APPLICATION OF A FOURIER RESTRICTION THEOREM TO CERTAIN FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

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Abstract. We use a restriction theorem for Fourier transforms of fractal measures to study projections onto families of planes in $\mathbb{R}^3$ whose normal directions form nondegenerate curves.

1. Introduction and statement of results

Suppose that $\gamma : [0,1] \to S^2$ is $C^{(2)}$. Following K. Fässler and T. Orponen [4] we say that γ is *nondegenerate* if

$$\operatorname{span}\{\gamma(t), \gamma'(t), \gamma''(t)\} = \mathbb{R}^3,\ \ t \in [0,1].$$

Let π_t be the orthogonal projection of $\mathbb{R}^3$ onto the plane $\gamma(t)^\perp$ and let $B \subset \mathbb{R}^3$ be a compact set with Hausdorff dimension $\dim(B) = \alpha$. One of the problems treated in [4] is to say something about the dimension of $\pi_t(B)$ for generic $t \in [0,1]$. Fässler and Orponen prove that

(a) if $\alpha \leq 1$ then

$$\dim(\pi_t(B)) = \dim(B)$$

for almost all $t \in [0,1]$, and

(b) if $\alpha > 1$ then there exists $\sigma = \sigma(\alpha) > 1$ such that the packing dimension of $\pi_t(B)$ exceeds σ for almost all t.

In a subsequent paper, [6], Orponen considers the particular γ given by

$$\gamma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, 1) \tag{1.1}$$

and establishes the analog of (b) for Hausdorff dimension. (We mention that, in addition to other interesting results of a similar nature, the papers [4] and [6] provide a nice account of the history of these problems.) The purpose of this note is to prove the following result:

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Theorem 1.1. *With notation as above, suppose that B is a compact subset of $\mathbb{R}^3$ and that $\dim(B) = \alpha \geq 1$. Then, for almost all $t \in [0,1]$,*

$$\dim\big(\pi_t(B)\big) \geq 3\alpha/4 \text{ if } 1 \leq \alpha \leq 2 \text{ and} \qquad \dim\big(\pi_t(B)\big) \geq \alpha - 1/2 \text{ if } 2 \leq \alpha \leq 3. \tag{1.2}$$

The proof uses the potential-theoretic method introduced in [5], which we approach using the Fourier transform as in [3]. In the model case (1.1) a critical role in the proof of Theorem 1.1 is played by the following result of Erdoğan, which can be extracted from [1]:

Theorem 1.2. *Suppose that μ is a nonnegative and compactly supported Borel probability measure on $\mathbb{R}^3$ satisfying*

$$\mu\big(B(x,r)\big) \leq c\, r^{\alpha} \tag{1.3}$$

for $x \in \mathbb{R}^3$ and $r > 0$. If $\alpha' < \alpha$ then there is C (depending only on c, α', and the diameter of the support of μ) such that

$$\int_0^{2\pi}\int_{1/2}^{1} |\hat{\mu}\big(R\,\rho\,(\cos t, \sin t, 1)\big)|^2 d\rho\, dt \leq C\, R^{-\beta(\alpha')}, R \geq 1,$$

where $\beta(\alpha') = \alpha'/2$ if $1 \leq \alpha \leq 2$ and $\beta(\alpha') = \alpha' - 1$ if $2 \leq \alpha \leq 3$.

To prove Theorem 1.1 we require the following generalization of Theorem 1.2:

Theorem 1.3. *Suppose $\phi : [-1/2, 1/2] \to \mathbb{R}$ is $C^{(2)}$ and satisfies*

$$M/2 \leq |\phi'(t)| \leq M, \quad |\phi''(t)| \geq m > 0$$

for $t \in [0,1]$ and for positive constants M and m. Suppose that μ is a nonnegative and compactly supported Borel probability measure on $\mathbb{R}^3$ satisfying

$$\mu\big(B(x,r)\big) \leq c\, r^{\alpha}$$

for $x \in \mathbb{R}^3$ and $r > 0$. If $\alpha' < \alpha$ then there is C (depending only on m, M, c, α', and the diameter of the support of μ) such that

$$\int_{-1/2}^{1/2}\int_{1/2}^{1} |\hat{\mu}\big(R\,\rho\,(t, \phi(t), 1)\big)|^2 d\rho\, dt \leq C\, R^{-\beta(\alpha')}, R > 1, \tag{1.4}$$

where $\beta(\alpha') = \alpha'/2$ if $1 \leq \alpha \leq 2$ and $\beta(\alpha') = \alpha' - 1$ if $2 \leq \alpha \leq 3$.

(For a similar generalization of Wolff's result in [7] on decay of circular means of Fourier transforms of measures on $\mathbb{R}^2$, see [2].)

This note is organized as follows: §2 contains the proof of Theorem 1.1, §3 contains the proof of Theorem 1.3, and §4 contains the proof of a technical lemma used in §3.

2. Proof of Theorem 1.1

Suppose $\alpha' < \tilde{\alpha} < \alpha$ so that we can find a probability measure μ on B satisfying

$$\mu\big(B(x,r)\big) \le C\, r^{\tilde{\alpha}},\ x \in \mathbb{R}^3,\ r > 0 \text{ and} \tag{2.1}$$
$$\int_{\mathbb{R}^3} \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{3-\tilde{\alpha}}}\, d\xi < \infty.$$

Write $\pi_t(\mu)$ for the measure which is the push-forward of μ onto $\pi_t(\mathbb{R}^3)$ under the projection π_t. For a function f on $\gamma(t)^\perp$ we have

$$\int_{\gamma(t)^\perp} f\, d\pi_t(\mu) = \int_{\mathbb{R}^3} f\big(x - [x\cdot\gamma(t)]\gamma(t)\big)\, d\mu(x)$$

so for $\xi \in \gamma(t)^\perp$ we have

$$\widehat{\pi_t(\mu)}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i\langle \xi, (x-[x\cdot\gamma(t)]\gamma(t)\rangle}\, d\mu(x) = \hat{\mu}(\xi).$$

To establish (1.2) it is therefore enough to show that for each $t_0 \in (0,1)$ there is some closed interval $I = I_{t_0}$ containing t_0 in its interior such that

$$\int_I \int_{\gamma(t)^\perp} \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{2-\tau(\alpha')}}\, d\xi\, dt < \infty, \tag{2.2}$$

where

$$\tau(\alpha') = 3\alpha'/4 \text{ if } 1 \le \alpha \le 2,\quad \tau(\alpha') = \alpha' - 1/2 \text{ if } 2 < \alpha \le 3. \tag{2.3}$$

Without loss of generality we assume that γ is parametrized by arclength. Let $u(t) = \gamma'(t)$, $v(t) = \gamma(t) \times \gamma'(t)$. We parametrize $\gamma(t)^\perp$ by

$$(u,v) \mapsto u \cdot u(t) + v \cdot v(t)$$

and part of $\mathbb{R}^3$ by

$$(t,u,v) \mapsto u \cdot u(t) + v \cdot v(t). \tag{2.4}$$

If we now parametrize (u,v)-space by polar coordinates

$$u = r\sin\theta,\ v = r\cos\theta$$

then (2.2) becomes

$$\int_0^{2\pi} \int_I \int_0^\infty \frac{|\hat{\mu}\big(r(\sin\theta\, u(t) + \cos\theta\, v(t)\,)\big)|^2}{r^{1-\tau(\alpha')}}\, dr\, dt\, d\theta < \infty. \tag{2.5}$$

To establish (2.5) for every $\alpha' < \bar{\alpha}$ it is enough to show that

(2.6)
$$\int_0^{2\pi} \int_I \int_R^{2R} \frac{|\hat{\mu}\big(r(\sin\theta\, u(t) + \cos\theta\, v(t)\,)\big)|^2}{r^{1-\tau(\alpha')}}\, dr\, dt\, d\theta \le C(I,\alpha'),\ R \ge R(I) > 1$$

for every $\alpha' < \bar{\alpha}$. We will focus, without loss of generality, on the part of the integral in (2.6) corresponding to the range $0 \le \theta \le \pi/2$. We write

$$\eta(R) = R^{-\alpha'/4} \text{ if } 1 \le \alpha \le 2,\quad \eta(R) = R^{-1/2} \text{ if } 2 < \alpha \le 3 \tag{2.7}$$

and then split the integral:

$$\int_0^{\eta(R)}\int_I\int_R^{2R}\frac{|\hat\mu\big(r(\sin\theta\, u(t)+\cos\theta\, v(t)\,)\big)|^2}{r^{1-\tau(\alpha')}}\,dr\,dt\,d\theta+ \tag{2.8}$$
$$\int_{\eta(R)}^{\pi/2}\int_I\int_R^{2R}\frac{|\hat\mu\big(r(\sin\theta\, u(t)+\cos\theta\, v(t)\,)\big)|^2}{r^{1-\tau(\alpha')}}\,dr\,dt\,d\theta\doteq$$
$$\mathcal{I}_1+\mathcal{I}_2.$$

We begin with the second of these and will use the change of variable (2.4). The Jacobian factor $J=J(t,u,v)$ associated with (2.4) is

$$|\det\big(u(t),v(t),u\cdot u'(t)+v\cdot v'(t)\big)|=|\langle u\cdot u'(t)+v\cdot v'(t),\gamma(t)\rangle|,$$

where the last equality follows because $u(t)\times v(t)=\pm\gamma(t)$. Since $u(t)\perp\gamma(t)$ implies

$$\langle u'(t),\gamma(t)\rangle=-\langle u(t),\gamma'(t)\rangle$$

and similarly for $v(t)$, we see that

$$J=|u|. \tag{2.9}$$

To use (2.9) we need some information about the multiplicity of the change of variables (2.4). To obtain this information we will impose a first restriction on the size of the interval $I=I_{t_0}$. (When we deal with with the second integral in (2.8) we will need to impose further restrictions on I.) Fix t_0 and choose coordinates for $\mathbb{R}^3$ so that $\gamma(t_0)=(0,0,1)$ and $\gamma'(t_0)=(1,0,0)$ and then write $\gamma=(\gamma_1,\gamma_2,\gamma_3)$. Let $\bar\gamma(t)$ be the curve in $\mathbb{R}^2$ given, in a neighborhood of t_0, by

$$\bar\gamma(t)=\Big(\frac{\gamma_1(t)}{\gamma_3(t)},\frac{\gamma_2(t)}{\gamma_3(t)}\Big) \tag{2.10}$$

and let $\tilde\gamma$ be the curve in $\mathbb{R}^3$ given by

$$\tilde\gamma(t)=\gamma(t)/\gamma_3(t)=\big(\bar\gamma(t);1\big). \tag{2.11}$$

We will need that fact that if $\kappa(\bar\gamma;t)$ is the curvature of $\bar\gamma$ at t_0 then

$$\kappa(\bar\gamma;t_0)=|\det\big(\gamma(t_0),\gamma'(t_0),\gamma''(t_0)\big)|>0,$$

where the inequality is a consequence of the non degeneracy of γ. To see the equality we begin by computing

$$\bar\gamma'=\Big(\frac{\gamma_1'\gamma_3-\gamma_1\gamma_3'}{\gamma_3^2},\ \frac{\gamma_2'\gamma_3-\gamma_2\gamma_3'}{\gamma_3^2}\Big)$$

and

$$\bar\gamma''=\Big(\frac{\gamma_3^2(\gamma_1''\gamma_3-\gamma_1\gamma_3'')-2\gamma_3\gamma_3'(\gamma_1'\gamma_3-\gamma_1\gamma_3')}{\gamma_3^4},\frac{\gamma_3^2(\gamma_2''\gamma_3-\gamma_2\gamma_3'')-2\gamma_3\gamma_3'(\gamma_2'\gamma_3-\gamma_2\gamma_3')}{\gamma_3^4}\Big).$$

When $t=t_0$ we have

$$\bar\gamma(t_0)=(1,0),\ \bar\gamma''(t_0)=\big(\gamma_1''(t_0),\gamma_2''(t_0)\big).$$

Thus $\kappa(\bar{\gamma}; t_0) = |\gamma_2''(t_0)| = |\det\big(\gamma(t_0), \gamma'(t_0), \gamma''(t_0)\big)|$ as desired. Now choose I small enough so that

$$\kappa(\bar{\gamma}; t) > 0, \;\; t \in I.$$

After possibly shrinking I again one sees that if $t_1, t_2, t_3 \in I$, then the vectors $\{\tilde{\gamma}(t_1), \tilde{\gamma}(t_2), \tilde{\gamma}(t_3)\}$ are linearly independent. Since

$$\tilde{\gamma}(t_i) \perp \big(u \cdot u(t_i) + v \cdot v(t_i)\big),$$

it follows that (2.4) is at most three-to-one on $I \times (\mathbb{R}^2 \sim \{0\})$. Therefore, with

$$\xi = r\big(\sin\theta\, u(t) + \cos\theta\, v(t)\big)$$

and using (2.9) to write $J = |u| = |r\sin\theta|$ we have

$$\text{(2.12)} \quad \mathcal{I}_2 = \int_{\eta(R)}^{\pi/2} \int_I \int_R^{2R} \frac{|\hat{\mu}\big(r(\sin\theta\, u(t) + \cos\theta\, v(t)\,)\big)|^2}{r^{1-\tau(\alpha')}}\, dr\, dt\, d\theta \lesssim \frac{1}{\eta(R)} \int_{R \le |\xi| \le 2R} \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{3-\tau(\alpha')}}\, d\xi \le C\, R^{\alpha' - \bar{\alpha}}$$

by (2.1), (2.3), and (2.7).

We now obtain a similar estimate for the term $\mathcal{I}_1$. Lemma 3.2 from [4] states that the function $v(t) = \gamma(t) \times \gamma'(t)$ satisfies the same hypotheses as $\gamma(t)$:

$$\operatorname{span}\{v(t), v'(t), v''(t)\} = \mathbb{R}^3, \;\; t \in [0, 1].$$

We proceed as above, beginning by choosing coordinates for $\mathbb{R}^3$ so that $v(t_0) = (0, 0, 1)$, $v'(t_0) = (1, 0, 0)$. It follows that if $\bar{v}$ and $\tilde{v}$ are defined as in (2.10) and (2.11) but with $v(t)$ in place of $\gamma(t)$, then $\kappa(\bar{v}; t_0) > 0$. For small θ we will also need the perturbations of $\bar{v}$ and $\tilde{v}$ given by taking

$$\text{(2.13)} \qquad v_\theta(t) = \cos\theta\, v(t) + \sin\theta\, u(t)$$

instead of γ in (2.10) and (2.11). Using $\kappa(\bar{v}; t_0) > 0$ we choose $\theta_0 > 0$ such that $\kappa(\bar{v}_\theta; t_0) > 0$ for $0 \le \theta \le \theta_0$. We then further restrict θ_0 and the interval $I = I_{t_0}$ so that

$$\text{(2.14)} \qquad |v_\theta(t) - (0, 0, 1)| \le 1/10 \text{ for } t \in I, \;\; 0 \le \theta \le \theta_0$$

and, for some $\tilde{m} > 0$, we have

$$\kappa(\bar{v}_\theta; t) \ge \tilde{m} \text{ for } 0 \le \theta \le \theta_0, \;\; t \in I.$$

After a suitable linear change of coordinates in $\mathbb{R}^2$ we choose positive numbers M and m such that (after possibly diminishing θ_0 and I) the curves $\{\tilde{v}_\theta(t) : t \in I\}$, $0 \le \theta \le \theta_0$, can be written as

$$\{\big(\tilde{t}, \phi_\theta(\tilde{t}), 1\big) : \tilde{t} \in [-1/2, 1/2]\}$$

with

$$M/2 \le |\phi_\theta'(\tilde{t})| \le M, \quad |\phi_\theta''(\tilde{t})| \ge m > 0, \quad -1/2 \le \tilde{t} \le 1/2, \quad 0 \le \theta \le \theta_0.$$

It follows from Theorem 1.3 that for $0 \leq \theta \leq \theta_0$ we have

$$\int_I \int_{9R/10}^{22R/10} |\hat{\mu}\big(r(\tilde{v}_\theta(t)\,)\big)|^2\, dr\, dt \leq C\, R^{1-\beta(\alpha')}.$$

So if $R(I)$ is chosen to have $\eta\big(R(I)\big) = \theta_0$ then it follows from (2.3), (2.7), (2.13), (2.14), and the definition of $\beta(\alpha')$ that

$$\mathcal{I}_1 = \int_0^{\eta(R)} \int_I \int_R^{2R} \frac{|\hat{\mu}\big(r(\sin\theta\, u(t) + \cos\theta\, v(t)\big)|^2}{r^{1-\tau(\alpha')}}\, dr\, dt\, d\theta \leq C(I, \alpha'),\ R \geq R(I).$$

With (2.12) and (2.8) this gives (2.6) and therefore completes the proof of Theorem 1.1.

3. Proof of Theorem 1.3

For $2 < \alpha \leq 3$, (1.4) follows directly from Theorem 1 in [1]. For $1 \leq \alpha \leq 2$, the proof is an adaptation of ideas from [1] and [7]. Specifically, we will write

$$\sigma(\rho, t) = \rho\,\big(t, \phi(t), 1\big),$$
$$\Gamma_R = \{R\,\sigma(t, \rho) : -1/2 \leq t \leq 1/2,\ 1/2 \leq \rho \leq 1\},$$
$$\Gamma_{R,\delta} = \Gamma_R + B(0, R^\delta),\ R \geq 2,\ \delta > 0$$

and, with μ as in Theorem 1.3, we will show that (1.4) follows from the estimate

$$\int_{\Gamma_{R,\delta}} |\widehat{\mu}(y)|^2\, dy \lesssim R^{2-\alpha/2+2\delta},\ 0 < \delta \leq 1. \tag{3.1}$$

We will then adapt a bilinear argument from [1] to prove (3.1). (Throughout this proof the constants implied by the symbol $\lesssim$ can be chosen to depend only δ and on the parameters mentioned for C in the statement of Theorem 1.1.)

So, arguing as in [7], if $\kappa \in C_c^\infty(\mathbb{R}^3)$ is equal to 1 on the support of μ, then

(3.2)

$$\int_{-1/2}^{1/2}\int_{1/2}^{1} |\hat{\mu}\big(R\,\sigma(\rho, t)\big)|^2 d\rho\, dt = \int_{-1/2}^{1/2}\int_{1/2}^{1} \Big| \int \widehat{\kappa}\big(R\,\sigma(\rho, t) - y\big)\, \widehat{\mu}(y) dy\Big|^2 d\rho\, dt \lesssim$$
$$\int \int_{-1/2}^{1/2}\int_{1/2}^{1} \big|\widehat{\kappa}\big(R\,\sigma(\rho, t) - y\big)\big|\, d\rho\, dt\ |\widehat{\mu}(y)|^2\, dy.$$

Let $\epsilon = \alpha - \alpha'$. Choose a large p_1 such that if $y \notin \Gamma_R + B(0, R^{p_1})$ then $\mathrm{dist}(y, \Gamma_{R,\epsilon/4}) \geq |y|/2$. Choose a large p_2 such that if $y \in \Gamma_R + B(0, R^{p_1})$ then $|y| \leq R^{p_2}$. Finally, choose a large K such that $(K-4)\epsilon/4 \geq 3p_2$. If

$y=(y_1,y_2,y_3)$, then

$$\int_{-1/2}^{1/2}\int_{1/2}^{1}\left|\widehat{\kappa}(R\,\sigma(\rho,t)-y)\right|d\rho\,dt \lesssim \int_{-1/2}^{1/2}\int_{1/2}^{1}\frac{1}{\big(1+|R\,\sigma(\rho,t)-y|\big)^{K}}\,d\rho\,dt \lesssim$$
$$\frac{1}{\big(1+\mathrm{dist}(\Gamma_R,y)\big)^{K-4}}\int_{1/2}^{1}\int_{-1/2}^{1/2}\frac{1}{\big(1+|R\,\rho\,\phi(t)-y_2|\big)^2}\,dt\,\frac{1}{\big(1+|R\,\rho-y_3|\big)^2}\,d\rho.$$

Estimating the last two integrals (we use the hypothesized lower bound on ϕ'), we see from (3.2) that

$$\int_{-1/2}^{1/2}\int_{1/2}^{1}|\widehat{\mu}\big(R\,\sigma(\rho,t)\big)|^2 d\rho\,dt \lesssim \frac{1}{R^2}\int\frac{|\widehat{\mu}(y)|^2}{\big(1+\mathrm{dist}(\Gamma_R,y)\big)^{K-4}}\,dy. \tag{3.3}$$

Now

$$\int\frac{|\widehat{\mu}(y)|^2}{\big(1+\mathrm{dist}(\Gamma_R,y)\big)^{K-4}}\,dy \;=\; \int_{\Gamma_{R,\epsilon/4}}+\int_{B(0,R^{p_2})\sim\Gamma_{R,\epsilon/4}}+\int_{\{|y|\geq R^{p_2}\}}.$$

The first integral, the principal term, is $\lesssim R^{2-\alpha'/2}$ by (3.1). Since $y\notin\Gamma_{R,\epsilon/4}$ implies $\mathrm{dist}(\Gamma_R,y)\geq R^{\epsilon/4}$, the second integral is $\lesssim 1$ by the fact that $(K-4)\epsilon/4\geq 3p_2$. Since $|y|\geq R^{p_2}$ implies $y\notin\Gamma_{R,p_1}$ and so implies $\mathrm{dist}(\Gamma_R,y)\geq |y|/2$, the last integral is also $\lesssim 1$. Thus, given (3.1), (1.4) follows from (3.3).

Turning to the proof of (3.1), we note that by duality (and the fact that μ is finite) it is enough to suppose that f, satisfying $\|f\|_2=1$, is supported on $\Gamma_{R,\delta}$ and then to establish the estimate

$$\int|\widehat{f}(y)|^2\,d\mu(y)\lesssim R^{2-\alpha/2+2\delta}. \tag{3.4}$$

The argument we will give differs from the proof of Theorem 5 in [1] only in certain technical details. But, because those details are not always obvious, we will give the complete proof.

For $y\in\mathbb{R}^3$, write y' for a point $R\sigma(\rho',t')$ ($\rho'\in[1/2,1]$, $t'\in[-1/2,1/2]$) on the surface Γ_R which minimizes $\mathrm{dist}(y,\Gamma_R)$. For a dyadic interval $I\subset[-1/2,1/2]$, define

$$\Gamma_{R,\delta,I}=\{y\in\Gamma_{R,\delta}: t'\in I\},\; f_I=f\cdot\chi_{\Gamma_{R,\delta,I}}\,.$$

For dyadic intervals $I,J\subset[-1/2,1/2]$, we write $I\sim J$ if I and J have the same length and are not adjacent but have adjacent parent intervals. The decomposition

$$[-1/2,1/2]\times[-1/2,1/2]=\bigcup_{n\geq 2}\Big(\bigcup_{\substack{|I|=|J|=2^{-n}\\ I\sim J}}(I\times J)\Big) \tag{3.5}$$

leads to

$$\int |\widehat{f}(y)|^2 \, d\mu(y) \leq \sum_{n\geq 2} \sum_{\substack{|I|=|J|=2^{-n} \\ I\sim J}} \int |\widehat{f_I}(y)\widehat{f_J}(y)| \, d\mu(y). \tag{3.6}$$

Truncating (3.5) and (3.6) gives

$$\int |\widehat{f}(y)|^2 \, d\mu(y) \leq \sum_{4\leq 2^n \leq R^{1/2}} \sum_{\substack{|I|=|J|=2^{-n} \\ I\sim J}} \int |\widehat{f_I}(y)\widehat{f_J}(y)| \, d\mu(y) + \sum_{I\in\mathcal{I}} \int |\widehat{f_I}(y)|^2 \, d\mu(y), \tag{3.7}$$

where $\mathcal{I}$ is a finitely overlapping set of dyadic intervals I with $|I| \approx R^{-1/2}$.

To estimate the integrals on the right hand side of (3.7), we begin with two geometric observations. The first of these is that it follows from the hypotheses on ϕ that if $I \subset [-1/2, 1/2]$ is an interval with length ℓ, then

$$\Gamma_{R,I} \doteq \{R\,\sigma(\rho, t) : t \in I,\ 1/2 \leq \rho \leq 1\}$$

is contained in a rectangle D with side lengths $\lesssim R, R\ell, R\ell^2$, which we will abbreviate by saying that D is an $R \times (R\ell) \times (R\ell^2)$ rectangle. Secondly, we observe that if $\ell \gtrsim R^{-1/2}$, then an R^δ neighborhood of an $R \times (R\ell) \times (R\ell^2)$ rectangle is contained in an $R^{1+\delta} \times (R^{1+\delta}\ell) \times (R^{1+\delta}\ell^2)$ rectangle. It follows that if I has length $2^{-n} \gtrsim R^{-1/2}$, then the support of f_I is contained in a rectangle D with dimensions $R^{1+\delta} \times (R^{1+\delta}2^{-n}) \times (R^{1+\delta}2^{-2n})$.

The next lemma is part of Lemma 4.1 in [1]. To state it, we introduce some notation: ϕ is a nonnegative Schwartz function such that $\phi(x) = 1$ for x in the unit cube Q, $\phi(x) = 0$ if $x \notin 2Q$, and, for each $M > 0$,

$$|\widehat{\phi}| \leq C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}.$$

For a rectangle $D \subset \mathbb{R}^3$, ϕ_D will stand for $\phi \circ b$, where b is an affine mapping which takes D onto Q. If D is a rectangle with dimensions $a_1 \times a_2 \times a_3$, then a dual rectangle of D is any rectangle with the same axis directions and with dimensions $a_1^{-1} \times a_2^{-1} \times a_3^{-1}$.

Lemma 3.1. *Suppose $1 \leq \alpha \leq 2$ and that μ is a non-negative Borel measure on $\mathbb{R}^3$ satisfying (1.3). Suppose D is a rectangle with dimensions $R_1 \times R_2 \times R_3$, where $R_3 \lesssim R_2 \lesssim R_1 \lesssim R$, and let D_{dual} be the dual of D centered at the origin. Then, if $\widetilde{\mu}(E) = \mu(-E)$,*

$$(\widetilde{\mu} * |\widehat{\phi_D}|)(y) \lesssim R_2^{2-\alpha} R_1,\ y \in \mathbb{R}^3 \tag{3.8}$$

and, if $K \gtrsim 1$, $y_0 \in \mathbb{R}^3$, then

$$\int_{K\cdot D_{dual}} (\widetilde{\mu} * |\widehat{\phi_D}|)(y_0 + y) \, dy \lesssim K^\alpha R_2^{1-\alpha} R_3^{-1}. \tag{3.9}$$

Now if $I \in \mathcal{I}$ and $\text{supp} f_I \subset D$ as above, the identity $\widehat{f_I} = \widehat{f_I} * \widehat{\phi_D}$ implies that

$$|\widehat{f_I}| \le (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2} \|\widehat{\phi_D}\|_1^{1/2} \lesssim (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2}$$

and so

$$\int |\widehat{f_I}(y)|^2 \, d\mu(y) \lesssim \int (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)(y) \, d\mu(y) = \int |\widehat{f_I}(y)|^2 (\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \, dy \lesssim \|f_I\|_2^2 \, R^{2-\alpha/2+2\delta}, \tag{3.10}$$

where the last inequality follows from (3.8), the fact that D has dimensions $R^{1+\delta} \times R^{1/2+\delta} \times R^{\delta}$ since $2^{-n} \approx R^{-1/2}$, and the inequalities $1 \le \alpha \le 2$. Thus the estimate

$$\sum_{I \in \mathcal{I}} \int |\widehat{f_I}(y)|^2 \, d\mu(y) \lesssim R^{2-\alpha/2+2\delta} \sum_{I \in \mathcal{I}} \|f_I\|_2^2 \lesssim R^{2-\alpha/2+2\delta} \tag{3.11}$$

follows from $\|f\|_2 = 1$ and the finite overlap of the intervals $I \in \mathcal{I}$ (which implies finite overlap for the supports of the $f_I, I \in \mathcal{I}$).

To bound the principal term of the right hand side of (3.7), fix n with $4 \le 2^n \le R^{1/2}$ and a pair I, J of dyadic intervals with $|I| = |J| = 2^{-n}$ and $I \sim J$. Since $I \sim J$, the support of $f_I * f_J$ is contained in a rectangle D with dimensions $R^{1+\delta} \times (R^{1+\delta} 2^{-n}) \times (R^{1+\delta} 2^{-2n})$. For later reference, let u, v, w be unit vectors in the directions of the sides of D with u parallel to the longest side and w parallel to the shortest side. As in (3.10),

$$\int |\widehat{f_I}(y) \widehat{f_J}(y)| \, d\mu(y) \lesssim \int (|\widehat{f_I} \, \widehat{f_J}| * |\widehat{\phi_D}|)(y) \, d\mu(y) = \int |\widehat{f_I}(y) \widehat{f_J}(y)| \, (\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \, dy. \tag{3.12}$$

Now tile $\mathbb{R}^3$ with rectangles P having exact dimensions $(C2^{-2n}) \times (C2^{-n}) \times C$ for some large $C > 0$ to be chosen later and having shortest side in the direction of u and longest side in the direction of w. Let ψ be a fixed nonnegative Schwartz function satisfying $1 \le \psi(y) \le 2$ if $y \in Q$, $\widehat{\psi}(x) = 0$ if $x \notin Q$, and

$$\psi \le C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}. \tag{3.13}$$

Since $\sum_P \psi_P^3 \approx 1$, it follows from (3.12) that if $f_{I,P}$ is defined by

$$\widehat{f_{I,P}} = \psi_P \cdot \widehat{f_I}$$

then

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\,d\mu(y) \lesssim \sum_P \Big(\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\,dy\Big)^{1/2}\Big(\int \big|(\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\big|^2\,dy\Big)^{1/2}. \tag{3.14}$$

To estimate the first integral in this sum, we begin by noting that the support of $f_{I,P}$ is contained in $\mathrm{supp}(f_I) + P_{\mathrm{dual}}$, where P_{dual} is a rectangle dual to P and centered at the origin. Let $\widetilde{I}$ be the interval with the same center as I but lengthened by $2^{-n}/10$ and let $\widetilde{J}$ be defined similarly. Since $I \sim J$, it follows that $\mathrm{dist}(\widetilde{I}, \widetilde{J}) \geq 2^{-n}/2$. Now the support of f_I is contained in $\Gamma_{R,I} + B(0, R^\delta)$ and P_{dual} has dimensions $(2^{2n}C^{-1}) \times (2^nC^{-1}) \times C^{-1}$ and side in the direction of v at an angle $\lesssim 2^{-n}$ to any of the tangents to the curve $\big(t, \phi(t)\big)$ for $t \in \widetilde{I}$ (or $t \in \widetilde{J}$). Recalling that $2^n \lesssim R^{1/2}$, one can check that, if C is large enough,

$$\mathrm{supp}(f_{I,P}) \subset \Gamma_{R,\widetilde{I}} + B(0, CR^\delta)$$

and, similarly,

$$\mathrm{supp}(f_{J,P}) \subset \Gamma_{R,\widetilde{J}} + B(0, CR^\delta).$$

The next lemma follows from Lemma 4.2 in §4 by scaling:

Lemma 3.2. *Suppose that the closed intervals $\tilde{I}, \tilde{J} \subset [0,1]$ satisfy dist$(\tilde{I}, \tilde{J}) \geq c\,2^{-n}$. Then, for $\delta > 0$ and $x \in \mathbb{R}^3$, there is the following estimate for the three-dimensional Lebesgue measure of the intersection of translates of neighborhoods of $\Gamma_{R,\tilde{I}}$ and $\Gamma_{R,\tilde{J}}$:*

$$\big|\, x + \Gamma_{R,\tilde{I}} + B(0, CR^\delta) \;\cap\; \Gamma_{R,\tilde{J}} + B(0, CR^\delta)\,\big| \lesssim R^{1+2\delta}2^n.$$

It follows from Lemma 3.2 that for $x \in \mathbb{R}^3$ we have

$$\big|\, x + \mathrm{supp}(f_{I,P}) \;\cap\; \mathrm{supp}(f_{J,P})\,\big| \lesssim R^{1+2\delta}2^n. \tag{3.15}$$

Now

$$\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\,dy = \int |\widetilde{f_{I,P}} * f_{J,P}(x)|^2\,dx$$

and

$$|\widetilde{f_{I,P}} * f_{J,P}(x)| \leq \int |f_{I,P}(w-x)\, f_{J,P}(w)|\,dw \leq |x + \mathrm{supp}(f_{I,P}) \cap \mathrm{supp}(f_{J,P})|^{1/2}\,\big(|\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x)\big)^{1/2}.$$

Thus, by (3.15),

$$\Big(\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\,dy\Big)^{1/2} \lesssim R^{1/2+\delta}2^{n/2}\Big(\int |\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x)\,dx\Big)^{1/2} = R^{1/2+\delta}2^{n/2}\|f_{I,P}\|_2\|f_{J,P}\|_2. \tag{3.16}$$

To estimate the second integral in the sum (3.14) we use (3.13) to observe that

$$\psi_P \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j P}.$$

Thus

$$\int (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\, dy \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \int_{2^j P} (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\, dy.$$

Noting that $2^j P \subset y_P + K D_{\text{dual}}$ for some $K \lesssim R^{1+\delta} 2^{-2n+j}$ and some $y_P \in \mathbb{R}^3$, we apply (3.9) to obtain

$$\int (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\, dy \lesssim$$
$$\sum_{j=1}^{\infty} 2^{-Mj} (R^{1+\delta} 2^{-2n+j})^{\alpha} (R^{1+\delta} 2^{-n})^{1-\alpha} (R^{1+\delta} 2^{-2n})^{-1} \lesssim 2^{-n(\alpha-1)}.$$

Since

$$(\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \lesssim (R^{1+\delta} 2^{-n})^{2-\alpha}\, R^{1+\delta}$$

by (3.8) and since $\psi_P(y) \lesssim 1$, it follows that

$$\Big(\int \big((\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y) \big)^2\, dy \Big)^{1/2} \lesssim R^{(1+\delta)(3-\alpha)/2}\, 2^{-n/2}. \tag{3.17}$$

Now (3.16) and (3.17) imply, by (3.14), that

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\, d\mu(y) \lesssim R^{(1+\delta)(2-\alpha/2)} \Big(\sum_P \|f_{I,P}\|_2^2 \Big)^{1/2} \Big(\sum_P \|f_{J,P}\|_2^2 \Big)^{1/2}.$$

Since

$$\sum_P \|\widehat{f_{I,P}}\|_2^2 = \int |\widehat{f_I}(y)|^2 \sum_P |\psi_P(y)|^2\, dy,$$

it follows from $\sum_P \psi_P^2 \lesssim 1$ that

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\, d\mu(y) \lesssim R^{(1+\delta)(2-\alpha/2)} \|f_I\|_2 \|f_J\|_2.$$

Thus

$$\sum_{\substack{|I|=|J|=2^{-n} \\ I\sim J}} \int |\widehat{f_I}(y)\widehat{f_J}(y)|\, d\mu(y) \lesssim R^{(1+\delta)(2-\alpha/2)} \sum_{\substack{|I|=|J|=2^{-n} \\ I\sim J}} \|f_I\|_2 \|f_J\|_2 \lesssim R^{(1+\delta)(2-\alpha/2)} \|f\|_2^2. \tag{3.18}$$

Now (3.4) follows from (3.7), (3.11), (3.18), and the fact that the first sum in (3.7) has $\lesssim \log R$ terms. This completes the proof of Theorem 1.3.

4. Two lemmas

As mentioned in §3, Lemma 3.2 follows from Lemma 4.2 below. The proof of Lemma 4.2 will use the following fact:

Lemma 4.1. *Suppose ϕ_1, ϕ_2 are functions on $[c,d]$ with $|\phi_1'(u_1) - \phi_2'(u_2)| \geq a > 0$ for all $u_1, u_2 \in [c,d]$ and $|\phi_i'| \leq b$. For $\delta' > 0$ let*

$$C_{i,\delta'} = \{(u, \phi_i(u)) : u \in [c,d]\} + B(0,\delta').$$

Then

$$|C_{1,\delta'} \cap C_{2,\delta'}| \leq 16(1+b)^2 \delta'^2/a. \tag{4.1}$$

Proof. To prove the lemma we begin by noting that we may the extend the ϕ_i so that they are defined on $[c-\delta', d+\delta']$ and satisfy the lemma's hypotheses on this larger interval. We may also assume that the intersection in (4.1) is nonempty and then choose $u_0 \in [c,d]$ with

$$|\phi_1(u_0) - \phi_2(u_0)| < (2+2b)\delta'.$$

Because of the assumptions on the ϕ_i it follows that

$$\text{if } |u - u_0| \geq (4+4b)\delta'/a \text{ then } |\phi_2(u) - \phi_1(u)| \geq (2+2b)\delta'. \tag{4.2}$$

Now assume that (x_1, x_2) is in the intersection in (4.1) and so

$$|(x_1,x_2) - (u_i, \phi_i(u_i))| < \delta', \ i = 1,2$$

for some $u_1, u_2 \in [c,d]$. Then

$$|x_1 - u_i|, \ |x_2 - \phi_i(u_i)| < \delta', \ i = 1,2. \tag{4.3}$$

Now

$$|x_2 - \phi_i(x_1)| \leq |x_2 - \phi_i(u_i)| + |\phi_i(u_i) - \phi_i(x_1)| < (1+b)\delta'$$

by (4.3) and $|\phi_i'| \leq b$ and so $|\phi_1(x_1) - \phi_2(x_1)| < (2+2b)\delta'$. Thus (4.2) shows that

$$|x_1 - u_0| < (4+4b)\delta'/a. \tag{4.4}$$

Since $|x_2 - \phi_1(x_1)| < (1+b)\delta'$, it follows from (4.4) (and the fact that (x_1,x_2) is a generic point of the intersection in (4.1)) that (4.1) holds, proving the lemma. □

Lemma 4.2. *Suppose the real-valued function ϕ on $[-1/2, 1/2]$ satisfies estimates $|\phi(t)|, |\phi'(t)| \leq M$, $|\phi''(t)| \geq m > 0$ for $-1/2 \leq t \leq 1/2$. For $I \subset [-1/2,1/2]$ and $0 < \delta' < 1$ define*

$$\Sigma_{I,\delta'} \doteq \{\rho(t, \phi(t), 1) : t \in I, \ 1/2 \leq \rho \leq 1\} + B(0,\delta').$$

There is a positive constant C depending only on M and m such that if $I, J \subset [-1/2,1/2]$ are ℓ-separated subintervals of $[0,1]$ and $x \in \mathbb{R}^3$ then

$$\big|(x + \Sigma_{I,\delta'}) \cap \Sigma_{J,\delta}\big| \leq C\delta'^2/\ell. \tag{4.5}$$

Proof. We can extend ϕ to $[-20,20]$ with estimates $|\phi(t)|$, $|\phi'(t)| \leq M'$, $|\phi''(t)| \geq m' > 0$ for $-3 \leq t \leq 3$ and with M', m' depending only on M and m. Our strategy will be to estimate the two-dimensional Lebesgue measure of certain sections of $(x+\Sigma_{I,\delta'})\cap\Sigma_{J,\delta'}$. Recall the notation $\sigma(t,\rho) = \rho\big(t,\phi(t),1\big)$ and assume that $y, y' \in (x+\Sigma_{I,\delta'})\cap\Sigma_{J,\delta'}$. Then there are $\rho, t, \tilde\rho, \tilde t$ with $t \in I$, $\tilde t \in J$ such that

$$|y - x - \sigma(t,\rho)|,\ |y - \sigma(\tilde t,\tilde\rho)| < \delta' \tag{4.6}$$

and there are $\rho', t', \tilde\rho', \tilde t'$ with $t' \in I$, $\tilde t' \in J$ such that

$$|y' - x - \sigma(t',\rho')|,\ |y' - \sigma(\tilde t',\tilde\rho')| < \delta'. \tag{4.7}$$

Write $x = (x_1, x_2, x_3)$ and similarly for y and y'. We are interested in the two-dimensional measure of the section defined by "third coordinate $= c$" - in fact, (4.5) will follow when we show that the two-dimensional Lebesgue measure of this section is $\leq C\delta'^2/\ell$ - and so we assume that $y_3 = y_3' = c$. It follows from (4.6) and (4.7) that

$$|c - x_3 - \rho|, |c - x_3 - \rho'| < \delta'$$

and thus that $|\rho - \rho'| < 2\delta'$. Similarly, $|\tilde\rho - \tilde\rho'| < 2\delta'$. Thus there are fixed $\rho, \tilde\rho$ such that if $y' \in (x+\Sigma_{I,\delta'})\cap\Sigma_{J,\delta'}$ then there are $t' \in I$, $\tilde t' \in J$ such that

$$|y' - x - \sigma(t',\rho)|,\ |y' - \sigma(\tilde t',\tilde\rho)| < C\delta'. \tag{4.8}$$

where C denotes, as it always will in this proof, a positive constant depending only on m and M. It follows from (4.8) that

$$|(y_1', y_2') - (x_1, x_2) - \big(\rho\, t', \rho\,\phi(t')\big)|,\ |(y_1', y_2') - \big(\tilde\rho\,\tilde t', \tilde\rho\,\phi(\tilde t')\big)| < C\delta'. \tag{4.9}$$

Define

$$\phi_1(u) = x_2 + \rho\,\phi\big((u - x_1)/\rho\big), \phi_2(u) = \tilde\rho\,\phi(u/\tilde\rho),$$
$$[c_1, d_1] = x_1 + \rho I, [c_2, d_2] = \tilde\rho J, [c, d] = [c_1, d_1] \cap [c_2, d_2].$$

Then (4.9) implies that

$$|(y_1', y_2') - \big(u_1, \phi_1(u_1)\big)|,\ |(y_1', y_2') - \big(u_2, \phi_2(u_2)\big)| < C\delta' \quad \text{for some } u_1 \in [c_1, d_1], u_2 \in [c_2, d_2]. \tag{4.10}$$

Recall that our goal is to show that the set of all (y_1', y_2') for which (4.10) holds has two-dimensional Lebesgue measure $\leq C\delta'^2/\ell$. In the case $u_1, u_2 \in [c, d]$ this follows from Lemma 4.1 with $a = m'\ell$. (The derivative separation requirement in Lemma 4.1 is a consequence of our hypothesis $|\phi''| \geq m'$.) Of the remaining cases, $u_1 \in [c_1, d_1], u_2 \notin [c_1, d_1]$ is typical: allowing C to increase from line to line, $|u_1 - u_2| < C\delta'$ follows from (4.10). So, since $u_1 \in [c_1, d_1], u_2 \notin [c_1, d_1]$, we have in succession that

$$\operatorname{dist}\big(u_2, \{c_1, d_1\}\big) \leq C\delta', \quad \operatorname{dist}\big(y_1', \{c_1, d_1\}\big) \leq C\delta',$$
$$\text{and } \operatorname{dist}\big(y_2', \{\phi_2(c_1), \phi_2(d_1)\}\big) \leq C\delta'.$$

(Note to the very careful reader: the extension of ϕ to the interval $[-20, 20]$ guarantees that both ϕ_1 and ϕ_2 are defined on an interval which contains both $[c_1, d_1]$ and $[c_2, d_2]$.) Thus the set of all (y_1', y_2') for which (4.10) holds with $u_1 \in [c_1, d_1], u_2 \notin [c_1, d_1]$ has two-dimensional Lebesgue measure $\leq C\delta'^2$. This completes the proof of Lemma 4.2.

$\square$

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