DIGRAPHS AND CYCLE POLYNOMIALS FOR FREE-BY-CYCLIC GROUPS.

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Abstract. Let ϕ be a free group outer automorphism that can be represented by an expanding, irreducible train-track map. The automorphism ϕ determines a free-by-cyclic group Γ and a homomorphism α_ϕ from Γ to $\mathbb{Z}$. Dowdall, Kapovich and Leininger showed that there exists an open cone neighborhood $\mathcal{A}$ of α_ϕ in $\mathrm{Hom}(\Gamma; \mathbb{R})$ with the property that for every primitive integral $\alpha \in \mathcal{A}$ there is a new decomposition of Γ as a free-by-cyclic group inducing the homomorphism α so that the corresponding free group outer automorphism also has an expanding and irreducible train track representation. In this paper, we define an analog Θ of McMullen's Teichmüller polynomial for fibered cones of 3-manifolds that is independent of the choice of train track representation of ϕ and computes the dilatations of ϕ_α for α ranging in primitive integral points in $\mathcal{A}$.

1. Introduction

There is continually growing evidence of a powerful analogy between the mapping class group $\mathrm{Mod}(S)$ of a closed oriented surfaces S of finite type, and the group of outer automorphisms $\mathrm{Out}(F_n)$ of free groups F_n. A recent advance in this direction can be found in work of Dowdall, Kapovich, and Leininger [DKL13.1] who developed an analog of the fibered face theory of surface homeomorphisms due to Thurston [Thu86] and Fried [Fri82]. In this paper we use their construction and combinatorial properties of labeled digraphs to define an analog of McMullen's polynomial for surface automorphisms defined in [McM00].

Motivation from pseudo-Anosov mapping classes on surfaces. Let S be a compact oriented surface of negative finite Euler characteristic. A *mapping class* $\phi = [\phi_\circ]$ is an isotopy class of homeomorphisms

$$\phi_\circ : S \to S$$

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where the isotopy fixes the boundary of S pointwise. The mapping torus $X_{(S,\phi)}$ of the pair (S, ϕ) is the quotient space

$$X_{(S,\phi)} = S \times [0,1]/(x,1) \sim (\phi_\circ(x), 0).$$

Its homeomorphism type is independent of the choice of representative $\phi_\circ$ for ϕ. The mapping torus $X_{(S,\phi)}$ has a distinguished fibration $\rho_\phi : X_{(S,\phi)} \to S^1$ defined by projecting $S \times [0,1]$ to its second component and identifying endpoints. Conversely, any fibration $\rho : X \to S^1$ of a 3-manifold X over a circle can be written as the mapping torus of a unique mapping class (S, ϕ), with $\rho = \rho_\phi$. The mapping class (S, ϕ) is called the *monodromy* of ρ.

Thurston's fibered face theory [Thu86] gives a parameterization of the fibrations of a 3-manifold X over the circle with connected fibers by the primitive integer points on a finite union of disjoint convex cones in $H^1(X; \mathbb{R})$, called *fibered cones*. A mapping class is pseudo-Anosov if there is a pair of transverse measured invariant foliations that ϕ stretches and contracts by $\lambda^{\pm 1}$, for some $\lambda > 1$. The expansion factor λ is called the *dilatation* of ϕ and is denoted by $\lambda(\phi)$. Thurston showed that the mapping torus of any pseudo-Anosov mapping class is hyperbolic, and the monodromy of any fibered hyperbolic 3-manifold is pseudo-Anosov. It follows that the set of all pseudo-Anosov mapping classes partitions into subsets corresponding to integral points on fibered cones of hyperbolic 3-manifolds.

By results of Fried [Fri82] (cf. [M87] [McM00]) the function $\log \lambda(\phi)$ defined on integral points of a fibered cone $\mathcal{T}$ extends to a continuous convex function

$$\mathcal{Y} : \mathcal{T} \to \mathbb{R},$$

that is a homogeneous of degree -1, and goes to infinity toward the boundary of any affine planar section of $\mathcal{T}$. The *Teichmüller polynomial of a fibered cone* $\mathcal{T}$ defined in [McM00] is an element Θ_{Teich} in the group ring $\mathbb{Z}G$ where $G = H_1(X; \mathbb{Z})/\text{Torsion}$. The group ring $\mathbb{Z}G$ can be thought of as a ring of Laurent polynomials in the generators of G considered as a multiplicative group, and in this way, we can think of Θ_{Teich} as a polynomial. The Teichmüller polynomial Θ_{Teich} has the property that the dilatation $\lambda(\phi_\alpha)$ of every mapping class ϕ_α associated to an integral point $\alpha \in \mathcal{T}$, $\lambda(\phi_\alpha)$ can be obtained from Θ_{Teich} by a process known as specialization (defined later in this section). Furthermore, the cone $\mathcal{T}$ and the function $\mathcal{Y}$ are determined by Θ_{Teich}.

Fibered face theory for free-by-cyclic groups. A free-by-cyclic group

$$\Gamma = F_n \rtimes_\phi \mathbb{Z},$$

is a semi-direct product defined by an element $\phi \in \text{Out}(F_n)$. If $x_1, \ldots, x_n$ are generators of F_n, and $\phi_\circ$ is a representative automorphism in the class ϕ, then Γ has a finite presentation

$$\langle x_1, \ldots, x_n, s \mid s x_i s^{-1} = \phi_\circ(x_i) \quad i = 1, \ldots, n \rangle.$$

Given a free group automorphism $\phi \in \mathrm{Out}(F_n)$, there is a corresponding free-by-cyclic group

$$\Gamma = F_n \rtimes_\phi \mathbb{Z}$$

endowed with a distinguished homomorphism $\alpha_\phi : \Gamma \to \mathbb{Z}$ induced by projection to the second coordinate. train-track maps, defined by Bestvina and Handel in [BH92], are topological graph maps that are homotopy equivalences, and have the property that forward iterations of edges do not backtrack (see Definition 4.4). A train-track is *Perron-Frobenius* (or *PF*) if it has a Perron-Frobenius transition matrix. Automorphisms that are representable by Perron-Frobenius train-track maps are generic since every fully irreducible outer automorphism has this property [BH92] (see section 4.1 for definitions of fully irreducible and hyperbolic automorphisms).

If ϕ is representable by an expanding irreducible train-track map (see [Kap13] [DKL13.1], and Section 4.1 for definitions), for example if the map is a PF train-track map, then the growth rate of cyclically reduced word-lengths of $\phi^k(\gamma)$, for nontrivial $\gamma \in F_n$ is exponential, with a base $\lambda(\phi) > 1$ that is independent of γ. The constant $\lambda(\phi)$ is called the *dilatation* (or *expansion factor*) of ϕ. In [DKL13.1], Dowdall, Kapovich and Leininger show the following (see [DKL13.1], Theorem A (5),(6)).

Theorem 1.1 (Dowdall-Kapovich-Leininger [DKL13.1])**.** *If $\phi \in Out(F_n)$ can be represented by an expanding irreducible train-track map then there is an open cone neighborhood $\mathcal{A}$ of α_ϕ in $\mathrm{Hom}(\Gamma; \mathbb{R})$, where $\Gamma = F_n \rtimes_\phi \mathbb{Z}$, such that, for all primitive integral elements $\alpha \in \mathcal{A}$, there is a corresponding free-by-cyclic decomposition*

$$\Gamma = F_{n_\alpha} \rtimes_{\phi_\alpha} \mathbb{Z}$$

so that $\alpha = \alpha_{\phi_\alpha}$ and $\phi_\alpha \in Out(F_{n_\alpha})$ is also representable by an expanding irreducible train-track map. Furthermore, the log dilatation function for the primitive integral points on $\mathcal{A}$ extends to a continuous, convex, homogeneous of degree -1 function on $\mathcal{A}$. If in addition, $\phi \in Out(F_n)$ is fully irreducible and hyperbolic, then for any primitive integral $\alpha \in \mathcal{A}$, ϕ_α is also fully irreducible and hyperbolic.

We call $\mathcal{A}$ a *DKL cone* associated to ϕ.

Remark 1.2. An automorphism $\phi \in \mathrm{Out}(F_n)$ may admit many train-track representatives f. Every train-track representative can be decomposed into a sequence of folds f [Sta83] which is also non-unique. The construction of $\mathcal{A}$ in [DKL13.1] depends on the choice of f and f. Hereafter, we will denote $\mathcal{A}$ by $\mathcal{A}_{\mathsf{f}}$.

Main Result. Let G be a finitely generated free abelian group of rank k, and let

$$\theta = \sum_{g\in G} a_g g, \qquad a_g \in \mathbb{Z}$$

be an element of the group ring $\mathbb{Z}G$.

Definition 1.3. Given $\theta = \sum_{g\in G} a_g g$, and $\alpha \in \mathrm{Hom}(G;\mathbb{Z})$, the *specialization* of θ at α is the single variable integer polynomial

$$\theta^{(\alpha)}(x) = \sum_{g\in G} a_g x^{\alpha(g)} \in \mathbb{Z}[x].$$

Remark 1.4. Thinking of G as an abelian group generated by $t_1, \dots, t_k$, we can identify the elements of G with monomials in the symbols $t_1, \dots, t_k$, and hence $\mathbb{Z}G$ with Laurent polynomials in $\mathbb{Z}(t_1, \dots, t_k)$. Thus, we can associate to $\theta \in \mathbb{Z}G$ a polynomial $\theta(t_1, \dots, t_k) \in \mathbb{Z}(t_1, \dots, t_k)$. Identifying $\mathrm{Hom}(G;\mathbb{Z})$ with $\mathbb{Z}^k$, each element $\alpha = (a_1, \dots, a_k)$ defines a specialization of $\theta = \theta(t_1, \dots, t_k)$ by

$$\theta^{(\alpha)}(x) = \theta(x^{a_1}, \dots, x^{a_k}).$$

For ease of notation, we mainly use the group ring notation through most of this paper.

Definition 1.5. The *house* of a polynomial $p \in \mathbb{Z}[x]$ is given by

$$|p| = \max\{|\mu| \;\mid \mu \in \mathbb{C},\ p(\mu) = 0\}.$$

Theorem 1.6. *Let $\phi \in Out(F_n)$ be an outer automorphism that can be represented by a PF train-track map $f\colon \tau \to \tau$ and let f be a folding decomposition of f. Let $\Gamma = F_n \rtimes_\phi \mathbb{Z}$ and let $G = \Gamma^{ab}/Torsion$. Let $\mathcal{A}_{\mathsf{f}}$ be the DKL cone neighborhood of α_ϕ in $\mathrm{Hom}(\Gamma;\mathbb{R})$. Then there exists an element $\Theta \in \mathbb{Z}G$ with the following properties.*

(1) For any $\alpha \in \mathcal{A}_{\mathsf{f}}$,

$$|\Theta^{(\alpha)}| = \lambda(\phi_\alpha).$$

(2) There is an open cone $\mathcal{T}_\phi \subset \mathrm{Hom}(G;\mathbb{R})$ dual to a vertex of the Newton polygon of Θ so that $\mathcal{A}_{\mathsf{f}} \subset \mathcal{T}_\phi$ and the function

$$L(\alpha) = \log|\Theta^{(\alpha)}|$$

that is defined on the primitive integral points of $\mathcal{A}_{\mathsf{f}}$ extends to a real analytic, convex function on $\mathcal{T}$ that is homogeneous of degree -1 and goes to infinity toward the boundary of affine planar sections of $\mathcal{T}$.

(3) The element Θ is minimal with respect to property (1), that is, if $\theta \in \mathbb{Z}G$ satisfies

$$|\theta^{(\alpha)}| = \lambda(\phi_\alpha)$$

on the integral elements of some open subcone of $\mathcal{T}$ then Θ divides θ.

(4) The element Θ is well-defined up to units in $\mathbb{Z}G$ and is independent of the choice of train-track map f and its folding decomposition f.

Remark 1.7. If $\phi \in \mathrm{Out}(F_n)$ is fully irreducible, then a result of Bestvina and Handel [BH92] implies that ϕ has a PF train-track map representation. Thus, Theorem 1.6 could be stated so that existence of a PF train-track representation is replaced by the condition that ϕ is fully irreducible. If ϕ is fully irreducible and atoroidal then all primitive integral elements in the DKL cone $\mathcal{A}_{\mathsf{f}}$ are fully irreducible (and atoroidal) [DKL13.1], while the existence of a PF train-track does not necessarily persist in $\mathcal{A}_{\mathsf{f}}$. However, checking that a map if a PF train-track map is straightforward, while checking that an automorphism is irreducible and atoroidal is generally difficult.

Remark 1.8. Dowdall, Kapovich, and Leininger give an independent definition of an element $\Theta_{\mathrm{DKL}} \in \mathbb{Z}G$ with the property that $\lambda(\phi_\alpha) = |\Theta_{\mathrm{DKL}}^{(\alpha)}|$ for $\alpha \in A_{\mathsf{f}}$ in [DKL13.2]. Property (3) of Theorem 1.6 implies that Θ_ϕ divides Θ_{DKL}.

Organization of paper. In Section 2 we establish some preliminaries about Perron-Frobenius digraphs D with edges labeled by a free abelian group G. Each digraph D determines a cycle complex C_D and cycle polynomial θ_D in the group ring $\mathbb{Z}G$. Under certain extra conditions, we define a cone $\mathcal{T}$, which we call the McMullen cone, and show that

$$L(\alpha) = \log |\theta_D^{(\alpha)}|$$

defined for integral elements of $\mathcal{T}$ extends to a homogeneous function of degree -1 that is real analytic and convex on $\mathcal{T}$ and goes to infinity toward the boundary of affine planar sections of $\mathcal{T}$. Furthermore, we show the existence of a distinguished factor Θ_D of θ_D with the property that

$$|\Theta_D^{(\alpha)}| = |\theta_D^{(\alpha)}|,$$

and Θ_D is minimal with this property. Our proof uses a key result of McMullen (see [McM00], Appendix A).

In Section 3 we define branched surfaces $(X, \mathfrak{C}, \psi)$, where X is 2-complex with a semiflow ψ, and cellular structure $\mathfrak{C}$ satisfying compatibility conditions with respect to ψ. To a branched surface we associate a dual digraph D and a G-labeled cycle complex C_D, where $G = H_1(X;\mathbb{Z})/\text{torsion}$, and a cycle function $\theta_D \in \mathbb{Z}G$. We show that θ_D is invariant under certain allowable cellular subdivisions and homotopic modifications of $(X, \mathfrak{C}, \psi)$.

In Sections 4 and 5 we study the branched surfaces associated to the train-track map f and folding sequence f defined in [DKL13.1], called respectively the mapping torus, and folded mapping torus. We use the invariance under allowable cellular subdivisions and modifications established in Section 2 and Section 3 to show that the cycle functions for these branched surfaces are equal. The results of Section 2 applied to the mapping torus for f implies the existence of Θ_ϕ and $\mathcal{T}_\phi$ in Theorem 1.6. Applying an argument in [DKL13.1], we show that further subdivisions with the folded mapping torus give rise to mapping tori for train-track maps corresponding to ϕ_α, and use this to show that for $\alpha \in \mathcal{A}_\mathsf{f}$, we have $\lambda(\phi_\alpha) = |\Theta_\phi^{(\alpha)}|$. We further compare the definition of the DKL cone $\mathcal{A}_\mathsf{f}$ and $\mathcal{T}_\phi$ to show inclusion $\mathcal{A}_\mathsf{f} \subset \mathcal{T}_\phi$, and thus complete the proof of Theorem 1.6.

We conclude in Section 6 with an example where A_f is a proper subcone of $\mathcal{T}_\phi$.

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2. Digraphs, their cycle complexes and eigenvalues of G-matrices

This section contains basic definitions and properties of digraphs, and a key result of McMullen that will be useful in our proof of Theorem 1.6.

2.1. Digraphs, cycle complexes and their cycle polynomials.

We recall basic results concerning digraphs (see, for example, [Gan59] and [CR90] for more details).

Definition 2.1. A *digraph* D is a finite directed graph with at least two vertices. Given an ordering $v_1, \dots, v_m$ of the vertices of D, the *adjacency matrix* of D is the matrix

$$M_D = [a_{i,j}],$$

where $a_{i,j} = m$ if there are m directed edges from v_i to v_j. The *characteristic polynomial* P_D is the characteristic polynomial of M_D and the *dilatation* $\lambda(D)$ of D is the spectral radius of M_D

$$\lambda(D) = \max\{|e| \mid e \text{ is an eigenvalue of } M_D\}.$$

Any square $m \times m$ matrix M with non-negative integer entries determines a digraph.

Definition 2.2. A non-negative matrix M with real entries is called *Perron-Frobenius* if for some n, the entries of M^n are positive. A digraph D is *Perron-Frobenius* if its directed adjacency matrix M_D is Perron Frobenius.

An eigenvalue of M is *simple* if there is no other eigenvalue of the same norm. The following theorem is well-known (see, for example, [Gan59]).

Theorem 2.3 (Perron-Frobenius Theorem)**.** *If M is Perron-Frobenius, then it has a positive simple eigenvalue with maximum norm.*

Definition 2.4. If M is Perron-Frobenius, the eigenvalue $\lambda_{\mathrm{PF}}(M)$ with maximum norm is called the *Perron-Frobenius eigenvalue* of M.

Definition 2.5. A *simple cycle* α on a digraph D is an isotopy class of embeddings of the circle S^1 to D oriented compatibly with the directed edges of D. A *cycle* is a disjoint union of simple cycles. The *cycle complex* C_D of a digraph D is the collection of cycles on D thought of as a simplicial complex, whose vertices are the simple cycles.

The cycle complex C_D has a measure which assigns to each cycle its length in D, that is, if γ is a cycle on C_D, then its *length* $\ell(\gamma)$ is the number of vertices (or equivalently the number of edges) of D on γ, and, if $\sigma = \{\gamma_1, \dots, \gamma_s\}$, then

$$\ell(\sigma) = \sum_{i=1}^{s} \ell(\gamma_i).$$

Let $|\sigma| = s$ be the *size* of σ. The *cycle polynomial* of a digraph D is given by

$$\theta_D(x) = 1 + \sum_{\sigma \in C_D} (-1)^{|\sigma|} x^{-\ell(\sigma)}.$$

Theorem 2.6 (Coefficient Theorem for Digraphs [CR90])**.** *Let D be a digraph with m vertices, and P_D the characteristic polynomial of the directed adjacency matrix M_D for D. Then*

$$P_D(x) \;=\; x^m \theta_D(x)$$

Proof. Let $\mathrm{M}_D = [a_{i,j}]$ be the adjacency matrix for D. Then

$$P_D(x) \;=\; \det(xI - \mathrm{M}_D).$$

Let S_{V} be the group of permutations of the vertices V of D. For $\pi \in S_V$, let $\mathrm{fix}(\pi) \subseteq \mathsf{V}$ be the set of vertices fixed by π, and let $\mathrm{sign}(\pi)$ be -1 if π is an odd permutation and 1 if π is even. Then

$$P_D(x) \;=\; \sum_{\pi \in S_{\mathsf{V}}} \mathrm{sign}(\pi) A_\pi$$

where

$$A_\pi \;=\; \prod_{v \notin \mathrm{fix}(\pi)} (-a_{v,\pi(v)}) \prod_{v \in \mathrm{fix}(\pi)} (x - a_{v,v}) \tag{1}$$

There is a natural map $\Sigma : C_D \to S_{\mathsf{V}}$ from the cycle complex C_D to the permutation group S_{V} on the set V defined as follows. For each simple cycle γ in D passing through the vertices $\mathsf{V}_\gamma \subset \mathsf{V}$, there is a corresponding cyclic permutation $\Sigma(\gamma)$ of V_γ. That is, if $\mathsf{V}_\gamma = \{v_1, \dots, v_\ell\}$ contains more than one vertex and is ordered according to their appearance in the cycle, then then $\Sigma(\gamma)(v_i) = v_{i+1(\mathrm{mod}\ell)}$. If V_γ contains one vertex, we say γ is a *self-edge*. For self-edges γ, $\Sigma(\gamma)$ is the identity permutation.

Let $\sigma = \{\gamma_1, \dots, \gamma_s\}$ be a cycle on D. Then we define $\Sigma(\sigma)$ to be the product of disjoint cycles

$$\Sigma(\sigma) = \Sigma(\gamma_1) \circ \cdots \circ \Sigma(\gamma_\ell).$$

The polynomial A_π in Equation (1) can be rewritten in terms of the cycles σ of C_D with $\Sigma(\sigma) = \pi$. First we rewrite A_π as

$$A_\pi = \sum_{\nu \subset \mathrm{fix}(\pi)} x^{|\mathrm{fix}(\pi) - \nu|} \prod_{v \notin \mathrm{fix}(\pi)} (-a_{v,\pi(v)}) \prod_{v \in \nu} (-a_{v,v}). \tag{2}$$

Let $\pi \in S_{\mathsf{V}}$ be in the image of Σ. For a cycle $\sigma \in C_D$, let $\nu(\sigma) \subset \mathsf{V}$ be the subset vertices at which σ has a self-edge.

For $\nu \subset \mathrm{fix}(\pi)$ let

$$P_{\pi,\nu} = \{\sigma \in C_D \mid \Sigma(\sigma) = \pi \text{ and } \nu(\sigma) = \nu\}.$$

The we claim that the number of elements in P_{π_ν} is

$$\prod_{v \notin \mathrm{fix}(\pi)} a_{v,\pi(v)} \prod_{v \in \nu} a_{v,v}. \tag{3}$$

Let $\sigma \in C_D$ be such that $\Sigma(\sigma) = \pi$. Then for each $v \in \mathsf{V} \smallsetminus \mathrm{fix}(\pi)$, there is a choice of $a_{v,\pi(v)}$ edges from v to $\pi(v)$, and for each $v \in \mathrm{fix}(\pi)$ σ either contains no self-edge, or one of $a_{v,v}$ possible self-edges at v. This proves (3).

For each $\sigma \in C_D$, we have

$$\ell(\sigma) = m - |\mathrm{fix}(\Sigma(\sigma))| + |\nu(\sigma)|.$$

Thus, the summand in (2) associated to $\pi \in S_{\mathsf{V}} \smallsetminus \mathrm{id}$, and $\nu \subset \mathrm{fix}(\pi)$ is given by

$$\begin{aligned} x^{|\mathrm{fix}(\pi)| - |\nu|} \prod_{v \notin \mathrm{fix}(\pi)} (-a_{v,\pi(v)}) \prod_{v \in \nu} (-a_{v,v}) &= (-1)^{m - |\mathrm{fix}(\pi)| + |\nu|} \sum_{\sigma \in P_{\pi,\nu}} x^{m - \ell(\sigma)} \\ &= \sum_{\sigma \in P_{\pi,\nu}} (-1)^{\ell(\sigma)} x^{m - \ell(\sigma)}, \end{aligned}$$

and similarly for $\pi = \mathrm{id}$ we have

$$\begin{aligned} A_\pi &= \prod_{v \in \mathsf{V}} (x - a_{v,v}) \\ &= x^m + \sum_{\sigma \in P_{\pi,\nu}} (-1)^{\ell(\sigma)} x^{m - \ell(\sigma)}. \end{aligned}$$

For each $\sigma \in C_D$, $\mathrm{sign}(\Sigma(\sigma)) = (-1)^{\ell(\sigma)-|\sigma|}$. Putting this together, we have

$$\begin{aligned} P_D(x) &= \sum_{\pi \in S_{\mathsf{V}}} \mathrm{sign}(\pi) A_\pi \\ &= x^m + \sum_{\pi \in S_{\mathsf{V}}} \sum_{\sigma \in C_D\colon \Sigma(\sigma)=\pi} (-1)^{\ell(\sigma)-|\sigma|}(-1)^{\ell(\sigma)} x^{m-\ell(\sigma)} \\ &= x^m + \sum_{\sigma \in C_D} (-1)^{|\sigma|} x^{m-\ell(\sigma)}. \end{aligned}$$

□

2.2. **McMullen Cones.** Given an element $\theta \in \mathbb{Z}G$, there is an associated cone in $\mathrm{Hom}(G;\mathbb{R})$.

Definition 2.7. (cf. [McM02]) Let G be a finitely generated free abelian group. Given an element $\theta = \sum_{g\in G} a_g g \in \mathbb{Z}G$, the *support* of θ is the set

$$\mathrm{Supp}(\theta) = \{g \in G \mid a_g \neq 0\}.$$

Let $\theta \in \mathbb{Z}G$ and $g_0 \in \mathrm{Supp}(\theta)$ the *McMullen cone* of θ for g_0 is the set

$$\mathcal{T}_\theta(g_0) = \{\alpha \in \mathrm{Hom}(G;\mathbb{R}) \mid \alpha(g_0) > \alpha(g) \text{ for all } g \in \mathrm{Supp}(\theta) \smallsetminus \{g_0\}\}.$$

Remark 2.8. The elements of G can be identified with a subset of the dual space

$$\widehat{\mathrm{Hom}(G;\mathbb{R})} = \mathrm{Hom}(\mathrm{Hom}(G;\mathbb{R}),\mathbb{R})$$

to $\mathrm{Hom}(G;\mathbb{R})$. Let $\theta \in \mathbb{Z}G$ be any element. The convex hull of $\mathrm{Supp}(\theta)$ in $\widehat{\mathrm{Hom}(G;\mathbb{R})}$ is called the *Newton polyhedron* $\mathcal{N}$ of θ. Let $\widehat{\mathcal{N}}$ be the dual of $\mathcal{N}$ in $\mathrm{Hom}(G;\mathbb{R})$. That is, each top-dimensional face of $\widehat{\mathcal{N}}$ corresponds to a vertex $g \in \mathcal{N}$, and each α in the cone over this face has the property that $\alpha(g) > \alpha(g')$ where g' is any vertex of $\mathcal{N}$ with $g \neq g'$. Thus, the McMullen cones $\mathcal{T}_\theta(g_0)$, for $g_0 \in \mathrm{Supp}(\theta)$ are the cones over the top dimensional faces of the dual to the Newton polyhedron of θ.

2.3. **A coefficient theorem for H-labeled digraphs.** Throughout this section let H be the free abelian group with k generators and let $\mathbb{Z}H$ be its group ring. Let $G = H \times \langle s\rangle$, where s is an extra free variable. Then the Laurent polynomial ring $\mathbb{Z}H(u)$ is canonically isomorphic to $\mathbb{Z}G$, by an isomorphism that sends s to u.

We generalize the results of Section 2.1 to the setting of H-labeled digraphs.

Definition 2.9. Let C be a simplicial complex. An *H-labeling* of C is a map

$$h\colon C \to H$$

compatible with the simplicial complex structure of H, i.e.,

$$h(\sigma) = \sum_{i=1}^{\ell} h(v_i)$$

for $\sigma = \{v_1, \dots, v_\ell\}$. An *$H$-complex* $\mathcal{C}^H$ is an abstract simplicial complex together with a H-labeling.

Definition 2.10. The *cycle function* of an H-labeled complex $\mathcal{C}^H$ is the element of $\mathbb{Z}H$ defined by

$$\theta_{\mathcal{C}^H} = 1 + \sum_{\sigma \in \mathcal{C}^H} (-1)^{|\sigma|} h(\sigma)^{-1}.$$

Definition 2.11. An *H-digraph* $\mathcal{D}^H$ is a digraph D, along with a map

$$h\colon\ \mathcal{E}_D \to H,$$

where $\mathcal{E}_D$ is the set of edge of D. The digraph D is the *underlying digraph* of $\mathcal{D}^H$.

An H-labeling on a digraph induces an H-labeling on its cycle complex. Let γ be a simple cycle on D. Then up to isotopy, γ can be written as

$$\gamma = e_0 \cdots e_{k-1},$$

for some collection of edge $e_0, \dots, e_{k-1}$ cyclically joined end to end on D. Let

$$h(\gamma) = h(e_0) + \ldots + h(e_{k-1}),$$

and for $\sigma = \{\gamma_1, \dots, \gamma_\ell\}$, let

$$h(\sigma) = \sum_{i=1}^{\ell} h(\gamma_i).$$

Denote the labeled cycle complex by $\mathcal{C}_{\mathcal{D}}^H$. The *cycle polynomial* $\theta_{\mathcal{D}^H}$ of $\mathcal{D}^H$

$$\theta_{\mathcal{D}^H}(u) = 1 + \sum_{\sigma \in \mathcal{C}_{\mathcal{D}}^H} (-1)^{|\sigma|} h(\sigma)^{-1} u^{-\ell(\sigma)} \in \mathbb{Z}H[u].$$

The cycle polynomial of $\theta_{\mathcal{D}^H}(u)$ contains both the information about the associated labeled complex $\mathcal{C}_{\mathcal{D}}^H$ and the length functions on cycles on D. One observes the following by comparing Definition 2.10 and Definition 2.11.

Lemma 2.12. *The cycle polynomial of the H-labeled digraph $\mathcal{D}^H$, and the cycle function of the labeled cycle complex $\mathcal{C}_{\mathcal{D}}^H$ are related by*

$$\theta_{\mathcal{C}_{\mathcal{D}}^H} = \theta_{\mathcal{D}^H}(1).$$

Definition 2.13. An element $\theta \in \mathbb{Z}H$ is *positive*, denoted $\theta > 0$, if

$$\theta = \sum_{h \in H} a_h h,$$

where $a_h \geq 0$ for all $h \in H$, and $a_h > 0$ for at least one $h \in H$. If θ is positive or 0 we say that it is non-negative and write $\theta \geq 0$.

A matrix M^H with entries in $\mathbb{Z}H$ is called an *H-matrix.* If all entries are non-negative, we write $M^H \geq 0$ and if all entries are positive we write $M^H > 0$.

Lemma 2.14. *There is a bijective correspondence between H-digraphs $\mathcal{D}^H$ and non-negative H-matrices $M^H_{\mathcal{D}}$, so that $M^H_{\mathcal{D}}$ is the directed incidence matrix for $\mathcal{D}^H$.*

Proof. Given a labeled digraph $\mathcal{D}^H$, let E_{ij} be the set of edges from the i–th vertex to the j–th vertex. We form a matrix $M^H_{\mathcal{D}}$ with entries in $\mathbb{Z}H$ by setting

$$a_{ij} = \sum_{e \in E_{ij}} h(e),$$

where $h(e)$ is the H-label of the edge e.

Conversely, given an $n \times n$ matrix M^H with entries in $\mathbb{Z}H$, let $\mathcal{D}^H$ be the H-digraph with n vertices $v_1, \ldots, v_n$ and, for each i, j with $m_{i,j} = \sum_{h \in H} a_g g \geq 0$, it has a_h directed edges from v_i to v_j labeled by h. The directed incidence matrix $M^H_{\mathcal{D}}$ equals M as desired. ☐

The proof of the next theorem is similar to that of the Theorem 2.6 and is left to the reader.

Theorem 2.15 (The Coefficients Theorem for H-labeled digraphs)**.** *Let $\mathcal{D}^H$ be an H-labeled digraph with m vertices, and $P_{\mathcal{D}}(u) \in \mathbb{Z}H[u]$ be the characteristic polynomial of its incidence matrix. Then,*

$$P_{\mathcal{D}}(u) = u^m \theta_{\mathcal{D}^H}(u).$$

2.4. **Perron Frobenius H-matrices.** In this section we recall a key theorem of McMullen on leading eigenvalues of specializations of Perron-Frobenius H-matrices (see [McM00], Appendix A).

Definition 2.16. A non-negative H-matrix M^H is called a *Perron-Frobenius H-matrix* if all entries of M^H are non-negative (see Def. 2.13), and all entries of $(M^H)^k$ are positive for some $k \geq 1$.

Recall the definition of a Perron-Frobenius digraph (Definition 2.1).

Lemma 2.17. *If D is a Perron-Frobenius digraph, then any choice of labeling of the edges of D by elements of H determines a Perron-Frobenius H-matrix.*

Proof. Let M be the adjacency matrix for the unlabeled digraph D, and let $n \geq 1$ be such that $M^n > 0$. Then for any i, j, and $n \geq N$, there is some directed edge-path on D of length n from v_i to v_j. Each of the edges in the edge-path corresponds to an element of H. The edge-path thus contributes a positive summand in the i, jth entry of the labeled adjacency matrix of D. □

The ring $\mathrm{Hom}(H, \mathbb{R})$ can be identified with $\mathbb{R}^k$. Let $\mathsf{w} = (w_1, \ldots w_k)$ be the coordinates of this space. Let $\mathsf{t} = (t_1, \ldots t_k)$ be the coordinates of $\mathrm{Hom}(H, \mathbb{R}_+) \cong \mathbb{R}^k_+$. There is a natural map $e\colon \mathrm{Hom}(H, \mathbb{R}) \to \mathrm{Hom}(H, \mathbb{R}_+)$ defined by $\mathsf{w} \mapsto e^{\mathsf{w}}$. A PF matrix with coefficients in $\mathbb{Z}H$ gives rise to a PF matrix with coefficients in the variables $\mathsf{t} = (t_1, \ldots, t_k)$, this is done by identifying elements of H with monomials in t. For any $\mathsf{t} \in \mathbb{R}^k_+$, $M(\mathsf{t})$ is a Perron-Frobenius matrix, and has a Perron-Frobenius eigenvalue $E(\mathsf{t}) = \lambda_{\mathrm{PF}}(\mathsf{t})$.

Theorem 2.18 (McMullen [McM00], Theorem A.1)**.** *Let $E(\mathsf{t})$ be the leading eigenvalue of a Perron-Frobenius $M(\mathsf{t})$, considered as a function of t the coordinates in $\mathbb{R}^k_+$, and let $P(\mathsf{t}, u)$ be the characteristic polynomial of $M(\mathsf{t})$. Then the following holds.*

(1) The real analytic function $\delta(\mathsf{w}) = \log(E(e^{\mathsf{w}}))$ is convex as a function of $\mathsf{w} \in \mathbb{R}^k$.
(2) The graph of $s = \delta(\mathsf{w})$ meets every ray through the origin of $\mathbb{R}^k \times \mathbb{R}$ at most once.
(3) Let $d = \deg(Q)$. The set of rays passing through the graph of $s = \delta(\mathsf{w})$ in $\mathbb{R}^k \times \mathbb{R}$ coincides with the McMullen cone $\mathcal{T}_Q(u^d)$, for any factor Q of P with $Q(E(e^{\mathsf{w}})) = 0$ for all $\mathsf{w} \in \mathbb{R}^k$.

Definition 2.19. An *affine plane* in $\mathbb{R}^k$, is a translated linear subspace of $\mathbb{R}^k$ that does not pass through the origin.

Theorem 2.20 (McMullen [McM00])**.** *Let M be a Perron-Frobenius H-matrix, and $P \in \mathbb{Z}H[u]$ its characteristic polynomial. Let $\mathcal{T} = \mathcal{T}_P(u^d)$, where d is the degree of P. Then the map $L : \mathrm{Hom}(H \times \langle s \rangle; \mathbb{Z}) \to \mathbb{R}$ defined by*

$$L(\alpha) = \log |P^{(\alpha)}|,$$

extends to a homogeneous of degree -1, real analytic, convex function on $\mathcal{T}$ that goes to infinity toward the boundary of affine planar sections of $\mathcal{T}$.

Definition 2.21. The McMullen cone $\mathcal{T}_P(u^d)$ for the element $P \in \mathbb{Z}H[u]$ is also called the *McMullen cone* for the H-matrix M.

Theorem 2.20 summarizes results taken from [McM00] given in the context of mapping classes on surfaces. For the convenience of the reader, we give a proof here.

Proof. The function L is real analytic since the house of a Perron polynomial is an algebraic function in its coefficients. Homogeneity of $L(z)$ follows from the observation: ρ is a root of $Q(x^{\mathsf{w}}, x^{s})$ if and only if $\rho^{1/c}$ is a root of $Q(x^{c\mathsf{w}}, x^{cs})$. Thus

$$L(cz) = \log |Q(x^{c\mathsf{w}}, x^{cs})| = c^{-1} \log |Q(x^{\mathsf{w}}, x^{s})| = c^{-1} L(z).$$

By homogeneity of L, the values of L are determined by the values at any level set, one of which is the graph of $E(\mathsf{t})$. To prove convexity of L, we show that level sets of L are convex i.e. the line connecting two points on a level set lies above the level set. Let $\Gamma = \{z = (\mathsf{w}, s) \mid L(z) = 1\}$ and $\Gamma' = \{z = (\mathsf{w}, s) \mid s = \delta(\mathsf{w})\}$. We show that $\Gamma = \Gamma'$. It then follows that, since Γ' is a graph of a convex function by Theorem 2.18, Γ is convex.

We begin by showing that $\Gamma' \subset \Gamma$ (cf. [McM00], proof Theorem 5.3). If $\beta = (\mathsf{a}, b) \in \Gamma'$ then $\delta(\mathsf{a}) = b$, hence $Q(e^{\mathsf{a}}, e^{b}) = 0$ and $|Q(e^{\mathsf{a}}, e^{b})| \geq e$. Let $r = L(\beta) = \log |Q(e^{\mathsf{a}}, e^{b})|$. Since $b = \delta(\mathsf{a})$, by the convexity of the function δ, we have $rb \geq \delta(r\mathsf{a})$. On the other hand, $Q(e^{r\mathsf{a}}, e^{rb}) = 0$ hence e^{rb} is an eigenvalue of $M(e^{r\mathsf{a}})$ so $rb \leq \log E(e^{r\mathsf{a}}) = \delta(r\mathsf{a})$. We get that $rb = \delta(r\mathsf{a})$. The points $(\mathsf{a}, b), (r\mathsf{a}, rb)$ both lie on the same line through the origin so by Theorem 2.18 part (2), they are equal. Thus $r = 1 = L(\beta)$, and hence $\beta \in \Gamma$.

To show that $\Gamma \subset \Gamma'$ in $\mathcal{T}$, note that every ray in $\mathcal{T}$ initiating from the origin intersects Γ because it intersects Γ' by part (3) of Theorem 2.18. Because L is homogeneous, level sets of L intersect every ray from the origin at most once. Therefore, in $\mathcal{T}$, $\Gamma = \Gamma'$ and the latter is the graph of a convex function.

We now show that if L is a homogeneous function of degree -1, and has convex level sets then L is convex (cf. [McM00] Corollary 5.4). This is equivalent to showing that $1/L(z)$ is concave on $\mathcal{T}$. Let $z_1, z_2 \in T$ lie on distinct rays through the origin, and let

$$z_3 = sz_1 + (1-s)z_2.$$

Let c_i, $i = 1, 2, 3$, be constants so that $z_i' = c_i^{-1} z_i$ is in the level set $L(c_i^{-1} z_i) = 1$. Let p lie on the line $[z_1', z_2']$ and on the ray through z_3. Then p has the form

$$p = rz_1' + (1-r)z_2'$$

for $0 < r < 1$. If

$$r = \frac{sc_1}{sc_1 + (1-s)c_2}$$

then we have

$$p = \frac{z_3}{sc_1' + (1-s)c_2'}.$$

Since the level set for $L(z) = 1$ is convex, p is equal to or above z_3/c_3, and we have

$$1/(sc_1 + (1-s)c_2) \geq 1/c_3. \tag{4}$$

Thus

$$1/L(z_3) = c_3 \geq sc_1 + (1-s)c_2 = s/L(z_2) + (1-s)/L(z_3). \tag{5}$$

Thus $1/L(z)$ is concave, and hence $L(z)$ is convex.

Let z_n be a sequence of points on an affine planar section of $\mathcal{T}$ approaching the boundary of $\mathcal{T}$. Let c_n be such that $c_n^{-1}z_n$ is in the level set $L(z) = 1$. Then $L(z_n) = c_n^{-1}$ for all n. But z_n is bounded, while the level set $L(z) = 1$ is asymptotic to the boundary of $\mathcal{T}$. Therefore, $1/L(z_n)$ goes to 0 as n goes to infinity. □

Remark 2.22. If the level set $L(z) = 1$ is strictly convex, then $L(z)$ is strictly convex. Indeed, if $L(z) = 1$ is strictly convex, then the inequality in (4) is strict, and hence the same holds for (5).

2.5. **Distinguished factor of the characteristic polynomial.** We define a distinguished factor of the characteristic polynomial of a Perron-Frobenius G-matrix.

Proposition 2.23. *Let P be the characteristic polynomial of a Perron-Frobenius G-matrix. Then P has a factor Q with the properties:*

(1) for all integral elements α in the McMullen cone $\mathcal{T}$,

$$|P^{(\alpha)}| = |Q^{(\alpha)}|,$$

(2) minimality: if $Q_1 \in \mathbb{Z}G[u]$ satisfies $|Q^{(\alpha)}| = |Q_1^{(\alpha)}|$ for all α ranging among the integer points of an open subcone of $\mathcal{T}$, then Q divides Q_1, and

(3) if r is the degree of Q the cones $\mathcal{T}_P(u^d)$ and $\mathcal{T}_Q(u^r)$ are equal, where d is the degree of P and r is the degree of Q as elements of $\mathbb{Z}G[u]$.

Definition 2.24. Given a Perron-Frobenius G-matrix M^G, the polynomial Q is called the *distinguished factor* of the characteristic polynomial of M^G.

Lemma 2.25. *Let $F(\mathsf{t})\colon \mathbb{R}^k \to \mathbb{R}$ be a function, consider the ideal*

$$I_F = \{\theta \in \mathbb{Z}(\mathsf{t})[u] \mid \theta(\mathsf{t}, F(\mathsf{t})) = 0 \text{ for all } \mathsf{t} \in \mathbb{R}^k\}$$

I_F is a principal ideal.

Proof. Let $\overline{\mathbb{Z}(\mathsf{t})}[u]$ be the ring of polynomials in the variable u over the quotient field of $\mathbb{Z}(\mathsf{t})$. Since $\overline{\mathbb{Z}}(\mathsf{t})[u]$ is a principal ideal domain, I_F generates a principal ideal $\overline{I}_F$ in $\overline{\mathbb{Z}(\mathsf{t})}[u]$.

Let $\overline{\theta}_1$ be a generator of $\overline{I}_F$, then $\overline{\theta_1} = \frac{\theta_1(\mathsf{t},u)}{\sigma(\mathsf{t})}$ with $\theta_1 \in I_F$. Thus $\overline{\theta}_1(\mathsf{t}, F(\mathsf{t})) = 0$ for all t. If I_F is the zero ideal then there is nothing to prove, therefore we suppose it is not. Let $\overline{\theta}_1(\mathsf{t}, u) = \frac{\nu(\mathsf{t},u)}{\delta(\mathsf{t})}$, where ν and δ are relatively prime in $\overline{\mathbb{Z}(\mathsf{t})}[u]$, a unique factorization domain. Since $\theta_1(\mathsf{t}, F(\mathsf{t})) = 0$ for all t then $\nu(\mathsf{t}, F(\mathsf{t})) = 0$ for all t and $\nu \in I_F$.

Since I_F is not the zero ideal then $\overline{I}_F$ is not the zero ideal, hence $\bar{\theta}_1 \neq 0$ which implies that $\nu \neq 0$. Let $\theta \in I_F$ be any polynomial. Since $\overline{\theta}_1$ divides θ, then ν divides $\theta\delta(\mathtt{t})$ but since ν and δ are relatively prime, ν divides θ. Therefore, ν is a generator of I_E. □

3. Branched surfaces with semiflows

In this section we associate a digraph and an element $\theta_{X,\mathfrak{C},\psi} \in \mathbb{Z}G$ to a branched surface $(X, \mathfrak{C}, \psi)$. We show that this element is invariant under certain kinds of vertical and transversal subdivisions of $\mathfrak{C}$.

3.1. The cycle polynomial of a branched surface with a semiflow.

Definition 3.1. Given a 2-dimensional CW-complex X, a *semiflow* on X is a continuous map $\psi : X \times \mathbb{R}_+ \to X$ satisfying

(i) $\psi(\cdot, 0)\colon\ X \to X$ is the identity,
(ii) $\psi(\cdot, t)\colon\ X \to X$ is a homotopy equivalence for every $t \geq 0$, and
(iii) $\psi(\psi(x, t_0), t_1) = \psi(x, t_0 + t_1)$ for all $t_0, t_1 \geq 0$.

A cell-decomposition $\mathfrak{C}$ of X is *ψ-compatible* if the following hold.

(1) Each 1–cell is either contained in a flow line (*vertical*), or transversal to the semiflow at every point (*transversal*).
(2) For every vertex $p \in \mathfrak{C}^{(0)}$, the image of the *forward flow* of p,
$$\{\psi(p, t) \mid t \in \mathbb{R}_{>0}\},$$
is contained in $\mathfrak{C}^{(1)}$.

A *branched surface* is a triple $(X, \mathfrak{C}, \psi)$, where X is a 2-complex with semi-flow ψ and a ψ-compatible cellular structure $\mathfrak{C}$.

Remark 3.2. We think of branched surfaces as flowing downwards. From this point of view, Property (2) implies that every 2–cell $c \in \mathfrak{C}^{(2)}$ has a unique *top* 1*–cell*, that is, a 1–cell e such that each point in c can be realized as the forward orbit of a point on e.

Definition 3.3. Let e be a 1–cell on a branched surface $(X, \mathfrak{C}, \psi)$ that is transverse to the flow at every point. A *hinge* containing e is an equivalence class of homeomorphisms $\kappa\colon\ [0, 1] \times [-1, 1] \hookrightarrow X$ so that:

(1) the half segment $\Delta = \{(x, 0) \mid x \in I\}$ is mapped onto e,
(2) the image of the interior of the Δ intersects $\mathfrak{C}^{(1)}$ only in e, and
(3) the vertical line segments $\{x\} \times [-1, 1]$ are mapped into flow lines on X.

Two hinges κ_1, κ_2 are *equivalent* if there is an isotopy rel Δ between them. The 2–cell on $(X, \mathfrak{C}, \psi)$ containing $\kappa([0, 1] \times [0, 1])$ is called the *initial cell* of κ and the 2–cell containing the point $\kappa([0, 1] \times [-1, 0])$ is called the *terminal cell* of κ.

An example of a hinge is illustrated in Figure 1.

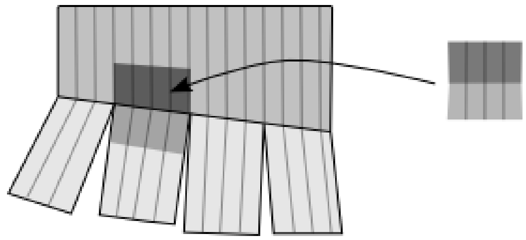

FIGURE 1. A hinge on a branched surface.

Definition 3.4. Let $(X, \mathfrak{C}, \psi)$ be a branched surface. The *dual digraph* D of $(X, \mathfrak{C}, \psi)$ is the digraph with a vertex for every 2–cell and an edge for every hinge κ from the vertex corresponding to its initial 2–cell to the vertex corresponding to its terminal 2–cell. The dual digraph D for $(X, \mathfrak{C}, \psi)$ embeds into X

$$D \hookrightarrow X$$

so that each vertex is mapped into the interior of the corresponding 2–cell, and each directed edge is mapped into the union of the two-cells corresponding to its initial and end vertices, and intersects the common boundary of the 2–cells at a single point. The embedding is well defined up to homotopies of X to itself.

An example of an embedded dual digraph is shown in Figure 2. In this example, there are three edges emanating from v with endpoints at w_1, w_2 and w_3. It is possible that $w_i = w_j$ for some $i \neq j$, of that $w_i = v$ for some i. These cases can be visualized using Figure 2, where we identify the corresponding 2–cells.

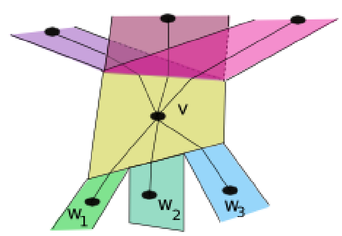

FIGURE 2. A section of an embedded dual digraph.

Let $G = H_1(X; \mathbb{Z})/\text{torsion}$, thought of as the integer lattice in $H_1(X, \mathbb{R})$. The embedding of D in X determines a G-labeled cycle complex $\mathcal{C}_D^G$ where for each $\sigma \in \mathcal{C}_D^G$ and $g(\sigma)$ is the homology class of the cycle σ considered as a 1-cycle on X.

Definition 3.5. Given a branched surface $(X, \mathfrak{C}, \psi)$, the *cycle function* of $(X, \mathfrak{C}, \psi)$ is the group ring element

$$\theta_{X,\mathfrak{C},\psi} = 1 + \sum_{\sigma \in \mathcal{C}_D^G} (-1)^{|\sigma|} g(\sigma)^{-1} \in \mathbb{Z}G.$$

Then we have

$$\theta_{X,\mathfrak{C},\psi} = \theta_{\mathcal{C}_D^G}(1)$$

where $\theta_{\mathcal{C}_D^G}(u)$ is the cycle polynomial of $\mathcal{C}_D^G$.

3.2. **Subdivision.** We show that the cycle function of $(X, \mathfrak{C}, \psi)$ is not invariant under certain kinds of cellular subdivisions.

Definition 3.6. Let $p \in \mathfrak{C}^{(1)}$ be a point in the interior of a transversal edge in $\mathfrak{C}^{(1)}$. Let $x_0 = p$ and inductively define $x_i = \psi(x_{i-1}, s_i)$, for $i = 1, \ldots, r$, so that

$$s_i = \min\{s \mid \psi(x_{i-1}, s) \text{ has endpoint in } \mathfrak{C}^{(1)}\}.$$

The *vertical subdivision of X along the forward orbit of p* is the cellular subdivision $\mathfrak{C}'$ of $\mathfrak{C}$ obtained by adding the edges $\psi(x_{i-1}, [0, s_i])$, for $i = 1, \ldots, r$, and subdividing the corresponding 2–cells. If x_r is a vertex in the original skeleton $\mathfrak{C}^{(0)}$ of X, then we say the vertical subdivision is *allowable*.

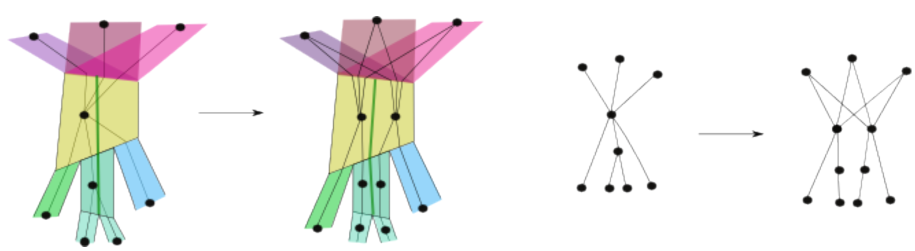

FIGURE 3. An allowable vertical subdivision, and effect on the directed dual digraph.

Figure 1 illustrates an allowable vertical subdivision with $r = 2$.

Proposition 3.7. *Let $(X, \mathfrak{C}', \psi)$ be obtained from $(X, \mathfrak{C}, \psi)$ by allowable vertical subdivision. Then the cycle function $\theta_{X,\mathfrak{C},\psi}$ and $\theta_{X,\mathfrak{C}',\psi}$ are equal.*

We establish a few lemmas before proving Proposition 3.7.

Lemma 3.8. *Let $(X, \mathfrak{C}', \psi)$ be obtained from $(X, \mathfrak{C}, \psi)$ by allowable vertical subdivision. Let D' and D be the dual digraphs for $(X, \mathfrak{C}', \psi)$ and $(X, \mathfrak{C}, \psi)$. There is a* *quotient map* $q : D' \to D$ *that is induced by a continuous map from* X *to itself that is homotopic to the identity, and in particular the diagram*

$$\begin{array}{ccc} H_1(D';\mathbb{Z}) & \xrightarrow{q_*} & H_1(D;\mathbb{Z}) \\ & \searrow & \downarrow \\ & & H_1(X;\mathbb{Z}) \end{array}$$

commutes.

Proof. Working backwards from the last vertically subdivided cell to the first, each allowable vertical subdivision decomposes into a sequence of allowable vertical subdivisions that involve only one 2–cell. An illustration is shown in Figure 4.

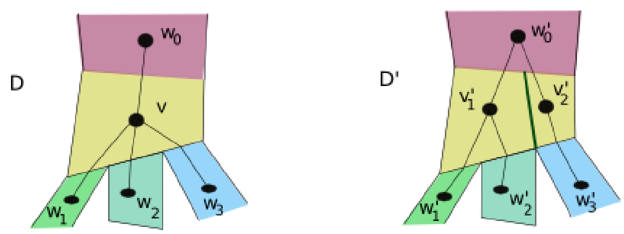

FIGURE 4. Vertical subdivision of one cell.

Let v be the vertex of D corresponding to the cell c of X that contains the new edge. The digraph D' is constructed from D by the following steps:

1. Each vertex $u \neq v$ in D lifts to a well-defined vertex u' in D'. The vertex $v \in D$ lifts to two vertices v_1', v_2' in D'.
2. For each edge ε of D neither of whose endpoints u and w equal v, the quotient map is 1-1 over ε, and hence there is only one possible lift ϵ' from u' to w'.
3. For each edge ε from $w \neq v$ to v there are two edges $\varepsilon_1', \varepsilon_2'$ where ε_i' begins at w' and ends at v_i'.
4. For each outgoing edge ε from v to w (where v and w are possibly equal), there is a representative κ of the hinge corresponding to ε that is contained in the union of two 2–cells in the $\mathfrak{C}'$. This determines a unique edge ε' on D' that lifts ε.

There is a continuous map homotopic to the identity from X to itself that restricts to the identity on every cell other than c or c_w, where c_w corresponds to a vertex w with an edge from w to v in D. On $c \cup c_w$ the map merges the edges $\varepsilon_1', \varepsilon_2'$ so that their endpoints v_i' merge to the one vertex v. ☐

Lemma 3.9. *The quotient map* $q : D \to D'$ *induces an inclusion*

$$q^* : C_D \hookrightarrow C_{D'},$$

which preserves lengths, sizes, and labels, so that for $\sigma \in C_D$, $q(q^*(\sigma)) = \sigma$.

Proof. Again we may assume that the subdivision involves a vertical subdivision of one 2–cell c corresponding to the vertex $v \in D$ and then use induction. It is enough to define lifts of simple cycles on D to a simple cycle in D'. All edges in D from u to w with $w \neq v$ have a unique lift in D'. Thus, if γ does not contain v then there is a unique γ' in D' such that $q(\gamma') = \gamma$. Assume that γ contains v. If γ consists of a single edge ε, then ε is a self-edge from v to itself, and ε has two lifts: a self-edge from v_1' to v_1' and an edge from v_1' to v_2', where v_1' is the vertex corresponding to the initial cell of the hinge containing ε. Thus, there is a well-defined self-edge γ' lifting γ (see Figure 5).

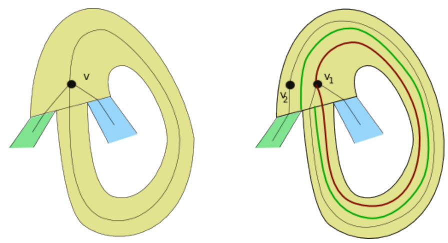

FIGURE 5. Vertical subdivision when digraph has a self edge.

Now suppose γ is not a self-edge and contains v. Let $w_1, \ldots, w_{\ell-1}$ be the vertices in γ other than v in their induced sequential order. Let ε_i be the edge from w_{i-1} to w_i for $i = 2, \ldots, \ell-1$. Then since none of the ε_i have initial or endpoint v, they have unique lifts ε_i' in D'. Since the vertical subdivision is allowable, there is one vertex, say v_1', above v with an edge ε_1' from v_1' to w_2'. Let ε_ℓ' be the edge from $w_{\ell-1}'$ to v_1' (cf. Figure 4). Let γ' be the simple cycle with edges $\varepsilon_1', \ldots, \varepsilon_\ell'$.

Since the lift of a simple cycle is simple, the lifting map determines a well-defined map $q^* : C_D \to C_{D'}$ that satisfies $q \circ q^* = \text{id}$ and preserves size. The commutative diagram in Lemma 3.8 implies that the images of σ and $q^*(\sigma)$ in G are the same, and hence their labels are the same. □

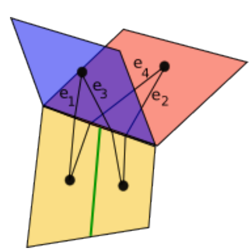

FIGURE 6. A switching locus.

Lemma 3.10. *Let D' be obtained from D by an allowable vertical subdivision on a single 2–cell. The set of edges of each $\sigma \in C_{D'} \smallsetminus q^*(C_D)$ contains exactly one matched pair.*

Proof. Since $\sigma' \notin q^*(C_D)$, the quotient map q is not injective on σ'. Thus $q(\sigma')$ must contain two distinct edges $\varepsilon_1, \varepsilon_2$ with endpoint v, and these have lifts ε'_1 and ε'_2 on σ'. Since σ' is a cycle, ε'_1 and ε'_2 must have distinct endpoints, hence one is v'_1 and one is v'_2. There cannot be more than one matched pair on σ', since σ' can pass through each v'_i only once. □

Definition 3.11. Let D' be obtained from D by an allowable vertical subdivision on a single 2–cell. Let v be the vertex corresponding to the subdivided cell, and let v'_1 and v'_2 be its lifts to D'.

For any pair of edges $\varepsilon'_1, \varepsilon'_2$ with endpoints at v'_1 and v'_2 and distinct initial points w'_1 and w'_2, there is a corresponding pair of edges η'_1, η'_2 from w'_1 to v'_2 and from w'_2 to v'_1. Write

$$\mathrm{op}\{\varepsilon'_1, \varepsilon'_2\} = \{\eta'_1, \eta'_2\}.$$

We call the pair $\{\varepsilon'_1, \varepsilon'_2\}$ a *matched pair*, and $\{\eta'_1, \eta'_2\}$ its *opposite*. (See Figure 6).

Lemma 3.12. *If $\sigma' \in C_{D'}$ contains a matched pair, the edge-path obtained from σ' by exchanging the matched pair with its opposite is a cycle.*

Proof. It is enough to observe that the set of endpoints and initial points of a matched pair and its opposite are the same. □

Define a map $\mathfrak{r} : C_{D'} \to C_{D'}$ be the map that sends each $\sigma \in C_{D'}$ to the cycle obtained by exchanging each appearance of a matched pair on $\sigma' \in C_{D'}$ with its opposite.

Lemma 3.13. *The map $\mathfrak{r}$ is a simplicial map of order two that preserves length and labels. It also fixes the elements of $q^*(C_D)$, and changes the parity of the size of elements in $C_{D'} \smallsetminus q^*(C_D)$.*

Proof. The map $\mathfrak{r}$ sends cycles to cycles, and hence simplicies to simplicies. Since op has order 2, it follows that $\mathfrak{r}$ has order 2. The total number of vertices does not change under the operation op. It remains to check that the homology class of σ' and $\mathfrak{r}(\sigma')$ as embedded cycles in X are the same, and that the size switches parity.

There are two cases. Either the matched edges lie on a single simple cycle γ' or on different simple cycles γ'_1, γ'_2 on σ'.

In the first case, $\mathfrak{r}(\{\gamma'\})$ is a cycle with 2 components $\{\gamma'_1, \gamma'_2\}$. As one-chains we have

$$\beta \;=\; \mathfrak{r}(\sigma') - \sigma' = \gamma'_1 + \gamma'_2 - \gamma' = \eta'_1 + \eta'_2 - \varepsilon'_1 - \varepsilon'_2. \tag{6}$$

In X, β bounds a disc (see Figure 6), thus $g(\gamma') = g(\gamma_1') + g(\gamma_2')$, and hence

$$g(\sigma') \quad = \quad g(\mathfrak{r}(\sigma')). \tag{7}$$

The one component cycle γ' is replaced by two simple cycles γ_1' and γ_2', and hence the size of σ' and $\mathfrak{r}(\sigma')$ differ by one.

Now suppose σ' contains two cycles γ_1' and γ_2', one passing through v_1' and the other passing through v_2'. Then $\mathfrak{r}(\sigma')$ contains a simple cycle γ' in place of $\gamma_1' + \gamma_2'$, so the size decreases by one. By (6) we have (7) for σ' of this type. □

Proof of Proposition 3.7. By Lemma 3.9, the quotient map $q : D' \to D$ induces an injection of $q^* : C_D \hookrightarrow C_{D'}$ defined by the lifting map, and this map preserves labels. We thus have

$$\theta_{X,\mathfrak{C},\psi} = 1 + \sum_{\sigma \in C_D} (-1)^{|\sigma|} g(\sigma)^{-1} = 1 + \sum_{\sigma' \in q^*(C_D)} (-1)^{|\sigma'|} g(\sigma')^{-1}.$$

The cycles in $C_{D'} \setminus q^*(C_D)$ partition into $\sigma', \mathfrak{r}(\sigma')$, and by Lemma 3.13 the contributions of these pairs in $\theta_{X,\mathfrak{C}',\psi}$ cancel with each other. Thus, we have

$$\theta_{X,\mathfrak{C}',\psi} = 1 + \sum_{\sigma' \in C_{D'}} (-1)^{|\sigma'|} g(\sigma')^{-1} = 1 + \sum_{\sigma' \in q^*(C_D)} (-1)^{|\sigma'|} g(\sigma')^{-1} = \theta_{X,\mathfrak{C},\psi}.$$

□

Definition 3.14. Let $(X, \mathfrak{C}, \psi)$ be a branched surface and c a 2–cell. Let p, q be two points on the boundary 1-chain ∂c of c that do not lie on the same 1–cell of $\mathfrak{C}$. Assume that p and q each have the property that

(i) it lies on a vertical edge, or
(ii) its forward flow under ψ eventually lies on a vertical 1–cell of $(X, \mathfrak{C})$.

The *transversal subdivision* of $(X, \mathfrak{C}, \psi)$ at $(c; p, q)$ is the new branched surface $(X, \mathfrak{C}', \psi)$ obtained from $\mathfrak{C}$ by doing the (allowable) vertical subdivisions of $\mathfrak{C}$ defined by p and q, and doing the additional subdivision induced by adding a 1–cell from p to q.

Lemma 3.15. *Let $(X, \mathfrak{C}, \psi)$ be a branched surface, and let $(X, \mathfrak{C}', \psi)$ be a transversal subdivision. Then the corresponding cycle functions are the same.*

Proof. By first vertically subdividing $\mathfrak{C}$ along the forward orbits of p and q if necessary, we may assume that p and q lie on different vertical 1–cells on the boundary of c. Let v be the vertex of D corresponding to c. Then D' is obtained from D by substituting the vertex v by a pair v_1', v_2' that are connected by a single edge. Each edge ε from $w \neq v$ to v is replaced by an edge ε' from w' to v_1' and edge ε from v to $u \neq v$ is replaced by an edge from v_2' to u'. Each edge from v to itself is substituted by an edge from v_2 to v_1. The cycle complexes of D and D' are the same, and their labelings are identical. Thus the cycle function is preserved. □

3.3. **Folding.** Let $(X, \mathfrak{C}, \psi)$ be a branched surface with a flow. Let c_1 and c_2 be two cells with the property that their boundaries ∂c_1 and ∂c_2 both contain the segment $e_1 e_2$, where e_1 is a vertical 1–cell and e_2 is a transversal 1–cell of $\mathfrak{C}$. Let p be the initial point of e_1 and q the end point of e_2. Then p and q both lie on vertical 1–cells, and hence $(c_1; p, q)$ and $(c_2; p, q)$ define a composition of transversal subdivisions $\mathfrak{C}_1$ of $\mathfrak{C}$. For $i = 1, 2$, let e_3^i, be the new 1–cell on c_i, and let $\Delta(e_1, e_2, e_3^i)$ be the triangle c_i bounded by the 1–cells e_1, e_2 and e_3^i.

Definition 3.16. The quotient map $F : X \to X'$ that identifies $\Delta(e_1, e_2, e_3^1)$ and $\Delta(e_1, e_2, e_3^2)$ (see Figure 7) is called the *folding map* of X. The quotient X' is endowed with the structure of a branched surface $(X', \mathfrak{C}', \psi')$ induced by $(X, \mathfrak{C}_1, \psi)$.

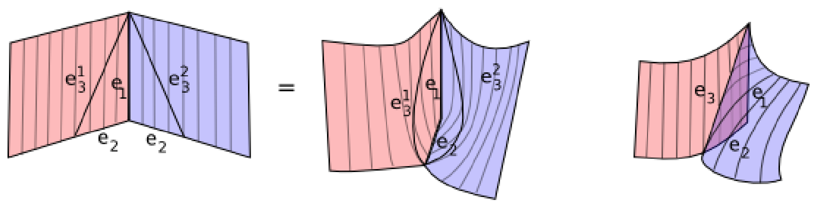

FIGURE 7. The left and middle diagrams depict the two 2–cells sharing the edges e_1 and e_2; the right diagram is the result of folding.

The following Proposition is easily verified (see Figure 7).

Proposition 3.17. *The quotient map F associated to a folding is a homotopy equivalence, and the semi-flow $\psi : X \times \mathbb{R}_+ \to X$ induces a semi-flow $\psi' : X \times \mathbb{R}_+ \to X$.*

Definition 3.18. Given a folding map $F : X \to X'$, there is an induced branched surface structure $(X', \mathfrak{C}', \psi')$ on X given by taking the minimal cellular structure on X' for which the map F is a cellular map and deleting the image of e_2 if there are only two hinges containing e_2 on X.

Remark 3.19. In the case that c_1, c_2 are the only cells above e_2, then folding preserves the dual digraph D.

Lemma 3.20. *Let $F : X \to X'$ be a folding map, and let $(X', \mathfrak{C}', \psi')$ be the induced branch surface structure of the quotient. Then*

$$\theta_{X,\mathfrak{C},\psi} = \theta_{X',\mathfrak{C}',\psi'}.$$

Proof. Let D be the dual digraph of $(X, \mathfrak{C}, \psi)$ and D' the dual digraph of $(X', \mathfrak{C}', \psi')$. Assume that there are at least three hinges containing e_2. Then D' is obtained from D by gluing two adjacent half edges (see Figure 8), a homotopy equivalence. Thus, $\mathcal{C}_{\mathcal{D}}^G = \mathcal{C}_{\mathcal{D}'}^G$, and the cycle polynomials are equal. ◻

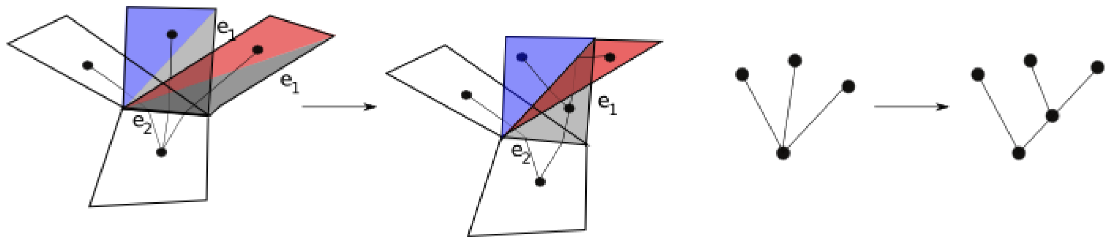

FIGURE 8. Effect of folding on the digraph.

4. Branched surfaces associated to a free group automorphism

Throughout this section, let $\phi \in \mathrm{Out}(F_n)$ be an element that can be represented by an expanding irreducible train-track map $f : \tau \to \tau$. We describe the mapping torus X_f of f the structure of a branched surface $(X_f, \mathfrak{C}_f, \psi_f)$ and realize the folded mapping tori (defined in [DKL13.1]) as vertically equivalent branched surface.

Let $\Gamma = F_n \rtimes_\phi \mathbb{Z}$, and $G = \Gamma^{\mathrm{ab}}/\mathrm{torsion}$. In this section we define the mapping torus $(Y_f, \mathfrak{C}, \psi)$ associated f. We prove that its cycle polynomial $\theta_{Y_f,\mathfrak{C},\psi}$ has a distinguished factor Θ with a distinguished McMullen cone $\mathcal{T}$. We show that the logarithm of the house of Θ specialized at integral elements in the cone extends to a homogeneous of degree -1, real analytic concave function L on an open cone in $\mathrm{Hom}(G, \mathbb{R})$, and satisfies a universality property. Later we relate Θ to dilatations of elements in the DKL cone A_{f}, for any folding decomposition f of f, and show that Θ only depends on ϕ.

4.1. **Free group automorphisms and train-tracks maps.** We give some background definitions for free group automorphisms, and their associated train-tracks following [DKL13.1]. We also recall some sufficient conditions for a free group automorphism to have an expanding irreducible train-track map due to work of Bestvina-Handel [BH92].

Definition 4.1. A *topological graph* is a finite 1-dimensional cellular complex. For each edge e, an orientation on e determines and *initial* and *terminal* point of e. Given an oriented edge e, we denote by $\overline{e}$, the edge e with opposite orientation. Thus the initial and terminal points of e are respectively the terminal and initial points of $\overline{e}$. An *edge path* on a graph is an ordered sequence of edges $e_1 \cdots e_\ell$, where the endpoint of e_i is the initial point of e_{i+1}, for $i = 1, \dots, \ell - 1$, and e_i is not equal to $\overline{e_{i+1}}$. The length of an edge path $e_1 \cdots e_\ell$ is ℓ.

Definition 4.2. A *graph map* $f : \tau \to \tau$ is a continuous map from a graph τ to itself that sends vertices to vertices, and is a local embedding on edges. A graph map assigns to each edge $e \in \tau$ an edge path $f(e) = e_1 \cdots e_\ell$. Identify $\pi_1(\tau)$ with a free group F_n. A graph map f *represents* an element $\phi \in \mathrm{Out}(F_n)$ if ϕ is conjugate to f_* as an element of $\mathrm{Out}(F_n)$.

Remark 4.3. In many definitions of graph map one is also allowed to collapse an edge, but for this exposition, graph maps send edges to nontrivial edge-paths.

Definition 4.4. A graph map $f\colon\ \tau \to \tau$ is a *train-track map* if

(i) f is a homotopy equivalence, and
(ii) f^k has no *back-tracking* for all $k \geq 1$, that is, if

$$f^k(e) = e_1 \cdots e_i e_{i+1} \cdots e_\ell,$$

for some $k \geq 1$, then $e_i \neq \overline{e_{i+1}}$ for all $i = 1, \dots, \ell - 1$.

Definition 4.5. Given a train-track map $f\colon\ \tau \to \tau$, let $\{e_1, \dots, e_k\}$ be an ordering of the edges of τ, and let D_f be the digraph whose vertices v_e correspond to the undirected edges e of τ, and whose edges from e_i to e_j correspond to each appearance of e_j in the edgepath $f(e_i)$. The *transition matrix* M_f of D_f is the directed adjacency matrix

$$M_f = [a_{i,j}],$$

where $a_{i,j}$ is equal to the number of edges from v_{e_i} to v_{e_j}.

Definition 4.6. If $f : \tau \to \tau$ be a train-track map, the *dilatation* of f is given by the spectral radius of M_f

$$\lambda(f) = \max\{|\mu| \mid \mu \text{ is an eigenvalue of } M_f\}.$$

In particular, if f is a PF train track map, then $\lambda(f) = \lambda_{\mathrm{PF}}(M_f)$.

Definition 4.7. A train-track map $f : \tau \to \tau$ is *irreducible* if its transition matrix M_f is irreducible, it is *expanding* if the lengths of edges of τ under iterations of f are unbounded, and it is *Perron-Frobenius* (*PF*) if M_f is a Perron-Frobenius map.

A PF train-track map is irreducible and expanding, but the converse is not necessarily true as seen in the following example.

Example 4.8. Let τ be the rose with four petals shown in figure 9. Let $f : \tau \to \tau$ be the train-track map associated to the free group automorphism

$$\begin{array}{rcl} a & \mapsto & cdc \\ b & \mapsto & cd \\ c & \mapsto & aba \\ d & \mapsto & ab \end{array}$$

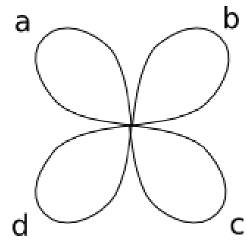

FIGURE 9. Four petal rose.

The train-track map f has transition matrix

$$M_f = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

which is an irreducible matrix, and hence f is irreducible. The train-track map is expanding, since its square is block diagonal, where each block is a 2×2 Perron-Frobenius matrix. On the other hand, f is clearly not PF, since no power of M_f is positive.

Definition 4.9. Fix a generating set $\Omega = \{\omega_1, \dots, \omega_n\}$ of F_n. Then each $\gamma \in F_n$ can be written as a *word* in Ω,

$$\gamma = \omega_{i_1}^{r_1} \cdots \omega_{i_\ell}^{r_\ell} \tag{8}$$

where $\omega_{i_1}, \dots, \omega_{i_\ell} \in \Omega$ and $r_j \in \{1, -1\}$. This representation is *reduced* if there are no cancelations, that is $\omega_{i_j}^{r_j} \neq \omega_{i_{j+1}}^{-r_{j+1}}$ for $j = 1, \dots, \ell - 1$. The word length $\ell_\Omega(\gamma)$ is the length ℓ of a reduced word representing γ in F_n. The *cyclically reduced word length* $\ell_{\Omega,\mathrm{cyc}}(\gamma)$ of γ represented by the word in (8) is the minimum word length of the elements

$$\gamma_j = \omega_{i_j}^{r_j} \omega_{i_{j+1}}^{r_{j+1}} \cdots \omega_{i_\ell}^{r_\ell} \omega_{i_1}^{r_1} \cdots \omega_{i_{j-1}}^{r_{j-1}},$$

for $j = 1, \dots, \ell - 1$.

Proposition 4.10. *Let $\phi \in Out(F_n)$ be represented by an expanding irreducible train-track map f, and let $\gamma \in F_n$ be a nontrivial element. Then either ϕ acts periodically on the conjugacy class of γ in F_n, or the growth rate satisfies*

$$\lambda_{\Omega,cyc}(\gamma) = \lim_k \ell_{\Omega,cyc}(\phi^k(\gamma))^{\frac{1}{k}} = \lambda(f),$$

and in particular, it is independent of the choice of generators, and of γ.

Proof. See, for example, Remark 1.8 in [BH92]. ☐

In light of Proposition 4.10, we make the following definition.

Definition 4.11. Let $\phi \in \mathrm{Out}(F_n)$ be an element that is represented by an expanding irreducible train-track map f. Then we define the *dilatation* of ϕ to be

$$\lambda(\phi) = \lambda(f).$$

In particular, if f is a PF train track map, then $\lambda(\phi) = \lambda_{\mathrm{PF}}(M_f)$.

Definition 4.12. An element $\phi \in \mathrm{Out}(F_n)$ is *hyperbolic* if $F_n \rtimes_\phi \mathbb{Z}$ is word-hyperbolic. It is *atoroidal* if there are no periodic conjugacy classes of elements of F_n under iterations of ϕ.

By a result of Brinkmann [Br00] (see also [BF92]), ϕ is hyperbolic if and only if ϕ is atoroidal.

Definition 4.13. An automorphism $\phi \in \mathrm{Out}(F_n)$ is *reducible* if ϕ leaves the conjugacy class of a proper free factor in F_n fixed. If ϕ is not reducible it is called *irreducible*. If ϕ^k is irreducible for all $k \geq 1$, then ϕ is *fully irreducible*.

Theorem 4.14 (Bestvina-Handel [BH92])**.** *If $\phi \in Out(F_n)$ is irreducible, then ϕ can be represented by an irreducible train track map, and if ϕ is fully irreducible, then it can be represented by a PF train track map.*

4.2. **The mapping torus of a train-track map.** In this section we define the branched surface $(X_f, \mathfrak{C}_f, \psi_f)$ associated to an irreducible expanding train-track map f.

Definition 4.15. The *mapping torus* (Y_f, ψ_f) associated to $f : \tau \to \tau$ is the branched surface where Y_f is the quotient of $\tau \times [0,1]$ by the identification $(t,1) \sim (f(t),0)$, and ψ_f is the semi-flow induced by the product structure of $\tau \times [0,1]$. Write

$$q : \tau \times [0,1] \to Y_f$$

for the quotient map. The map to the circle induced by projecting $\tau \times [0,1]$ to the second coordinate induces a map $\rho : Y_f \to S^1$.

Definition 4.16. The ψ_f-compatible cellular decomposition $\mathfrak{C}_f$ for Y_f is defined as follows. For each edge e, let v_e be the initial vertex of e (the edges e are oriented by the orientation on τ). The 0–cells of $\mathfrak{C}_f$ are $q(v_e \times \{0\})$, the 1–cells are of the form $s_e = q(v_e \times [0,1])$ or $t_e = q(e \times \{0\})$, and the 2–cells are $c_e = q(e \times [0,1])$, where e ranges over the oriented edges of τ. For this cellular decomposition of Y_f, the collection $\mathcal{V}$ of s_e is the set of *vertical* 1–cells and the collection $\mathcal{E}$ of 1–cells t_e is the set of *horizontal* 1–cells.

By this definition $(Y_f, \mathfrak{C}_f, \psi_f)$ is a branched surface. Let $\theta_{Y_f,\mathfrak{C}_f,\psi_f}$ be the associated cycle function (Definition 3.5).

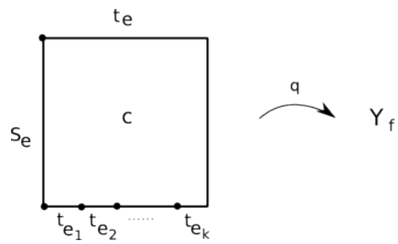

FIGURE 10. A cell of the mapping torus of a train-track map.

Proposition 4.17. *The digraph D_f for the train-track map f and the dual digraph of $(Y_f, \mathfrak{C}_f, \psi_f)$ are the same, and we have*

$$\lambda(\phi) = |\theta^{(\alpha)}_{Y_f,\mathfrak{C}_f,\psi_f}|,$$

where $\alpha : \Gamma \to \mathbb{Z}$ is the projection associated to ϕ.

Proof. Each 2–cell c of $(Y_f, \mathfrak{C}_f, \psi_f)$ is the quotient of one drawn as in Figure 10, and hence there is a one-to-one correspondence between 2–cells and edges of τ. One can check that for each time $f(e)$ passes over the edge e_i, there is a corresponding hinge between the cell $q(e \times [0,1])$ and the cell $q(e_i \times [0,1])$. This gives a one-to-one correspondence between the directed edges of D_f and the edges of the dual digraph.

Recall that $\lambda(\phi) = \lambda(f)$ is the spectral radius of M_f (Definition 4.11). By Theorem 2.6, the characteristic polynomial of D_f satisfies

$$P_{D_f}(x) = x^m \theta_{D_f}(x).$$

Each edge of D_f has length one with respect to the map α, and hence for each cycle $\sigma \in C_{D_f}$, the number of edges in σ equals $\ell_\alpha(\sigma)$. It follows that $\theta_{D_f}(x)$ is the specialization by α of the cycle function $\theta_{Y_f,\mathfrak{C}_f,\psi_f}$, and we have

$$\lambda_{\mathrm{PF}}(D_f) = |P_{D_f}| = |\theta_{D_f}| = |\theta^{(\alpha)}_{Y_f,\mathfrak{C}_f,\psi_f}|.b \qquad \square$$

In the following sections, we study the behavior of $|\theta^{(\alpha)}_{Y_f,\mathfrak{C}_f,\psi_f}|$ as we let α vary.

4.3. **Application of McMullen's theorem to cycle polynomials.** Fix a train-track map $f : \tau \to \tau$. Recall that $\theta_f = \theta_{Y_f,\mathfrak{C}_f,\pi_f} = 1 + \sum_{\sigma \in \mathcal{C}^G_{D_f}} (-1)^{|\sigma|} g(\sigma)^{-1}$. Thus the McMullen cone $\mathcal{T}_{\theta_f}(1)$ is given by

$$\begin{aligned}\mathcal{T}_{\theta_f}(1) &= \{\alpha \in \mathrm{Hom}(G;\mathbb{R}) \mid \alpha(g) > 0, \text{ for all } g \in \mathrm{Supp}(\theta)\} \\ &= \{\alpha \in \mathrm{Hom}(G;\mathbb{R}) \mid \alpha(g) > 0, \text{ for all } g \in G \text{ such that } a_g \neq 0\}.\end{aligned}$$

(see Definition 2.7). We write $\mathcal{T}_f = \mathcal{T}_{\theta_f}(1)$ for simplicity when the choice of cone associated to θ_f is understood.

Proposition 4.18. *Let $\mathcal{T}_f$ be the McMullen cone for θ_f. The map*

$$\delta : \mathrm{Hom}(G;\mathbb{R}) \to \mathbb{R}$$

defined by

$$\delta(\alpha) = \log|\Theta^{(\alpha)}|,$$

extends to a homogeneous of degree -1, real analytic, convex function on $\mathcal{T}_f$ that goes to infinity toward the boundary of affine planar sections of $\mathcal{T}_f$. Furthermore, θ_f has a factor Θ with the properties:

(1) for all $\alpha \in \mathcal{T}_f$,

$$|\theta_f^{(\alpha)}| = |\Theta^{(\alpha)}|,$$

and

(2) minimality: if $\theta \in \mathbb{Z}G$ satisfies $|\theta^{(\alpha)}| = |\theta_f^{(\alpha)}|$ for all α ranging among the integer points of an open subcone of $\mathcal{T}_f$, then Θ divides θ.

To prove Proposition 4.18 we write $G = H \times \langle s \rangle$ and identify θ_f with the characteristic polynomial P_f of a Perron-Frobenius H-matrix M_f. Then Proposition 4.18 follows from Theorem 2.20.

Let

$$G = H_1(Y_f;\mathbb{Z})/\mathrm{torsion} = \Gamma^{\mathrm{ab}}/\mathrm{torsion},$$

and let H be the image of $\pi_1(\tau)$ in G induced by the composition

$$\tau \to \tau \times \{0\} \hookrightarrow \tau \times [0,1] \xrightarrow{q} Y_f.$$

Let $\rho_* : G \to \mathbb{Z}$ be the map corresponding to $\rho : Y_f \to S^1$.

Lemma 4.19. *The group G has decomposition as $G = H \times \langle s \rangle$, where $\rho_*(s) = 1$.*

Proof. The map ρ_* is onto $\mathbb{Z}$ and its kernel equals H. Take any $s \in \rho_*^{-1}(1)$. Then since $s \notin H$, and G/H is torsion free, we have $G = H \times \langle s \rangle$. □

We call s a *vertical generator* of G with respect to ρ, and identify $\mathbb{Z}G$ with the ring of Laurent polynomials $\mathbb{Z}H(u)$ in the variable u with coefficients in $\mathbb{Z}H$, by the map $\mathbb{Z}G \to \mathbb{Z}H(u)$ determined by sending $s \in \mathbb{Z}G$ to $u \in \mathbb{Z}H(u)$.

Definition 4.20. Given $\theta \in \mathbb{Z}G$, the *associated polynomial* $P_\theta(u)$ of θ is the image of θ in $\mathbb{Z}H(u)$ defined by the identification $\mathbb{Z}G = \mathbb{Z}H(u)$.

The definition of support for an associated polynomial P_θ is analogous to the one for θ.

Definition 4.21. The *support* of an element $P_\theta \in \mathbb{Z}H(u)$ is given by

$$\mathrm{Supp}(P_\theta) = \{hu^r \mid h, r \text{ are such that } (h, s^r) \in \mathrm{Supp}(\theta)\}.$$

Let $P_{\theta_f} \in ZH(u)$ be the polynomial associated to θ_f. Instead of realizing P_{θ_f} directly as a characteristic polynomial of an H-labeled digraph, we start with a more natural labeling of the digraph D_f.

Let $C_1 = \mathbb{Z}^{\mathcal{V}\cup\mathcal{E}}$ be the free abelian group generated by the positively oriented edges of Y_f, which we can also think of as 1-chains in $\mathfrak{C}^{(1)}$ (see Definition 4.16). Let $Z_1 \subseteq C_1$ be the subgroup corresponding to closed 1-chains. The map ρ induces a homomorphism $\rho_* : C_1 \to \mathbb{Z}$.

Let $\nu : Z_1 \to G$ be the quotient map. The map ν determines a ring homomorphism

$$\begin{array}{rcl} \nu_* : \mathbb{Z}Z_1 & \to & \mathbb{Z}G \\ \displaystyle\sum_{g\in Z_1} a_g g & \mapsto & \displaystyle\sum_{g\in J} a_g \nu(g). \end{array}$$

This extends to a map from $\mathbb{Z}Z_1[u]$ to $\mathbb{Z}G[u]$.

Let $K_1 \subseteq Z_1$ be the kernel of $\rho_*|_{Z_1} : Z_1 \to \mathbb{Z}$. Then H is the subgroup of G generated by $\nu(K_1)$. Let ν^H be the restriction of ν to K_1. Then ν^H similarly defines

$$\nu_*^H : \mathbb{Z}K_1 \to \mathbb{Z}H,$$

the restriction of ν_* to $\mathbb{Z}K_1$, and this extends to

$$\nu_*^H : \mathbb{Z}K_1[u] \to \mathbb{Z}H[u].$$

Proposition 4.22. *There is a Perron-Frobenius K_1-matrix $M_f^{K_1}$, whose characteristic polynomial $P_f^{K_1}(u) \in \mathbb{Z}K_1[u]$ satisfies*

$$P_{\theta_f}(u) = u^{-m}\nu_*^H P_f^{K_1}(u).$$

To construct $M_f^{K_1}$, we define a K_1-labeled digraph with underlying digraph D_f. Let s be a vertical generator relative to ρ_*. Choose any element $s' \in Z_1$ mapping to the vertical generator $s \in G$. Write each $s_e \in \mathcal{V}$ as $s_e = s'k_e$, where $k_e \in K_1$. Label edges of the digraph D_f by elements of C_1 as follows. Let $f(e) = e_1 \cdots e_r$. Then for each $i = 1, \dots, r$, there is a corresponding hinge κ_i whose initial cell corresponds to e and whose terminal cell corresponds to e_i. Take any edge η on D_f emanating from v_e. Then η corresponds to one of the hinges κ_i, and has initial vertex v_e and terminal vertex v_{e_i}. For such an η, define

$$\begin{aligned} g(\eta) &= s_e t_{e_1} \cdots t_{e_{i-1}} \\ &= s' k_e t_{e_1} \cdots t_{e_{i-1}} \\ &= s' k(\eta) \end{aligned}$$

where $k(\eta) \in K_1$. This defines a map from the edges D_f to C_1 giving a labeling $\mathcal{D}_f^{C_1}$. It also defines a map from edges of D_f to K_1 by $\eta \mapsto k(\eta)$. Denote this labeling of D_f by $D_f^{K_1}$.

Definition 4.23. Given a labeled digraph $\mathcal{D}^G$, with edge labels $g(\varepsilon)$ for each edge ε of the underlying digraph D, the *conjugate digraph* $\widehat{\mathcal{D}}^G$ of $\mathcal{D}^G$ is the digraph with same underlying graph D, and edge labels $g(\varepsilon)^{-1}$ for each edge ε of D.

Let $\widehat{\mathcal{D}}_f^{K_1}$ be the conjugate digraph of $\mathcal{D}_f^{K_1}$, and let $\widehat{M}_f^{K_1}$ be the directed adjacency matrix for $\widehat{\mathcal{D}}_f^{K_1}$.

Lemma 4.24. *The cycle function $\theta_f \in \mathbb{Z}G$ and the characteristic polynomial $\widehat{P}_f(u) \in \mathbb{Z}K_1[u]$ of $\widehat{M}^{K_1}$ satisfy*

$$\nu_*^H(\widehat{P}_f(u)) = u^m P_{\theta_f}(u).$$

Proof. By the coefficient theorem for labeled digraphs (Theorem 2.15) we have

$$\widehat{P}_f(u) = u^m \theta_{\mathcal{D}_f^{K_1}} = u^m(1 + \sum_{\sigma \in C_{D_f}} (-1)^{|\sigma|} k(\sigma)^{-1} u^{-\ell(\sigma)}).$$

Since $g(\sigma) = k(\sigma)s^{\ell(\sigma)}$, a comparison of $\widehat{P}_f$ with θ_f gives the desired result. ◻

Proof of Proposition 4.18. Let M_f be the matrix with entries in $\mathbb{Z}H$ given by taking $\widehat{M}^{K_1}$ and applying ν^H to its entries. Then the characteristic polynomial P_f of M_f is related to the characteristic polynomial $\widehat{P}_f$ of $\widehat{M}^{K_1}$ by

$$P_f(u) = \nu_*^H(\widehat{P}_f(u)).$$

Thus, Lemma 4.24 implies

$$P_f(u) = u^m P_{\theta_f}(u),$$

and hence the properties of Theorem 2.18 applied to $\widehat{P}_f$ also hold for θ_f. ◻

5. The folded mapping torus and its DKL-cone

We start this section by defining a folded mapping torus and stating some results of Dowdall-Kapovich-Leininger on deformations of free group automorphisms. We then proceed to finish the proof of the main theorem.

5.1. **Folding maps.** In [Sta83] Stallings introduced the notion of a folding decomposition of a train-track map.

Definition 5.1 (Stallings [Sta83])**.** Let τ be a topological graph, and v a vertex on τ. Let e_1, e_2 be two distinct edges of τ meeting at v, and let q_1 and q_2 be their other endpoints. Assume that q_1 and q_2 are distinct vertices of τ. The *fold* of τ at v, is the image τ_1 of a quotient map $\mathsf{f}_{(e_1,e_2:v)} : \tau \to \tau_1$ where q_1 and q_2 are identified as a single vertex in τ_1 and the two edges e_1 and e_2 are identified as a single edge in τ_1. The map $\mathsf{f}_{(e_1,e_2:v)}$ is called a *folding map*

It is not hard to check the following.

Lemma 5.2. *Folding maps on train-tracks are homotopy equivalences.*

Definition 5.3. A *folding decomposition* of a graph map $f\colon \tau \to \tau$ is a decomposition

$$f = hf_k \cdots f_1$$

where τ_0 is the graph τ with a finite number of subdivisions on the edges, $f_i\colon \tau_{i-1} \to \tau_i$ for $i = 1, \dots, k$ are folding maps, and $h\colon \tau_k \to \tau_k$ is a homeomorphism. We denote the folding decomposition by $(f_1, \dots, f_k; h)$.

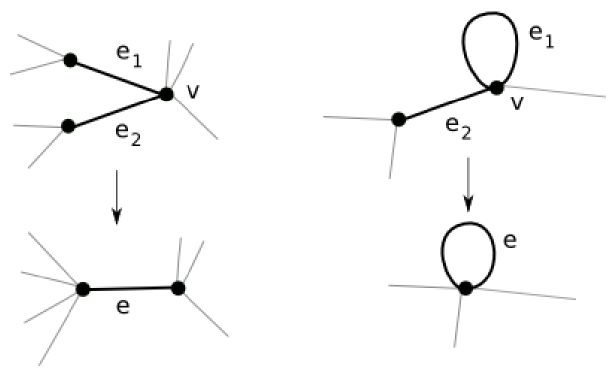

Figure 11. Two examples of folding maps.

Lemma 5.4 (Stallings [Sta83])**.** *Every homotopy equivalence of a graph to itself has a (non-unique) folding decomposition. Moreover, the homeomorphism at the end of the decomposition is uniquely determined.*

Decompositions of a train-track map into a composition of folding maps gives rise to a branched surface that is homotopy equivalent to Y_f.

Let $f : \tau \to \tau$ be a train-track map with a folding decomposition $\mathsf{f} = (f_1, \dots, f_k; h)$, where $f_i : \tau_{i-1} \to \tau_i$ is a folding map, for $i = 1, \dots, k$, $\tau = \tau_0 = \tau_k$, and $h : \tau \to \tau$ is a homeomorphism.

For each $i = 0, \dots, k$, define a 2-complex X_i and semiflow ψ_i as follows. Say f_i is the folding map on τ folding e_1 onto e_2 at their common endpoint v. Let q be the initial vertex of both e_1 and e_2, and q_i the terminal vertex of e_i. Let X_i be the quotient of $\tau_{i-1} \times [0, 1]$ obtained by identifying the triangles

$$\big[(q, 0), (q, 1), (q_1, 1)\big] \quad \text{on} \quad e_1 \times [0, 1]$$

with

$$\big[(q, 0), (q, 1), (q_2, 1)\big] \quad \text{on} \quad e_2 \times [0, 1].$$

The semi-flow ψ_i is defined by the second coordinate of $\tau_{i-1}\times[0,1]$. By the definitions, the image of $\tau_{i-1}\times\{1\}$ in X_i under the quotient map is τ_i.

Let X_{f} be the union of pieces $X_0\cup\cdots\cup X_k$ so that the image of $\tau_{i-1}\times\{1\}$ in X_{i-1} is attached to the image of $\tau_i\times\{0\}$ in X_i by their identifications with τ_i, and the image of $\tau_k\times\{1\}$ in X_k is attached to the image of $\tau_0\times\{0\}$ in X_0 by h.

Each X_i has a semiflow induced by its structure as the quotient of $\tau_i\times[0,1]$. This induces a semiflow ψ_{f} on X_{f}. The cellular structure on X_{f} is defined so that the 0–cells correspond to the images in X_i of $(q,0),(q,1),(q_1,1)$ and $(q_2,1)$. The transversal 1–cells of $\mathfrak{C}_{\mathsf{f}}$ correspond to the images in X_i of edges $[(q,0),(q_i,1)]$, for $i=1,2$. The vertical 1–cells of $\mathfrak{C}_{\mathsf{f}}$ are the forward flows of all the vertices of X_{f}. The vertical and transversal 1–cells form the boundaries of the 2–cells of $\mathfrak{C}_{\mathsf{f}}$.

Definition 5.5 (cf. [DKL13.1])**.** A *folded mapping torus* associated to a folding decomposition f of a train-track is the branched surface $(X_{\mathsf{f}},\mathfrak{C}_{\mathsf{f}},\psi_{\mathsf{f}})$ defined above.

Lemma 5.6. *If $(X_{\mathsf{f}},\mathfrak{C}_{\mathsf{f}},\psi_{\mathsf{f}})$ is a folded mapping torus, then there is a cellular decomposition of X_{f} so that the following holds:*

(i) The 1–skeleton $\mathfrak{C}_{\mathsf{f}}^{(1)}$ is a union of oriented 1–cells meeting only at their endpoints.
(ii) Each 1–cell has a distinguished orientation so that the corresponding tangent directions are either tangent to the flow (vertical case) or positive but skew to the flow (diagonal case).
(iii) The endpoint of any vertical 1–cell is the starting point of another vertical 1–cell.

Proof. The cellular decomposition of X_{f} has transversal 1–cells corresponding to the folds, and vertical 1–cells corresponding to the flow suspensions of the endpoints of the diagonal 1–cells. □

5.2. **Simple example.** We give a simple example of a train-track map, a folding decomposition and their associated branched surfaces.

Consider the train-track in Figure 12, and the train-track map corresponding to the free group automorphism $\phi\in\mathrm{Out}(F_2)$ defined by

$$\begin{array}{rcl} a & \mapsto & ba \\ b & \mapsto & bab \end{array}$$

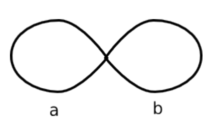

FIGURE 12. Two petal rose.

Then the corresponding train-track map $f\colon \tau \to \tau$ sends the edge a over b and a, and the edge b over b then a then b. The corresponding mapping torus is shown on the left of Figure 13. A folding decomposition is obtained from f by subdividing the

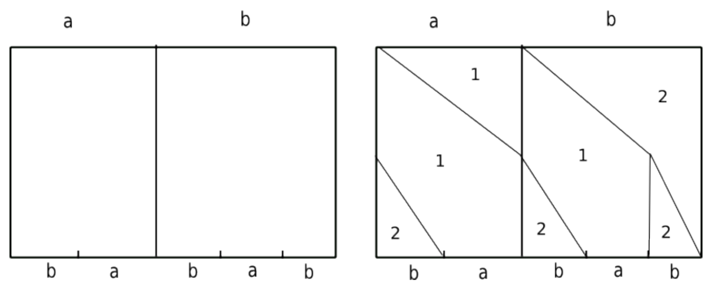

FIGURE 13. Mapping torus and folded mapping torus.

edge a twice and the edge b three times. The first fold identifies the entire edge a with two segments of the edge b. This yields a train-track that is homeomorphic to the original. The second fold identifies the edge b to one segment of the edge a. The resulting folded mapping torus is shown on the right of Figure 13. Here cells labeled with the same number are identified.

5.3. **Dowdall-Kapovich-Leininger's theorem.** Recall that elements $\alpha \in H^1(X_{\mathsf{f}};\mathbb{R})$ can be represented by cocycle classes $z\colon H_1(X_f;\mathbb{R}) \to \mathbb{R}$.

Definition 5.7. Given a branched surface $\mathsf{X} = (X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$, let $\mathcal{E}$ and $\mathcal{V}$ be the positively oriented transversal and vertical 1–cells in $\mathfrak{C}^{(1)}$. The associated *positive cone* for X in $H^1(X;\mathbb{R})$, denoted $\mathcal{A}_{\mathsf{f}}$, is given by

$$\mathcal{A}_{\mathsf{f}} = \left\{\alpha \in H^1(X_{\mathsf{f}};\mathbb{R}) \;\middle|\; \text{there is a } z \in \alpha \text{ so that } z(e) > 0 \text{ for all } e \in \mathcal{E} \cup \mathcal{V}\right\}.$$

Theorem 5.8 (Dowdall-Kapovich-Leininger [DKL13.1])**.** *Let f be an expanding irreducible train-track map, f a folding decomposition of f and $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$ the folded mapping torus associated to f. For every integral $\alpha \in A_{\mathsf{f}}$ there is a continuous map $\eta_\alpha\colon X_{\mathsf{f}} \to S^1$ with the following properties.*

(1) Identifying $\pi_1(X_{\mathsf{f}})$ with Γ and $\pi_1(S^1)$ with $\mathbb{Z}$, $(\eta_\alpha)_* = \alpha$,
(2) The restriction of η_α to a semiflow line is a local diffeomorphism. The restriction of η_α to a flow line in a 2–cell is a non-constant affine map.
(3) For all simple cycles c in X_{f} oriented positively with respect to the flow, $\ell(\eta_\alpha(c)) = \alpha([c])$ where $[c]$ is the image of c in G.
(4) Suppose $x_0 \in S^1$ is not the image of any vertex, denote $\tau_\alpha := \eta_\alpha^{-1}(x_0)$. If α is primitive τ_α is connected, and $\pi_1(\tau_\alpha) \cong \ker(\alpha)$.
(5) For every $p \in \tau_\alpha \cap (\mathfrak{C}_{\mathsf{f}})^{(1)}$, there is an $s \geq 0$ so that $\psi(p,s) \in (\mathfrak{C}_{\mathsf{f}})^{(0)}$.

(6) *The flow induces a map of first return* $f_\alpha\colon \tau_\alpha \to \tau_\alpha$*, which is an irreducible train-track map.*

(7) *The assignment that assoc»ciates to a primitive integral* $\alpha \in A_{\mathsf{f}}$ *the logarithm of the dilatation of* f_α *can be extended to a continuous and convex function on* A_{f}*.*

Proof. This is a compilation of results of [DKL13.1]. □

5.4. **The proof of main theorem.** In this section, we prove Theorem 1.6. A crucial step to our proof is that the mapping torus $\mathsf{Y} = (Y_f, \mathfrak{C}_f, \psi_f)$ and the folded mapping torus $\mathsf{X} = (X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$ both have the same cycle polynomial.

Proposition 5.9. *The cycle functions* θ_{Y} *of* $(Y_f, \mathfrak{C}_f, \psi_f)$ *and* θ_{X} *of* $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$ *coincide.*

Proof. We observe that $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$ can be obtained from the mapping torus of the train-track map $(Y_f, \mathfrak{C}_f, \psi_f)$ by a sequence of folds, vertical subdivisions and transversal subdivision, as defined in Sections 3.2 and 3.3. The reverse of these folds is shown in Figure 14.

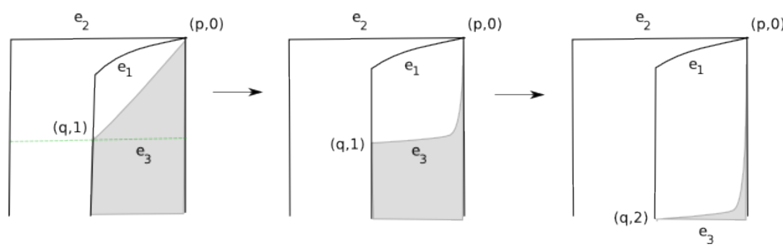

FIGURE 14. Vertical unfolding.

The proposition now follows from Proposition 3.7, Lemma 3.15 and Lemma 3.20, □

We also need to check that our theorems apply for vectors in the DLK-cone $\mathcal{A}_{\mathsf{f}}$.

Proposition 5.10. *Let* θ_{f} *be the cycle polynomial of the DKL mapping torus. Then*

$$\mathcal{A}_{\mathsf{f}} \subseteq \mathcal{T}_{\theta_{\mathsf{f}}}(1).$$

Proof. To see the second assertion, we need to show that, for every $\sigma \in \mathcal{C}_{X_{\mathsf{f}}}$ with $|\sigma| = 1$, we have $\alpha(g(\sigma)) > 0$. Then for all nontrivial $g \in \mathrm{Supp}(\theta_{\mathsf{f}})$, we have $\alpha(g) > 0$, and hence $\alpha \in \mathcal{T}_{\theta_{\mathsf{f}}}(1) = \mathcal{T}$. Let c be a closed loop in D. The embedding of D in X_{f} described in Def. 3.4 induces an orientation on the edges of D that is compatible with the flow ψ. For each edge μ of c, item (2) in Theorem 5.8 implies $\ell(\eta_\alpha(\mu)) > 0$ and tem (3) in Theorem 5.8 implies $\alpha([c]) = \ell(\eta_\alpha(c)) = \sum_{\mu \in c} \ell(\eta_\alpha(\mu)) > 0$. □

Proposition 5.11. *Let $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$ be the folded mapping torus, θ_{f} its cycle polynomial and $\mathcal{A}_{\mathsf{f}}$ the DKL-cone. For all primitive integral $\alpha \in A_{\mathsf{f}}$, we have*

$$\lambda(\phi_\alpha) = |\theta_{\mathsf{f}}^{(\alpha)}|$$

Proof. Embed τ_α in X_{f} transversally, and perform a vertical subdivision so that the intersections of τ_α with $(X_{\mathsf{f}})^{(1)}$ are contained in the 0–skeleton (we can do this by Theorem 5.8(5)). Perform transversal subdivisions to add the edges of τ_α to the 1-skeleton. Then perform a sequence of foldings and unfoldings to move the branching of the complex into τ_α, and remove the extra edges. Denote the new branched surface by $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}^{(\alpha)}, \psi_{\mathsf{f}})$. Let $f_\alpha \colon \tau_\alpha \to \tau_\alpha$ be the map induced by the first return map, and D_α its digraph. Then f_α defines a train-track map compatible with ϕ_α, and $\lambda(\phi_\alpha) = \lambda(D_\alpha)$.

The digraph $D_{\mathsf{f}}^{(\alpha)}$ of the new branched surface $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}^{(\alpha)}, \psi_{\mathsf{f}})$ is identical to D_α. Denote by θ_α the cycle function of $(X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}^{(\alpha)}, \psi_{\mathsf{f}})$, then by Lemma 3.15, Proposition 3.7 and Lemma 3.20,

$$\theta_\alpha = \theta_{\mathsf{f}}.$$

For every cycle c in D_α, let $\ell(c) = \#\tau_\alpha \cap (X_{\mathsf{f}})^{(1)} = \ell(\eta_\alpha(c)) = \alpha([c])$. Thus $\ell(\sigma) = \alpha(g(\sigma))$ for every $\sigma \in C_{D_\alpha}$. Let $P_\alpha(x)$ be the characteristic polynomial of the incidence matrix associated to D_α. By the coefficients theorem for digraphs (Theorem 2.6) we have:

$$\begin{aligned} P_\alpha(x) &= x^m + \sum_{\sigma \in C_D} (-1)^{|\sigma|} x^{m-\ell(\sigma)} \\ &= x^m \left(1 + \sum_{\sigma \in C_{D_\alpha}} (-1)^{|\sigma|} x^{\alpha(g(\sigma))} \right) \\ &= x^m (\Theta')^{(\alpha)} \end{aligned}$$

Therefore,

$$\lambda(\phi_\alpha) = |P_\alpha| = |\theta_\alpha^{(\alpha)}| = |\theta_{\mathsf{f}}^{(\alpha)}| \qquad \square$$

We are now ready to prove our main result.

Proof of Theorem 1.6. Choose a train-track representative f of ϕ, and a folding decomposition f of f. As before, let $\mathsf{Y} = (Y_f, \mathfrak{C}_f, \psi_f)$ be the mapping torus of f, and $\mathsf{X} = (X_{\mathsf{f}}, \mathfrak{C}_{\mathsf{f}}, \psi_{\mathsf{f}})$ the folded mapping torus. By Proposition 5.9 their cycle function $\theta_{\mathsf{Y}}, \theta_{\mathsf{X}}$ are equal, and we will call them θ.

Let Θ be the minimal factor of θ defined in Proposition 4.18, and let $\mathcal{T} = \mathcal{T}_\Theta(1)$ be the McMullen cone. By Proposition 5.10, $A_{\mathsf{f}} \subseteq \mathcal{T}$, and by Proposition 5.11, $\lambda(\phi_\alpha) = |\Theta^{(\alpha)}|$. By Proposition 4.18, $|\Theta_\phi^{(\alpha)}| = |\Theta^{(\alpha)}|$ in $\mathcal{T}$ so we have $\lambda(\alpha) = |\Theta_\phi^{(\alpha)}|$

for all $\alpha \in A_{\mathsf{f}}$. Item (2) of Proposition 4.18 implies part (2) of Theorem 1.6. If f' is another folding decomposition of another train-track representative f' of ϕ, we get another distinguished factor $\Theta_{f'}$. Since the cones $\mathcal{T}_{\mathsf{f}}$ and $\mathcal{T}_{\mathsf{f}'}$ must intersect, it follows by the minimality properties of Θ_{f} and $\Theta_{f'}$ in Proposition 4.18 that they are equal. Item (3) of Proposition 4.18 completes the proof. □

6. Example

Consider the rose with four directed edges a, b, c, d and the map:

$$f = \begin{cases} a \to & B \to & adb \\ c \to & D \to & cbd. \end{cases}$$

Capital letters indicate the relavent edge in the opposite orientation to the chosen one. It is well known (e.g. Proposition 2.6 in [AR]) that if $f : \tau \to \tau$ is a graph map, and τ is a graph with $2m$ directed edges, and for every edge e of τ, the path $f^{2m}(e)$ does not contain backtracking then f is a train-track map. Checking f^8 one verifies that f is a train-track map.

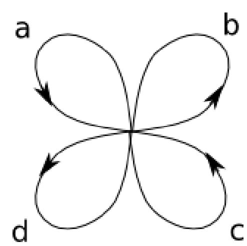

Figure 15. Four petal rose with directed edges.

The train-track transition matrix is given by

$$M_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The associated digraph is shown in figure 16.

The matrix M_f is non-negative and M_f^3 is positive. Thus M_f is a Perron-Frobenius matrix and f is a PF train-track map, hence an expanding irreducible train-track map (see Section 4.1 for definitions). By Theorem 1.1, α_ϕ has an open cone neighborhood, the DKL cone $\mathcal{A}_{\mathsf{f}} \subset \mathrm{Hom}(\Gamma; \mathbb{R})$ such that the primitive integral elements of $\mathcal{A}_{\mathsf{f}}$ correspond to free group automorphisms that can be represented by expanding irreducible train-track maps.

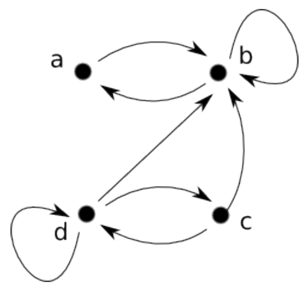

FIGURE 16. Digraph associated to the train-track map f.

Remark 6.1. The outer automorphism ϕ represented by f is reducible. Consider the free factor $\langle bA, ad, ac\rangle$ then

$$f(bA) = BDAb, \quad f(ad) = BDBC = aBDAaBCA, \quad f(ac) = BD = bAad.$$

Therefore this factor is invariant up to conjugacy. Thus ϕ is reducible, but f is a PF train-track map, and hence it is expanding and irreducible. Thus we can apply both Theorem 5.8 and Theorem 1.6.

Identifying the fundamental group of the rose with F_4 we choose the basis a, b, c, d of F_4. The free-by-cyclic group corresponding to $[f_*]$ has the presentation:

$$\Gamma = \langle a, b, c, d, s' \mid a^{s'} = B, b^{s'} = BDA, c^{s'} = D, d^{s'} = DBC\rangle.$$

Let $G = \Gamma^{\mathrm{ab}}$ and for $w \in \Gamma$ we denote by $[w]$ its image in G. Then

$$[a] = -[b] = [d] = -[c].$$

Thus $G = \mathbb{Z}^2 = \langle t, s\rangle$ where $t = [a]$ and $s = [s']$. We decompose f into four folds

$$\tau = \tau_0 \xrightarrow{f_1} \tau_1 \xrightarrow{f_2} \tau_2 \xrightarrow{f_3} \tau_3 \xrightarrow{f_4} \tau_4 \cong \tau,$$

where all the graphs τ_i are roses with four petals. f_1 folds all of a with the first third of b, to the edge a_1 of τ_1, the other edges will be denoted b_1, c_1, d_1. f_1 folds the edge c_1 with the first third of the edge d_1. With the same notation scheme, f_2 folds the edge c_2 with half of the edge b_2 and f_3 folds the edge a_3 with half of the edge d_3. Figure 17 shows the folded mapping torus $X_{\mathfrak{f}}$ for this folding sequence.

The cell structure $\mathfrak{C}_{\mathfrak{f}}$ has 4 vertices, 8 edges: $s_1, s_2, s_3, s_4, x, y, z, w$, and four 2-cells: c_x, c_y, c_z, c_w. The 2-cells are sketched in Figure 18.

Let C_1 be the free abelian group generated by the edges of $X_{\mathfrak{f}}$, and let F be the maximal tree consisting of the edges s_1, s_2, s_3, then $Z_1 \subset C_1$ is generated by x, y, z, w and $s_1 + s_2 + s_3 + s_4$. The quotient homomorphism $\nu\colon Z_1 \to G$ is given by collapsing the maximal tree and considering the relations given by the two cells. The map is

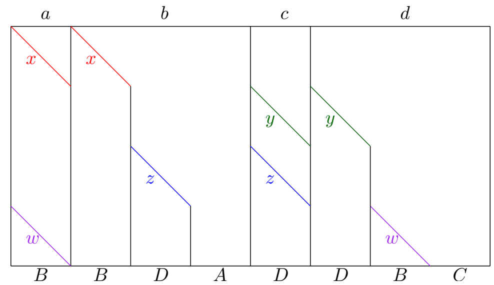

FIGURE 17. The complex X

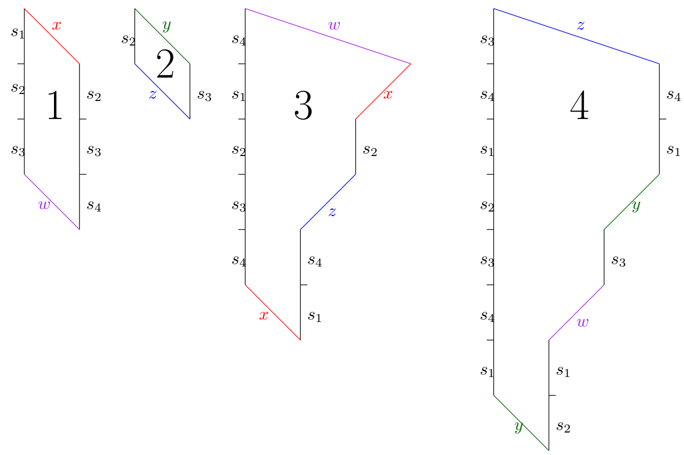

FIGURE 18. The discs in X

given by $\nu(s_1 + s_2 + s_3 + s_4) = s$ and

$$\nu(x) = t \quad \nu(y) = \nu(z) = -t \quad \nu(w) = t + s.$$

The dual digraph D to X is shown on the left of Figure 19. There are five cycles: ω_{13} and ω'_{13} the two distinct cycles containing 1 and 3, ω_{24} and ω'_{24} the two distinct cycles containing 2 and 4, and ω_{34} is the cycle containing 3 and 4. The cycle complex is shown on the right of figure 19.

$$\begin{aligned} \theta_f &= 1 - (s^{-2} + s^{-1}t^{-1} + s^{-2} + s^{-1}t + s^{-2}) + (s^{-3}t + s^{-2} + s^{-3}t^{-1} + s^{-4}) \\ &= 1 + s^{-4} - 2s^{-2} - s^{-1}t^{-1} - s^{-1}t + s^{-3}t + s^{-3}t^{-1} \end{aligned}$$

Note that Θ_ϕ might be a proper factor of this polynomial. However, for the sake of computing the support cone (and the dilatations of ϕ_α for different $\alpha \in \mathcal{A}_{\mathsf{f}}$) we may use θ_f.

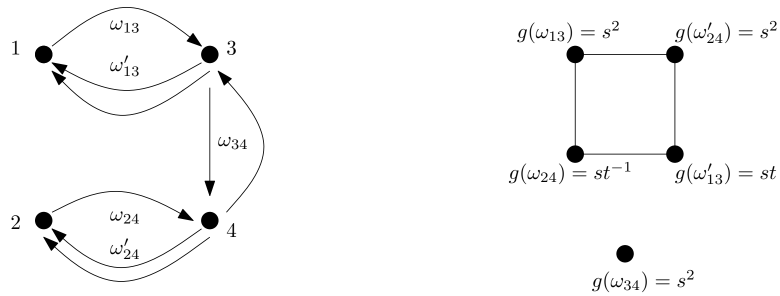

FIGURE 19. The vertical diagraph is on the left and the labeled cycle complex on the right.

Computing the McMullen cone: In order to simplify notation, for $\alpha \in \mathrm{Hom}(G, \mathbb{R})$ and $g \in G$ we denote $g^\alpha = \alpha(g)$. The cone $\mathcal{T}_\phi$ in $H^1(G, \mathbb{R})$ is given by

$$\begin{aligned} \mathcal{T}_\phi &= \{\alpha \in \mathrm{Hom}(G,\mathbb{R}) \mid g^\alpha < 0^\alpha \text{ for all } g \in \mathrm{Supp}(\theta_f)\} \\ &= \left\{ \alpha \in \mathrm{Hom}(G,\mathbb{R}) \;\middle|\; \begin{array}{l} (-4s)^\alpha, (-2s)^\alpha, (-s-t)^\alpha < 0 \\ (-s+t)^\alpha, (-3s+t)^\alpha, (-3s-t)^\alpha < 0 \end{array} \right\}. \end{aligned}$$

Therefore, the McMullen cone is

$$\mathcal{T}_\phi = \{\alpha \in \mathrm{Hom}(G,\mathbb{R}) \mid s^\alpha > 0 \quad \text{and} \quad |t^\alpha| < s^\alpha\}. \tag{9}$$

Computing the DKL cone: We now compute the DKL cone $\mathcal{A}_{\mathsf{f}}$. A cocycle a

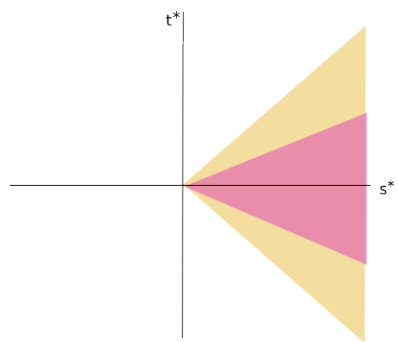

FIGURE 20. The McMullen cone $\mathcal{T}$ (outer) and DKL cone $\mathcal{A}_{\mathsf{f}}$ (inner).

represents an element in $\alpha \in \mathcal{A}_{\mathsf{f}}$ if it evaluates positively on all edges in X_{f}. We use the notation: $\mathsf{a}(e) = e^{\mathsf{a}}$. Thus for a a positive cocycle: we have

$$s_1^{\mathsf{a}}, s_2^{\mathsf{a}}, s_3^{\mathsf{a}} > 0$$

and

$$s_4^{\mathsf{a}} > 0 \implies s^{\mathsf{a}} - s_1^{\mathsf{a}} + s_2^{\mathsf{a}} + s_3^{\mathsf{a}} > 0 \implies s^{\mathsf{a}} > s_1^{\mathsf{a}} + s_2^{\mathsf{a}} + s_3^{\mathsf{a}} > 0.$$

Now by considering the cell structure given by all edges in Figure 20 and recalling that $[a] = [d] = t$ and $[b] = [c] = -t$ we have:

$$x = t + s_1 \quad w = t + s_4 \quad y = s_2 - t \quad z = s_3 - t.$$

The diagonal edges x, w give us:

$$0 < x^{\mathsf{a}} = t^{\mathsf{a}} + s_1^{\mathsf{a}} \text{ and } 0 < w^{\mathsf{a}} = t^{\mathsf{a}} + s_4^{\mathsf{a}},$$

so

$$t^\alpha > -\frac{s_1^{\mathsf{a}} + s_4^{\mathsf{a}}}{2} > -\frac{s^\alpha}{2}.$$

The other diagonal edges give us

$$0 < z^{\mathsf{a}} = s_3^{\mathsf{a}} - t^{\mathsf{a}} \text{ and } 0 < y^{\mathsf{a}} = s_2^{\mathsf{a}} - t^{\mathsf{a}},$$

hence

$$t^\alpha < \frac{s_2^{\mathsf{a}} + s_3^{\mathsf{a}}}{2} < \frac{s^\alpha}{2}.$$

We obtain the cone:

$$\left\{ s^\alpha > 0 \ \text{ and } \ |t^\alpha| < \frac{s^\alpha}{2} \right\}. \tag{10}$$

It is not hard to see that if α is in this cone we may find a positive cocycle a representing α. Therefore $\mathcal{A}_{\mathsf{f}}$ is equal to the cone in (10) and is strictly contained in the cone $\mathcal{T}_\phi$ (see (9) and Figure 20).

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