)\frac{dy(x)}{dx} + \frac{abx-q}{x(x-1)(x-t)}y(x) = 0 \tag{5.2}$$

are second order Fuchsian equations [24] with 3 or 4 singularities, respectively. The singular points are $z = 0, 1, \infty$ and $x = 0, 1, t, \infty$. If $C \notin \mathbb{Z}$, a basis of local solutions of (5.1) at $x = 0$ is given by the famous *Gauss hypergeometric series*:

$$z^0 \;\cdot\; {}_2\mathrm{F}_1\left(\left.\begin{array}{c} A,\, B \\ C \end{array}\right|\, z\right), \qquad z^{1-C} \;\cdot\; {}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1+A-C,\, 1+B-C \\ 2-C \end{array}\right|\, z\right). \tag{5.3}$$

The starting powers $0, 1-C$ of the local parameter z are the *local exponents* at $z = 0$. The local exponents at $z = 1$ are $0, C-A-B$, while the exponents at $z = \infty$ are A, B. The local exponents for Heun equation (5.2) are

$$\begin{array}{ll} \text{at } x = 0 : 0, 1-c; & \text{at } x = \infty : a, b; \\ \text{at } x = 1 : 0, 1-d; & \text{at } x = t : 0, c+d-a-b. \end{array}$$

The local solution at $x = 0$ with the exponent 0 is denoted by

$$\mathrm{Hn}\left(\begin{array}{c} t \\ q \end{array}\left|\begin{array}{c} a,\, b \\ c;\, d \end{array}\right|\, x\right). \tag{5.4}$$

The parameter q is an *accessory parameter*; it does not influence the local exponents. If $c \notin \mathbb{Z}$, then an independent local solution at $x = 0$ is

$$x^{1-c}\,\mathrm{Hn}\left(\begin{array}{c} t \\ q_1 \end{array}\left|\begin{array}{c} a-c+1,\, b-c+1 \\ 2-c;\, d \end{array}\right|\, x\right) \tag{5.5}$$

with $q_1 = q - (c-1)(a+b-c-d+d\,t+1)$.

A *pull-back transformation* of a differential equation for $y(z)$ in d/dz has the form

$$z \longmapsto \varphi(x), \qquad y(z) \longmapsto Y(x) = \theta(x)\, y(\varphi(x)), \tag{5.6}$$

where $\varphi(x)$ is a rational function, and $\theta(x)$ is a radical function (an algebraic root of a rational function). Geometrically, the transformation *pulls-back* a differential equation on $\mathbb{P}^1_z$ to a differential equation on $\mathbb{P}^1_x$, with respect to the covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ determined by the rational function $\varphi(x)$.

Pull-back transformations between hypergeometric and Heun equations give identities between the classical Gauss hypergeometric and Heun functions. For example, we have

$$\mathrm{Hn}\left(\begin{array}{c} 9 \\ 7/9 \end{array} \middle| \begin{array}{c} 1/3, 1 \\ 7/9; 2/3 \end{array} \middle| \, x \right) = \theta(x)\; {}_2\mathrm{F}_1\left(\begin{array}{c} 1/36, 13/36 \\ 8/9 \end{array} \middle| \, \varphi(x) \right), \tag{5.7}$$

where $\varphi(x)$ is as in Example 2.4, and $\theta(x) = (1-x)^{-1/36}\left(1 - x - \frac{8}{3}x^2 - \frac{8}{27}x^3\right)^{-1/12}$. The transformation of singularities and local exponents for Fuchsian equations is explained in [22, Lemma 2.1]. The prefactor $\theta(x)$ shifts the local exponents at some points, but does not change the exponent difference anywhere. The rational function $\varphi(x)$ multiplies the local exponents and their differences by the branching order at each point. If Q is a singularity of the starting Fuchsian equation in d/dz, a point P above Q will be non-singular for the pulled-back equation only if the branching order at P is n and the exponent difference at Q is equal to $1/n$ (and Q is not a logarithmic point when $n = 1$). For example, the ${}_2\mathrm{F}_1$ function in (5.7) solves a hypergeometric equation in $z, d/dz$ with the exponent differences $1/9, 1/2, 1/3$ at $z = 0, 1, \infty$, respectively, while the exponent differences for the pulled-back Heun equation are the branching fractions $2/9, 1/3, 1/9, 2/3$ at $x = 0, 1, t, \infty$, respectively. The roots of $8x^3 - 72x^2 - 27x + 27$ became non-singular after the proper choice of $\theta(x)$. By the way, the rational function $\varphi(x)$ of Example 2.4 is identified in our classification by the label B7 in Table 2.3.9 of §3.1 and in Appendix §B.

Recently, *parametric* transformations between Heun and hypergeometric equations without Liouvillian solutions[11] were classified in [22], [23]. They apply to hypergeometric equations where at least one exponent difference is not restricted to a value $1/n$ with $n \in \mathbb{N}$; hence a parameter. In total, there are 61 parametric transformations up to the well known symmetries of hypergeometric and Heun equations [23]. But the number of Galois orbits of utilized Belyi coverings (up to Möbius transformations) is 48. These Belyi functions are listed in [22, Table 4]. They satisfy condition *(i)* but not *(ii)* of Definition 1.3, because the parameter(s) could be specialized to satisfy the hyperbolic condition. The parametric transformations are labeled P1–P61 in [23], following similar criteria

[11] Liouvillian solutions [24] of second order linear differential equations are the "elementary" solutions: power, algebraic, exponential, trigonometric functions, their integrals (in particular, logarithmic and inverse trigonometric functions). They can be written in the form $y = \exp(\int r)\, {}_2\mathrm{F}_1(\varphi)$, where r, φ are rational functions, ${}_2F_1$ is Gauss' hypergeometric function with a reducible, dihedral or finite monodromy. There are algorithms to find Liouvillian solutions in this form [18], hence a table of pull-back transformations is not needed.

The hyperbolic restriction $1/k + 1/\ell + 1/m < 1$ gives a finite list of (k, ℓ, m)-minus-4-regular Belyi functions, while $1/k + 1/\ell + 1/m \geq 1$ would lead to infinitely many Belyi functions because of applications to Liouvillian functions.

as in Appendix §A here. The Belyi functions of this article complete the list of hypergeometric-to-Heun transformations when no Liouvillian solutions are involved.

Remark 5.1. *Non-existence of Belyi functions with some branching patterns can be proved by non-existence of implied transformations of Fuchsian equations [22, §5]. For example, there is no* $(2,3,10)$*-minus-*4 *Belyi function with the branching pattern* $9\,[2] = 6\,[3] = [10]+2+2+2+2$*, because it would pull-back a hypergeometric equation with the exponent differences* $1/2, 1/3, 1/2$ *to a non-existent Fuchsian equation with a single singularity where the exponent difference is not* 1 *(but* 5*). This example illustrates that there are no* (k,ℓ,m)*-minus-*1 *functions, unless* $1 \in \{k,\ell,m\}$*. In the exceptions, the implied hypergeometric equation must have a logarithmic singularity with the exponent difference* 1*. In particular, the polynomial* $(x^d+1)^k$ *is a* $(k,1,dk)$*-minus-*1 *Belyi function, and* $(x^3-3x)^{2k}/(x^2-2)^{3k}$ *is a* $(2k,1,3k)$*-minus-*1 *Belyi function.*

An interesting observation is that the pull-back covering $\varphi(x)$ can be recovered from local solutions of the related hypergeometric and Heun equations, if only an oracle would tell us one constant. Particularly, suppose that the point $x=0$ of Heun's equation lies above the singularity $z=0$ of hypergeometric equation. Let y_1, y_2 denote the hypergeometric local solutions in (5.3), respectively, and let Y_1, Y_2 denote the Heun local solutions in (5.4), (5.5), respectively. We have the formula $Y_1(x) = \theta(x)\,y_1(\varphi(x))$ like (5.7), and a similar formula [23, Lemma 3.1] relating y_2, Y_2 but normalized by a constant K that depends on the first power series term of $\varphi(x)$. The quotient $\psi_1(x) = Y_2/Y_1$ does not depend on the prefactor $\theta(x)$, and can be identified with the respective quotient $\psi_0(z) = y_2/y_1$ up to the constant multiple K. We have $\psi_1(x) = x^{1-c}\,(1+O(x))$ and $\psi_0(z) = z^{1-C}(1+O(z))$. The identification $\psi_0(z) = K\psi_1(x)$ gives $z = \psi_0^{-1}(K\psi_1(x))$. Therefore, the Belyi covering $\varphi(x)$ the composition of the inverse function ψ_0^{-1} with $\psi_1(x)$ multiplied by the proper constant K. For instance, the Belyi function of Example 2.4 can be computed by inverting the function

$$z^{1/9}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}5/36, 17/36\\ 10/9\end{array}\right|\, z\right) \Big/ {}_2\mathrm{F}_1\left(\left.\begin{array}{c}1/36, 13/36\\ 8/9\end{array}\right|\, z\right)$$

and composing with

$$(Kx^2)^{1/9}\,\mathrm{Hn}\left(\begin{array}{c}9\\ 187/81\end{array}\left|\begin{array}{c}5/9, 11/9\\ 11/9; 2/3\end{array}\right|\, x\right) \Big/ \mathrm{Hn}\left(\begin{array}{c}9\\ 7/9\end{array}\left|\begin{array}{c}1/3, 1\\ 7/9; 2/3\end{array}\right|\, x\right)$$

where $K = -64/3$. The ratio of two independent solutions of the same differential equation of order 2 is called a *Schwarz map* of the differential equation. We consider the Schwarz maps[12] again in Appendix §D.

[12] In the general context of Fuchsian equations related by a pull-back transformation, the pull-back covering can be similarly recovered by a proper identification (up to a constant multiple) of Schwarz maps as well. In fact, our implemented algorithms often assume a pull-back of a hypergeometric equation to a Fuchsian equation with 4 singularities (rather than canonically normalized Heun's equation), so to avoid extensions of the moduli field. This is done when two or more branching fractions are equal and represent points in the same fiber, as demonstrated by the polynomial W in Example 6.2. Instead of the constants t, q, K in §6.1, the constants j, q, K were generally used.

This observation is significant in a few ways. Firstly, a data base of our Belyi functions could be given by the data of Heun equations to which they apply (the exponent differences, the parameters q, t), the hyperbolic type (k, ℓ, m), and the constant K. The Belyi coverings would be then recovered by reconstructing a rational function from a power series. If d is the degree of a Belyi covering, $2d + 8$ power series terms would suffice (and exclude most of false rational reconstructions). Secondly, given a branching pattern (and thus the exponent differences of presumably related Heun and hypergeometric equations), the Belyi coverings $\varphi(x)$ could be computed by assuming undetermined constants t, q, K and finding algebraic restrictions between them for reconstruction of $\varphi(x)$ from the power series of $\psi_0^{-1}(K\psi_1(x))$. This approach does not appear practical, but §6.2 presents a deterministic algorithm that uses an implied Heun-to-hypergeometric transformation in a similarly general way, and eliminates all undetermined variables except 3 before calling Gröbner basis routines. And thirdly, our probabilistic algorithm §6.1 searches through all possible t, q, K in finite fields, reconstructs possible minus-4-hyperbolic Belyi functions over considered finite fields, and uses a version of Henzel lifting to produce Belyi functions in $\overline{\mathbb{Q}}(x)$.

6 The algorithms

The list of minus-4-hyperbolic Belyi functions was originally generated by a probabilistic algorithm by a thorough examination of Heun functions and their Schwarz maps over some finite fields, and lifting, identifying the obtained Belyi functions in $\overline{\mathbb{Q}}(x)$. This is explained in §6.1. The complete list was generated by considering at most 7 finite fields $\mathbb{F}_p = \mathbb{Z}/(p)$ for $p < 960$, though eventually we kept the algorithm running for total 100 primes. In principle, this does not ensure completeness of the list however.

The deterministic algorithm in §6.2 takes a branching pattern as an input, and produces the Belyi coverings with that branching pattern. By using the implied Heun-to-hypergeometric transformations, smaller degree algebraic systems for undetermined coefficients are obtained than with straightforward methods, and with far less *parasitic* solutions [11]. The deterministic algorithm produced the same Belyi maps (up to Möbius transformations) as the probabilistic one. Completeness of our results is proved assuming correct implementation of the deterministic algorithm.

As a practical matter of confidence, the completeness of results is foremost verified by the same output of the two independent algorithms. In addition, we did a combinatorial search to find all minus-4-hyperbolic dessins d'enfant, up to degree 36. This gives a verification of a large part ($\approx 95\%$) of relevant branching patterns, covering $\approx 91\%$ of obtained dessins. We also compared the list of Belyi functions with the r-field in $\mathbb{R}$ with Felikson's list [7] of *Coxeter decompositions* in the hyperbolic plane; see Appendix §D. This provides enough confidence in completeness of our results.

6.1 A probabilistic modular method

The used probabilistic algorithm is based on the expectation that a Belyi function with any realization field will be properly defined over a p-adic field $\mathbb{Q}_p$ for some prime p among a sequence of considered subsequent or random primes. Concretely, suppose that a Belyi function $\varphi(x)$ pulls-back a hypergeometric equation to Heun's equation with specific parameters t, q, and that respective Schwarz maps of both equations are identified by a constant K as described after Remark 5.1. If t, q, K are elements of a number field $\mathbb{Q}(\alpha)$, then $\varphi(x) \in \mathbb{Q}(\alpha)(x)$. By Chebotarev's theorem [24], the minimal polynomial for α has a root in $\mathbb{F}_p$ for a positive density of primes p. The density is at least $1/D$, where D is the degree of the number field $\mathbb{Q}(\alpha)$. For all but finitely many of those primes, we will have $\alpha \in \mathbb{Q}_p$ and $t, q, K \in \mathbb{Z}_p$ (the p-adic integers). The Belyi function $\varphi(x)$ can be found as follows:

- (i) Consider all possible values $\overline{t}, \overline{q}, \overline{C} \in \mathbb{F}_p$ of the reduction of t, q, K modulo an (eventually) suitable prime p;
- (ii) Reconstruct $\varphi(x)$ as a rational function in $\mathbb{F}_p[x]$ by identifying the Schwarz maps as described after Remark 5.1. We need the first $2d + 8$ terms in the Schwarz maps $\psi_0(x)$ and $\psi_1(x)$ to be in $\mathbb{F}_p$, so p has to be sufficiently large. For example, if a local exponent difference is $1/3$, then we need $p > 3(2d + 8)$ to ensure that local hypergeometric solutions have the first $2d + 8$ coefficients in $\mathbb{Z}_p$. For degree 60 coverings, the starting prime was $907 > 7(2 \cdot 60 + 8)$.
- (iii) Use Hensel lifting to obtain an expression of $\varphi(x)$ in $\mathbb{Q}_p$;
- (iv) Use LLL techniques to compute minimal polynomials of its coefficients, thus reconstructing $\varphi(x)$ as an element of $\mathbb{Q}(\alpha)(x)$.

Our strategy is as follows. Foe each branching pattern of Tables 2.3.7–3.4.4, we run through a sequence of primes p and the possible reduced values $\overline{t}, \overline{q}, \overline{C} \in \mathbb{F}_p[x]$. For each of the $O(p^2)$ pairs of $\overline{t}, \overline{q}$ we have to compute series expansions for the solutions of H_0 and H_1. This is done rapidly using linear recurrences for coefficients of these solutions; Maple has the command gfun[diffeqtorec] for getting the recurrences. We expect φ to be in $\mathbb{F}_p[[x]]$ for suitable primes p. If $\psi_0 \circ \varphi$ matches $K\psi_1$ in $\mathbb{Q}_p[[x]]$, then this poses certain necessary conditions[13] on the p-adic valuations of the coefficients of ψ_0 and ψ_1. We compute the series solutions of H_0 and H_1 to enough precision so that we can test these necessary conditions. This way, many pairs $\overline{t}, \overline{q}$ can be discarded, and we typically end up with $O(p^1)$ pairs. Thus, the rational reconstruction step *(ii)* "only" needs to be called for $O(p^2)$ combinations of $\overline{t}, \overline{q}, \overline{K}$.

If we find a φ mod p, we store it in a file. Another program will Hensel lift it, apply LLL reconstruction to $\mathbb{Q}(\alpha)(x)$, and compare with the already computed data base. Each Belyi map φ has a density δ_φ of suitable primes. The expectation number of times that the same φ will be found is then $100 \cdot \delta_\varphi$. Unless the density is tiny, the likelihood that φ will be found is very high.

[13] If $\varphi = \lambda x^m + \cdots \in \mathbb{Z}_p[[x]]$ is substituted into $\psi_0 = x^{d_1}(1 + a_1 x + a_2 x^2 + \ldots)$, with $a_i \in \mathbb{Q}_p$, and if the first $a_i \notin \mathbb{Z}_p$ is a_n, and if $\psi_1 = \lambda^{d_1} x^{d_1 m}(1 + b_1 x + b_2 x^2 + \ldots)$ is the result of the substitution, then the first $b_i \notin \mathbb{Z}_p$ must be b_{mn}.

The smallest δ_φ encountered was 1/6, for the H10–H14 coverings[14] with the realization field $\mathbb{Q}(\zeta_7)$. Most of the table was found after just two primes. The first 10 primes took about a week on Maple, running on 12 CPU's. Among the 100 primes, each Belyi function was found at least 16 times.

The modular method is quite slow, because $O(p^3)$ combinations of $\bar{t}, \bar{q}, \overline{C}$ have to be inspected for each p. But its advantage is low requirement of computer memory. This means that the computation can continue for weeks on end, without a risk that the computation will halt due to memory problems, and without human intervention (this is important, because if human intervention is needed in any of the steps, then, in a table with hundreds of cases, a gap would become likely).

6.2 A deterministic algorithm

A (k, ℓ, m)-minus-4 Belyi function is determined by a polynomial identity

$$P^k\,U + Q^\ell\,V = R^m\,W, \tag{6.1}$$

where P, Q, R are monic polynomials in $\mathbb{C}[x]$ whose roots are the regular branchings, and U, V, W are polynomials whose roots are exceptional points with correct multiplicities. The Belyi function is then expressed as

$$\varphi(x) = \frac{Q^\ell\,V}{R^m W}, \qquad 1 - \varphi(x) = \frac{P^k\,U}{R^m W}. \tag{6.2}$$

The polynomials P, Q, R should not have multiple roots; W may be monic. The 4 exceptional points may be immediately set to $x = 0, 1, t, \infty$, but that may introduce an extension of the moduli field if there are exceptional points in the same fiber with the same branching order. If there is a point above $\varphi \in \{0, 1, \infty\}$ that has a unique (possibly regular) branching order for that fiber, it is convenient to set that point to $x = \infty$ without extending the moduli field. The degrees of the polynomials in (6.1) are determined by the branching pattern and the assignment of $x = \infty$.

The most straightforward computational method is to assume undetermined coefficients of the polynomials in (6.1), and solve the resulting system of algebraic equations between the coefficients. This is not practical for Belyi functions of degree ≥ 12, mainly because of numerous *parasitic* [11] solutions where some polynomials in (6.1) have common roots.

A more restrictive set of equations for undetermined coefficients can be obtained by differentiating $\varphi(x)$, as in [22, §4.1]. In particular, the roots of $\varphi'(x)$ include the branching points above $\varphi = 1$ with the multiplicities reduced by 1. A factorized shape of logarithmic derivatives must be the following:

$$\frac{\varphi'(x)}{\varphi(x)} = h_1 \frac{P^{k-1}\,U}{Q\,R\,F}, \qquad \frac{\varphi'(x)}{\varphi(x) - 1} = h_2 \frac{Q^{\ell-1}\,V}{P\,R\,F}. \tag{6.3}$$

Here h_1, h_2 are constants, and F is the product of irreducible factors of $U\,V\,W$, each to the power 1. On the other hand,

$$\frac{\varphi'(x)}{\varphi(x)} = \ell\,\frac{Q'}{Q} + \frac{V'}{V} - m\,\frac{R'}{R} - \frac{W'}{W}, \tag{6.4}$$

[14] The estimate $\delta_\varphi \geq 1/\deg \mathbb{Q}(\alpha)$ is sharp when $\mathbb{Q}(\alpha) \supset \mathbb{Q}$ is a Galois extension. This is the case for $\mathbb{Q}(\zeta_7)$. Higher degree encountered number fields (such as for J28) had significantly higher $\delta_\varphi > 1/6$.

and similarly for the logarithmic derivative of $\varphi(x)-1$. Both logarithmic derivatives have two expressions. This gives a generally stronger[15] set of algebraic equations. If $k=2$, the polynomial P can be even eliminated symbolically. This setting does not use the location $\varphi=1$ of the first fiber. For convenience, we assume the polynomials U,V,W to be monic therefore. Then $\varphi(x)$ in (6.2) is determined up to a constant multiple, which can be found by at the latest stage by evaluating (6.1) at a root of Q or V. If $x=\infty$ is in the $\varphi=\infty$ fiber additionally, the constants h_1,h_2 in (6.3) are equal to the branching order at $x=\infty$, which is then the residue value of (6.4) at $x=\infty$.

To get an even more restrictive system of algebraic equations, we utilize the fact that our Belyi functions transform hypergeometric equations to Heun equations. The method bluntly uses the following lemma.

Lemma 6.1. *Let $\varphi(x)$ be a Belyi map as in* (6.2)*. Hypergeometric equation* (5.1) *with*

$$A=\frac{1}{2}\left(1-\frac{1}{k}-\frac{1}{\ell}-\frac{1}{m}\right),\quad B=\frac{1}{2}\left(1-\frac{1}{k}-\frac{1}{\ell}+\frac{1}{m}\right),\quad C=1-\frac{1}{\ell}.$$

is transformed to the following differential equation under the pull-back transformation $z\mapsto\varphi(x)$, $y(z)\mapsto\left(R^mW\right)^A\,Y(\varphi(x))$:

$$\begin{aligned}&\frac{d^2Y(x)}{dx^2}+\left(\frac{F'}{F}-\frac{U'}{k\,U}-\frac{V'}{\ell\,V}-\frac{W'}{m\,W}\right)\frac{Y(x)}{dx}+\\&+A\left[B\left(\frac{h_1h_2\,P^{k-2}Q^{\ell-2}\,U\,V}{R^2F^2}-\frac{m^2R'^2}{R^2}-\frac{W'^2}{W^2}\right)+\frac{mR''}{R}+\frac{W''}{W}+\right.\\&\quad\left.+\left(\frac{1}{k}+\frac{1}{\ell}\right)\frac{mR'W'}{R\,W}+\left(\frac{mR'}{R}+\frac{W'}{W}\right)\left(\frac{F'}{F}-\frac{U'}{kU}-\frac{V'}{\ell V}-\frac{W'}{W}\right)\right]Y(x)=0.\end{aligned}$$

Proof. A lengthy symbolic computation, using (6.2) and (6.4). □

The transformed equation is to be identified with the target Heun equation, or (if the roots of U,V,W are not normalized to $x=0,1,t,\infty$) with a Fuchsian equation with 4 singularities at the roots of UVW. The accessory parameter q is a new undetermined variable. The terms to $dY(x)/dx$ are always identical, but comparison of the terms to $Y(x)$ gives new algebraic equations between the undetermined variables. If $k=2$, $\ell=3$, not only P but also Q can be eliminated symbolically. The two expressions in (6.4) and Lemma 6.1 then give a non-linear differential equation for R, with q and the coefficients of (presumably monic) U,V,W as parametric variables. After substitution of general polynomial expressions for R and U,V,W, we collect to the powers of x and get a system of algebraic equation for undetermined coefficients.

Example 6.2. *Consider computation of degree 54 Belyi functions with the branching fractions* $1/7,1/7,1/7,2/7$. *We assign the branching fraction* $2/7$

[15] Particularly, the new system has less parasitic solutions in general. Let $G=\gcd(P^kU,Q^\ell V,R^mW)$ and $H=PQRF/G$. Then only those parasitic solutions remain for which H is a polynomial. Besides, a root of G is then a root of H if and only if it divides one of $P^kU,Q^\ell V,R^mW$ in a higher order than others, and then it must be a simple root of H. This follows from invariance of the rational functions in (6.3) in the hyperbolic and the reduced settings.

to $x=\infty$, so that $U=V=1$, and the polynomials P,Q,R,W are monic, without multiple roots, of degree $27,18,7,3$ respectively. If we would assume $W=x(x-1)(x-t)$, the Heun equation would have $a=9/14$, $b=13/14$ and $c=d=6/7$. To avoid increase of the moduli field, we rather assume $W=x^3+w_2x+w_3$. Here the x^2 term is zero-ed by a translation $x\to x+\beta$, so that only scaling Möbius transformations $x\to\alpha\,x$ are left to act. The transformed Fuchsian equation must have the following term to $Y(x)$: $ab(x-q)/W$. Symbolic elimination of P,Q on Maple *gives the following differential expression:*

$$\begin{aligned}&\frac{7R''''}{15R}+\frac{7R'''}{3R}\left(\frac{W'}{W}-\frac{R'}{R}\right)+\frac{(7R'')^2}{26R^2}+\frac{R''W'}{RW}\left(\frac{13W'}{7W}-\frac{35R'}{13R}\right)\\ &+\frac{3R''}{7RW}\left(115q-\frac{1033}{13}x\right)+\frac{R'^2}{7R^2W}\left(\frac{3}{2}(163x-247q)+\frac{16W'^2}{13W}\right)-\frac{13W'}{2W^2}\\ &+\frac{3R'}{2RW}\left(\left(\frac{183}{7}q-\frac{241}{13}x\right)\frac{W'}{W}+\frac{67}{21}\right)+\frac{18}{W^2}\left(2x-\frac{13}{5}q\right)\left(\frac{46}{13}x-\frac{27}{7}q\right)=0.\end{aligned}$$

Here the values $W''=6x$, $W'''=6$, $W''''=0$ are simplified. Substituting the explicit W, $R=x^7+r_1x^6+\ldots+r_6x+r_7$ and clearing the denominator, we obtain a polynomial expression of degree 15 in x. The leading term gives $q=-5r_1/52$. The next term gives nothing new[16]*. But the next 5 equations allow subsequent elimination of r_3,r_4,r_5,r_6,r_7 in terms of r_1,r_2,w_2,w_3. The 4 remaining variables are weighted-homogeneous, with weights $1,2,2,3$. Elimination of w_2,w_3 using the other 10 equations is done with the Gröbner basis routine of* Maple 15 *in about 35s. The algebraic system has 4 Galois orbits of solutions, 3 of them parasitic*[17]*. In the proper solution, we can take*

$$W=x^3-4899x-370078,\qquad R=x^7+28x^6+\frac{29063265}{512}x^5+\ldots.$$

The expression for $\varphi(x)$ is long. We looked for an optimizing Möbius transformation. The bit size of $\varphi(x)$ is reduced by the factor ≈ 2.26 after the Möbius substitution $x\mapsto(241x-212)/(x+4)$. Then

$$\varphi(x)=\frac{Q^3}{864(x-4)(3x^2+1)(x+4)^2R^7},$$

where $R=3x^7-7x^6-14x^5-98x^4+147x^3-7x^2+56x+16$ and

$$\begin{aligned}Q=&\,47x^{18}-2028x^{17}+5502x^{16}+54540x^{15}-263535x^{14}-32592x^{13}+2249268x^{12}\\ &-3436872x^{11}+14145x^{10}-1425900x^9-8774370x^8-1715652x^7-10594017x^6\\ &+2223144x^5-5284080x^4+1638144x^3-1306368x^2+239616x-135168.\end{aligned}$$

[16] We observed the following property of the polynomial system generated by just (6.4). Let E_1 denote the sequence of polynomial equations given by the subsequent leading x-terms from the identification of $\varphi'(x)/\varphi(x)$, and E_2 denote the same sequence from the other logarithmic derivative. If $x=\infty$ is in the fiber $\varphi=\infty$ and has branching order e, then the first e equations from E_1 and the first $e-1$ equations from E_2 are usually independent, but the eth equation from E_2 is always a consequence of those.

[17] The parasitic solutions are: the degree 18 covering mentioned in footnote 3; a degree 10 covering with the branching pattern $5[2]=3[2]+4[1]=7+2+1$; and the non-cyclic cubic Belyi covering. In all cases, the simplification of the numerator and the denominator of $\varphi(x)$ is by a linear polynomial to the maximal power (36, 44 or 51).

Even if $(k,\ell) \neq (2,3)$, most of the coefficients of P, Q, R can be eliminated subsequently from the two expressions of logarithmic derivatives and the $Y(x)$ term of the transformed Fuchsian equation. If $x = \infty$ is assigned as a point above $\varphi = \infty$, the highest order terms usually allow elimination of all but 3 variables (or all except 4 weighted homogeneous variables, if the scaling transformations $x \to \alpha x$ are allowed to act). Lemma 6.1 fails to give new algebraic relations when $A = 0$.

The order of k, ℓ, m (or the 3 fibers) is not essential, of course. In our algorithm realization, we sought to assign a point $x = \infty$ to an exceptional or regular point (preferably with a unique branching order in its fiber, and of maximal possible branching order e), and then assign that fiber as $\varphi = \infty$. With this we take advantage of the explicit constants $h_1 = h_2 = e$ in (6.4) and sooner eliminations. The hardest Gröbner basis computation is with the last 3 (or 4 if weighted homogeneous) variables. The strategy of using first a total degree, then elimination of 2 variables appears to be fastest for complicated examples. On Maple 15, the degree 60 Belyi maps were computed in 110s, the Galois orbit J28 in 274s, and the orbit pair H11, J26 in 830s.

Lemma 6.1 was already used with $k = \ell = 2$ in [21] basically, to derive linear differential expressions for the polynomial components of Belyi maps for Kleinian transformations of hypergeometric equations with a finite dihedral monodromy group.

6.3 Additional algorithms

The main work for this paper was to compute all Belyi functions (up to Möbius equivalence, and up to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugacy) for the Heun table. But we computed a lot of additional data as well, and that required implementing numerous additional algorithms. The following is a summary, sorted roughly w.r.t. the amount of work involved.

(1) Given a 4-hyperbolic Belyi map, compute its branching type, its t-value, j-invariant, the canonical realization field, and moduli field.

(2) Given (g_1, g_0, g_∞) and $(\tilde{g}_1, \tilde{g}_0, \tilde{g}_\infty)$, with $g_i, \tilde{g}_i \in S_d$, $\quad g_0 g_1 g_\infty = 1$ and $<g_0, g_1>$ transitive (and likewise for the $\tilde{g}_i$). Decide if they represent the same dessin or not (decide if $\exists_{h \in S_d}\ h^{-1} g_i h = \tilde{g}_i$ for $i \in \{0, 1, \infty\}$).

(3) Compute the obstruction conic and a conic-model (if it exists).

(4) Decompositions. Given (g_1, g_0, g_∞) representing a dessin, compute the subfield lattice of $\overline{\mathbb{Q}}(x)/\overline{\mathbb{Q}}(\varphi)$, where each subfield is represented by its dessin, by which is meant the following: If $\overline{\mathbb{Q}}(\varphi) \subseteq L \subseteq \overline{\mathbb{Q}}(x)$ is a subfield, then $L = \overline{\mathbb{Q}}(g)$ for some g by Lüroth's theorem, and $\varphi = h(g)$ for some Belyi map h. By dessin of L we mean the dessin of h.

(5) Given a Belyi map in $f \in K(x)$ and an embedding $K \to \mathbb{C}$ compute the dessin d'enfant of the image of f under this embedding.

(6) Given $\varphi \in K(x)$, if there exists a Möbius-equivalent $\tilde{\varphi}$ of substantially smaller bitsize, then find such $\tilde{\varphi}$.

Item (1) is done directly from the definition. Most of the time for item (1) was spent on optimizing the polynomials representing the number fields (this can be done with the polredabs command from GP/PARI, but we had to work around some efficiency problems, avoiding unnecessary integer factorization).

For item (2), suppose that $h(1) = b$ for some yet to be determined $b \in \{1, \ldots, d\}$. Then $h(g_0^{n_1} g_1^{n_2} \cdots g_1^{n_k}(1)) = \tilde{g}_0^{n_1} \tilde{g}_1^{n_2} \cdots \tilde{g}_1^{n_k}(b)$ for all $n_1, \ldots, n_k$. That determines h because $<g_0, g_1>$ acts transitively. So we can find h (if it exists) by checking d cases, $b = 1, 2, \ldots, d$.

Item (3) was explained in Section 4. For item (4) we computed the subgroups H of the monodromy group $G :=< g_0, g_1 >$ that contain $\{g \in G | g(1) = 1\}$. Given such H, writing down the action of g_1, g_0, g_∞ on the cosets of H produces the dessin of the subfield corresponding to H. We then identified the component Belyi maps (such as h of item (4)) by using the full table of known hypergeometric and hypergeometric-to-Heun transformations; This way we obtained all decompositions of all entries of our table; see [17, Decomposition_or_GaloisGroup] for used notation and detailed composition lattices.

Although algorithms for computing the monodromy exist [19] item (5) was still a considerable amount of work because we had to develop our own implementation, specifically optimized for rational functions of high degree that ramify over only 3 points.

Item (6) consists of a collection of algorithms, we will sketch a few. Suppose $\varphi \in K(x)$ and $S = \varphi^{-1}(\{0, 1, \infty\})$. If some $\alpha \in S$ has a minimal polynomial $f \in K[x]$ of degree 3, we can apply polredabs to find an optimized polynomial g, then compute a Möbius transformation over K that will send α to a root of g, and check if it makes φ smaller. A similar trick will work if we have three points of degree 1, or one of degree 1 and one of degree 2.

Given a factor $f = a_n x^n + \cdots a_0 x^0$ we can also try scaling, i.e. multiply x by primes that appear in a_n, a_0 to see if that makes φ smaller (implementing this for number fields is work). Scaling is generally most effective after applying other steps first. For example, if S has just one point in $K \bigcup \{\infty\}$, then first shift that to ∞, then select a factor $f = a_n x^n + \cdots a_0 x^0$, clear its second highest term with $x \mapsto x - a_{n-1}/(n a_n)$, and then apply scaling.

A Appendix: Sorting criteria

In §3.1, the minus-4-hyperbolic Belyi functions were grouped into 10 classes A–J. We order the Belyi functions inside those classes by the following criteria:

(a) the first criterium is the j-invariant;

(b) the second criterium is the branching fractions[18];

(c) the last criterium is the degree of the covering.

The sort of j-invariants lexicographically adheres to the following criteria:

(a1) the j-field;

[18] The first two criteria establish that our list is basically sorted by Heun equations. To identify the Heun equations, invariants describing accessory parameters should be added [23, §D].

(a2) the t-field;

(a3) the leading coefficient of the minimal polynomial in $\mathbb{Z}[x]$ for the j-invariant.

The order of j-fields and t-fields is settled by the following criteria:

(f1) the field degree;

(f2) if the field is a quadratic extension of $\mathbb{Q}$ then:

- *(f1a)* real quadratic fields have precedence over $\mathbb{Q}(\sqrt{a})$ with $a < 0$;
- *(f1b)* the fields $\mathbb{Q}(\sqrt{a})$ with the same sign of a are ordered by the increasing order of $|a|$.

(f3) if the field is of higher degree, then the criterium is the field discriminant.

The integers in *(a3)* and *(f3)* are ordered as follows:

(i1) the product of the primes dividing the integer;

(i2) the absolute value.

The numbers in *(i1)*, *(i2)*, *(f1b)* and *(c)* are ordered in the increasing order. The tuples of branching fractions are ordered as follows

(b1) in each tuple, the four branching fractions are ordered in the increasing order of their denominators, then secondarily the numerators.

(b2) the tuples are compared lexicographically, from their first elements, and the elements are matched first by their denominators then numerators.

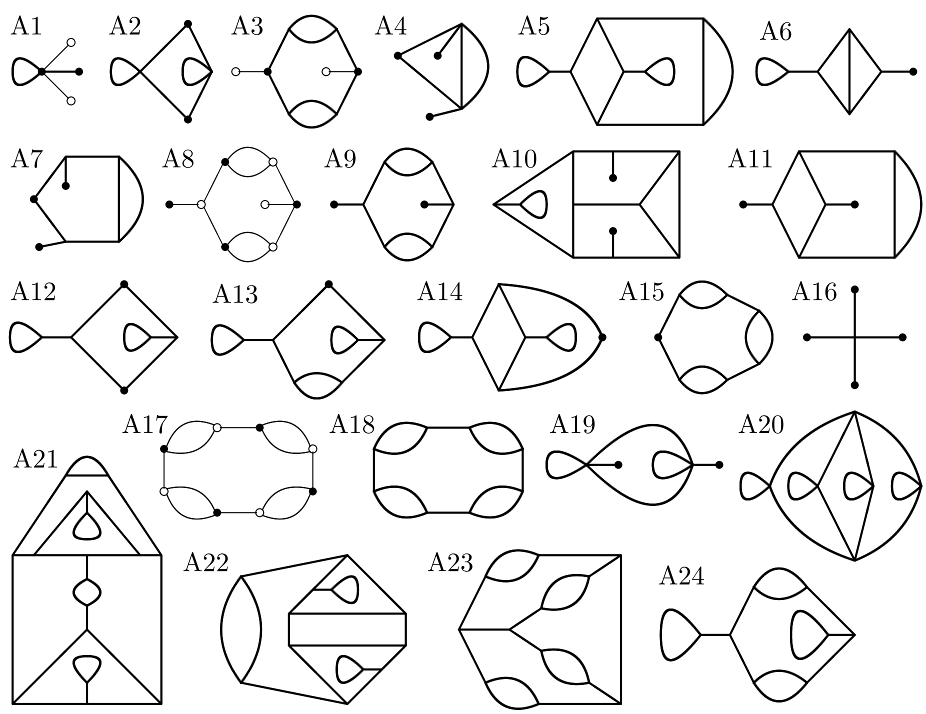
These criteria break all ties in our list of Belyi functions. Due to *(i1)*, the fields or t-values that ramify or degenerate modulo the same set of primes are placed next to each other. The leading coefficient in *(a3)* gives information about the primes where the covering is ramified. In particular, for $j \in \mathbb{Q}$ the leading coefficient is the denominator of j.

B Appendix: The A-J tables

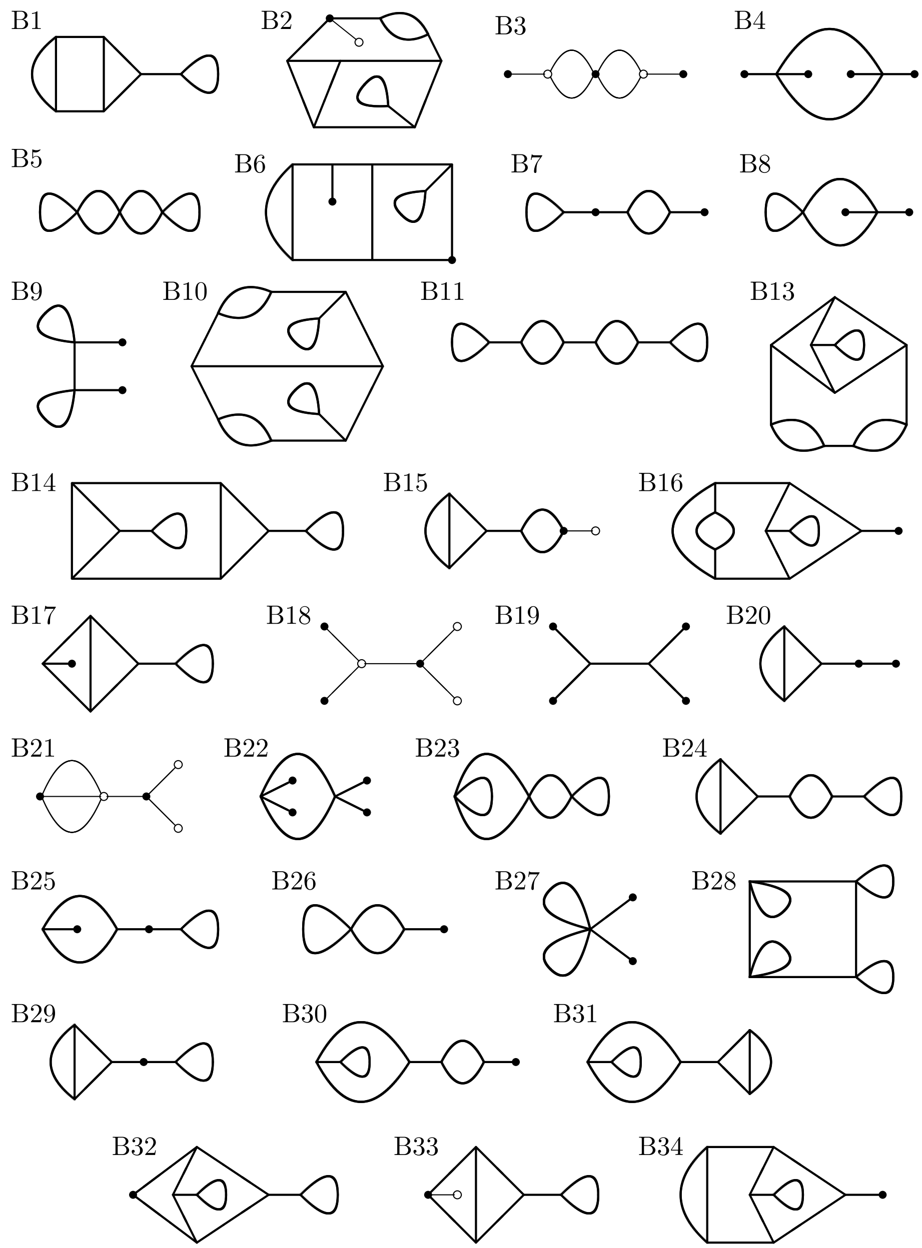
The following pages display tables of Galois orbits of minus-4-hyperbolic Belyi functions, grouped as specified in §3.1 and ordered by the criteria in §A. All tables contain the following columns:

- Id: the label from A1 to J18;
- Branching frac.: the branching fractions of a minus-4-hyperbolic function;
- d: the degree of a Belyi function;
- $[k\ell m]$: the values of k, ℓ, m written compactly. For $k = 2, \ell = 3, m \geq 10$, only the value of $m \in [10, 14]$ is given.
- Monodromy/comp. or Mndr/cmp.: The monodromy group $G = \ldots$ is given for indecomposable coverings, and compositions are indicated otherwise. The composition notation is explained in §C.

Id	Branching frac.	d	$[k\ell m]$	Monodromy/comp.	m-$\sqrt{\ }$	r-$\sqrt{\ }$
A1	1/2, 1/2, 1/5, 1/5	6	[255]	$G = A_5$	—	−1
A2		12	[245]	$2\,[445] \circ 6,\ \mathrm{A1} \circ 2$	—	−1
A3	1/2, 1/2, 2/7, 2/7	18	[237]	$2\,[277] \circ 9$	—	−7
A4	1/2, 1/4, 1/4, 2/5	12	[245]	$2\,[445] \circ 6$	−1	−1
A5	1/2, 1/4, 1/8, 1/8	24	[238]	$2\,[488] \circ 2\,[288] \circ 2\,[248] \circ 3$	—	—
A6	1/3, 1/3, 1/3, 1/9	16	[239]	$4\,[339] \circ 4$	—	—
A7	1/3, 1/3, 2/3, 2/7	16	[237]	$2\,[337] \circ 8$	−3	−3
A8	1/3, 1/3, 1/4, 1/4	10	[334]	$G = A_6$	—	−2
A9		20	[238]	$2\,[388] \circ 10,\ \mathrm{A8} \circ 2$	—	−2
A10	1/3, 1/3, 1/7, 3/7	32	[237]	$4\,[337] \circ 8$	—	−3
A11	1/3, 1/3, 2/7, 4/7	20	[237]	$2\,[377''] \circ 10$	—	—
A12	2/3, 2/3, 1/7, 1/7	16	[237]	$2\,[337] \circ 8$	—	−3
A13	2/3, 1/4, 1/8, 1/8	20	[238]	$2\,[388] \circ 10$	−2	−2
A14	2/3, 1/7, 1/7, 4/7	20	[237]	$2\,[377''] \circ 10$	—	—
A15	2/3, 2/7, 2/7, 2/7	20	[237]	$2\,[377''] \circ 10$	—	—
A16	1/4, 1/4, 1/4, 1/4	8	[248]	$2_H \circ 2\,[444] \circ 2$	—	−1
A17		12	[334]	$2_H \circ 2\,[444] \circ 3$	—	3
A18		24	[238]	see diagram (C.1)	—	—
A19	1/5, 1/5, 1/5, 1/5	12	[255]	$2_H \circ \mathrm{A1}$	—	—
A20		24	[245]	$2_H \circ \mathrm{A2}\,\{\mathrm{A19}, [445]\}$	—	—
A21	1/7, 1/7, 2/7, 2/7	48	[237]	$2\,[777] \circ 3\,[337] \circ 8$	—	—
A22	1/7, 1/7, 2/7, 4/7	36	[237]	$2\,[777''] \circ 2\,[277] \circ 9$	−7	−7
A23	2/7, 2/7, 2/7, 2/7	36	[237]	$4\,\{\mathrm{A3}, [277'']^{\times}\}[277] \circ 9$	—	−7
A24	1/9, 1/9, 2/9, 2/9	24	[239]	$2\,[999] \circ 3\,[339] \circ 4$	—	−3



Id	j-invariant	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
B1	$2^4 13^3/3^2$	$1/2, 1/4, 1/8, 3/8$	18	[238]	$3\,[288] \circ 2\,[248] \circ 3$
B2		$1/2, 1/7, 2/7, 3/7$	27	[237]	$3\,[277] \circ 9$
B3		$1/4, 1/4, 1/4, 1/4$	6	[344]	$2_H \circ 3$
B4			12	[246]	$4_H\{\text{D25}, \text{P22}\{\text{B3}\}, \text{P39}\} \circ 3$
B5	$2^2 73^3/3^4$	$1/3, 1/3, 1/6, 1/6$	12	[246]	$6\,\{\text{P16}, [366]\}[266] \circ 2$
B6		$1/3, 2/3, 1/7, 2/7$	24	[237]	$3\,[337] \circ 8$
B7		$1/3, 2/3, 1/9, 2/9$	12	[239]	$3\,[339] \circ 4$
B8		$1/4, 1/4, 1/5, 4/5$	10	[245]	$G = S_{10}$
B9		$1/4, 1/4, 1/8, 1/8$	10	[248]	$G = A_6 : C_2$
B10			30	[238]	$3\,[388] \circ 10$, $\text{B9} \circ 3$
B11		$1/6, 1/6, 1/12, 1/12$	18	[12]	$6\,\{\text{P17}, [3\,12\,12]\}[2\,6\,12] \circ 3$
B12		$1/7, 1/7, 3/7, 3/7$	36	[237]	$4\,[277] \circ 9$
B13		$1/7, 2/7, 2/7, 4/7$	30	[237]	$3\,[377''] \circ 10$
B14		$1/8, 1/8, 3/8, 3/8$	24	[238]	$8\,\{\text{D9}, [288]\}[248] \circ 3$
B15	$6481^3/3^8 5^2$	$1/2, 1/4, 1/4, 3/8$	15	[238]	$5\,[248] \circ 3$
B16	$2^6 7^3 97^3/3^6 5^4$	$1/3, 1/7, 2/7, 4/7$	28	[237]	$G = A_{28}$
B17		$1/3, 1/7, 3/7, 5/7$	16	[237]	$G = A_{16}$
B18	$7^3 127^3/2^2 3^6 5^2$	$1/3, 1/3, 1/3, 1/3$	5	[335]	$G = A_5$
B19			10	[10]	$2_H \circ 5$, $\text{B18} \circ 2$
B20	$7^3 2287^3/2^6 3^2 5^6$	$1/3, 2/3, 2/7, 3/7$	12	[237]	$G = S_{12}$
B21		$1/4, 1/4, 1/4, 1/4$	6	[344]	$G = S_5$
B22			12	[246]	$2_H \circ \text{C24}$, $\text{B21} \circ 2$
B23		$1/5, 1/5, 2/5, 3/5$	12	[245]	$6\,[255] \circ 2$
B24		$1/5, 1/5, 1/10, 3/10$	18	[10]	$6\,[2\,5\,10] \circ 3$
B25	$2^6 7^3 31^3 271^3/3^{10} 11^4$	$1/3, 2/3, 1/7, 4/7$	12	[237]	$G = S_{12}$
B26	$4993^3/2^2 3^8 7^4$	$1/4, 3/4, 1/5, 2/5$	8	[245]	$G = S_8$
B27		$1/6, 1/6, 1/6, 1/6$	8	[266]	$G = \mathrm{PSL}(3,2) : C_2$
B28			16	[246]	$2_H \circ \text{C3}$, $\text{B27} \circ 2$
B29	$2^4 3^3 7^6 103^3/5^6 11^4$	$2/3, 1/4, 1/8, 3/8$	14	[238]	$G = S_{14}$
B30	$2^4 181^3 2521^3/3^6 5^4 13^4$	$1/3, 1/4, 1/8, 5/8$	16	[238]	$G = A_{16}$
B31	$73^3 193^3 409^3/2^2 3^2 5^4 7^8$	$1/7, 2/7, 3/7, 5/7$	18	[237]	$G = S_{18}$
B32	$49201^3/2^8 3^6 5^2 11^4$	$2/3, 1/7, 1/7, 4/7$	20	[237]	$G = S_{20}$
B33	$2^4 106791301^3/3^{14} 5^2 7^8 11^6$	$1/2, 1/7, 3/7, 4/7$	15	[237]	$G = S_{15}$
B34	$829^3 30469^3/3^6 5^6 7^8 19^4$	$1/3, 1/7, 2/7, 5/7$	22	[237]	$G = S_{22}$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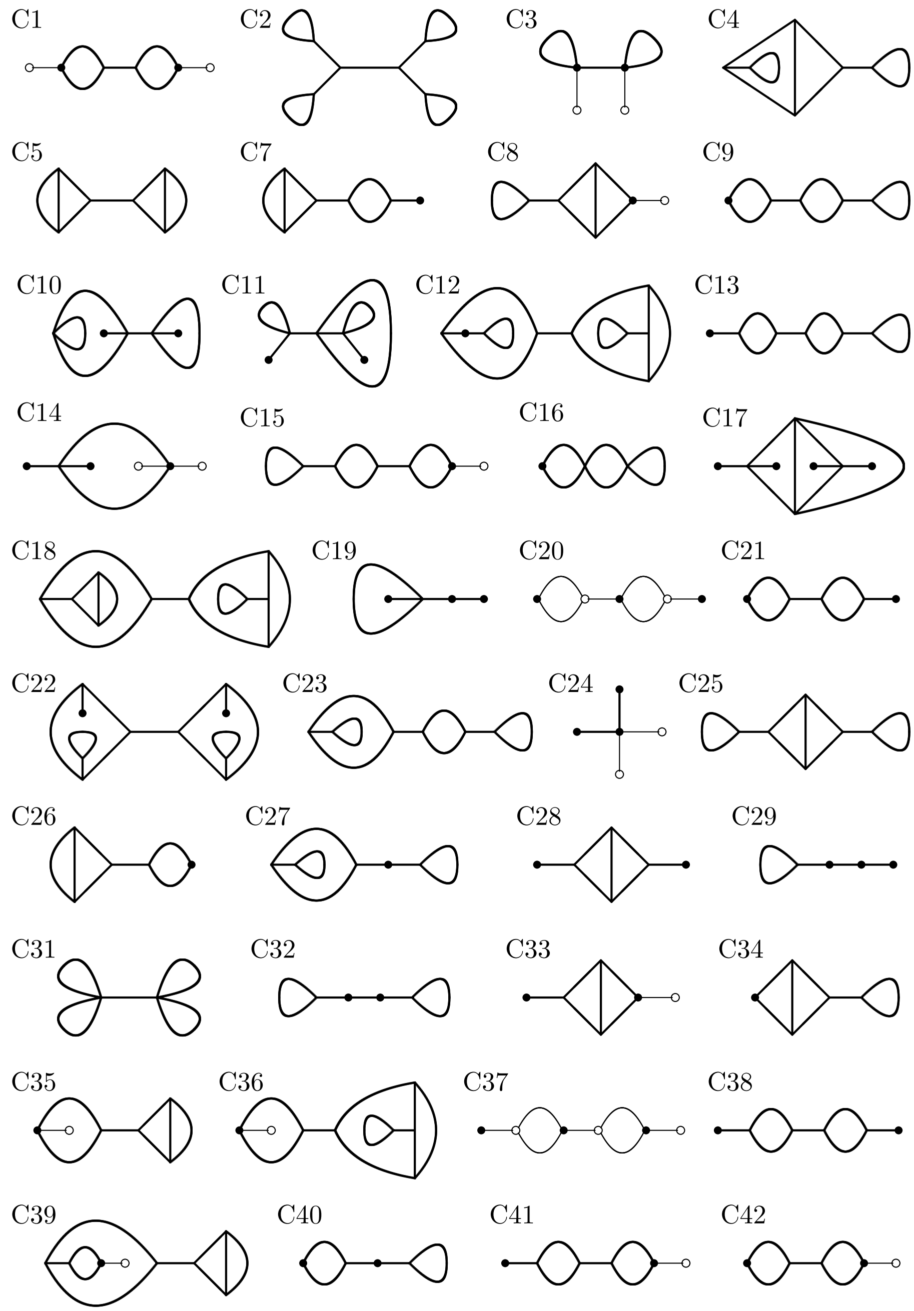
Id	$\sqrt{\ }$	j-invariant	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
C1	2	$2^3 3^3 11^3$	$1/2, 1/2, 1/4, 1/4$	12	[238]	$2_H \circ 2\,[248] \circ 3$
C2		$5^3 11^3 31^3/2^3 7^6$	$1/14, 1/14, 1/14, 1/14$	18	[14]	$2_H \circ 9$
C3		$73^3 601^3/2{\cdot}3^4 7^8$	$1/2, 1/2, 1/6, 1/6$	8	[246]	$G = \mathrm{PSL}(3,2){:}C_2$
C4			$1/7, 1/7, 3/7, 6/7$	18	[237]	$G = S_{18}$
C5	3	$2^2 193^3/3$	$1/4, 1/4, 3/8, 3/8$	18	[238]	$2_H \circ 3\,[248] \circ 3$
C6		$2^7 53^3/3^3$	$1/3, 1/3, 2/7, 2/7$	32	[237]	$4\,[337] \circ 8$
C7			$1/3, 1/3, 2/9, 2/9$	16	[239]	$4\,[339] \circ 4$
C8		$3^3 5^3 157^3/2^2 11^6$	$1/2, 1/8, 3/8, 3/8$	15	[238]	$G = S_{15}$
C9		$13^3 541^3/3^3 11^4$	$2/3, 1/9, 2/9, 2/9$	14	[239]	$G = S_{14}$
C10		$109^3 9133^3/2^4 3^5 13^4$	$1/4, 1/4, 1/5, 3/5$	14	[245]	$G = S_{14}$
C11			$1/4, 1/4, 1/6, 1/6$	14	[246]	$G = \mathrm{PSL}(2,13){:}C_2$
C12		$3^3 37^3 192637^3/11^6 17^4$	$2/3, 1/7, 1/7, 3/7$	26	[237]	$G = S_{26}$
C13		$2^7 5^3 1301^3 43889^3/3^{17} 11^{10} 13^4$	$1/3, 1/11, 2/11, 2/11$	16	[11]	$G = A_{16}$
C14	5	2^{11}	$1/2, 1/2, 1/4, 1/4$	10	[245]	$2_H \circ 5$
C15		$2^4 17^3$	$1/2, 1/5, 1/5, 1/10$	15	[10]	$5\,[2\,5\,10] \circ 3$
C16			$1/2, 1/5, 2/5, 2/5$	10	[245]	$5\,[255] \circ 2$
C17			$1/4, 1/4, 1/4, 1/4$	20	[245]	$4_H\{\mathrm{F13}\times, \mathrm{C14}\} \circ 5$
C18		$103681^3/3^4 5$	$1/7, 2/7, 3/7, 3/7$	30	[237]	$G = S_{30}$
C19	6	$2^6 971^3/3^5$	$1/2, 1/4, 1/4, 3/5$	8	[245]	$G = A_8$
C20		$2^6 19^3 467^3/3^7 5^6$	$1/3, 2/3, 1/4, 1/4$	6	[334]	$G = S_6$
C21				12	[238]	$\mathrm{C20} \circ 2$
C22		$11^3 1259^3/2{\cdot}3^3 5^4$	$1/3, 1/3, 1/8, 1/8$	26	[238]	$G = \mathrm{PSL}(2,25){:}C_2$
C23			$1/9, 1/9, 2/9, 5/9$	18	[239]	$G = S_{18}$
C24		$11^3 1979^3/2^3 3{\cdot}5^{12}$	$1/2, 1/2, 1/4, 1/4$	6	[246]	$G = S_5$
C25			$1/10, 1/10, 3/10, 3/10$	18	[10]	$2_H \circ 9$
C26	7	$3^3 5^3 17^3$	$2/3, 2/7, 2/7, 3/7$	14	[237]	$G = S_{14}$
C27		$2^4 37^3 271^3/3^6 5^4$	$2/3, 1/7, 1/7, 5/7$	14	[237]	$G = S_{14}$
C28		$2^2 11^3 107^3/3^{12} 7$	$1/3, 1/3, 3/8, 3/8$	14	[238]	$2_H \circ 7$
C29		$2^7 5^6 1607^3/3^{16} 7^5$	$1/3, 2/3, 2/3, 1/7$	8	[237]	$G = A_8$
C30	10	$7949^3/2^5 3^{10}$	$1/2, 1/2, 1/4, 1/4$	10	[245]	$G = A_6$
C31			$1/6, 1/6, 1/6, 1/6$	10	[256]	$2_H \circ 5$
C32		$11^3 13^3 23^3/2{\cdot}3^{12} 5$	$2/3, 2/3, 1/8, 1/8$	10	[238]	$2_H \circ 5$
C33	13	$112297^3/2^4 3^{20} 13$	$1/2, 1/3, 3/7, 3/7$	13	[237]	$G = A_{13}$
C34	21	$3^3 127^3/5^6$	$2/3, 1/7, 3/7, 3/7$	14	[237]	$G = S_{14}$
C35		$3^3 1367^3/2^4 5^2$	$1/2, 2/7, 3/7, 3/7$	15	[237]	$G = S_{15}$
C36		$757^3 11827^3/2^4 3^7 17^6$	$1/2, 1/7, 3/7, 3/7$	21	[237]	$G = A_{21}$
C37		$37^3 5653^3/2^2 3^3 5^{12} 7$	$1/3, 1/3, 1/5, 1/5$	7	[335]	$G = A_7$
C38				14	[10]	$2_H \circ 7$, $\mathrm{C37} \circ 2$
C39	105	$2^4 3^6 8681^3/5^7$	$1/2, 2/7, 2/7, 3/7$	21	[237]	$G = A_{21}$
C40		$3^3 27329^3/2^{14} 5{\cdot}7^5$	$2/3, 2/3, 1/7, 2/7$	10	[237]	$G = S_{10}$
C41	273	$5^3 349^3 8516873^3/2^{30} 3^3 7^9 11^6 13$	$1/2, 1/3, 2/9, 2/9$	13	[239]	$G = A_{13}$
C42	385	$3^3 289189^3/2^{18} 5^7 11$	$1/2, 2/3, 2/7, 2/7$	11	[237]	$G = S_{11}$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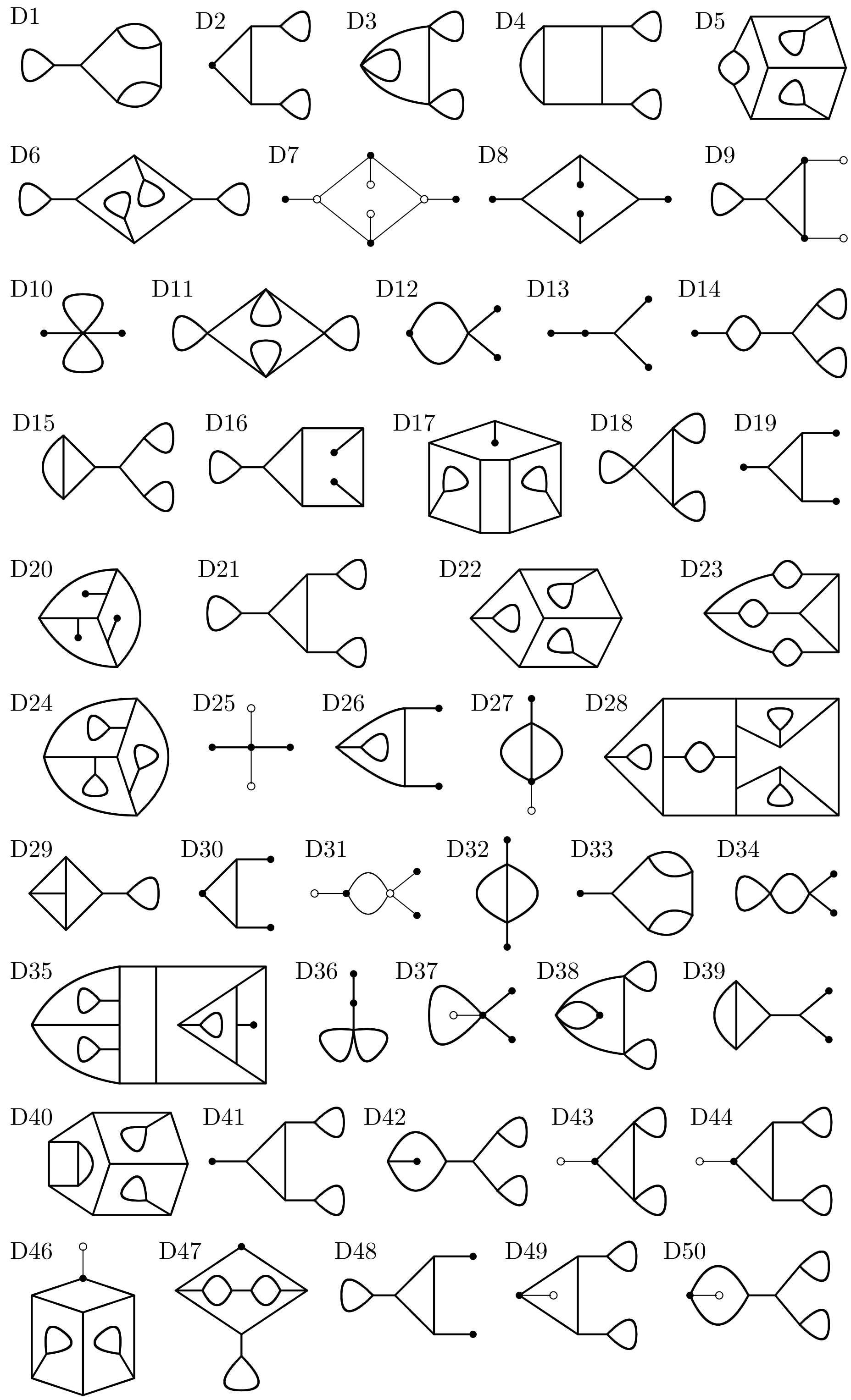
Id	$\sqrt{\ }$	j-invariant	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
D1	-1	$2^2 3^3 13^3/5^4$	$1/4, 1/4, 1/8, 5/8$	18	[238]	$6\,[248] \circ 3$
D2		$-2^4 109^3/5^6$	$1/3, 2/3, 1/9, 1/9$	14	[239]	$G = S_{14}$
D3			$1/5, 1/5, 1/5, 4/5$	12	[245]	$6\,[255] \circ 2$
D4			$1/5, 2/5, 1/10, 1/10$	18	[10]	$6\,[2\,5\,10] \circ 3$
D5			$1/7, 1/7, 2/7, 5/7$	30	[237]	$G = S_{30}$
D6			$1/10, 1/10, 1/10, 1/10$	24	[10]	$4_H\{\mathrm{G5}\times, \mathrm{P43}\} \circ 6$
D7	-2	$-2^5 19^3/3^6$	$1/3, 1/3, 1/3, 1/3$	8	[334]	$2_H \circ 4$
D8				16	[238]	$4_H\{\mathrm{G13}\times, \mathrm{P41}\{\mathrm{D7}\}\}\circ 4$
D9		$2{\cdot}47^3/3^8$	$1/2, 1/2, 1/8, 3/8$	12	[238]	$4\,[248] \circ 3$
D10			$1/6, 1/6, 1/6, 1/6$	8	[266]	$2_H \circ 4$
D11				16	[246]	$4_H\{\mathrm{G11}\times, \mathrm{P23}\{\mathrm{D10}\}\}\circ 4$
D12		$-2^6 239^3/3^{10}$	$1/2, 1/3, 1/4, 1/4$	8	[246]	$4\,[344] \circ 2$
D13			$1/3, 1/3, 1/3, 2/3$	8	[238]	$4\,[334] \circ 2$
D14			$1/3, 1/6, 1/12, 1/12$	16	[12]	$4\,[3\,4\,12] \circ 4$
D15		$-482641^3/2{\cdot}3^2 11^{10}$	$1/11, 1/11, 2/11, 3/11$	18	[11]	$G = S_{18}$
D16		$-254977^3/2^5 3^{12} 19^4$	$1/3, 1/3, 1/8, 3/8$	20	[238]	$G = A_{20}$
D17		$7607^3 1753^3/2^7 3^{20} 5^4 11^4$	$1/3, 1/7, 1/7, 4/7$	34	[237]	$G = S_{34}$
D18	-3	0	$1/2, 1/6, 1/6, 1/6$	12	[246]	$3\,[366] \circ 2\,[266] \circ 2$
D19			$1/3, 1/3, 1/3, 1/3$	12	[239]	$3\,[339] \circ 4$
D20			$1/3, 1/3, 1/3, 3/7$	24	[237]	$3\,[337] \circ 8$
D21			$1/4, 1/12, 1/12, 1/12$	18	[12]	$3\,[3\,12\,12] \circ 2\,[2\,6\,12] \circ 3$
D22			$1/7, 1/7, 1/7, 6/7$	30	[237]	$3\,[377''] \circ 10$
D23			$2/7, 2/7, 2/7, 3/7$	30	[237]	$3\,[377''] \circ 10$
D24			$1/8, 1/8, 1/8, 3/8$	30	[238]	$3\,[388] \circ 10$
D25		$2^{11}/3$	$1/2, 1/2, 1/4, 1/4$	6	[246]	$2_H \circ 3$
D26		$-2^{17} 3^3 7^3/13^4$	$1/3, 1/3, 1/7, 6/7$	14	[237]	$G = S_{14}$
D27		$-23^3 71^3/3 \cdot 7^8$	$1/2, 1/4, 2/5, 2/5$	9	[245]	$G = A_9$
D28			$1/7, 1/7, 1/7, 2/7$	54	[237]	$G = S_{54}$
D29			$1/7, 3/7, 3/7, 4/7$	18	[237]	$G = S_{18}$
D30	-5	$-5281^3/3^{16} 5$	$1/3, 1/3, 2/3, 3/7$	10	[237]	$G = S_{10}$
D31			$1/3, 1/3, 1/4, 1/4$	5	[344]	$G = S_5$
D32				10	[246]	$2_H \circ 5,\ \mathrm{D31} \circ 2$
D33		$2^7 91423^3/3^6 5^7 7^8$	$1/3, 2/7, 2/7, 5/7$	16	[237]	$G = A_{16}$
D34		$-11^3 88811^3/2^6 3^4 5{\cdot}7^{12}$	$1/4, 1/4, 1/7, 2/7$	10	[247]	$G = S_{10}$
D35		$-11^3 23830621091^3/2^8 3^{20} 5^3 7^8 43^4$	$1/3, 1/7, 1/7, 1/7$	52	[237]	$G = A_{52}$
D36	-6	$-2^3 6359^3 2999^3/3^7 5^{16} 7^4$	$1/5, 2/5, 1/6, 1/6$	8	[256]	$G = S_8$
D37	-7	$-5^3 1637^3/2^{18} 7$	$1/2, 1/5, 1/5, 2/5$	7	[255]	$G = S_7$
D38				14	[245]	$\mathrm{D37} \circ 2$
D39			$1/3, 1/3, 1/3, 2/9$	14	[239]	$G = S_{14}$
D40		$-5^3 37^3 167^3/2^8 3^4 11^4$	$1/7, 1/7, 2/7, 4/7$	36	[237]	$G = A_{36}$
D41		$-2^6 5^3 14411^3/3^6 7^3 11^{10}$	$1/3, 1/11, 1/11, 3/11$	16	[11]	$G = A_{16}$
D42	-14	$-2^5 199287631^3/3^{26} 5^6 7^3$	$1/3, 2/5, 1/10, 1/10$	16	[10]	$G = A_{16}$
D43	-15	$-269^3/2^{10} 3^5$	$1/2, 1/5, 1/5, 3/5$	10	[245]	$5\,[255] \circ 2$
D44			$1/2, 1/10, 1/10, 3/10$	15	[10]	$5\,[2\,5\,10] \circ 3$
D45			$1/4, 1/4, 1/4, 1/4$	20	[245]	$2_H \circ \mathrm{C30}$
D46		$-11^3 59^3/2^{12} 3{\cdot}5^3$	$1/2, 1/7, 1/7, 4/7$	27	[237]	$G = S_{27}$
D47		$-3^3 335089^3/2^{14} 5^7 23^4$	$2/3, 1/7, 2/7, 2/7$	26	[237]	$G = S_{26}$
D48	-35	$1685104151^3/2^6 3^{32} 5^7 7{\cdot}13^4$	$1/3, 1/3, 1/10, 3/10$	14	[10]	$G = S_{14}$
D49	-39	$-17^3 29^3 5197^3/2^{30} 3^3 5^2 13^3$	$1/2, 1/9, 1/9, 4/9$	15	[239]	$G = S_{15}$
D50			$1/2, 1/10, 1/10, 3/10$	15	[10]	$G = S_{15}$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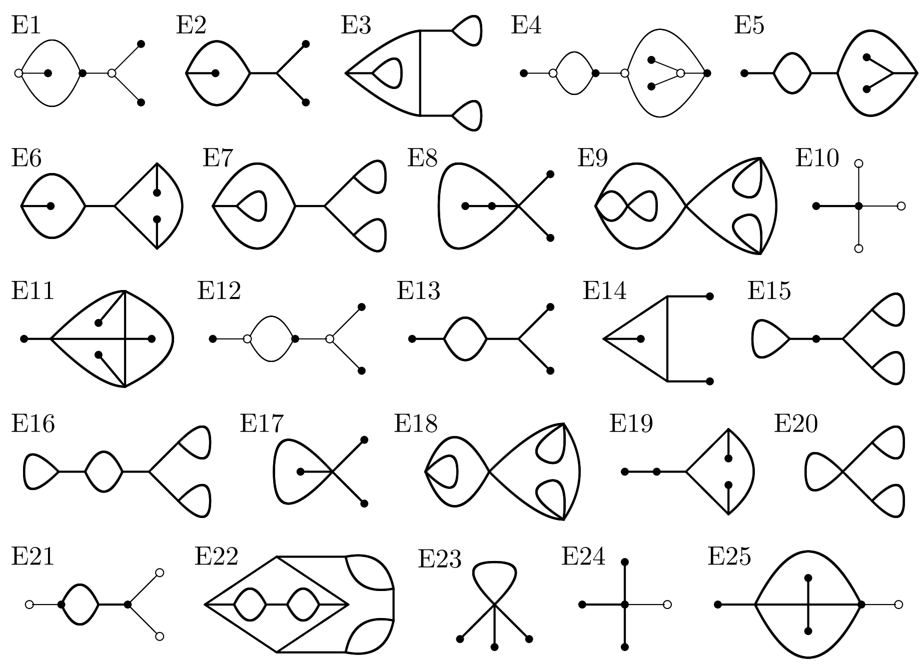
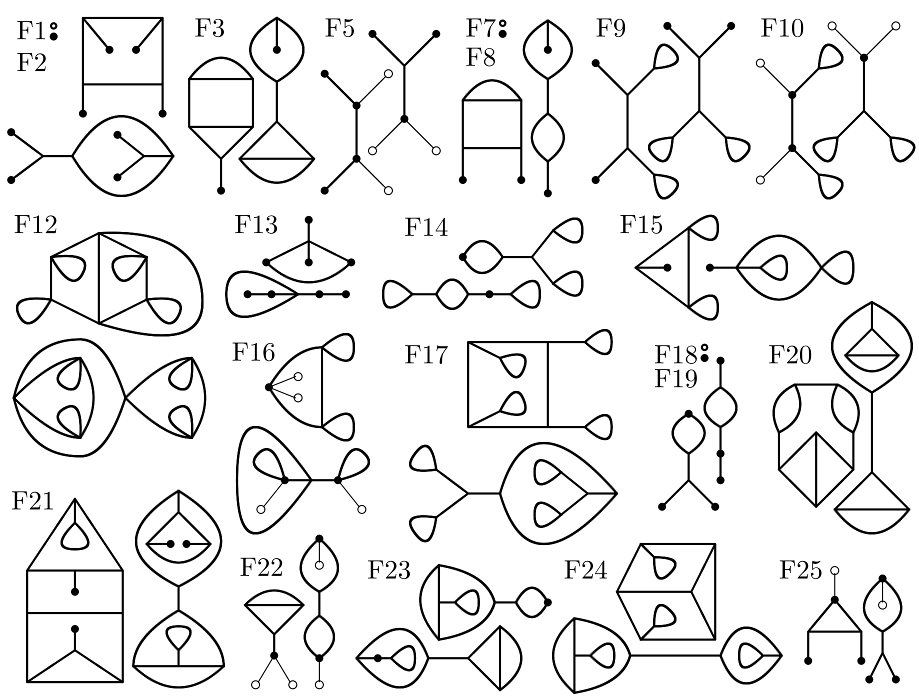
D6 D7 D8 D9
D10 D11 D12 D13 D14
D15 D16 D17 D18 D19
D20 D21 D22 D23
D24 D25 D26 D27 D28
D29 D30 D31 D32 D33 D34
D35 D36 D37 D38 D39
D40 D41 D42 D43 D44
D46 D47 D48 D49 D50

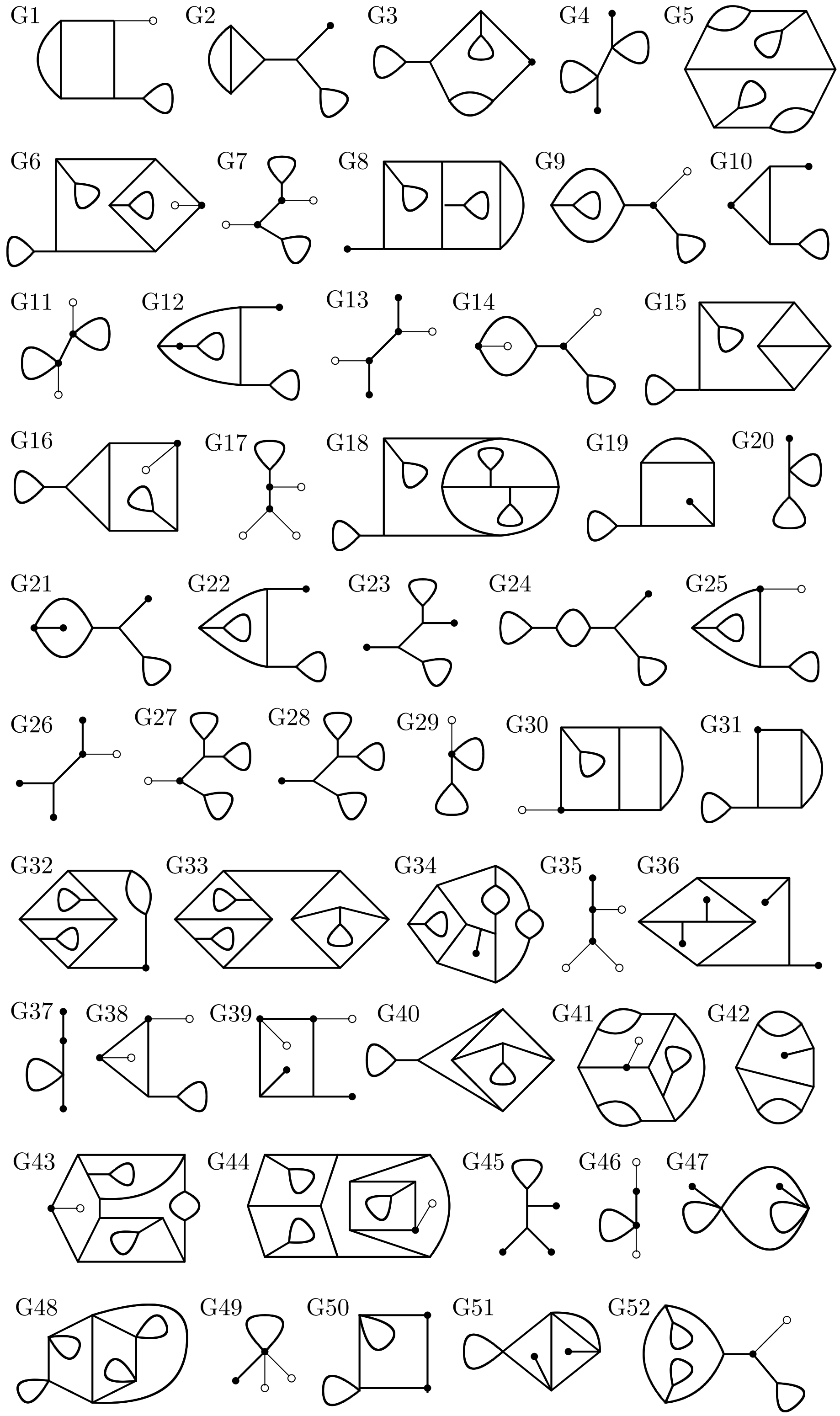
Id	disc $\mathbb{Q}(t)$	j-invariant	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
E1	$-2^9 3^7$	$-2^6 3^3 23^3$	$1/2, 1/3, 1/3, 1/3$	6	[334]	$G = A_6$
E2				12	[238]	E1 $\circ$ 2
E3		$-3^3 17^3/2$	$2/3, 1/9, 1/9, 1/9$	18	[239]	$G = S_{18}$
E4	$-2^9 3^8$	$-3^3 5^3 383^3/2^7$	$1/3, 1/3, 1/3, 1/4$	9	[334]	$G = ((C_3 \times C_3) : Q_8) : C_3$
E5				18	[238]	E4 $\circ$ 2
E6			$1/3, 1/3, 1/3, 4/7$	18	[237]	$G = S_{18}$
E7	$-2^9 5^4$	$-5^2 241^3/2^3$	$1/2, 1/10, 1/10, 1/10$	18	[10]	$G = S_{18}$
E8			$1/5, 1/5, 1/5, 2/5$	10	[255]	$G = S_{10}$
E9				20	[245]	E8 $\circ$ 2
E10		$5 \cdot 211^3/2^{15}$	$1/2, 1/2, 1/2, 1/4$	5	[245]	$G = S_5$
E11			$1/4, 1/4, 1/4, 1/4$	20	[245]	$4_H\{\mathrm{H17}^{\times\times}\} \circ$ E10
E12	$-3^7 5^3$	$-2^8 3^3 61^3/5^7$	$1/3, 1/3, 1/3, 1/5$	6	[335]	$G = A_6$
E13				12	[10]	E12 $\circ$ 2
E14			$1/3, 1/3, 1/3, 5/7$	12	[237]	$G = A_{12}$
E15	$-2^6 11^4$	$-2^4 11^2 13^3/3^6$	$2/3, 1/11, 1/11, 1/11$	14	[11]	$G = S_{14}$
E16	$-2^9 13^3$	$-3^3 41^3 83^3/2 \cdot 13^7$	$1/13, 1/13, 1/13, 2/13$	18	[13]	$G = S_{18}$
E17	$-2^4 3^3 5^4$	$-2^9 5^4 11^3/3^5$	$1/5, 1/5, 1/5, 3/5$	8	[255]	$G = A_8$
E18				16	[245]	E17 $\circ$ 2
E19	$-2^4 3^3 7^4$	$-2^{12} 7^1 17^3 23^3/3^{13}$	$1/3, 1/3, 1/3, 2/3$	14	[237]	$G = S_{14}$
E20	$-2^9 3^3 7^4$	$-2 \cdot 3^3 7^2$	$4/3, 1/7, 1/7, 1/7$	10	[237]	$G = S_{10}$
E21	$-2^6 3^7 7^3$	$-3 \cdot 223^3/2^8$	$1/2, 1/2, 1/2, 2/7$	9	[237]	$G = S_9$
E22			$2/7, 2/7, 2/7, 2/7$	36	[237]	$4_H\{\mathrm{H37}^{\times\times}\} \circ$ E21
E23	$-2^4 5^3 7^3$	$2^9 3^3 3739^3/5^{11} 7^5$	$1/5, 1/5, 1/5, 1/7$	8	[257]	$G = A_8$
E24	$-2^6 5^3 7^4$	$3^3 7 \cdot 2099^3/2^{14} 5^7$	$1/2, 1/4, 1/4, 1/4$	7	[247]	$G = S_7$
E25	$-2^6 5^4 13^3$	$-5 \cdot 3410909^3/2^{20} 3^{10} 13^5$	$1/2, 1/4, 1/4, 1/4$	15	[245]	$G = S_{15}$



Id	$\sqrt{\ }$	disc $\mathbb{Q}(t)$	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
F1	2	as j	$1/3, 1/3, 1/3, 1/3$	8	[334]	$G=$PSL(3,2)
F2				16	[238]	$2_H \circ$ F5, F1 $\circ$ 2
F3			$1/3, 2/7, 3/7, 4/7$	16	[237]	$G = A_{16}$
F4		2^{11}	$1/2, 1/2, 1/8, 1/8$	18	[238]	$G=$PSL(2,17)
F5		$2^{11}3^2$	$1/2, 1/2, 1/3, 1/3$	8	[238]	$G=$PSL(3,2)$:C_2$
F6		$2^6 17$	$1/8, 1/8, 1/8, 1/8$	36	[238]	$2_H \circ$ F4
F7		$-2^6 3^2 7$	$1/2, 1/3, 1/3, 1/4$	7	[334]	$G=$PSL(3,2)
F8				14	[238]	F7 $\circ$ 2
F9	3	as j	$1/3, 1/3, 1/12, 1/12$	14	[12]	$G=$PSL(2,13)$:C_2$
F10	5	as j	$1/2, 1/2, 1/10, 1/10$	12	[10]	$G=$PSL(2,11)$:C_2$
F11			$1/5, 1/5, 1/5, 1/5$	12	[255]	$G=$PSL(2,11)
F12				24	[245]	$2_H \circ$ F16, F11 $\circ$ 2
F13		$-2^4 5^2$	$1/2, 1/2, 1/4, 1/4$	10	[245]	$2_H \circ 5$
F14		$-2^4 5^3$	$2/3, 1/5, 1/10, 1/10$	14	[10]	$G = S_{14}$
F15			$1/4, 3/4, 1/5, 1/5$	12	[245]	$G = A_{12}$
F16		$2^6 5^3$	$1/2, 1/2, 1/5, 1/5$	12	[245]	$G=$PSL(2,11)$:C_2$
F17		$2^4 5^2 11$	$1/10, 1/10, 1/10, 1/10$	24	[10]	$2_H \circ$ F10
F18	6	$-2^6 3^3 5$	$1/3, 1/3, 2/3, 1/4$	5	[334]	$G = S_5$
F19				10	[238]	F18 $\circ$ 2
F20	7	$2^4 3^2 7^2$	$2/7, 2/7, 3/7, 3/7$	24	[237]	$2_H \circ$ F22
F21		$-2^6 3 \cdot 7^2$	$1/3, 1/3, 1/7, 3/7$	32	[237]	$G = A_{32}$
F22		$-2^8 3 \cdot 7^2$	$1/2, 1/2, 2/7, 3/7$	12	[237]	$G = S_{12}$
F23	21	as j	$2/3, 1/7, 2/7, 3/7$	20	[237]	$G = S_{20}$
F24		$-3^3 7^2$	$1/7, 1/7, 3/7, 5/7$	24	[237]	$G = A_{24}$
F25	22	$2^6 3 \cdot 11^3$	$1/2, 1/3, 1/3, 3/8$	11	[238]	$G = S_{11}$



Id	$\sqrt{\ }$	disc $\mathbb{Q}(t)$	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
G1	-1	as j	$1/2, 1/2, 1/4, 1/8$	15	[238]	$5\,[248] \circ 3$
G2			$1/3, 1/5, 1/10, 3/10$	16	[10]	$G = A_{16}$
G3		2^8	$2/3, 1/4, 1/8, 1/8$	20	[238]	$G = S_{20}$
G4		$2^6 5$	$1/4, 1/4, 1/8, 1/8$	10	[248]	$2_H \circ 5$
G5				30	[238]	$2_H \circ$ G1: 5 :G4 $\circ\, 3$
G6			$1/2, 1/8, 1/8, 1/8$	27	[238]	$G = S_{27}$
G7		$2^8 5$	$1/2, 1/2, 1/10, 1/10$	12	[10]	$2_H \circ 6$
G8		$2^6 13$	$1/3, 1/4, 1/8, 1/8$	28	[238]	$G = A_{28}$
G9		$2^4 5 \cdot 13$	$1/2, 1/8, 1/8, 5/8$	15	[238]	$G = S_{15}$
G10	-2	as j	$1/3, 2/3, 1/8, 3/8$	12	[238]	$G = S_{12}$
G11		2^{11}	$1/2, 1/2, 1/6, 1/6$	8	[246]	$2_H \circ 4$
G12		$2^{10} 3$	$1/3, 2/3, 1/8, 1/8$	18	[238]	$G = S_{18}$
G13		$2^{11} 3$	$1/2, 1/2, 1/3, 1/3$	8	[238]	$2_H \circ 4$
G14			$1/2, 1/2, 1/8, 3/8$	12	[238]	$G = S_{12}$
G15		$2^6 3 \cdot 11$	$1/8, 1/8, 3/8, 3/8$	24	[238]	$2_H \circ$ G14
G16		$2^6 3 \cdot 19$	$1/2, 1/8, 1/8, 3/8$	21	[238]	$G = A_{21}$
G17		$\underline{12}{:}\, 2^{33} 3^7$	$1/2, 1/2, 1/2, 1/8$	9	[238]	$G = (((C_3)^2 : Q_8) : C_3) : C_2$
G18			$1/8, 1/8, 1/8, 1/8$	36	[238]	$4_H\{\mathrm{J1}^{\times\times}\} \circ$ G17
G19	-3	as j	$1/3, 1/7, 2/7, 6/7$	16	[237]	$G = A_{16}$
G20		$2^4 3^3$	$1/4, 3/4, 1/6, 1/6$	8	[246]	$G = S_8$
G21		$3^3 7$	$1/3, 1/3, 1/9, 4/9$	14	[239]	$G = S_{14}$
G22		$2^4 3^2 7$	$1/3, 3/4, 1/8, 1/8$	16	[238]	$G = A_{16}$
G23			$1/3, 1/3, 1/12, 1/12$	14	[12]	$2_H \circ 7$
G24			$1/3, 1/6, 1/12, 1/12$	16	[12]	$G = A_{16}$
G25		$3^3 7 \cdot 13$	$1/2, 1/7, 1/7, 6/7$	15	[237]	$G = S_{15}$
G26		$\underline{12}{:}\, 3^{21} 7^3$	$1/2, 1/3, 1/3, 1/3$	9	[239]	$G =$PSL(2,8)$: C_3$
G27		$\underline{12}{:}\, 2^{12} 3^{13} 13^3$	$1/2, 1/12, 1/12, 1/12$	15	[12]	$G = S_{15}$
G28		$\underline{12}{:}\, 3^9 7^3 13^7$	$1/3, 1/13, 1/13, 1/13$	16	[13]	$G = A_{16}$
G29	-5	$2^4 3 \cdot 5^3 7$	$1/2, 3/4, 1/5, 1/5$	7	[245]	$G = S_7$
G30	-7	as j	$1/2, 1/7, 2/7, 4/7$	21	[237]	$G = A_{21}$
G31			$2/3, 1/7, 2/7, 4/7$	14	[237]	$G = S_{14}$
G32			$2/3, 1/7, 1/7, 2/7$	32	[237]	$G = S_{32}$
G33		$2^3 7^2$	$1/7, 1/7, 1/7, 4/7$	42	[237]	$G = S_{42}$
G34		$2^4 7^2$	$1/3, 1/7, 2/7, 2/7$	40	[237]	$G = A_{40}$
G35		$2^2 7^3$	$1/2, 1/2, 1/2, 1/3$	7	[237]	$G =$PSL(3,2)
G36			$1/3, 1/3, 1/3, 1/3$	28	[237]	$4_H\{\mathrm{I3}^{\times}$, G39$\} \circ$ G35
G37		$2^5 7^2$	$1/2, 1/4, 1/4, 1/7$	8	[247]	$G = (C_2)^3$:PSL(3,2)
G38			$1/2, 1/2, 1/7, 4/7$	12	[237]	$G = S_{12}$
G39		$2^2 3^2 7^3$	$1/2, 1/2, 1/3, 1/3$	14	[237]	$2_H \circ$ G35
G40		$2^2 7^2 11$	$1/7, 1/7, 4/7, 4/7$	24	[237]	$2_H \circ$ G38
G41			$1/2, 1/7, 2/7, 2/7$	33	[237]	$G = A_{33}$
G42			$1/3, 2/7, 2/7, 4/7$	22	[237]	$G = S_{22}$
G43		$7^2 37$	$1/2, 1/7, 1/7, 2/7$	39	[237]	$G = S_{39}$
G44		$\underline{12}{:}\, 2^6 7^{10} 43^3$	$1/2, 1/7, 1/7, 1/7$	45	[237]	$G = A_{45}$
G45	-11	$\underline{12}{:}\, 3^7 11^9$	$1/3, 1/3, 1/3, 1/11$	12	[11]	$G = M_{12}$
G46	-15	$2^3 3^3 5^2$	$1/2, 1/2, 1/2, 1/5$	6	[245]	$G = A_6$
G47			$1/5, 1/5, 1/5, 1/5$	12	[255]	$2_H \circ$ G49
G48				24	[245]	$4_H\{\mathrm{I11}^{\times}$, G50{G47}$\} \circ$ G46
G49		$2^2 3^3 5^3$	$1/2, 1/2, 1/5, 1/5$	6	[255]	$G = A_6$
G50				12	[245]	$2_H \circ$ G46, G49 $\circ\, 2$
G51		$2^5 3^3 5^2$	$1/4, 1/4, 1/5, 2/5$	18	[245]	$G = S_{18}$
G52		$\underline{12}{:}\, 2^9 3^{18} 5^9 19^3$	$1/2, 1/9, 1/9, 1/9$	21	[239]	$G = A_{21}$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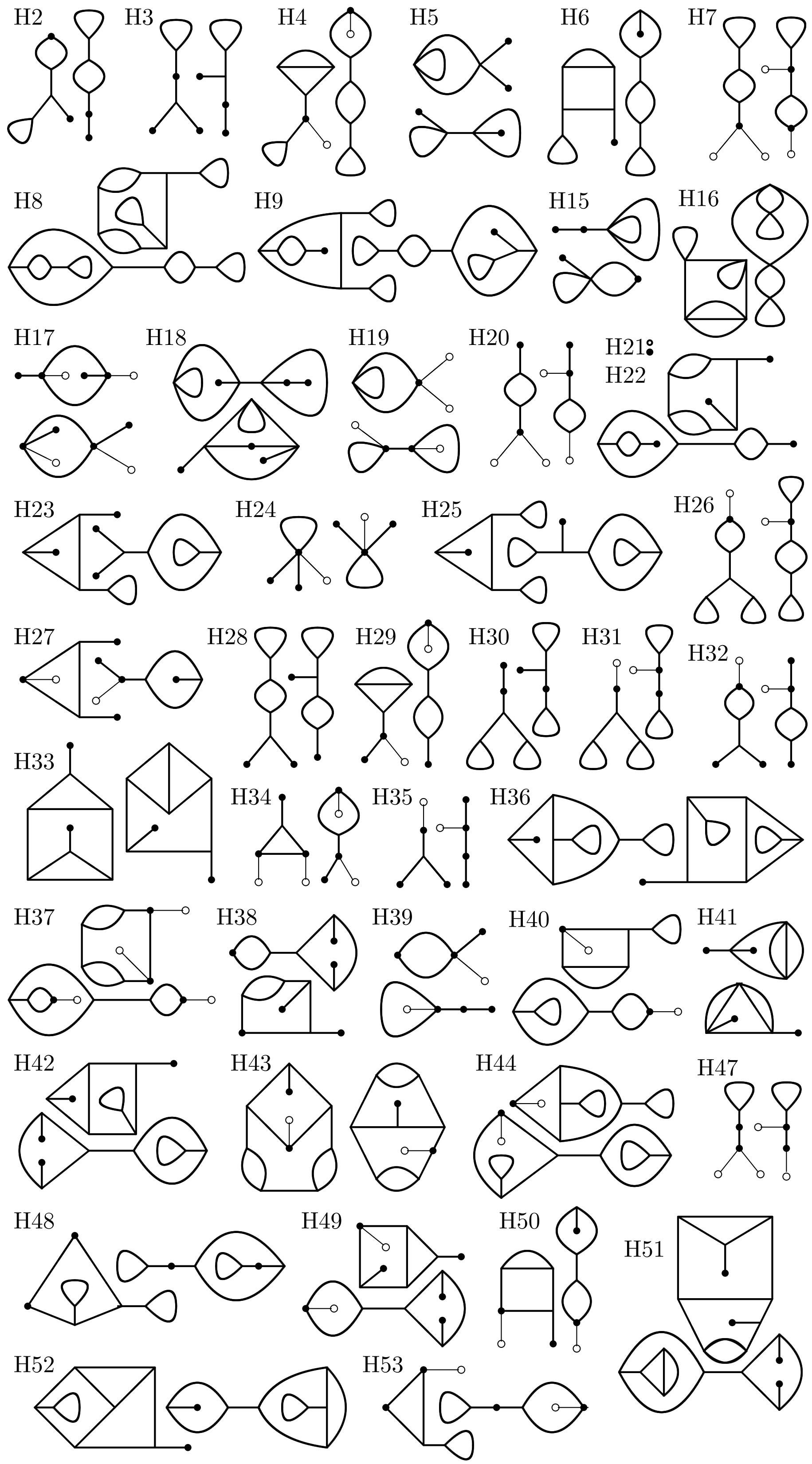
Id	disc $\mathbb{Q}(j)$	disc $\mathbb{Q}(t)$	Branching frac.	d	$[k\ell m]$	Monodromy/comp.
H1	3^4	as j	$1/3, 1/3, 1/9, 1/9$	20	[239]	$G =$PSL(2,19)
H2	-3^5	as j	$1/3, 2/3, 1/9, 2/9$	12	[239]	$G = S_{12}$
H3		$-2^2 3^{10} 5$	$1/3, 1/3, 2/3, 1/9$	10	[239]	$G = S_{10}$
H4	$-2^2 3^3$	as j	$1/2, 1/3, 1/9, 2/9$	15	[239]	$G = S_{15}$
H5		$-2^8 3^6 5$	$1/2, 1/4, 1/4, 1/6$	10	[246]	$G = (A_6 : C_2) : C_2$
H6	$-2^3 3^3$	as j	$1/3, 1/9, 2/9, 4/9$	16	[239]	$G = A_{16}$
H7	$-2^2 3^4$	$-2^8 3^9$	$1/2, 1/2, 1/9, 2/9$	12	[239]	$G = S_{12}$
H8		$2^7 3^8 11$	$1/9, 1/9, 2/9, 2/9$	24	[239]	$2_H \circ$ H7
H9		$-2^8 3^8 11$	$1/3, 1/9, 1/9, 2/9$	22	[239]	$G = S_{22}$
H10	7^2	$2^6 7^5$	$1/2, 1/2, 1/7, 1/7$	30	[237]	$G =$PSL(2,29)
H11		$3^3 7^5$	$1/3, 1/3, 1/7, 1/7$	44	[237]	$G =$PSL(2,43)
H12		$2^6 3^3 7^4$	$1/2, 1/2, 1/3, 1/3$	14	[237]	$G =$PSL(2,13)
H13		$7^4 13$	$1/3, 1/3, 1/3, 1/3$	28	[237]	$2_H \circ$ H12
H14		$7^4 29$	$1/7, 1/7, 1/7, 1/7$	60	[237]	$2_H \circ$ H10
H15	$-2^3 5^2$	as j	$1/5, 1/5, 2/5, 2/5$	8	[255]	$G = S_8$
H16				16	[245]	$2_H \circ$ H19, H15 $\circ\, 2$
H17		$2^{11} 5^4$	$1/2, 1/2, 1/4, 1/4$	10	[245]	$2_H \circ$ E10
H18		$2^{12} 5^4$	$1/2, 1/4, 1/4, 1/5$	16	[245]	$G = ((C_2)^4 : A_5) : C_2$
H19		$-2^{14} 5^4$	$1/2, 1/2, 1/5, 2/5$	8	[245]	$G = S_8$
H20		$-2^{14} 5^5$	$1/2, 1/2, 1/3, 1/4$	10	[238]	$G = A_{10}$
H21		$2^9 3^3 5^5$	$1/3, 1/3, 1/4, 1/4$	10	[334]	$G = A_{10}$
H22				20	[238]	$2_H \circ$ H20, H21 $\circ\, 2$
H23		$-2^8 3^3 5^4 7$	$1/3, 1/3, 1/8, 5/8$	14	[238]	$G = S_{14}$
H24	$-3^3 5$	$-3^6 5^3 7$	$1/2, 1/5, 1/5, 1/6$	7	[256]	$G = S_7$
H25			$1/3, 1/9, 1/9, 5/9$	16	[239]	$G = A_{16}$
H26	$-2^2 11$	$-2^4 11^3 13$	$1/2, 1/11, 1/11, 2/11$	15	[11]	$G = S_{15}$
H27		$-2^4 3^3 7 \cdot 11^3$	$1/2, 1/3, 1/3, 4/7$	11	[237]	$G = S_{11}$
H28			$1/3, 1/3, 1/11, 2/11$	14	[11]	$G = S_{14}$
H29	$-2^3 13$	as j	$1/2, 1/3, 1/4, 3/8$	13	[238]	$G = A_{13}$
H30	$-2^2 3 \cdot 5^2$	$-2^6 3^3 5^5$	$1/3, 2/3, 1/10, 1/10$	12	[10]	$G = S_{12}$
H31	$-2^2 3^4 5$	$-2^4 3^8 5^3 11$	$1/2, 2/3, 1/9, 1/9$	11	[239]	$G = S_{11}$
H32	$-3^4 11$	$-3^8 11^3$	$1/2, 1/3, 1/3, 2/9$	11	[239]	$G = S_{11}$
H33	$-5^2 7$	$5^5 7^2$	$1/3, 1/3, 3/7, 3/7$	20	[237]	$2_H \circ$ H34
H34		$-2^6 5^5 7^2$	$1/2, 1/2, 1/3, 3/7$	10	[237]	$G = A_{10}$
H35	$-2^2 3 \cdot 7^2$	$-2^4 3^3 5 \cdot 7^4$	$1/2, 1/3, 1/3, 2/3$	7	[237]	$G = S_7$
H36	$-2^2 3^3 7$	$2^8 3^6 7^2$	$1/3, 1/7, 1/7, 6/7$	22	[237]	$G = S_{22}$
H37		$2^8 3^7 7^3$	$1/2, 1/2, 2/7, 2/7$	18	[237]	$2_H \circ$ E21
H38	$-2^3 3 \cdot 7^2$	$-2^8 3^3 7^4$	$1/3, 1/3, 2/3, 2/7$	16	[237]	$G = S_{16}$
H39	$-2^2 5 \cdot 7$	as j	$1/2, 1/2, 1/4, 2/5$	7	[245]	$G = S_7$
H40			$1/2, 1/7, 2/7, 5/7$	15	[237]	$G = S_{15}$
H41		$2^8 5^2 7^3$	$1/4, 1/4, 2/5, 2/5$	14	[245]	$2_H \circ$ H39
H42		$-2^4 3^3 5^3 7^2$	$1/3, 1/3, 1/7, 5/7$	20	[237]	$G = A_{20}$
H43		$-2^4 5^3 7^2 19$	$1/2, 1/3, 2/7, 2/7$	25	[237]	$G = A_{25}$
H44		$2^4 5^3 7^3 19$	$1/2, 1/7, 1/7, 5/7$	21	[237]	$G = A_{21}$
H45	$-2^2 5 \cdot 7^2$	$\underline{18}: -2^{30} 5^{10} 7^{12}$	$1/2, 1/2, 1/2, 1/7$	15	[237]	$G = A_{15}$
H46			$1/7, 1/7, 1/7, 1/7$	60	[237]	$4_H\{\text{J19}^{\times\times}\} \circ$H45
H47	$-2^3 5 \cdot 7^2$	$-2^{13} 3 \cdot 5^2 7^4$	$1/2, 1/2, 2/3, 1/7$	8	[237]	$G = S_8$
H48		$2^8 3 \cdot 5^3 7^5$	$2/3, 2/3, 1/7, 1/7$	16	[237]	$2_H \circ$ H47
H49	$-7 \cdot 17^2$	$-3 \cdot 5 \cdot 7^2 17^5$	$1/2, 1/3, 1/3, 3/7$	17	[237]	$G = A_{17}$
H50	$-2^2 7 \cdot 13$	as j	$1/2, 1/3, 2/7, 4/7$	13	[237]	$G = A_{13}$
H51		$-2^4 3 \cdot 7^2 13^3$	$1/3, 1/3, 2/7, 3/7$	26	[237]	$G = S_{26}$
H52	$-3 \cdot 7 \cdot 11$	as j	$1/3, 1/7, 3/7, 4/7$	22	[237]	$G = S_{22}$
H53	$-2^2 3 \cdot 7^2 11$	as j	$1/2, 2/3, 1/7, 3/7$	11	[237]	$G = S_{11}$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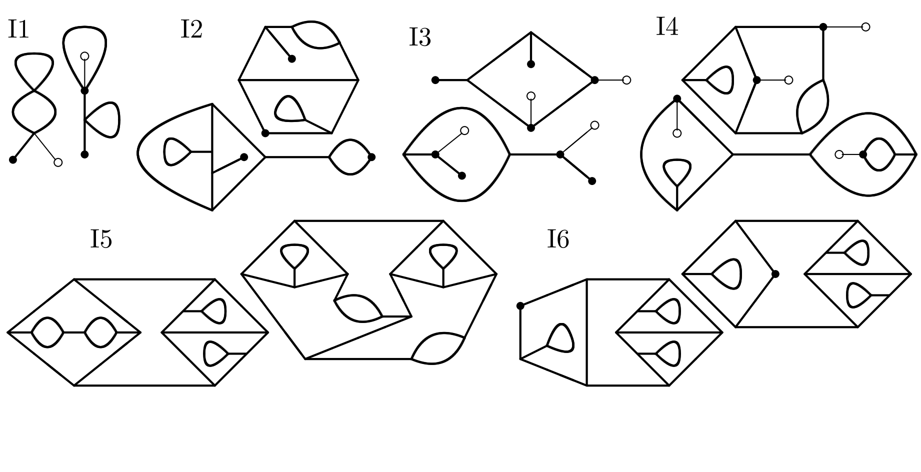
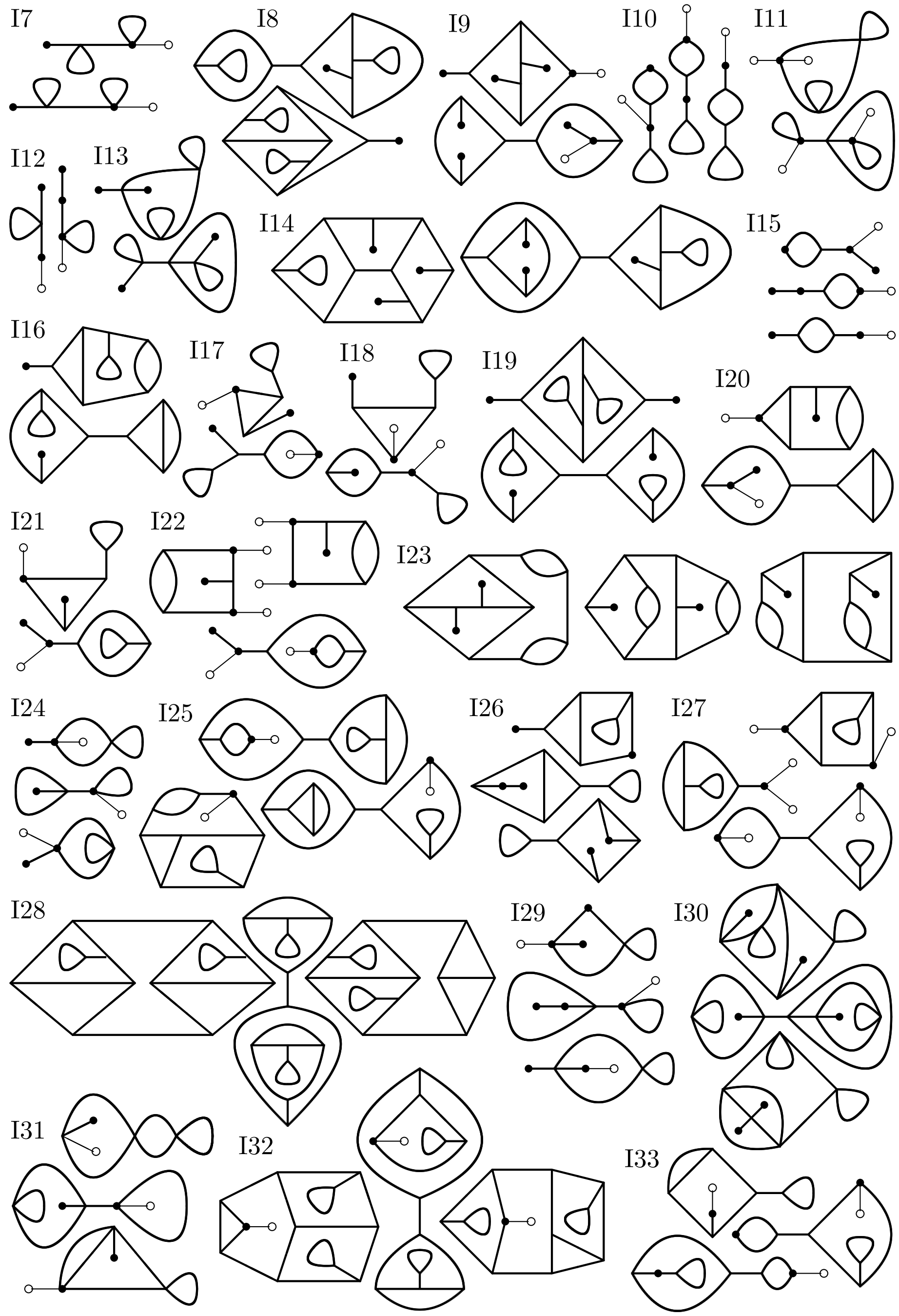
Id	d_j	$\operatorname{disc}\mathbb{Q}(j)$	$\operatorname{disc}\mathbb{Q}(t)$	Branching frac.	d	$[k\ell m]$	Mndr./cmp.
I1	4	$2^6 3^4$	as j	$1/2, 1/3, 1/4, 1/6$	9	[246]	$G = A_9$
I2		$2^2 7^3$	as j	$1/3, 2/3, 1/7, 2/7$	24	[237]	$G = S_{24}$
I3			$2^{12} 3^2 7^6$	$1/2, 1/2, 1/3, 1/3$	14	[237]	$2_H \circ$ G35
I4		$2^3 7^3$	$2^{13} 7^6$	$1/2, 1/2, 1/7, 2/7$	24	[237]	$G = S_{24}$
I5			$2^{10} 7^6 23$	$1/7, 1/7, 2/7, 2/7$	48	[237]	$2_H \circ$ I4
I6			$\underline{24}{:}\, 2^{37} 5^6 7^{22}$	$2/3, 1/7, 1/7, 1/7$	38	[237]	$G = S_{38}$
I7		$2^6 7^2$	$2^{12} 3 \cdot 5 \cdot 7^5$	$1/2, 1/4, 1/7, 1/7$	9	[247]	$G = A_9$
I8		$3^3 7$	$3^6 5^2 7^2 13$	$1/3, 1/7, 1/7, 5/7$	28	[237]	$G = A_{28}$
I9			$\underline{24}{:}\, 3^{30} 7^{17} 19^3$	$1/2, 1/3, 1/3, 1/3$	21	[237]	$G = A_{21}$
I10		$-2^8 11$	as j	$1/2, 2/3, 1/4, 1/8$	11	[238]	$G = S_{11}$
I11		$2^3 3^3 5^2$	$2^{17} 3^6 5^5$	$1/2, 1/2, 1/5, 1/5$	12	[245]	$2_H \circ$ G46
I12		$2^4 3^3 7$	as j	$1/2, 1/2, 1/4, 1/6$	7	[246]	$G = S_7$
I13			$2^{12} 3^7 7^3$	$1/4, 1/4, 1/6, 1/6$	14	[246]	$2_H \circ$ I12
I14		$2^2 3^3 7^3$	$\underline{24}{:}\, 2^{18} 3^{33} 7^{18} 13^3$	$1/3, 1/3, 1/3, 1/7$	36	[237]	$G = A_{36}$
I15		$-2^2 3^5 7^2$	as j	$1/2, 1/3, 2/3, 2/7$	9	[237]	$G = S_9$
I16		$2^6 3 \cdot 11$	as j	$1/3, 1/4, 1/8, 3/8$	22	[238]	$G = S_{22}$
I17		$2^2 3^4 13$	as j	$1/2, 1/3, 1/3, 1/9$	13	[239]	$G = A_{13}$
I18		$2^5 3^3 13$	as j	$1/2, 1/2, 1/3, 1/8$	13	[238]	$G = A_{13}$
I19			$2^{14} 3^6 13^3$	$1/3, 1/3, 1/8, 1/8$	26	[238]	$2_H \circ$ I18
I20		$7^2 19^2$	as j	$1/2, 1/3, 2/7, 3/7$	19	[237]	$G = S_{19}$
I21		$2^2 3^3 5 \cdot 7 \cdot 13$	as j	$1/2, 1/3, 1/7, 5/7$	13	[237]	$G = A_{13}$
I22	5	$2^{11} 7^3$	$-2^{31} 7^6$	$1/2, 1/2, 1/3, 2/7$	16	[237]	$G = S_{16}$
I23			$2^{26} 3^3 7^6 13$	$1/3, 1/3, 2/7, 2/7$	32	[237]	$2_H \circ$ I22
I24		$2^6 3^4 5^2$	as j	$1/2, 1/4, 1/5, 3/5$	9	[245]	$G = A_9$
I25		$2^4 3^5 7$	as j	$1/2, 1/7, 2/7, 3/7$	27	[237]	$G = S_{27}$
I26		$2^2 3^5 7^2$	as j	$1/3, 2/3, 1/7, 3/7$	18	[237]	$G = S_{18}$
I27		$3^3 5 \cdot 7^3$	$-2^{10} 3^6 5^2 7^6$	$1/2, 1/2, 1/7, 3/7$	18	[237]	$G = A_{18}$
I28			$3^8 5^3 7^6 17$	$1/7, 1/7, 3/7, 3/7$	36	[237]	$2_H \circ$ I27
I29		$2^4 5^3 11^2$	as j	$1/2, 1/2, 1/4, 1/5$	11	[245]	$G = S_{11}$
I30			$2^{15} 3 \cdot 5^6 11^5$	$1/4, 1/4, 1/5, 1/5$	22	[245]	$2_H \circ$ I29
I31		$2^6 5^3 13^2$	as j	$1/2, 1/4, 1/5, 2/5$	13	[245]	$G = S_{13}$
I32		$2^4 3^2 7^3 11$	$-2^8 3^6 5^2 7^6 11^2 31$	$1/2, 1/7, 1/7, 3/7$	33	[237]	$G = A_{33}$
I33		$2^2 3 \cdot 7^3 17^2$	as j	$1/2, 2/3, 1/7, 2/7$	17	[237]	$G = S_{17}$




I7
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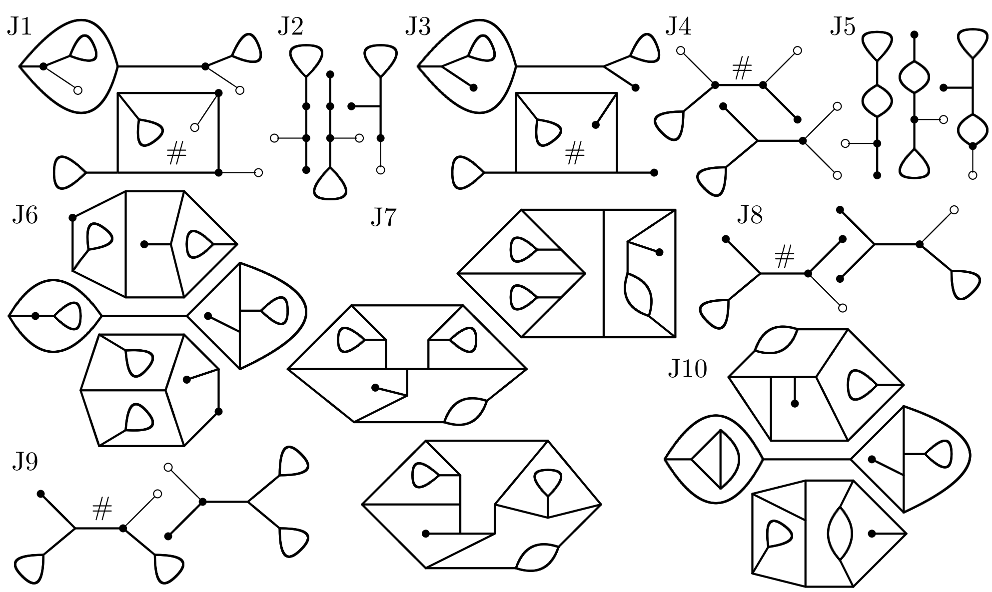
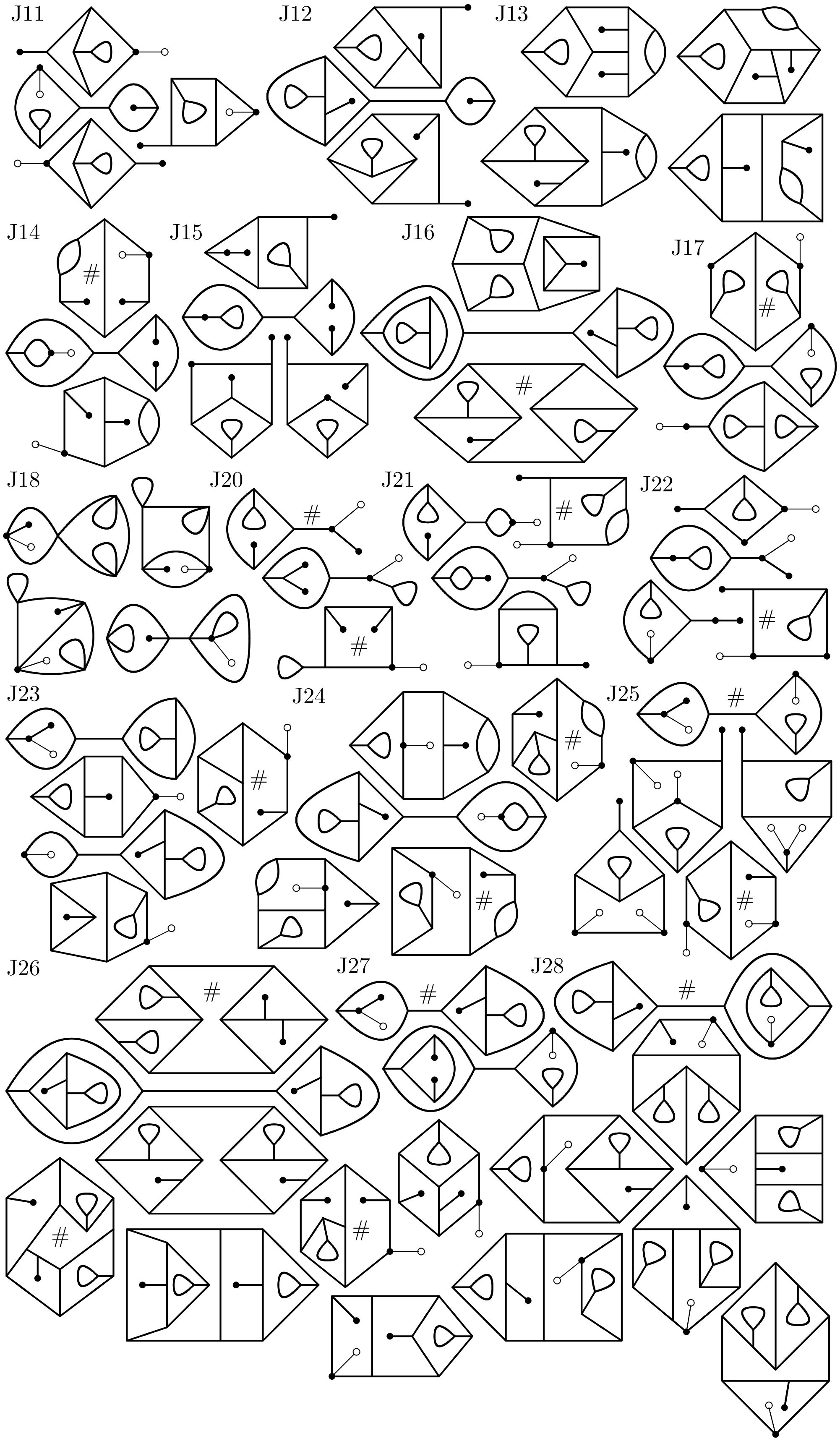
I33

Id	d_j	$\operatorname{disc}\mathbb{Q}(j)$	$\operatorname{disc}\mathbb{Q}(t)$	Branching frac.	d	$[k\ell m]$	Mndr./cmp.
J1	6	$-2^{14}3^3$	$2^{37}3^7$	$1/2, 1/2, 1/8, 1/8$	18	[238]	$2_H \circ$ G17
J2		$-2^{11}3^6 5^2$	as j	$1/2, 1/3, 2/3, 1/8$	9	[238]	$G = S_9$
J3		$-3^8 5^2 7$	$2^3 3^{16} 5^5 7^3$	$1/3, 1/3, 1/9, 1/9$	20	[239]	$2_H \circ$ J4
J4			$2^{10} 3^{17} 5^5 7^2$	$1/2, 1/2, 1/3, 1/9$	10	[239]	$G = A_{10}$
J5		$-3^3 5^5 13$	as j	$1/2, 1/3, 1/5, 1/10$	13	[10]	$G = A_{13}$
J6		$-2^4 3^3 5{\cdot}7^4$	$2^{12} 3^7 5^2 7^{10}$	$1/3, 2/3, 1/7, 1/7$	30	[237]	$G = S_{30}$
J7		$-2^4 7^4 23$	$2^{12} 7^8 23^3$	$1/3, 1/7, 1/7, 2/7$	46	[237]	$G = S_{46}$
J8		$-5^5 7^2 11$	$3^5 5^{11} 7^4 11^3$	$1/2, 1/3, 1/3, 1/10$	11	[10]	$G = S_{11}$
J9		$-2^4 3^4 7{\cdot}11^4$	$2^8 3^8 7^3 11^9 13$	$1/2, 1/3, 1/11, 1/11$	13	[11]	$G = A_{13}$
J10		$-2^2 3^3 7^2 17^2$	as j	$1/3, 1/7, 2/7, 3/7$	34	[237]	$G = S_{34}$
J11		$2^5 3^4 7^3 19^2$	as j	$1/2, 1/3, 1/7, 4/7$	19	[237]	$G = S_{19}$
J12		$-2^3 7^2 11^2 13$	$2^6 3^6 7^4 11^4 13^3$	$1/3, 1/3, 1/7, 4/7$	26	[237]	$G = S_{26}$
J13	7	$-2^2 7^5 19^2$	$-2^4 3^6 7^{10} 19^5$	$1/3, 1/3, 1/7, 2/7$	38	[237]	$G = S_{38}$
J14		$-2^2 7^4 23^3$	$-2^4 3^5 7^9 23^7$	$1/2, 1/3, 1/3, 2/7$	23	[237]	$G = S_{23}$
J15		$-2^2 3^7 7^5 11$	$-2^4 3^{16} 5{\cdot}7^{10} 11^3$	$1/3, 1/3, 2/3, 1/7$	22	[237]	$G = S_{22}$
J16		$-3^3 5^2 7^4 17$	$-3^6 5^5 7^8 17^3 19$	$1/3, 1/7, 1/7, 3/7$	40	[237]	$G = A_{40}$
J17		$-2^4 5{\cdot}7^4 17{\cdot}23$	$2^8 5^4 7^{11} 17^3 23^3$	$1/2, 2/3, 1/7, 1/7$	23	[237]	$G = S_{23}$
J18	8	$2^{14} 5^4 17$	$2^{28} 5^{11} 13{\cdot}17^3$	$1/2, 1/4, 1/5, 1/5$	17	[245]	$G = A_{17}$
J19	9	$2^{12} 5^5 7^6$	$2^{36} 5^{10} 7^{14}$	$1/2, 1/2, 1/7, 1/7$	30	[237]	$2_H \circ$ H45
J20		$2^{20} 13^2 17^2$	$-2^{43} 3^8 13^4 17^5$	$1/2, 1/3, 1/3, 1/8$	17	[238]	$G = A_{17}$
J21	10	$-2^{23} 3^4 5^2 19^2$	as j	$1/2, 1/3, 1/4, 1/8$	19	[238]	$G = S_{19}$
J22		$-2^6 3^6 5^6 7^7 11^2$	as j	$1/2, 1/3, 2/3, 1/7$	15	[237]	$G = S_{15}$
J23	11	$-2^6 3^3 5^9 7^6$	as j	$1/2, 1/3, 1/7, 3/7$	25	[237]	$G = A_{25}$
J24	13	$2^{10} 3^5 7^9 31^4$	as j	$1/2, 1/3, 1/7, 2/7$	31	[237]	$G = S_{31}$
J25		$2^2 3^6 7^9 11^5 19$	$-2^{28} 3^{12} 7^{18} 11^{11} 19^2$	$1/2, 1/2, 1/3, 1/7$	22	[237]	$G = A_{22}$
J26			$2^6 3^{19} 7^{20} 11^{11} 13^2 19^3$	$1/3, 1/3, 1/7, 1/7$	44	[237]	$2_H \circ$ J25
J27	14	$-2^8 5^2 7^{10} 19{\cdot}29^4$	$2^{18} 3^{13} 5^4 7^{20} 13{\cdot}19^2 29^9$	$1/2, 1/3, 1/3, 1/7$	29	[237]	$G = A_{29}$
J28	15	$-2^{12} 3^9 7^{10} 11^2 31{\cdot}37^2$	$-2^{24} 3^{18} 7^{24} 11^4 19^2 31^3 37^5$	$1/2, 1/3, 1/7, 1/7$	37	[237]	$G = A_{37}$




J11
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J26
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J28

Other occasional columns:

- j-invariant: given if it is in $\mathbb{Q} \setminus \{1728\}$, in a factorized form;
- d_j: the degree of the j-field (in tables I, J);
- disc $\mathbb{Q}(j)$, disc $\mathbb{Q}(t)$: the field discriminants. If the extension $\mathbb{Q}(t) \supset \mathbb{Q}(j)$ is of degree 6, the degree of the the t-field is indicated in the disc $\mathbb{Q}(t)$ column in a small underlined font.
- $\sqrt{\ }$: indicates the quadratic extension of either the t-field (in Tables C, D) or of the j-field (in Tables F, G);
- m-$\sqrt{\ }$: the quadratic extension for the moduli field (only in table A);
- r-$\sqrt{\ }$: the quadratic extension for the r-field (only in table A).

The tables are supplemented by pictures of respective minus-4-hyperbolic dessins d'enfant. We put $\overset{\circ}{\bullet}$ next to a few F, H-labels to indicate that their dessins are obtained by bi-coloring the vertices (and thereby making the edges thin). The J-pictures marked by the symbol # represent 4 dessins each, obtainable by reflecting (with respect to a horizontal axis) their left and right parts independently. The dessins of the following Galois orbits are displayed elsewhere:

- H14, H46: in Figure 1;
- H45, J19: in Figure 2;
- B12, C6, C30, D45, F4, F6, H1, H10–H13: in Figure 3;
- F1, F7, F11: in Figure 4.

C Appendix: Composite Belyi functions

Composition of a Belyi function $\varphi(x)$ into smaller degree rational functions can be decided from the function field lattice between $\mathbb{C}(x)$ and $\mathbb{C}(\varphi)$, as described in [12, Section 1.7.2] and §6.3 here. The subfield lattices are listed in our online table [17, Decomposition_or_GaloisGroup].

On the other hand, composite minus-4-hyperbolic Belyi functions induce composite hypergeometric-to-Heun transformations. Thereby special cases of the parametric transformations P1–P61 of [23, §2.2] and the Heun-to-Heun transformations 2_H, 4_H of [23, §4.3] often occur as composition parts. The quadratic transformation 2_H acts on the exponent differences as $(1/2, 1/2, \alpha, \beta) \leftarrow (\alpha, \alpha, \beta, \beta)$ and changes the j-invariant to a 2-isogenous j-invariant. The transformation $4_H = 2_H \circ 2_H$ transforms $(1/2, 1/2, 1/2, \alpha) \leftarrow (\alpha, \alpha, \alpha, \alpha)$ and does not change the j-invariant. The composite transformations could be figured out by a careful consideration of possible compositions of hypergeometric-to-hypergeometric, indecomposable hypergeometric-to-Heun (parametric or some newly implied), and Heun-to-Heun transformations. That would constitute yet another check[19] of our list of Belyi functions.

[19] In particular, any transformation to Heun's equation with 2 (or 3) exponent differences equal to 1/2 can be composed with 2_H (or 4_H, respectively). Further, any Belyi function of the $[k\ell m]$-type [344] or [266] gives rise to a type-[246] composition (with the degree doubled),

The most complicated decomposition lattice is for A18:

$$
\begin{tikzcd}
 & {[488]^{\times}{:}\mathrm{P1}} \arrow[r, dash] & {[288]{:}\mathrm{P2}} \arrow[dr, dash] & {[248]^{\times}{:}\mathrm{A16}} \arrow[dr, dash] & \\
{[4444]} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & {\mathrm{P12}{:}[2244]} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & {[444]{:}\mathrm{P9}} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & {[248]{:}\mathrm{P10}} \arrow[r, dash] & {[238]} \\
 & {\mathrm{C1}{:}[2244]^{\times}} \arrow[r, dash] & {\mathrm{P4}{:}[2224]} \arrow[ur, dash] & {[334]{:}\mathrm{A17}} \arrow[ur, dash] &
\end{tikzcd}
\tag{C.1}
$$

In the square brackets, we see the $[k\ell m]$ triples of intermediate hypergeometric equations, or similar indication of intermediate Heun equations. The transformation from [238] to the Heun equations is indicated before their square brackets. Similarly, the $[k\ell m]$ triples are followed by the indication of transformations from them to the final [4444]. The $\times$-power indicates two copies of that intermediate function field. The diagram includes P10 and P12, the most complicated parametric compositions [23, §C]. The components $[238]-[248]$ and $[334]-[444]$ are cubic transformations, while the other lines represent quadratic ones (possibly 2_H).

In Tables of §B, we indicate the components either by an A-J label from our list (if applicable), or by the degree otherwise. In the latter case, we give intermediate hypergeometric equations in the $[k\ell m]$ notation. Intermediate Heun equations are clear, hence no extras to 2_H. Deeper branching is indicated by {}. The A-J, P labels inside them either mean a transformation from a starting $[k\ell m]$ to an intermediate Heun equation (after 4_H) or to the target Heun equation (otherwise). The $\times\times$-power indicates three copies of an intermediate function field. A label inside nested {} refers to a composition string avoiding the merging point of the outer {}. These hints should be enough to recover the composition lattices.

D Appendix: Coxeter decompositions

If a minus-4-hyperbolic Belyi function in a canonical form (of Definition 2.2) is defined over $\mathbb{R}$, the Schwarz maps[20] of the related hypergeometric and Heun equations fit together nicely. Particularly, the quadrangle of Heun's equation is then tessellated into congruent (in the hyperbolic metric) triangles

while all [334], [248]-type functions give type-[238] compositions, with the degree 2 or 3 times larger. In the same way, the [335], [255] Belyi functions give type [2 3 10], [245] (respectively) compositions. Quadratic transformation P1 can be composed to C1 and all compositions in Table A of §B, as its j-invariant 1728 is 2-isogenous to itself and the j-value of C1.

[20]We already considered Schwarz maps in the paragraph after Remark 5.1. If a hypergeometric equation has real local exponent differences α, β, γ in the interval $[0, 1]$, the image of the upper half plane $\subset \mathbb{C}$ under its Schwarz map is a curvilinear triangle with the angles $\pi\alpha, \pi\beta, \pi\gamma$. A nice illustration can be found in [3, pg. 38]. Analytic continuation of Schwarz maps follows the Schwarz reflection principle. Hodgkinson [9] first observed that pull-back transformations of hypergeometric equations induce tessellations of Schwarz triangles into smaller congruent Schwarz triangles. Similarly, if a Heun equation has real local exponent differences $\alpha, \beta, \gamma, \delta$ in the interval $[0, 1]$, the image of its Schwarz map is a curvilinear quadrangle with the angles $\pi\alpha, \pi\beta, \pi\gamma, \pi\delta$.

of the hypergeometric equation. The degree formula in Lemma 3.1*(ii)* can be interpreted as the area ratio between the hyperbolic quadrangle and the triangles, if we multiply both the numerator and the denominator by π.

Subdivisions of hyperbolic quadrangles (or triangles) into congruent hyperbolic triangles are called *Coxeter decompositions* in [7]. The list of Coxeter decompositions can be compared with our list of Belyi maps with the r-field $\subset \mathbb{R}$, providing a mutual check of completeness. All Coxeter decompositions from our Belyi maps can be discerned in Figure 5. The similar pictures for Coxeter decompositions from parametric hypergeometric-to-Heun transformations are given in [22, Figure 2].

The Belyi functions of Tables D, E, G of §B would certainly give no Coxeter decompositions, as their r-fields are not real. As shown in the last column of Table A, most of its Belyi functions have imaginary quadratic r-fields. Here is the list[21] of other extensions of the r-field over the t-field:

$\sqrt{-1}$: B21, B22, C11, C22, C24, F9 (and C6, F4, F6, H10, H12);
$\sqrt{-2}$: B9, B10, D24, F16 (and C30);
$\sqrt{-3}$: B6, B27, B28, C3, F2, F5 (and F1, H1, H11, H13);
$\sqrt{-7}$: B2, C2 (and B12, H14);
$\sqrt{-15}$: B18, B19, C31;
$\sqrt{\frac{-5+\sqrt{5}}{2}}$: F10, F12, F17 (and F11);

(and $\sqrt{3}$ for D45). In the parentheses we list the obstructed Galois orbits of §4. Evidently, the obstructed Belyi functions have no real r-fields. Among the non-mentioned Galois orbits of Belyi functions:

- All other cases of the A, B, C-tables give a Coxeter decomposition each; $10 + 23 + 34$ in total.
- Each entry of the F-table with discrim $\mathbb{Q}(t) < 0$ gives one Coxeter decomposition; 10 in total.
- The entries F3, F23 give pairs[22] of Coxeter decompositions as $\mathbb{Q}(t) = \mathbb{Q}(j)$; F20 gives another pair with the t-field $\mathbb{Q}(\sqrt{7}, \sqrt{3})$; and F25 gives none with the t-field $\mathbb{Q}(\sqrt{4\sqrt{22} - 22})$.

[21]The cases where the moduli field is an extension of the j-field are: A4, A7, A13, A22 (as shown in Table A of §B) and B2, B6, D20, D24, F7. The coverings D20 and D24 have $j = 0$, but their moduli fields are $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-2})$, respectively. The function F7 is considered in §4.5. It does give one Coxeter decomposition, into symmetric $(\pi/3, \pi/3, \pi/4)$ triangles.

The list of cases with additional extensions for the r-field correlates well with the list of Belyi coverings with interesting monodromy groups (such as PSL in tables of §B) and the list of multiple Galois orbits with the same branching pattern (as one can inspect empty entries in the first columns in tables of §3.1).

[22]Dessins d'enfant and Coxeter decompositions are different geometric representations of a Belyi covering. The difference is twofold: the decompositions represent only a half of the Riemann sphere, and their vertices are the points not just above $z \in \{0, 1\}$ but above $z = \infty$ as well. To get a corresponding (real) dessin, two parallel copies of a Coxeter decomposition have to be glued along the edges to a topological sphere, and the vertices above $z = \infty$, the incident edges and triangles have to be removed. A real conjugation in $\mathbb{Q}(t) \supset \mathbb{Q}(j)$ acts on a Coxeter decomposition as a Möbius symmetry.

Among the Galois orbits A1–J28 without an obstruction, the F-orbits and I10, I15, J11 have two real dessins, while other orbits have at most one real dessin.

- Each entry of the H-table with $\mathbb{Q}(t) = \mathbb{Q}(j)$ or discrim $\mathbb{Q}(t) > 0$ gives a Coxeter decomposition; 11+11 in total.
- Similarly, the odd degree I, J-orbits with $\mathbb{Q}(t) = \mathbb{Q}(j)$ or discrim $\mathbb{Q}(t) > 0$ give Coxeter decompositions; 6+3 among I22–I33 and $2+3$ in the J-table.
- The entries I10, I15, J11 with pairs of real dessins and $\mathbb{Q}(t) = \mathbb{Q}(j)$ give pairs of Coxeter decompositions.

In total, we must have 125 Coxeter decompositions, just as listed in [7, Figures 10 (5)–(11), 12, 13, 15–18]. There is a caveat, however. The decompositions 24 and 36 in [7, Figure 18] coincide, while one triangulated quadrangle with the angles $\pi/3, 2\pi/3, \pi/7, 3\pi/7$ is missing. We identify the repeated decomposition as C4, and the missing one as I26.

All Coxeter decompositions can be recognized in Figure 5. To assign Belyi functions (with the r-field in $\mathbb{R}$) to Coxeter decompositions, we multiplying the branching fractions by π and look for quadrangles with the same angles. The pictures *(a)*, *(b)* show the Coxeter decompositions 7, 6 in [7, Figure 15] into triangles with the angles $\pi/2, \pi/3$ and $\pi/12$ or $\pi/11$. They represent the Belyi functions B11 and C13, respectively. Picture *(c)* contains two hyperbolic quadrangles subdivided into twelve $(\pi/2, \pi/4, \pi/6)$-triangles. They represent the Belyi functions B5 and B4, and coincide with the triangulations 3, 4 in [7, Figure 12], respectively. Picture *(d)* contains the first five triangulations in [7, Figure 15], into $(\pi/2, \pi/3, \pi/10)$-triangles. Here are the labels of Belyi maps and the quadrangles:

C38: *BCFK,* F14: *ACEG,* C15: *ACFH,* C25: *ACDG,* B24: *ACFG.*

Picture *(e)* contains the $(\pi/2, \pi/3, \pi/9)$-triangulations in [7, Figure 16]. Here is the respective sequence of Belyi maps and the quadrangles:

H2: *DFKM,* B7: *EGKM,* C41: *EHLM,* C7: *EHKM,* C9: *CFKM,*
H4: *CFKN,* H6: *CFKP,* A6: *EFKM,* C23: *ACFK,* H8: *ABFK.*

There is initial ambiguity for assigning B7 and H2 because of the same branching fractions. But B7 is a composition $3\,[339] \circ 4$ as shown in the B-table, and its Coxeter decomposition splits[23] into 3 triangles with the angles $\pi/3, \pi/3, \pi/9$ (each formed by 4 smaller triangles). Further, picture *(f)* contains the $(\pi/2, \pi/3, \pi/8)$-triangulations in [7, Figure 17]. Here is the respective sequence of Belyi maps and the quadrangles:

B1: *ABEK,* A18: *BEGK,* B14: *BEKN,* A5: *BGKN,* C1: *ADST,*
I10: *FGKS,* I10: *ACET,* C32: *CEKQ,* F19: *DFUQ,* C21: *ADFU,*
H29: *ADFV,* C28: *ABRH,* B29: *ACEK,* B15: *ADSK,* F8: *ADFL,*
B30: *ADFM,* C8: *ABET,* H22: *PDFM,* C5: *ADEK.*

Picture *(g)* contains the $(\pi/2, \pi/4, \pi/5)$-triangulations in [7, Figure 13]. Here is the respective sequence of Belyi maps and the quadrangles:

B26: *FKPS,* H39: *VYQT,* C19: *VKQT,* C16: *ACOS,* B23: *ACQS,*
H16: *ACQR,* I24: *FHPS,* H17: *DHPZ,* F13: *VKPT,* C14: *BXZT,*
B8: *BFST,* H41: *BFQT,* I29: *OLNZ,* F15: *OLNS,* C10: *VLNS,*
I31: *WUYQ,* H18: *WUKP,* C17: *BGPT,* A20: *AEMR,* I30: *WULN.*

[23] Coxeter decompositions do not always split according to (all) compositions of their Belyi functions, because smaller degree components do not necessarily have Coxeter decompositions. For example, consider A18 = A16 ∘ 3, A19 = $2_H \circ$ A1, B4 = $2_H \circ$ D25, B14 = $2_H \circ$ D9, J19 = $2_H \circ$ H45, J26 = $2_H \circ$ J25, etc.

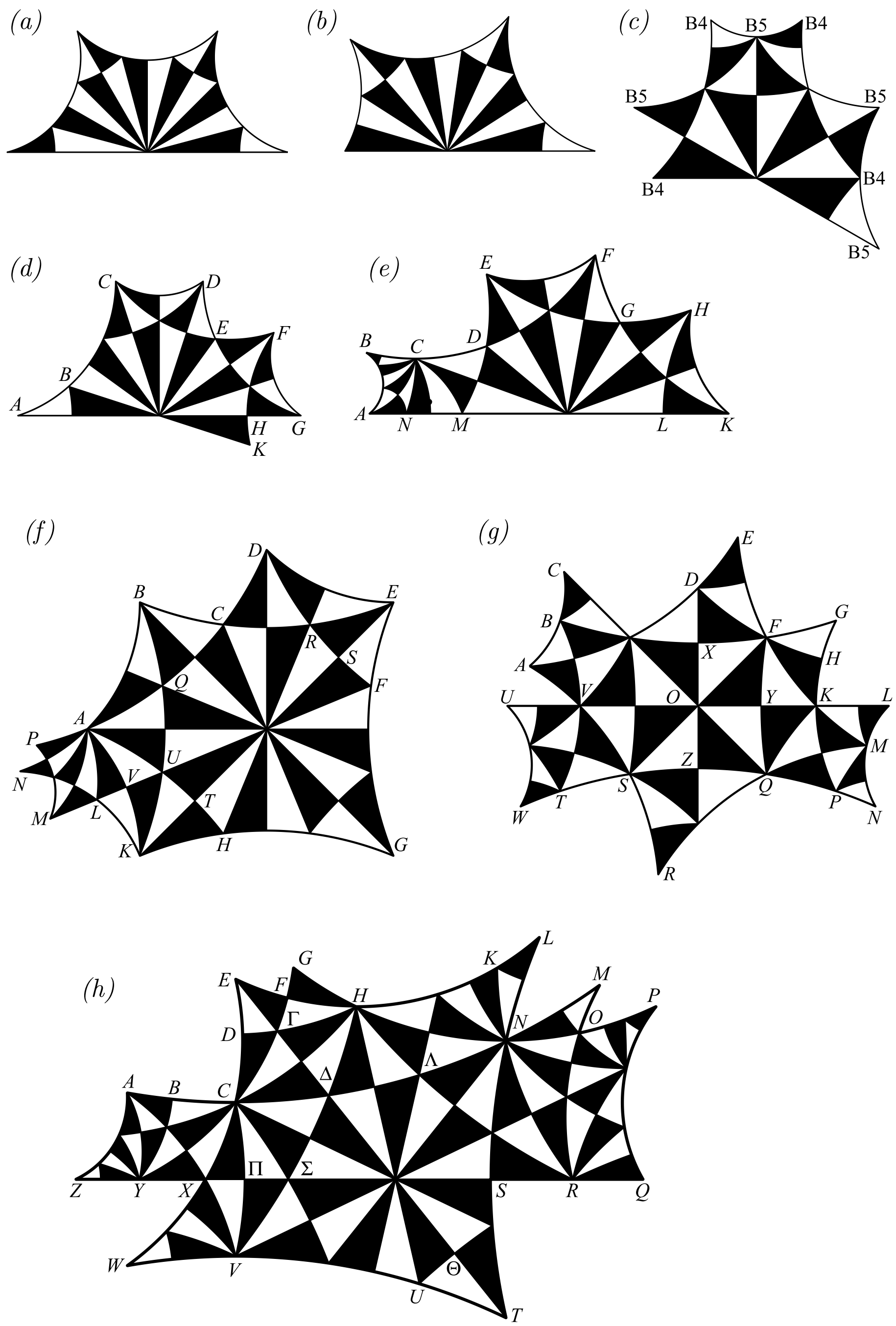

Figure 5: The Coxeter decompositions of Felikson

There is the same ambiguity between C14 and H17 due to the same branching fractions. It is resolved by the composition $\mathrm{C17} = 2_H \circ \mathrm{C14}$. Picture *(h)* contains the $(\pi/2, \pi/3, \pi/7)$-triangulations in [7, Figure 18]. Here is the respective sequence of Belyi maps and the quadrangles, with the repeated decomposition (36) replaced by the one for I26:

C34: $CHR\Sigma$,	F3: $HMR\Sigma$,	A15: $MTV\Delta$,	A11: $HKR\Sigma$,	H33: $HOR\Sigma$,
A14: $A\Delta TV$,	A21: $ELTW$,	C29: $R\Sigma\Delta\Lambda$,	I15: $MS\Sigma\Delta$,	C42: $CMS\Sigma$,
F22: $CMS\Pi$,	I15: $UV\Delta\Theta$,	B33: $CHR\Pi$,	C40: $C\Lambda R\Sigma$,	H53: $C\Lambda R\Pi$,
B20: $MR\Sigma\Delta$,	B17: $CHRX$,	B25: $C\Lambda RX$,	H50: $CMSX$,	C33: $HMS\Sigma$,
C26: $CMR\Sigma$,	C35: $CMR\Pi$,	C27: $C\Lambda RY$,	C4: $CHRY$,	H48: $A\Lambda RX$,
F3: $CMRX$,	H40: $CMSY$,	I33: $AMSX$,	H37: $BMSY$,	B32: $HRY\Gamma$,
B31: $CMRY$,	B34: $EHRX$,	F23: $AMRX$,	C39: $BMRY$,	F20: $AMRY$,
I26: $NR\Sigma\Delta$,	F23: $CNR\Sigma$,	C36: $CNR\Pi$,	H52: $CNRX$,	F24: $CNRY$,
F20: $CMTV$,	B13: $CMTW$,	I23: $EKRX$,	J11: $DHRX$,	J11: $HLS\Sigma$,
H44: $EHRX$,	I25: $BNRY$,	J24: $FKRY$,	J19: $FLSY$,	H36: $GHRY$,
C12: $ANRX$,	J26: $GMSY$,	B16: $GMRY$,	C18: $ANRY$,	J17: $E\Theta UW$,
J23: $GPQY$,	I28: $ANRZ$,	F21: $GORY$.		

The ambiguity between A14 and B32 is resolved by the reflection symmetry of $\mathrm{A14} = 2 \circ 10$. The non-parametric decompositions (5)–(11) of [7, Figure 10] and the decompositions (1), (2) of [7, Figure 12] represent the Galois orbits F18, B3, C20, C37, F7, H21, A17, H15, A19, respectively. They can be obtained from our listed quadrangles of (respectively) F19, B4, C21, C38, F8, H22, A18, H16, A20 by pairing their triangles to larger triangles with the requisite angles $(\pi/3, \pi/3, \pi/4)$, $(\pi/3, \pi/4, \pi/4)$, $(\pi/3, \pi/3, \pi/5)$ or $(\pi/2, \pi/5, \pi/5)$.

E Appendix: Arithmetic observations

As observed in [23, §2.3], the t-parameters of Heun equations reducible to hypergeometric equations by a pul-back transformation are arithmetically interesting. The whole orbit (2.1) of t-values can be encoded by an arithmetic identity $A + B = C$ with algebraic integers A, B, C (as "co-prime" as possible), as the set $\{A/C, B/C, C/A, C/B, -A/B, -B/A\}$. Here are these identities for a few t-orbits in $\mathbb{Q}$:

$$\begin{array}{lll} \mathrm{B25}: 1 + 2{\cdot}11^2 = 3^5, & \mathrm{B29}: 2^2 + 11^2 = 5^3, & \mathrm{B30}: 1 + 3^3 5^2 = 2^2 13^2, \\ \mathrm{B31}: 1 + 2^5\, 3{\cdot}5^2 = 7^4, & \mathrm{B33}: 11^3 + 2^2 7^4 = 3^7\, 5, & \mathrm{B34}: 7^4 + 3^3 5^3 = 2^4 19^2. \end{array}$$

The terms in these identities involve only small primes, usually in some power. Correspondingly, the t-values factorize nicely in $\mathbb{Q}$. These identities are interesting in the context of the ABC conjecture [24] and S-unit equations [24]. The "factorization" pattern holds for the t-values in algebraic extensions of $\mathbb{Q}$ as well, though arithmetic quality is then measured more technically [16] by the prime places and arithmetic height in $\mathbb{P}^2(\overline{\mathbb{Q}})$. The underlying reason is that the Belyi coverings (of pull-back transformations) tend to degenerate only modulo a few small primes [2]. Hence the t-orbit (2.1) degenerates only modulo those bad primes.

Amidst the encountered examples, we find the following well-known identities $A+B=C$ in quadratic fields:

$$\text{C18}: \left(\tfrac{\sqrt{5}-1}{2}\right)^{12} + 2^4 3^2\sqrt{5} = \left(\tfrac{\sqrt{5}+1}{2}\right)^{12},$$
$$\text{D37/D39}: \left(\tfrac{1+\sqrt{-7}}{2}\right)^{13} + \sqrt{-7} = \left(\tfrac{1-\sqrt{-7}}{2}\right)^{13}.$$

They are among top 12 known examples of remarkable ABC identities [16] in algebraic number fields. Their ABC-quality is $\approx$ 1.697794, 1.707222, respectively, while Nitaj's table [16] includes examples with the quality > 1.5. The Belyi function D42 gives a new example in $\mathbb{Q}(\sqrt{-14})$ with the quality $\log(3^{13}5^3)/\log(56\cdot 2\cdot 7\cdot 3^2\cdot 5^2) \approx 1.581910$. However, the class number of $\mathbb{Q}(\sqrt{-14})$ is equal to 4, hence an explicit arithmetic identity is less impressive, without 13th powers:

$$\left(5-2\sqrt{-14}\right)\left(11+\sqrt{-14}\right)^3 + \left(\sqrt{-14}\right)^3 = \left(5+2\sqrt{-14}\right)\left(11-\sqrt{-14}\right)^3. \quad \text{(E.1)}$$

Less symmetric quadratic identities arrise from the F, G-cases with $\mathbb{Q}(t)=\mathbb{Q}(j)$. For example, G30 gives

$$\left(\tfrac{1+\sqrt{-7}}{2}\right)^{10} + \left(\tfrac{1-\sqrt{-7}}{2}\right)^{5} + \left(2+\sqrt{-7}\right)^3 = 0. \quad \text{(E.2)}$$

The Belyi coverings E10/E11give the following $A+B=C$ example in a number field of degree 6. Let ζ denote a root of $z^6+4z^4-3z^2+2$. Then

$$\zeta^{23} + \left(\frac{\zeta+\zeta^2}{2} - \frac{5\zeta^3+\zeta^5}{4}\right)^{23}\left(\frac{1-\zeta}{2} - \frac{3\zeta^2-3\zeta^3+\zeta^4-\zeta^5}{4}\right)^{-6}$$
$$= \left(\frac{-\zeta+\zeta^2}{2} + \frac{5\zeta^3+\zeta^5}{4}\right)^{23}\left(\frac{1+\zeta}{2} - \frac{3\zeta^2+3\zeta^3+\zeta^4+\zeta^5}{4}\right)^{-6}.$$

The numbers under the 23rd power have the norm 2, while the numbers in the (-6)th power are units.

References

[1] N.M. Adrianov, N.Y. Amburg, V.A. Dremov, Y.Y. Kochetkov, E.M. Kreines, Y. A. Levitskaya, V.F. Nasretdinova, and G. B. Shabat. Catalog of dessins d'enfant with no more than 4 edges. *J. Math. Sciences*, 158:22–80, 2009.

[2] S. Beckmann. Ramified primes in the field of moduli of branched coverings of curves. *J. Algebra*, 125:236–255, 1989.

[3] F. Beukers. Gauss' hypergeometric function. In R.-P. Holzapfel et al, editor, *Arithmetic and Geometry Around Hypergeometric Functions*, volume 260 of *Progress in Mathematics*, pages 23–42. Birkhauser, Basel, 2007.

[4] F. Beukers and H. Montanus. Explicit calculation of elliptic fibrations of k3-surfaces and their belyi maps. In *Number theory and polynomials*, pages 33–51. Cambridge University Press, 2008.

[5] J.-M. Couveignes and L. Granboulan. Dessins from a geometric point of view. In *The Grothendieck theory of dessins d'enfant*, pages 79–113. Cambridge Univ. Press, 1994.

[6] L. Pharamond dit d'Costa. Comparaison de deux notions de rationalit d'un dessin d'enfant. *Journal de Théorie des Nombres de Bordeaux*, 13(2):529–538, 2001.

[7] A. A. Felikson. Coxeter decompositions of hyperbolic polygons. *Europ. J. Combinatorics*, 19:801–817, 1998.

[8] A. Grothednieck. Esquisse dún Programme (1984). In L. Schneps and P. Lochak, editors, *Geometric Galois Actions*, volume 242 of *London Math. Soc. Lecture Notes*, pages 5–48; English transl. 243–283. Cambridge University Press, 1997.

[9] J. Hodgkinson. An application of conformal representation to certain hypergeometric series. *Proc. London Math. Soc.(2)*, 17:17–24, 1918.

[10] F. Klein. Uber lineare differentialgleichungen II. *Math. Annalen*, 12:167–179, 1878.

[11] E. Kreines. On families of geometric parasitic solutions for Belyi systems of genus zero. *Fundamentalnaya i Priklandaya Matematika*, 9(1):103–111, 2003.

[12] S.K. Lando and A.K. Zvonkin. *Graphs on Surfaces and their Applications*, volume 141 of *Encyclopedia of Mathematical Sciences*. Springer-Verlag, 2004.

[13] L.G. Luttmer. Using dessins d'enfant to understand the absolute Galois group. 2012.

[14] R. E. Macrae. On rational points on conics. *Proc. AMS*, 67(1):38–40, 1977.

[15] N. Magot and A. Zvonkin. Belyi functions for archimedean solids. *Disc. Math.*, 217:249–271, 2000.

[16] A. Nitaj. The abc conjecture home page. www.math.unicaen.fr/nitaj/abc.html.

[17] M. van Hoeij and R. Vidunas. Online data for "belyi functions for hyperbolic hypergeometric-to-Heun transformations". http://www.math.fsu.edu/~hoeij/Heun.

[18] M. van Hoeij and J.-A. Weil. Solving second order linear differential equations with Klein's theorem. Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation (ISSAC), 2005.

[19] J. Verschelde. Phcpack: a general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Softw.*, Alg. 795, 1999.

[20] R. Vidūnas. Algebraic transformations of Gauss hypergeometric functions. *Funkcialaj Ekvacioj*, 52(2):139–180, 2009. Available at http://arxiv.org/math.CA/0408269.

[21] R. Vidunas. Transformations and invariants for dihedral Gauss hypergeometric functions. Accepted by Kyushu J. Math. Available at http://arxiv.org/abs/1101.3688., 2011.

[22] R. Vidūnas and G. Filipuk. A classification of coverings yielding heun-to-hypergeometric reductions. Accepted by *Osaka J. Math.* Available at arxiv:1204.2730[math/CA], 2012.

[23] R. Vidūnas and G. Filipuk. Parametric transformations between the Heun and Gauss hypergeometric functions. Accepted by *Funkcialaj Ekvacioj*. Available at arxiv:0910.3087v2[math/CA], 2012.

[24] Wikipedia. Dessin d'enfant, Möbius transformation, Fuchsian equation, Liuovillian extension, Elliptic curve, j-invariant, Cross-ratio, ABC conjecture, S-units equation, Chebotarev's density theorem. http://en.wikipedia.org/.